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SFB 649 Discussion Paper 2011-017

The Law of Attraction

Bilateral Search and Horizontal Heterogeneity

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

http://sfb649.wiwi.hu-berlin.de
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin
Spandauer Straße 1, D-10178 Berlin

SFB 649 ECONOMIC RISK BERLIN

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Bilateral Search and Horizontal Heterogeneity

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March 15, 2011

Abstract
We study a matching model with heterogeneous agents, nontransferable utility and search frictions. Agents differ along a horizontal dimension (e.g. taste) and a vertical dimension (e.g. income). Agents' preferences coincide only in the vertical dimension. This approach introduces individual preferences in this literature as seems suitable in applications like labor markets (e.g. regional preferences). We analyze how the notion of assortativeness generalizes to integration or segregation outcomes depending on search frictions. Contrary to results from the purely vertical analysis, here, agents continuously adjust their reservation utility strategies to changing search frictions. The model is easily generalizable in the utility specification, the distribution of taste-related payoffs and the number of vertical types. Extreme utility specifications can be treated as a case of horizontal heterogeneity only.

Keywords: Matching, Horizontal Differentiation , Marriage Markets
JEL Classification Numbers: D13, D61, J12

*The authors are grateful to Helmut Bester, Tom Cusack, Kai Konrad and Roland Strausz for advice and suggestions. We are also grateful to audiences at the Free University Berlin, the Humboldt Universität Berlin, the Social Science Research Center Berlin (WZB), the 2010 Econometric Society World Congress (Shanghai) and the Max Planck Research Center for Tax Law and Public Finance Munich for helpful comments. This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

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1. Introduction

Finding a suitable match on the labor or marriage market is a complex task. Search frictions in the market force agents to trade off the costs of searching a suitable partner against the foregone utility due to early match formation. Quality is determined by a number of traits and agents usually agree on the ranking for a subset of the characteristics involved - the vertical dimension - and disagree in others - the horizontal dimension. In this paper, we study the different roles of vertical and horizontal heterogeneity among agents in matching models. The first parameter captures a discrete vertical dimension, for example income. All agents agree on the ranking along this vertical dimension. The second parameter represents a horizontal non-ordered taste. We refer to this trait as "taste" to highlight that agents do not necessarily agree on the ranking of agents along this dimension. Agents take the implications of both dimensions for their individual utility into account when deciding on their search strategy. The central aim of our paper is to analyze matching and sorting in this environment assuming that utility is nontransferable.

One recurring result of the pure vertical model is that assortative mating arises endogenously in equilibrium. In these models all individuals have identical preferences and prefer to be matched with the highest type. Yet, as the highest types know that all agents in the market propose to them, they can afford to be selective and therefore reject agents below some minimum standard (see, for example, Burdett and Coles 1997).

However, the interaction of horizontal and vertical preferences reveals some new insights in this strand of search and matching models. First, the model encompasses the prediction of assortative mating from the pure vertical model. If the level of search frictions is sufficiently low, the costs of rejecting offers are low and agents, especially the vertically "gifted", are selective and this leads high type agents to reject low types. This in turn generates positive assortativeness. Second, contrary to approaches of purely vertical heterogeneity all agents adjust their optimal reservation utility strategies in a continuous way. This seems appealing as agents do not fall victim to drops in their expected lifetime utility at (more or less) arbitrary marginal changes in decreasing search frictions. Third, we show that low type agents might benefit from a higher degree of search frictions as they suffer from high types' withdrawing from the marriage market concerning low types due to lower costs of searching. Fourth, our analysis shows that both kind of matching equilibria, complete segregation of types or integrating equilibria may emerge. Finally, we show that our approach is general in the sense that it allows for (i) different utility specifications concerning

the two traits, (ii) different type distributions along the horizontal dimension as well as (iii) for the number of vertical types. The analysis shows that extreme utility specifications (only money or only taste counts) are equivalent to a specification without vertical heterogeneity at all.

A large empirical literature in economics and sociology documents a positive association between traits of partners in existing unions. For example, the classic work by Becker (1973, 1974) finds a positive correlation between partners' education levels and partners' height. Recent examples in the sociology literature include Kalmijn and van Tubergen (2006) who find a correlation between ethnicity, while Kalmijn (2006) and Mare (1991) document assortative mating for religion and education.

