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# Continuous Fictitious Play in Zero-Sum Games

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## Abstract

Robinson (1951) showed that the learning process of Discrete Fictitious Play converges to Nash equilibrium in two-player zero-sum games for any initial condition. In several earlier works, Brown (1949, 1951) makes some heuristic arguments for a similar convergence result for the case of Continuous Fictitious Play (CFP). The standard reference for a formal proof is Harris (1998); his argument requires several technical lemmas, and moreover, involves the advanced machinery of Lyapunov functions. In this note we present a simple alternative proof. In particular, we show that Brown's convergence result follows easily from a result obtained by Monderer et al. (1997).

**JEL-Classification:** C72, D83

**Keywords:** Fictitious play; Zero-sum games.

## 1 Introduction

Fictitious Play is a procedure in which at each instance, players of a game myopically play their best replies against the opponents' past play. The origins of Fictitious Play lie in a series of papers by Brown (1949, 1951) and Robinson (1951). While today, the algorithm is usually interpreted as a myopic learning process, the initial objective of these studies was to develop an easy method to find the value of zero-sum games. In his 1949 article, Brown already argued heuristically that the continuous version of the procedure – Continuous Fictitious Play (CFP) – must converge at a linear speed to the set of Nash equilibria, and thus, the value of the game. This result is again mentioned – without proof – in Brown (1951). Brown's to a large extent heuristic approach towards CFP may be explained by the fact that he was primarily interested in a discrete algorithm, and thus, only considered convergence of CFP in zero-sum games

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as support for his conjecture that Discrete Fictitious Play (DFP) in zero-sum games converges to equilibrium.

The proof of this conjecture was obtained by Robinson (1951) in the early fifties. However, it was not until Hofbauer (1994) and Harris (1998) that the result for CFP was rigorously proven. Today, Harris' proof has become the standard reference. Essentially, he shows that the sum of players' instantaneous improvement steps in a zero-sum game, is a Lyapunov function. While this argument appears to be quite simple at first glance, it requires several rather technical lemmas<sup>1</sup>. Moreover, it makes use of non-trivial results on dynamical systems. Different versions of Harris' proof – often in a simplified form – have appeared in the literature. See a.o. Krishna and Sjöström (1997), Shamma and Arslan (2004), and Hofbauer and Sorin (2006).

In this note we provide a simple alternative argument. We show that the convergence of CFP in two-player zero-sum games follows trivially from a result obtained by Monderer et al. (1997) that says that for CFP in any game, each player's instantaneous expected payoff will in the long run coincide with the average payoff that player has realized so far. Using the same approach, convergence of DFP is established up to the condition of infrequent switching (Fudenberg and Levine, 1994; Monderer et al. 1997). That is, if for DFP in a zero-sum game players play each pure strategy profile for increasingly long periods of time, then the DFP converges to equilibrium. Simulations lead us to conjecture that infrequent switching is the engine behind Robinson's result<sup>2</sup>. Whether DFP in zero-sum games satisfies the condition of infrequent switching in general, remains an open question.

## 2 Preliminaries

### 2.1 Zero-sum Games

A zero-sum game is represented as an  $m \times n$  matrix  $A$ . There are two players – 1 and 2. Player 1 [2] has a pure strategy set  $I$  [ $J$ ] of cardinality  $m$  [ $n$ ]. If player 1 plays strategy  $i$  and 2 plays  $j$ , then player 1 receives a payoff of  $a_{ij}$  and 2 of  $-a_{ij}$ . Let  $\Delta^m$  and  $\Delta^n$  be the sets of probability distributions over sets of respectively  $m$  and  $n$  alternatives;  $\Delta^m$  and  $\Delta^n$  are players' respective sets of mixed strategies over  $I$  and  $J$ . A pure strategy  $i \in I$  is also denoted as the unit vector  $e^i \in \Delta^m$ ; we write  $\tilde{I} := \{e^i \mid i \in I\}$ . Analogously, we denote player 2's  $j$ -th pure strategy also as the  $j$ -th unit vector  $f^j \in \Delta^n$ , and  $\tilde{J} := \{f^j \mid j \in J\}$ . Let  $\beta_1 : \Delta^n \rightarrow \Delta^m$  be player 1's *best-reply correspondence*. That is,

$$\beta_1(\hat{q}) := \{p \in \Delta^m \mid pA\hat{q} \geq p'A\hat{q} \text{ for all } p' \in \Delta^m\}.$$

Player 2's best-reply correspondence,  $\beta_2 : \Delta^m \rightarrow \Delta^n$  is defined as

$$\beta_2(\hat{p}) := \{q \in \Delta^n \mid \hat{p}Aq \leq \hat{p}Aq' \text{ for all } q' \in \Delta^n\}.$$

<sup>1</sup>See also Hofbauer and Sorin (2006).

<sup>2</sup>Note that convergence of CFP implies convergence of DFP for a class of games that includes zero-sum games. See Harris (1998) and Hofbauer and Sorin (2006).

