View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Research Papers in Economics

THE CORRELATION BETWEEN THE CONVERGENCE OF SUBDIVISION PROCESSES AND SOLVABILITY OF REFINEMENT EQUATIONS

VLADIMIR PROTASSOV

Econometric Institute, Erasmus University Rotterdam [1]

December 12, 2001

ECONOMETRIC INSTITUTE REPORT EI-2001-45

ABSTRACT. We consider the univariate two-scale refinement equation $\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x-k)$, where $c_0, \cdots, c_N$ are complex values and $\sum c_k = 2$.

This paper analysis the correlation between the existence of smooth compactly supported solutions of this equation and the convergence of the corresponding cascade algorithm/subdivision scheme. In the work [P2] we have introduced a criterion that expresses this correlation in terms of mask of the equation. It was shown that the convergence of subdivision scheme depends on values that the mask takes at the points of its *generalized cycles*. In this paper we show that the criterion is sharp in the sense that an arbitrary generalized cycle causes the divergence of a suitable subdivision scheme. To do this we construct a general method to produce divergent subdivision schemes having smooth refinable functions. The criterion therefore establishes a complete classification of divergent subdivision schemes.

Key words. refinement equations, cascade algorithm, subdivision process, rate of convergence, cycles.

AMS subject classification. 26C10, 39B32, 42A05, 42A38

I. Introduction.

Refinement equations have been studied by many authors in great detail in connection with their role in the theory of wavelets and of subdivision schemes in approximation theory and design of curves and surfaces (see References). In this paper we study a criterion of convergence of subdivision processes having smooth refinable functions. This criterion was presented in the work [P2]. In particular we

[1]Econometric Institute, Erasmus University Rotterdam, H 11-21, Postbus 1738, 3000 DR Rotterdam, The Netherlands, e-mail: protassov@few.eur.nl

show that the criterion is sharp in the sense that each if its cases is realized. To do this we provide a general procedure for constructing divergent subdivision schemes (or cascade algorithms) corresponding to smooth refinable functions.

We restrict ourselves to univariate equations with compactly supported mask. Throughout the paper we denote by $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the unit circle, by $\mathcal{H}$ the space of entire functions on $\mathbb{C}$, by $\mathcal{C}^l$ the space of l times continuously differentiable functions on $\mathbb{R}$, by $\mathcal{C}^0 = \mathcal{C}$ the space of continuous functions, by $\mathcal{C}_0^l$ the space of compactly supported functions from $\mathcal{C}^l$, and by $\mathcal{C}_0$ the space of compactly supported continuous functions on $\mathbb{R}$. A sequence $\{f_k\}$ converges to zero in $\mathcal{C}_0^l$ if it converges to zero in $\mathcal{C}^l$ and the supports of $f_k,\ k \in \mathbb{N}$ are uniformly bounded.

Consider a refinement equation

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x-k), \tag{1}$$

where $c_k \in \mathbb{C},\ \sum_k c_k = 2$. The trigonometric polynomial $m(\xi) = \frac{1}{2}\sum_{k=0}^{N} c_k e^{-ik\xi}$ is the *mask* of this equation. It is well known that a $\mathcal{C}_0$-solution of this equation (*refinable function*), if it exists at all, is unique up to normalization and has its support on the segment $[0, N]$. For a given mask m we denote by $[m]$ the corresponding refinement equation. Let us also define the following subspaces of the space $\mathcal{C}_0$:

$$\mathcal{M}^l = \{f \in \mathcal{C}_0 \mid \widehat{f}(\xi)(1-e^{-i\xi})^{-l-1} \in \mathcal{H}\}, \qquad \mathcal{L}^l = \{f \in \mathcal{C}_0^l \mid \widehat{f^{(l)}} \in \mathcal{M}^l\},\ l \geq 0.$$

In other words the Fourier transform of a function from $\mathcal{M}^l$ has zeros of order $\geq l+1$ at all the points $2\pi k,\ k \in \mathbb{Z}$. The Fourier transform of a function from $\mathcal{L}^l$ has zero at the point $\xi = 0$ and has zeros of order $\geq l+1$ at all the points $2\pi k,\ k \in \mathbb{Z} \setminus \{0\}$.
Let us also denote $\mathcal{L} = \mathcal{L}^0 = \mathcal{M}^0$.