However, the theoretical explanations for these correlations at hand are mostly centered on different preferences regarding (vertically) ordered traits as can be seen for example in the survey article of Burdett and Coles (1999). The classic example for such a trait is wage or income. For married individuals, spousal wages are clearly payoff-relevant. Yet, the documented positive assortative mating might result from two very distinct underlying preferences. In the first case, all single individuals prefer to be matched with the highest type, but competition among singles leads to positive assortative mating. In the second case, agents have "horizontal" preferences and prefer to be matched with individuals who have a similar income.

By contrast, examples for non-ordered traits are religion, location or "taste" as mentioned above. For these traits it is only possible to define horizontal preferences. For example, agents might prefer singles of the same religion. There is also a recent economic literature which indicates that such traits play a major role in marriage decisions (e.g. Lundberg and Pollak 2007, Hitsch, Hortacsu and Ariely 2010). However, the search-theoretic literature on matching and sorting has largely focused on vertical characteristics and therefore provides little guidance to the equilibrium patterns when horizontal preferences matter. It is this dearth, which motivates our analysis of individuals' strategies and the equilibrium outcome in a dynamic model where agents are characterized by both vertical and horizontal traits.

Our framework builds on the existing literature that deals with dynamic bilateral search and matching. As mentioned above, the seminal static model by Becker (1973, 1974) assumes that utility is transferable and that no search frictions are present. Starting from Becker's observations, a large literature has emerged which mainly focuses on vertical heterogeneity. For example Bloch and Ryder (2000), Shimer and Smith (2000) and Atakan (2006) consider transferable utility and vertical heterogeneity. Burdett and Coles (1997), Eeckhout (1999) and Smith (2006) analyze

vertical heterogeneity in a nontransferable-utility model including search frictions, while Legros and Newman (2007) analyze a setting without search frictions. This paper extends the latter strand of the literature, while the results do not change qualitatively as long as part of the intra-marital gain is non-transferable.

Clark (2007) studies bilateral sorting with horizontal heterogeneity in a frictionless assignment model. We depart from Clark (2007) as in his model agents prefer similar agents, but the trait considered for matching is ordinal, e.g. height. We borrow the assumption of a horizontal trait from Konrad and Lommerud (2010), who study the interplay of matching and redistributive taxation in a static model. In their model, agents are matched only for a single period in which they have to either accept a match or stay single forever.

We proceed as follows. In the following section we develop the marriage model. In the next section we solve for the equilibria in the model for one or two income-levels and discuss the comparative statics of the approach. Section 4 provides an example for specific income and taste distance functions. These permit an explicit model solution. Section 5 discusses three extensions of the model. First, we introduce weights for the two parts of utility into the model. This allows us to analyze the full range between two polar cases: in the first polar case agents marry for love and in the second they marry for money. Second, we introduce a reformulation that permits one to choose almost any continuous probability distribution for the utility along the horizontal dimension. Third, we show that our framework can be easily modified to account for more than two income levels. Finally, we discuss the role of assortativeness. Section 6 concludes. All proofs are relegated to the appendix.

2. The Model

There is a unit mass of individuals. Each individual i is characterized by two parameters $(y_i, t_i) \in Y \times T$. The first parameter captures vertical ex-ante heterogeneity of agents. For concreteness, we refer to this trait as income y_i. We assume that there are two income levels such that $y_i \in Y = \{y_L, y_H\}$ with $y_H \geq y_L \geq 0$. The second parameter t_i captures horizontal heterogeneity and for concreteness we refer to it as taste t_i. We assume a continuum of taste parameters such that $t_i \in T = [0, 2]$. In this taste dimension we follow the common approach from the industrial economics literature as introduced by Schmalensee (1978) and Salop (1979), i.e. individuals

are located around a circle of circumference 2.[1] This approach allows to avoid end point effects.[2] We assume that individuals are uniformly distributed on $Y \times T$; both income levels are equally likely and the location on the circle for each agent is drawn independently of her income.

Preferences of all agents are identical regarding the relative locations of potential partners to their own position. In the vertical dimension agents prefer to be matched with partners of higher income. In the horizontal dimension agents prefer to be matched with individuals of similar taste.[3] If two agents i and j decide to marry each other, both enjoy a utility gain depending on a measure of the distance between their respective locations on the circle. Formally the distance in taste x between two agents i and j is given by

$$x := \begin{cases} \min\{t_i - t_j, 2 + t_j - t_i\}, & \text{if } t_i \geq t_j \\ \min\{t_j - t_i, 2 + t_i - t_j\}, & \text{if } t_i < t_j \end{cases} .$$

Defined by this formula the taste difference of two randomly drawn agents x is the realization of a random variable X. Clearly, as both agents' taste parameters are drawn independently and are ex-ante uniformly distributed, the random variable X is uniformly distributed on $[0, 1]$.