Define  $\beta : \Delta^m \times \Delta^n \rightarrow \Delta^m \times \Delta^n$  as  $\beta(p, q) := (\beta_1(q), \beta_2(p))$ . The set of Nash equilibria of a game  $A$  is defined as

$$NE(A) := \{(p, q) \in \Delta^m \times \Delta^n \mid (p, q) \in \beta(p, q)\}.$$

That is,  $NE(A)$  is the set of fixed points of the best-reply correspondence  $\beta$ . Define the function  $W : \Delta^m \times \Delta^n \rightarrow \mathbb{R}$  as

$$W(\hat{p}, \hat{q}) := H(\hat{q}) - L(\hat{p}),$$

where  $H(\hat{q}) := \max_{p \in \Delta^m} pA\hat{q}$  and  $L(\hat{p}) := \min_{q \in \Delta^n} \hat{p}Aq$ .

**Theorem 2.1**  $W(p, q) \geq 0$ , with equality if and only if  $(p, q) \in NE(A)$ .

This is a classic result that follows from von Neumann's (1928) maximin theorem. For an exact proof, see a.o. Myerson (1991).

## 2.2 Continuous Fictitious Play

A Continuous Fictitious Play (CFP) in  $A$  is a pair  $(x(t), y(t))$  of Lebesgue measurable functions  $x : [0, \infty) \rightarrow \tilde{I}$  and  $y : [0, \infty) \rightarrow \tilde{J}$ , such that for almost all  $t \geq 1$ , we have

$$(x(t), y(t)) \in \beta \left( \frac{1}{t} \int_0^t x(\tau) d\tau, \frac{1}{t} \int_0^t y(\tau) d\tau \right).$$

Players' *beliefs*  $(p(t), q(t))$  are defined as functions

$$p(t) := \frac{1}{t} \int_0^t x(\tau) d\tau, \text{ and } q(t) := \frac{1}{t} \int_0^t y(\tau) d\tau$$

on the domain  $[1, \infty)$ . Furthermore, we define the set  $\Omega$  of limit points of  $(p(t), q(t))$  as

$$\Omega := \bigcap_{T \geq 1} \text{cl}(\{(p(t), q(t)) \mid t \geq T\}),$$

where  $\text{cl}$  denotes the topological closure. We say  $(p, q)$  converges to Nash equilibrium if  $\Omega \subseteq NE(A, B)$ . Expected and average payoffs are defined for  $t \in [1, \infty)$  as

$$E_1(t) := x(t)Aq(t), \text{ and } E_2(t) := -p(t)Ay(t),$$

and

$$P_1(t) := \frac{1}{t} \int_0^t x(\tau)Ay(\tau) d\tau, \text{ and } P_2(t) := -P_1(t).$$

Monderer et al. (1997) introduce the concept of belief affirmation; a CFP is said to be *belief affirming* if

$$\lim_{t \rightarrow \infty} (E_i(t) - P_i(t)) = 0$$

for  $i = 1, 2$ . In a belief affirming process players' expected payoffs and average realized payoffs coincide in the long run.

### 3 CFP Convergence in Zero-Sum Games

The aim of this note is to prove the following.

**Theorem 3.1** *CFP in zero-sum games converges to Nash equilibrium.*

The proof is divided into three simple lemmas. The first is due to Monderer et al. (1997). For completeness, we include the proof.

**Lemma 3.2 (Monderer et al. 1997)** *Every CFP is belief affirming.*

**Proof.** Each function  $f_i : [1, \infty) \rightarrow \mathbb{R}$ ,  $i \in I$ , defined as

$$f_i(t) := \int_0^t e^i Ay(\tau) d\tau,$$

is continuous on  $[1, \infty)$ . Since  $tE_1(t) = \max\{f_i(t) \mid i \in I\}$  for all  $t \geq 1$ , also  $tE_1(t)$  is continuous on  $[1, \infty)$ . The function  $tP_1(t)$  is also continuous on  $[1, \infty)$ ; it follows that  $tE_1(t) - tP_1(t)$  is continuous on  $[1, \infty)$ .

Suppose  $(x(\cdot), y(\cdot))$  is continuous at  $t$ . Since  $x(\cdot)$  and  $y(\cdot)$  are by definition pure strategies, there is an  $\varepsilon > 0$  such that  $x(\tau)$  and  $y(\tau)$  are constant for all  $\tau \in (t - \varepsilon, t + \varepsilon)$ . This implies that the derivatives of  $tE_1(t)$  and  $tP_1(t)$  at  $t$  are both equal to  $x(t)Ay(t)$ . It follows that  $tE_1(t) - tP_1(t)$  is constant on each interval on which  $(x(\cdot), y(\cdot))$  is continuous. Continuity of  $tE_1(t) - tP_1(t)$  on  $[1, \infty)$  then implies

$$tE_1(t) - tP_1(t) = K,$$

for all  $t \in [1, \infty)$ , where  $K := E_1(1) - P_1(1)$ . Thus,

$$\lim_{t \rightarrow \infty} (E_1(t) - P_1(t)) = \lim_{t \rightarrow \infty} \frac{1}{t} (tE_1(t) - tP_1(t)) = \lim_{t \rightarrow \infty} \frac{K}{t} = 0. \quad (1)$$