The cascade algorithm for refinement equations is the construction of the sequence $f_n = Tf_{n-1}$ for some initial function $f_0 \in \mathcal{C}_0$, where $Tf(x) = \sum_k c_k f(2x-k)$ is the *subdivision operator* associated to equation (1). This operator is defined on the space $\mathcal{C}_0$ and preserves all the subspaces $\mathcal{C}^l$, $\mathcal{L}^l$. If f_n converges in the space $\mathcal{C}_0^l$ to a function $\varphi \in \mathcal{C}_0^l$ $(l \geq 0)$, then obviously it converges in $\mathcal{C}_0^l$ and φ is the solution of (1). Moreover, in that case the function $g = f_0 - \varphi$ necessarily belongs to $\mathcal{L}^l$ (see [CDM], [Du]). Thus we say that the cascade algorithm converges in $\mathcal{C}^l$ if $T^n g \to 0,\ n \to \infty$ for any $g \in \mathcal{L}^l$. Properties of the cascade algorithms have been studied by many authors in various contexts. This algorithm gives a simple way for approximation of refinable functions and wavelets. On the other hand the convergence of the cascade algorithm is equivalent to the convergence of the corresponding subdivision scheme ([DL2]). For a given mask $m(\xi)$ we say that the *subdivision process* $\{m\}$ *converges in* $\mathcal{C}^l$ if the corresponding cascade algorithm or the corresponding subdivision scheme converges in that space.

It is clear that if a subdivision process converges in $\mathcal{C}^l$, then the corresponding refinement equation has a $\mathcal{C}_0^l$-solution. In general the converse is not true, corresponding examples are well-known (see [CDM], [CH], [W], [RS] for general discussions of this aspect). A natural question arises under which extra conditions the solvability of a refinement equation implies the convergence of the subdivision process ?

1) A necessary condition (first introduced in [DGL]):

If a subdivision process $\{m\}$ converges in $\mathcal{C}^l$, then its mask can be factored as

$$m(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{l+1} a(\xi) \tag{2}$$

for some trigonometric polynomial $a(\xi)$. In particular the condition

$$m(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big) a(\xi) \quad \Leftrightarrow \sum_k c_{2k} = \sum_k c_{2k+1} = 1 \tag{3}$$

is necessary for the convergence of the subdivision process in $\mathcal{C}$. Let us remember that for the *existence* of smooth solutions of refinement equation this condition is not necessary (there is a weaker condition for this, see [P1]).

For a given mask m denote by $\mathbf{l}(m)$ the maximal integer l such that condition (2) is satisfied. So if a subdivision process $\{m\}$ converges in $\mathcal{C}^k$, then $k \le \mathbf{l}(m)$.

2) A sufficient condition (introduced in [CDM], developed in [JW],[Z],[He],[N]):

Suppose a mask m satisfying (2) *for some $l \ge 0$ has neither symmetric roots nor cycles; then if the equation* $[m]$ *has a $\mathcal{C}^l_0$-solution, then the process $\{m\}$ converges in $\mathcal{C}_l$.*

Let us recall the notation used in this statement. If for a trigonometric polynomial $p(\xi)$ and for some $\alpha \in \mathbb{T}$ we have $p(\alpha/2) = p(\pi + \alpha/2) = 0$, then $\{\alpha/2, \pi + \alpha/2\}$ is a pair of symmetric roots for $p(\xi)$. In order to be defined we set that for any $\alpha \in \mathbb{T}$ the element $\alpha/2 \in \mathbb{T}$ has the corresponding real value from the half-interval $[0, \pi)$. Further, a given set $\mathbf{b} = \{\beta_1, \cdots, \beta_n\} \subset \mathbb{T}$, where $n \ge 2$, is called cyclic if $2\mathbf{b} = \mathbf{b}$, i.e., $2\beta_j = \beta_{j+1}$ for $j = 1, \cdots, n$ (we set $\beta_{n+1} = \beta_1$). We consider only irreducible cyclic sets, for which all the elements are different. Note that if two cyclic sets do not coincide, then they are disjoint. A cyclic set $\mathbf{b}$ is called a *cycle* of a trigonometric polynomial p if $p(\mathbf{b} + \pi) = 0$, i.e., $p(\beta + \pi) = 0$ for all $\beta \in \mathbf{b}$.

It is well known that sufficient condition 2) for a mask m is equivalent to the stability of the corresponding refinable function (i.e., integer translates of the refinable function possess Riesz basis property in $L_2(\mathbb{R})$). It is also equivalent to say that the mask satisfies Cohen's criterion (see for example [V, proposition 2.4]).