If a match is formed, individuals enjoy a utility flow depending on both partners' incomes and the taste difference x between them for the rest of time.[4] The utility function is modeled in an additively separable manner. One part of an agent's utility is the gain induced by household income. A second source of utility is the intra-marital gain through the difference, or rather the similarity in taste.[5] By this choice the two dimensions of utility generation become substitutes for the agents. If agent i marries agent j she gets a per-period-utility of

$$U(y_i, t_i, y_j, t_j) = f(y_i, y_j) + g(x),$$

[1] In contrast to Schmalensee (1978) and Salop (1979) we re-scale the circumference of the individual's location to 2. Hence, we have a maximum distance of 1 between two individuals on the circle.

[2] This differs from the approach by Clark (2007) who analyzes a model in which agents prefer agents of similar height. In such a model agents with "extreme" (small or large) height and agents with more common realizations behave differently.

[3] The only paper known to us with a similar framework is Konrad and Lommerud (2010) where a static approach is conducted.

[4] For simplification we assume that there is no divorce. This is a crucial assumption as in a framework including divorce agents would still be on the search after a marriage which they are not in our model.

[5] This additively separable utility specification is motivated and justified by the latter result that there is interaction between these different sources of utility in the optimal search policy *although* they are treated separately in the utility function. Clearly, the interaction would be more direct if the two sources are modeled for example in a multiplicative way.

where x is determined by the definition given above. Function f, which captures the part of agent i's utility induced by the vertical trait income, is assumed to be increasing in both arguments. Function g, which captures the (intra-marital) utility gain through the horizontal trait taste, is assumed to be differentiable in $(0, 1)$. Furthermore, we assume $g' < 0$, as agents prefer partners which are located closer to themselves. We simplify $g(1) = 0$.[6] As long as staying single the per-period-utility of agent i only depends on her income y_i and is given by

$$U(y_i, t_i) = f(y_i, y_i).^7$$

This definition of a single agent's utility implies that agent i's utility derived from income within marriage is only affected if she marries a partner with a different income level. For ease of notation and reading the utility of a single agent is sometimes written as $f(y_i, -) := f(y_i, y_i)$. We assume that marital surplus is not transferable.[8]

The matching institution is characterized by search frictions, i.e. individuals face difficulties finding a spouse, a job or in general any type of matching partner. As commonly used in the matching literature with search frictions (e.g. Burdett and Coles 1997) we use α as the arrival rate of potential partners where α is the parameter of a Poisson process. The type of a potential partner j an individual i meets is assumed to be independent of individual i's type. If two singles meet they first observe each others traits and then decide either to propose to the other or to continue searching. If both singles propose to each other a match is formed and the two singles leave the market forever, otherwise the agents continue searching. Two individuals forming a match are replaced by (unmarried) clones. Hence, the distribution of single agents is stationary across time. Future per-period utility flows are discounted at rate r.[9] Both the discount rate and the arrival rate of singles determine the extent of the market, which we denote by $\theta := \alpha/r$.

A strategy of a (single) individual in this framework is a subset of all potential

[6] This normalization implies that there are no negative utilities induced by the maximum taste difference. This assumption does not affect the results qualitatively.

[7] Note that depending on the matching status of the individual utility U is defined as a function of 4 arguments when being matched and of only 2 arguments when being single. This will cause no confusion in the further text.

[8] The qualitative results do not change if only some part of the surplus is nontransferable. It seems reasonable to assume that in both labor and marriage market settings some fraction of the surplus is nontransferable. A possible microfoundation are cooperative bargaining models with non-cooperative actions as threat-points (e.g. Konrad and Lommerud 2000, Lundberg and Pollak 1993, 1994); these models result in some degree of nontransferable utility.