A similar argument applies to player 2. ■

**Lemma 3.3**  $\lim_{t \rightarrow \infty} W(p(t), q(t)) = 0$ .

**Proof.** It follows from Lemma 3.2 that CFP is belief affirming. This implies

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} (E_1(t) - P_1(t)) + \lim_{t \rightarrow \infty} (E_2(t) - P_2(t)) \\ &= \lim_{t \rightarrow \infty} (E_1(t) - P_1(t) + E_2(t) - P_2(t)) \\ &= \lim_{t \rightarrow \infty} (E_1(t) + E_2(t)) \\ &= \lim_{t \rightarrow \infty} W(p(t), q(t)), \end{aligned}$$

where the third equality follows from  $P_1(t) = -P_2(t)$  for all  $t$ , and the fourth from  $W(p(t), q(t)) = E_1(t) + E_2(t)$  for all  $t$ . ■

**Proof of Theorem 3.1** Take some  $(p^*, q^*) \in \Omega$  and a sequence of times  $(t_k)_{k=1}^\infty$  such that  $\lim_{k \rightarrow \infty} (p(t_k), q(t_k)) = (p^*, q^*)$ . Note that the existence of such a sequence is implied by  $(p^*, q^*) \in \Omega$ . Since  $W$  is continuous in  $(p, q)$  we have

$$\lim_{k \rightarrow \infty} W(p(t_k), q(t_k)) = W(p^*, q^*).$$

By Lemma 3.3 it follows that  $W(p^*, q^*) = 0$ , which by Theorem 2.1 implies  $(p^*, q^*) \in NE(A)$ . Hence,  $\Omega \subseteq NE(A)$ . ■

**Remark 3.5** From Equation (1) it follows that  $(E_1(t) - P_1(t))$  converges at rate  $\frac{1}{t}$ ; this also holds for player 2. It follows that  $(p(t), q(t))$  converges to equilibrium at a rate of  $\frac{1}{t}$ , confirming Harris' (1998) result.

**Remark 3.6** Our proof of Theorem 3.1 seems to be closer to Brown's heuristic argument than the one usually presented in the literature. For instance, Brown (1951, pp. 375) says:<sup>3</sup> "*In the system of differential equations the convergence rests on the fact that  $t\bar{V}(t)$  and  $t\underline{V}(t)$  maintain a constant difference between them.*"

## 4 DFP in Zero-Sum Games

A Discrete Fictitious Play (DFP) in  $A$  is a sequence  $(x(t), y(t))$  in  $\tilde{I} \times \tilde{J}$ ,  $t \in \mathbb{N}$ , with  $(x(0), y(0)) \in \tilde{I} \times \tilde{J}$  and for all  $t \in \mathbb{N} \setminus \{0\}$ ,

$$(x(t), y(t)) \in \beta \left( \frac{1}{t} \sum_{\tau=0}^{t-1} (x(\tau), y(\tau)) \right).$$

The sequence of beliefs  $(p(t), q(t)) : \mathbb{N} \setminus \{0\} \rightarrow \Delta^m \times \Delta^n$ , is given by

$$\frac{1}{t} \sum_{\tau=0}^{t-1} (x(\tau), y(\tau)).$$

Like before, we define  $E_1(t) := x(t)Aq(t)$ ,  $E_2(t) := -p(t)Ay(t)$ ,

$$P_1(t) := \frac{1}{t} \sum_{\tau=0}^{t-1} x(\tau)Ay(\tau), \text{ and } P_2(t) := -P_1(t).$$

The following result was proven by Robinson (1951).

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<sup>3</sup>Here,  $\bar{V}(t) \equiv E_1(t)$  and  $\underline{V}(t) \equiv -E_2(t)$ .

**Theorem 4.1 (Robinson 1951)** *DFP in zero-sum games converges to Nash equilibrium.*

A DFP satisfies the condition of *infrequent switching*<sup>4</sup> if

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t M_i(\tau) = 0,$$

for  $i = 1, 2$ , where

$$M_1(t) := \begin{cases} 1 & \text{if } x(t) \neq x(t-1) \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad M_2(t) := \begin{cases} 1 & \text{if } y(t) \neq y(t-1) \\ 0 & \text{otherwise.} \end{cases}$$

for all  $t \in \mathbb{N} \setminus \{0\}$ . Monderer et al. (1997), and Fudenberg and Levine (1994) established the following result.

**Lemma 4.2** *Every DFP satisfying infrequent switching is belief affirming.*

Hence, if the DFP process satisfies infrequent switching, then its convergence in zero-sum games follows along the lines of the previous section. Whether DFP in zero-sum games satisfies infrequent switching remains an open question.

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<sup>4</sup>Monderer et al. (1997) call DFP's that satisfy infrequent switching “smooth” Fictitious Plays.

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