Actually condition 2) was formulated for the case $l = 0$ only, but it can be easily extended to general l. It is seen, for instance, from Theorem 1 of this paper.

Thus we have one necessary and one sufficient condition for the convergence of subdivision processes having smooth refinable functions. It was a natural problem to fulfill this gap and to elaborate a criterion in terms "if and only if". In 1998 two attempts were made independently from each other and almost simultaneously. They were the work [N] by M.Neamtu and my work [P2]. Those two criteria were very similar, but different. Moreover, it turned out that our results actually incompatible. We will discuss this aspect after formulating the main result of the work [P2].

II. A criterion of convergence.

We give a criterion of convergence of a subdivision process under the condition that the corresponding refinement equation has a smooth solution. We will see that symmetric roots of mask do not influence the convergence of subdivision processes. This means in particular that the stability of solutions is not necessary for the convergence. The convergence entirely depends on values of the mask at the points of so-called *generalized cycles*.

Everywhere below we consider trigonometric polynomials without positive powers, i.e., polynomials of the form $p(\xi) = \sum_{k=0}^{N} a_k e^{-ik\xi}$. Us usual we set deg $p = N$ (assuming $a_0 a_N \neq 0$). To a given value $\alpha \in \mathbb{T}$ we assign a binary tree denoted in the sequel by $\mathcal{T}_\alpha$. To every vertex of this tree we associate a value from $\mathbb{T}$ as follows: put α at the root, then put $\alpha/2$ and $\pi + \alpha/2$ at the vertices of the first level (the *level* of the vertex is the distance from this vertex to the root. The root has level 0). If a value γ is associated to a vertex on the n-th level, then the values $\gamma/2$ and $\pi + \gamma/2$ are associated to its neighbors on the $(n+1)$-st level. Thus there are the values $\frac{\alpha}{2^n} + \frac{2k\pi}{2^n},\ k = 0, \cdots, 2^n - 1$ on the n-th level of the tree $\mathcal{T}_\alpha$. A set of vertices $\mathcal{A}$ of the tree $\mathcal{T}_\alpha$ is called a *minimal cut set* if every infinite path (all the paths are without backtracking) starting at the root includes exactly one element of $\mathcal{A}$. For instance the one-element set $\mathcal{A} = \{root\}$ is a minimal cut set. Every minimal cut set is finite.

Definition 1. *A set $\{\beta_1, \cdots, \beta_n\} \subset \mathbb{T}$ is called a generalized cycle of a polynomial $p(\xi)$ if this set is cyclic and for any $j = 1, \cdots, n$ the tree $\mathcal{T}_{\beta_j + \pi}$ possesses a minimal cut set $\mathcal{A}_j$ such that $p(\mathcal{A}_j) = 0$.*

The family $\{\mathcal{A}_1, \cdots, \mathcal{A}_n\}$ is said to be sets of zeros of the generalized cycle $\mathbf{b}$. Let us remark that for a given generalized cycle the set of zeros may not be defined in a unique way. Any (regular) cycle of $p(\xi)$ is also a generalized cycle, in this simplest case each minimal cut set $\mathcal{A}_j$ is the root of the corresponding tree $\mathcal{T}_{\beta_j + \pi}$. On the other hand, not any generalized cycle is a regular cycle. For example, the polynomial $p(\xi) = \left(e^{-i\xi} - e^{-\frac{\pi i}{3}}\right)\left(e^{-2i\xi} - e^{\frac{\pi i}{3}}\right)$ has no regular cycles, but is has a generalized cycle $\mathbf{b} = \{\beta_1, \beta_2\} = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$. Indeed, this polynomial has three zeros on the period: $\frac{\pi}{3}, -\frac{\pi}{6}, \frac{5\pi}{6} \in \mathbb{T}$. The set $\mathcal{A}_1 = \left\{-\frac{\pi}{6}, \frac{5\pi}{6}\right\}$ is a minimal cut set for the point $\beta_1 + \pi,\ \mathcal{A}_2 = \left\{\frac{\pi}{3}\right\}$ is a minimal cut set for $\beta_2 + \pi$, and $p(\mathcal{A}_1) = p(\mathcal{A}_2) = 0$. Roughly speaking, each cyclic set $\{\beta_1, \ldots, \beta_n\}$ has a unique corresponding cycle (the family of zeros is $\{\beta_1 + \pi, \ldots, \beta_n + \pi\}$) and a variety of generalized cycles (all possible sets of zeros $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$, where $\mathcal{A}_j$ is an arbitrary minimal cut set of the tree $\mathcal{T}_{\beta_j + \pi},\ j = 1, \ldots, n$.) Note, that if at least one set $\mathcal{A}_j$ differs from the root $\beta_j + \pi$, then it necessarily contains a pair of symmetric roots of p. Therefore, if the polynomial p has no symmetric roots, then all its generalized cycles, if there are any, are regular cycles.