[9] The life-time-utility of an individual marrying someone with per-period-utility k at period 0 when using discount factor r is $\int_0^\infty k \cdot e^{-rt} dt = \frac{k}{r}$.

partners for any point in time. Such a subset can be interpreted in the following way: The subset contains all partners this particular individual would like to marry and hence she would propose to at a certain point in time.[10] Such a strategy of a particular agent depends on her belief about the distribution of singles who propose to her. In turn, her strategy determines the offer distribution which other singles face. In equilibrium, beliefs and offer strategies must be compatible with each other. For example, consider for a moment two types of agents, females and males. Females have beliefs about the offers they receive from males and calculate a list of acceptable males based on these beliefs. This determines the offer distribution of females proposing to male agents. In equilibrium, females must have correct beliefs about the offers they receive from males. Likewise, male singles must have correct beliefs about the offers received by females.

In order to keep the model as tractable as possible, we focus entirely on a stationary environment. To be more precise, we assume individuals' strategies are independent of the time and the ongoing duration of their search. Furthermore, the matching framework is stationary, as the distribution of singles is stationary over time due to the cloning assumption. This generates a stationary environment where agents face the same search problem at any point in time and hence this simplification seems forgivable.

3. Optimal Search Policies

In the following we first analyze the model without vertical heterogeneity to establish some intermediate results. This analysis helps to get an intuitive understanding of the effects due to the horizontal heterogeneity part which is the contribution of this model to the existing literature. Having done this we move to the more general case when agents are additionally characterized by two distinct income classes. Finally, we discuss the comparative statics of the model.

3.1. No vertical heterogeneity

In this section we discuss our model in absence of any vertical heterogeneity aspects and concentrate on the effects of horizontal heterogeneity only. Therefore, we assume all agents in the marriage market have the same income $y_H = y_L \equiv y$. As introduced

[10] Note that agents *end* their game when marrying another agent as there is no divorce and consequently there are no strategic considerations for married agents.

above, taste parameter t is distributed uniformly on $T = [0, 2]$ with the geometric interpretation that individuals are located on a circle with circumference 2.

By the definition of agents' utility in the case of being matched and being single we exclude intra-marital income effects when excluding vertical heterogeneity as $f(y, -) = f(y, y)$. As a consequence, agents effectively incorporate only the taste parameter of their potential partners into their marriage decisions. Furthermore, when choosing their optimal strategies agents take the search frictions of the matching institution into account. They balance the costs of waiting for the next proposal against the foregone utility due to marrying an ordinary partner.

The optimal strategy of an individual i can be determined by using a dynamic programming approach. Agents maximize their maximum expected lifetime utility V_i when being single by choosing optimally the partners they would propose to.[11] For the next short time period Δ lifetime utility V_i is composed of three things. For the duration of Δ the agent stays single and obtains $\Delta f(y, -)$. For the second term let α_i denote the arrival rate of singles who are proposing to agent i. Moreover, let $H_{-i}(x)$ denote the cumulative distribution function that the difference in taste of a proposing agent is equal to x or less. Then, over this time interval of Δ she receives at least one proposal with probability $\Delta\alpha_i$. In this case the individual decides whether to accept the proposal (i.e. proposing) or to reject (i.e. not proposing). As V_i is the maximum of lifetime utility from the perspective of a single she will only accept proposals which yield a higher lifetime utility than V_i. Formally, this means that the utility out of the marriage with an agent at taste distance x, which is $(f(y, y) + g(x))/r$, exceeds V_i, i.e. agents apply a reservation utility strategy. The third and last term of V_i captures the unlucky event that the agent did not meet anyone proposing to her in the time span of Δ and is stuck with her lifetime utility V_i. Finally including discounting we get:

$$V_i = \frac{1}{1 + \Delta r}\left[\Delta f(y, -) + \Delta\alpha_i E_{-i}\left(\max\left\{V_i, \frac{f(y, y) + g(x)}{r}\right\}\right) + (1 - \Delta\alpha_i)V_i\right] + o(\Delta) \tag{1}$$

In equation (1) one has to use the expectation operator for the event of meeting another proposing agent.[12] This is because ex-ante the taste difference x of a randomly

[11] This approach is widely used in dynamic matching models. For a deeper introduction we refer to the survey article on search and matching by Burdett and Coles (1999).