For any trigonometric polynomial p and any finite subset $Y = \{\alpha_1, \cdots, \alpha_n\} \subset \mathbb{T}$ we denote $\rho_p(Y) = (\prod_{q=1}^{n} |p(\alpha_q)|)^{1/n}$. This is a multiplicative function on the set of trigonometric polynomials.

Now we formulate the criterion of stability of subdivision process.

Theorem 1. *Suppose a refinement equation $[m]$ has a $\mathcal{C}_0^l$-solution for some $l \geq 0$; then the process $\{m\}$ converges in $\mathcal{C}^l$ if and only if the mask m satisfies (2) and for any generalized cycle $\mathbf{b}$ of the mask m we have $\rho_m(\mathbf{b}) < 2^{-l}$.*

In particular, for $l = 0$, this means that a subdivision process $\{m\}$, whose refinement equation has a continuous solution, converges if and only if $\rho_m(\mathbf{b}) < 1$ for every generalized cycle $\mathbf{b}$ of the mask. Another corollary is condition 2) from the Introduction. Indeed, if a mask has neither symmetric roots nor cycles, then it has no generalized cycles either. Hence, by Theorem 1, the subdivision process must converge.

Example 1. Consider a mask

$$m(\xi) = \Bigl(0.2 + 0.5e^{-i\xi} + 0.3e^{-2i\xi}\Bigr)\Bigl(e^{-i\xi} - e^{-\frac{\pi i}{3}}\Bigr)^2\Bigl(e^{-2i\xi} - e^{\frac{\pi i}{3}}\Bigr)^2 \qquad (4)$$

The corresponding equation $[m]$ has a $\mathcal{C}_0$-solution, this is shown in Example 2. The polynomial m has a unique generalized cycle $\mathbf{b} = \bigl\{\frac{2\pi}{3}, \frac{4\pi}{3}\bigr\}$, the same as in the previous example, with the same sets of zeros $\mathcal{A}_1 = \bigl\{-\frac{\pi}{6}, \frac{5\pi}{6}\bigr\}$, $\mathcal{A}_2 = \bigl\{\frac{\pi}{3}\bigr\}$. Actually this is not one, but two coinciding generalized cycles, if we count roots with multiplicity. We have $\Bigl(\rho_m(\mathbf{b})\Bigr)^2 =$

$$\Bigl|m(\frac{2\pi}{3})\Bigr|\cdot\Bigl|m(\frac{4\pi}{3})\Bigr| = \Bigl|(-0.2-0.1\sqrt{3}i)\cdot 1\cdot 1\Bigr|\cdot\Bigl|(-0.2+0.1\sqrt{3}i)\cdot 4e^{\frac{4\pi i}{3}}\cdot 4e^{-\frac{4\pi i}{3}}\Bigr| = 1.12 > 1.$$

Hence the subdivision process $\{m\}$ diverges.

III. A historical remark

Theorem 1 was obtained in 1998, then I presented this result in several conferences and seminars. By Spring 1999 the paper was ready and submitted to the SIAM J. Math Anal. Several months later I became aware that a newly published issue of The East Journal Approximation (the date of issue is Summer 1999) presented the paper [N] by M.Neamtu devoted to the same problem and containing a very similar result. M.Neamtu used a different approach, and his line of reasoning seems to me nicer than one in my paper. But unfortunately his result turned out to be not correct. Namely, the main theorem of Neamtu's work claims that the convergence of a subdivision process (having continuous refinable function) in the space $\mathcal{C}$ is equivalent to the the condition $\rho_m(\mathbf{b}) < 1$ for all *regular* cycles of the mask. In Theorem 1 this condition must be satisfied for all *generalized* cycles. The difference between these two statements is seen immediately. These theorems actually contradict each other. There exist masks that have generalized cycles and have no (regular) cycles. Moreover, there are masks, which should converge by the criterion from [N] and should diverge by Theorem 1. In reality the family of convergent subdivision schemes is wider, than that determined by the criterion of M.Neamtu. For instance, mask (4) from Example 1 has no regular cycles at all, but nevertheless the corresponding subdivision process diverges. This gap is caused by a mistake in the proof in the work [N], and that mistake is hardly removable. Nevertheless the general method developed by M.Neamtu seems to me correct and very interesting. His proof can be modified in order to deliver Theorem 1 as well.

IV. Statement of the problem.

Most examples of divergent subdivision schemes (having smooth refinable functions) are constructed for some special class of masks. These are either "unload" masks of the form $m(\xi) = p(n\xi)$ for some polynomial p and an odd integer n, or, at least, masks whose associated matrix $B = \{c_{2i-j}\}_{i,j\in\{0,\dots,N\}}$ have a multiple eigenvalue 1. The divergence of such schemes is well known and does not require any special criterion. A natural question arises whether one really needs the criterion of Theorem 1 to determine divergent processes ? May be the family of generalized cycles is too wide to describe unstable subdivision schemes? In general there is no evidence that the condition $\rho_m(\mathbf{b}) > 1$ can be combined with the existence of a smooth solution for the mask m. In this paper we are going to show that Theorem 1 indeed characterizes the family of unstable subdivision processes properly. We show that each generalized cycle can cause the divergence of a suitable scheme. On the other hand, we will see that every converging subdivision scheme can be "spoiled" by some generalized cycle.

V. Preliminary results. Reductions of masks.

To construct examples of divergent processes we need some auxiliary results. The first of them establishes two properties of cyclic sets. The proof of this Lemma is an easy exercise to the reader.

Lemma 1. ***a)*** *Let* $\mathbf{b}$ *be a cyclic set and* $\alpha \in \mathbb{T}$*. Then for the polynomials* $p_1(\xi) = e^{-i\xi} - e^{-i\alpha}$ *and* $p_2(\xi) = e^{-2i\xi} - e^{-i\alpha}$ *we have* $\rho_{p_1}(\mathbf{b}) = \rho_{p_2}(\mathbf{b})$*.*

b) *Let* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *be cyclic sets and* $p(\xi) = \prod_{\beta\in\mathbf{b}_1}(e^{-i\xi} + e^{-i\beta})$*. Then we have:* $\rho_p(\mathbf{b}_2) = 1$ *if* $\mathbf{b}_1 \neq \mathbf{b}_2$*, and* $\rho_p(\mathbf{b}_2) = 2$ *if* $\mathbf{b}_1 = \mathbf{b}_2$*.*

Now turn back to the subdivision schemes.

(A measure of the rate of convergence). For a given integer $l \geq 0$, a mask m, and a function $f \in \mathcal{L}^l$ denote

$$\nu_l(m, f) = -\lim_{n\to\infty} \frac{\log_2 \|T^n[f^{(l)}]\|_{\mathcal{C}}}{n},$$

where T is the subdivision operator associated to m (we set $\log_2 0 = -\infty$). The value $\nu_l(m) = \inf_{f\in\mathcal{L}^l} \nu_l(m, f)$ is the *degree of convergence of the process* $\{m\}$ *in the space* $\mathcal{C}^l$.

For every mask m we have $\nu_l(m) \leq l+1$ (see [DL1]). Furthermore, it was shown in [DL1] and [CH] that a process $\{m\}$ converges in $\mathcal{C}^l$ if and only if $\nu_l(m) > l$. In particular, the inequality $\nu_0(m) > 0$ means that $\{m\}$ converges in $\mathcal{C}$. Let L be the maximal integer such that $\{m\}$ converges in $\mathcal{C}^L$ (if the process $\{m\}$ does not converge in $\mathcal{C}$, then we nevertheless set $L = 0$). The value $\nu_L(m)$ is said to be the *degree of convergence of the process* $\{m\}$ and denoted in the sequel by $\nu(m)$. If $\nu(m_1) = \nu(m_2)$, then $\nu_l(m_1) = \nu_l(m_2)$ for any $l \geq 0$.

(A measure of smoothness of solutions). For a given refinement equation $[m]$ denote by $L(m)$ the maximal integer L such that the corresponding refinable function φ belongs to $\mathcal{C}_0^L$. If this equation has no continuous compactly-supported solution, we set $L(m) = -1$. The smoothness of the refinable function φ is the value $s(m) = L + h$, where h is the Holder exponent of the Lth derivative $\varphi^{(L)}$ on $\mathbb{R}$. It is well known that a refinable function belongs to $\mathcal{C}^l$ if and only if $s(m) > l$ (the equality $s(m) = l$ is impossible). In particular, a refinement equation has a $\mathcal{C}_0$-solution if and only if $s(m) > 0$.