[12] Note that we follow the convention to subscript the expectations operator according to the respective random variable. Since agent i applies the expectation operator on taste difference $x \sim H_{-i}(x)$, it is denoted as E_{-i}.

met and proposing partner is only determined by the cumulative distribution of agents proposing to agent i which is $F_{-i}(x)$. The o-function captures the small probability events of meeting more than one individual in time span Δ. Rearranging equation (1) and letting $\Delta \to 0$ yields

$$rV_i = f(y,-) + \frac{\alpha_i}{r} E_{-i} \left(\max \left\{ 0, f(y,y) + g(x) - rV_i \right\} \right). \tag{2}$$

As mentioned above agents apply a reservation utility strategy, a proposal is accepted if the utility out of marriage exceeds the lifetime utility when staying single. As $g(\cdot)$ is decreasing in the taste difference x we see from equation (2) that this corresponds to a cut-off strategy. Each individual i sets a certain minimal quality standard $\bar{x}_i$ on her future partner and accepts proposals if and only if $x \leq \bar{x}_i$. We will refer to such strategies as reservation quality strategies. By this observation we restrict the further analysis to agents' strategies which are of this type.

Rearranging equation (2) yields the following description of the agents' strategies:

Lemma 1 *Reservation utility strategies*

Given the expected distribution of offers by other agents $H_{-i}(x)$, agent i proposes to all singles with $x \leq \bar{x}_i$ where either

1. $rV_i = f(y,y) + g(\bar{x}_i)$ *and*

$$rV_i = f(y,-) + \frac{\alpha_i}{r} \int_0^{\bar{x}_i} \left(g(x) - g(\bar{x}_i) \right) dH_{-i}(x) \tag{3}$$

or

2. $rV_i < f(y,y) + g(\bar{x}_i)$, $H_{-i}(\bar{x}_i) = 1$ *and*

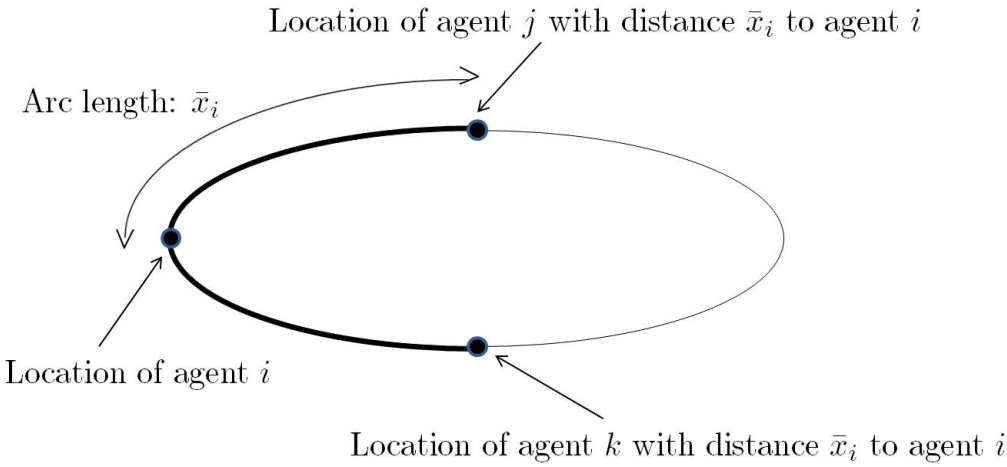
$$rV_i = f(y,-) + \frac{\alpha_i}{r} E_{-i} \left(f(y,y) + g(x) - rV_i \right). \tag{4}$$

Lemma 1 implicitly characterizes agent i's optimally chosen subset of acceptable agents, given her beliefs about the distribution of offers by other singles and about the search frictions in the market. In the first case agent i is decisive in the sense that she selects only a subset of all available offers. In the second case she is restricted by the offers made to her and is willing to accept all of them.

Figure 1 describes the type space and agent i's strategy.

As there is only one income level all agents are located on one circle. Agent i is

Figure 1: Geometry of the One-Circle-Model



located to the very left. Given a particular offer distribution $H_{-i}(x)$ she applies a cut-off-strategy as best response. In the figure the thick line represents agent i's cut-off-strategy. The partners she is willing to accept are symmetrically located around her own position. Which partnerships are possibly formed depends on the offer distribution. From Lemma 1 it follows that if she is decisive in the marriage market she is indifferent between marrying and staying single when meeting the proposing agents j or k. If she is not decisive agent i would like to marry agents which do not even propose to her and hence accepts all proposals she gets.