Now we can describe the procedure of reduction of subdivision schemes introduced in [P2]. This reduction makes it possible to get rid of both symmetric roots and cycles.

Eliminating of symmetric roots. Let $p(\xi)$ be a given trigonometric polynomial (let us remember that we consider polynomials without positive powers). Assume that p possesses a pair of symmetric roots $\{\alpha/2, \pi + \alpha/2\}$. The transfer from $p(\xi)$ to the polynomial $p_\alpha(\xi) = \frac{p(\xi)(e^{-i\xi} - e^{-i\alpha})}{e^{-2i\xi} - e^{-i\alpha}}$ is said to be a *transfer to the previous level*. The inverse transfer from p_α to p is a *transfer to the next level*. So a transfer to the previous level reduces a pair of symmetric roots $\{\alpha/2, \pi + \alpha/2\}$ to the one root α.

Proposition 1. *Let a mask $\tilde{m}$ be obtained from a mask m by a transfer to the previous level. Then $s(\tilde{m}) = s(m)$. Moreover, $\nu(\tilde{m}) = \nu(m)$, whenever $\mathbf{l}(\tilde{m}) = \mathbf{l}(m)$.*

(The constant $\mathbf{l}(m)$ responsible for condition (2) was defined in the Introduction). This implies, in particular, that the reduced equation $[\tilde{m}]$ possesses a smooth compactly supported solution if and only if the initial equation $[m]$ does; and the same true for the convergence of the corresponding subdivision schemes. Thus, a transfer to the next (previous) level does not change the smoothness of solutions. It also respects the rate of convergence of subdivision processes, unless this transfer does not violate condition (2) (a transfer to the previous level may increase the value $\mathbf{l}(m)$). Using this Proposition one can consequently eliminate all symmetric roots of a given mask.

Eliminating of regular cycles. Let a polynomial p possess a cycle $\mathbf{b}$. The transfer from $p(\xi)$ to the polynomial $\tilde{p}(\xi) = p(\xi)/\prod_{\beta \in \mathbf{b}}(e^{-i\xi} + e^{-i\beta})$ is called an *eliminating of a cycle.*

Proposition 2. *Let a mask $\tilde{m}$ be obtained from a mask m by eliminating of a cycle $\mathbf{b}$. Then $s(\tilde{m}) = s(m)$ and $\nu(m) = \max\{\nu(\tilde{m}), \rho_m(\mathbf{b})\}$.*

Thus the equation $[m]$ possesses a smooth compactly supported solution if and only if the equation $[\tilde{m}]$ does. Moreover, the process $\{m\}$ converges in $\mathcal{C}^l$ if and only if the process $\{\tilde{m}\}$ does and in addition $\rho_m(\mathbf{b}) < 2^{-l}$.

See [P2] for the proofs of Propositions 1 and 2. Now it becomes clear how to establish Theorem 1. First we consequently eliminate all symmetric roots. By Proposition 1 it does not change neither the smoothness of solution nor the rate of convergence (if the initial mask satisfied condition (2)). Moreover, by Lemma 1 this process respects the constants $\rho_m(\mathbf{b})$ for all cyclic sets $\mathbf{b}$. The final mask has no symmetric roots, hence it can have only regular cycles. Then we eliminate all regular cycles (refereeing to Proposition 1) and obtain a mask satisfying Cohen's criterion, whose subdivision process does converge. This line of reasoning also allow us to eliminate directly all generalized cycles as follows:

Eliminating of generalized cycles. Let a polynomial p possess a generalized cycle $\mathbf{b}$ with corresponding sets of zeros $\mathcal{A}_1, \dots, \mathcal{A}_n$. The transfer from $p(\xi)$ to the polynomial $\tilde{p}(\xi) = p(\xi)/\prod_{\alpha \in \mathcal{A}_j, j=1,\dots,n}(e^{-i\xi} - e^{-i\alpha})$ is called an *eliminating of a generalized cycle.*

Proposition 3. *Let a mask $\tilde{m}$ be obtained from a mask m by eliminating of a generalized cycle $\mathbf{b}$. Then $s(\tilde{m}) = s(m)$ and $\nu(m) = \max\{\nu(\tilde{m}), \rho_m(\mathbf{b})\}$.*

Proof. After a suitable sequence of transfers to the previous level all the sets of zeros $\mathcal{A}_1, \dots, \mathcal{A}_n$ drop to the corresponding roots $\beta_1 + \pi, \dots, \beta_n + \pi$, and $\mathbf{b}$ becomes a regular cycle. By Lemma 1 this does not change the value $\rho_m(\mathbf{b})$. Now it remains to apply Proposition 2. □

Example 2. Consider again the mask $m(\xi)$ from Example 1. After eliminating the generalized cycle $\mathbf{b} = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ we obtain the mask $\tilde{m}(\xi) = 0.2 + 0.5e^{-i\xi} + 0.3e^{-2i\xi}$. Since all the coefficients of $\tilde{m}$ are positive, it follows that the equation $[\tilde{m}]$ has a $\mathcal{C}_0$-solution and, moreover, the corresponding subdivision process $\{\tilde{m}\}$ converges (see, for instance [CDM]). Now applying Proposition 3 we see that the initial process $\{m\}$ diverges, since $\rho_m(\mathbf{b}) = \sqrt{1.12}$. Let us note, that the matrix B corresponding to the mask m ($B = \{c_{2i-j}\}_{i,j\in\{0,\dots,8\}}$) has the eigenvalue 1 with multiplicity one and has no other eigenvalues on the unit circle. So the divergence of the subdivision scheme in this case does not follow from the well-known argument of multiple eigenvalues.

VI. Unimprovability of the criterion. Examples of divergent schemes.

Now we are going to see that Theorem 1 gives a full description of divergent subdivision schemes having smooth refinable functions. This means that all possible cases of the criterion of convergence are realized on suitable masks. For the sake of simplicity we formulate this result for the convergence in the space $\mathcal{C}$, i.e., for the case $l = 0$.

Theorem 2. *Let $\mathbf{b} = \{\beta_1, \dots, \beta_n\}$ be a cyclic set and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be arbitrary minimal cut sets of the trees $\mathcal{T}_{\beta_1+\pi}, \dots, \mathcal{T}_{\beta_n+\pi}$ respectively. Then there exists a mask $m(\xi)$ such that*

1) $m(\mathcal{A}_j) = 0$, $j = 1, \dots, n$, i.e., $\mathbf{b}$ is a generalized cycle of the mask m, and A_j are its sets of zeros;

2) the equation $[m]$ has a $\mathcal{C}_0$-solution, but the subdivision process $\{m\}$ does not converge in $\mathcal{C}$;

3) after eliminating of the generalized cycle $\mathbf{b}$ this process becomes converging in $\mathcal{C}$.

Proof. Consider a mask $p(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)a(\xi)$ such that $\deg a \geq 2$, and the subdivision process $\{p\}$ converges in $\mathcal{C}$. To obtain such a mask it suffices to take an arbitrary polynomial $a(\xi)$ with positive coefficients such that $a(0) = 1$. Now we use the fact that if the process $\{p\}$ converges in $\mathcal{C}$, then it will still converge in this space after all sufficiently small perturbations of the coefficients of $a(\xi)$ preserving the condition $a(0) = 1$ (see [DL1]). Thus, with possible perturbation of the coefficients, we assume that the trigonometric polynomial a has no real roots and that the value $\rho_a(\mathbf{b})$ is irrational. Such a perturbation exists by the mean value theorem, because $\rho_a(\mathbf{b})$ is a continuous function of the coefficients of $a(\xi)$. This implies, in particular, that $\rho_a(\mathbf{b}) > 0$ and hence $\rho_p(\mathbf{b}) > 0$. Now take the polynomial $q(\xi) = \prod_{\alpha\in\mathcal{A}_j, j=1,\dots,n}(e^{-i\xi} - e^{-i\alpha})$. By Lemma 1 we have $\rho_{pq^r}(\mathbf{b}) = 2^r\rho_p(\mathbf{b})$ for every $r \geq 0$. Consequently there exists a nonnegative integer r such that $\rho_{pq^r}(\mathbf{b}) > 1$. Take the smallest such integer r_0 and denote $\tilde{a} = aq^{r_0-1}$ and $\tilde{p} = pq^{r_0-1}$ (if $r_0 = 0$, then we put $\tilde{a} = a, \tilde{p} = p$). Let us remark that the case $\rho_{\tilde{p}}(\mathbf{b}) = 1$ is impossible, because this value is not rational, therefore $\rho_{\tilde{p}}(\mathbf{b}) < 1$. Since $\mathbf{b}$ is the only generalized cycle of the polynomial $\tilde{p}$, therefore, by Proposition 3, the subdivision process $\{\tilde{p}\}$ converges. Now make a small perturbation of the coefficients of the polynomial $\tilde{a}$ after which the process $\{\tilde{p}\}$ still converges, and the value $\rho_{\tilde{p}q}(\mathbf{b})$ is still bigger than 1, but the polynomial $\tilde{a}$ does not have real roots.

Then denote $\tilde{m} = \tilde{p}, m = \tilde{m}q$. We see that the mask m has a unique generalized cycle $\mathbf{b}$, and this cycle has sets of zeros $\mathcal{A}_1, \dots, \mathcal{A}_n$. Since $\rho_m(\mathbf{b}) > 1$, the process $\{m\}$ diverges, however removing this generalized cycle we obtain the converging process $\{\tilde{m}\}$. This proves the Theorem. $\square$

Remark 1. One could take the initial polynomial in the form $p(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{l+1} a(\xi)$ and by the same argument construct a mask m that has a $\mathcal{C}^l$- solution, but the corresponding subdivision algorithm diverges even in $\mathcal{C}$. However in this case the mask m must have at least $l+1$ generalized cycles, may be coinciding (i.e., a generalized cycle with multiplicity $l+1$). In the last case this multiple cycle can be also given in advance, as in Theorem 2. Why is it impossible to manage by just one generalized cycle in this case ? The answer is given by Proposition 4 below.

Proposition 4. *If the solution of a refinement equation is in $\mathcal{C}^l$ and the corresponding subdivision process diverges in $\mathcal{C}$, then the mask of this equation possesses at least $l+1$ generalized cycles (counting with multiplicity).*

The proof can be found in [P3].

References

[CDM] D.Cavaretta,W.Dahmen,C.Micchelli, *Stationary subdivision,* Mem.Amer.Math.Soc.,93 (1991), pp.1-186.

[CH] D. Collela and C. Heil, *Characterization of scaling functions. I. Continuous solutions,* SIAM J. Matrix Anal. Appl. 15 (1994), 496–518.

[DL1] I. Daubechies and J. Lagarias, *Two-scale difference equations. I. Global regularity of solutions,* SIAM. J. Math. Anal. 22 (1991), 1388–1410.

[DL2] I. Daubechies and J.Lagarias, *Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals,* SIAM. J. Math. Anal. 23 (1992), 1031–1079.

[Du] S.Durand, *Convergence of the cascade algorithms introduced by I.Daubechies*, Numer.Algorithms, 4 (1993), pp.307-322

[DGL] N.Dyn,J.A.Gregory and D.Levin, *Analysis of linear binary subdivision schemes for curve design*, Constr.Approx., 7 (1991),127-147.

[He] L.Herve, *Régularité et conditions de bases de Riesz por les fonctions d'échelle,*, C.R.Acad.Sci.,Paris,Ser. I 335 (1992), pp.1029-1032.

[JW] R.Q.Jia and J.Wang, *Stability and linear independence associated with wavelet decomposition*, Proc.Amer.Math.Soc.,117 (1993), pp.1115-1124.

[N] M. Neamtu *Convergence of subdivisions versus solvability of refinement equations*, East J. Approx, 5 (June), 1999, pp.183-210.

[P1] V.Protasov, *A complete solution characterizing smooth refinable functions*, SIAM J. Math. Anal., Vol.31 No 6, pp.1332-1350.

[P2] V.Protasov, *The stability of subdivision operator at its fixed point*, SIAM J. Math. Anal., 33 (2001), No 2, pp. 448-460.

[P3] V.Protasov, *Univariate refinement equations. Basic constructions*, submitted

[RS] A.Ron, Z.Shen, *The Sobolev regularity of refinable functions*, J. Approx. Theory 106 (2000), no. 2, 185–225.

[V] L.Villemoes, *Wavelet analysis of refinement equations*, SIAM J.Math.Anal., 25 (1994), pp.1433-1460.

[W] Y.Wang, *Two-scale dilation equations and the cascade algorithm*, Random Comput.Dynamic,3 (1995),No 4, pp.289-307.

[Z] D.-X.Zhou, *Stability of refinable functions, multiresolution analysis, and Haar bases.* SIAM J.Math.Anal., 27(1996), No 3, pp.891-904.

Erasmus Universiteit Rotterdam, Fac. der Economische Wetenschappen, H 11-21, Postbus 1738, 3000 DR Rotterdam, The Netherlands, e-mail: protassov@few.eur.nl