Equilibrium

In this section we determine the equilibrium behavior of agents in the two-sided matching model without vertical heterogeneity. So far, we established that agents will apply a reservation utility strategy. In the framework without vertical heterogeneity this coincides with a cut-off-value concerning the maximal distance in taste an agent accepts when deciding on proposals. The reservation utility V_i, which determines this cut-off-value $\bar{x}_i$, is determined endogenously by the offer distribution an agent faces in equilibrium.

We will focus on symmetric equilibria in the sense that in equilibrium an agent never proposes to another agent who rejects the offer. This restriction is motivated by the fact that in the model without vertical heterogeneity all agents are ex-ante symmetric regarding their income and their relative location to others.

As agents apply a reservation utility strategy there are no holes in the subset of agents an individual proposes to; i.e. an agent proposes to all agents up to a certain

cut-off-value. Together with the symmetry restriction this shows that the offer distribution an agent faces in equilibrium does not have holes.[13] Otherwise some of this individual's proposals would be rejected. Formally, we denote by $H_{-i}(x)$ the offer distribution agent i faces, where μ_{-i} denotes the fraction of singles who will propose to agent i on contact. Agent i is now decisive in the sense of the first part of Lemma 1, which allows us to rewrite equation (2) as

$$rV_i = f(y,-) + \frac{\alpha\mu_{-i}}{r}\int_0^{\bar{x}_i} (g(x) - g(\bar{x}_i))\, h_{-i}(x)dx. \tag{5}$$

The offer distribution does not have holes and individuals are located uniformly on the circle. Therefore, the density $h_{-i}(x)$ of offer distribution $H_{-i}(x)$ is $h_{-i}(x) = 1/\mu_{-i}$ and one can rewrite equation (5) as[14]

$$rV_i = f(y,-) + \theta\left[\int_0^{\bar{x}_i} g(x)dx - \bar{x}g(\bar{x})\right]. \tag{6}$$

Furthermore, the first part of Lemma 1 yields

$$rV_i = f(y,y) + g(\bar{x}_i). \tag{7}$$

Equilibrium requires that equation (6) and equation (7) are satisfied for any agent i. Per construction of the model, V_i is the maximum value of lifetime utility for each agent i. Hence, all agents face the same decision problem and will apply the same cut-off-value $\bar{x} = \bar{x}_i$ and receive the same lifetime utility $V_i = V$.

The following proposition summarizes the solution.

Proposition 1 *Equilibrium without vertical heterogeneity*
In equilibrium, all agents marry the first single who yields at least the reservation

[13] More formally, having no *holes* in our context means two things: (i) Agents apply a reservation utility strategy. Therefore, if agent i proposes to an agent j within a certain distance x_j then he proposes to all agents which are located closer to her and hence there are no holes in agent i's acceptance region. (ii) If we say that offer distributions do not have holes this means that the density $h_{-i}(x)$ for a certain offer distribution is strictly positive on the full support.

[14] Note that $\int_0^{\bar{x}} g(x) - g(\bar{x})dx = \int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x})$.

utility rV, where rV is characterized by:

$$(I) \qquad rV = f(y,y) + g(\bar{x})$$
$$(II) \quad rV = f(y,-) + \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right]$$

Proposition 1 shows that there are two unknown parameters ($\bar{x}$ and V) for a given extent of the market θ to characterize the full solution.[15]

We will mainly focus on the threshold quality $\bar{x}$ to explain the main features of the model. This allows a more intuitive discussion of the model especially when we introduce vertical heterogeneity. The main properties are summarized as follows:

Corollary 1 *Properties of the reservation quality*

1. *The reservation quality $\bar{x}$ is characterized by*

$$g(\bar{x}) = \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right]. \tag{10}$$

2. *The reservation quality $\bar{x}$ is decreasing in θ and we have*

$$\lim_{\theta \to 0} \bar{x} = 1 \text{ and } \lim_{\theta \to \infty} \bar{x} = 0.$$

There is some intuition behind Corollary 1. As there is no intra-marital redistribution of income the reservation quality is unaffected by the income level y. Generally, all agents prefer to be matched with singles of similar taste, as such matches yield the largest utility gain. An extent θ close to zero implies that there are maximum search

[15] We can easily rewrite the model to a setup with two sets of agents, say females and males. In this case let $i \in \{w, m\}$ denote women and men respectively. The reservation utility for females and males is given by

$$rV_w = f(y,-) + \frac{\alpha\mu_m}{r} \int_0^{\bar{x}_w} (g(x) - g(\bar{x}_w))\, h_m(x)dx \tag{8}$$

and

$$rV_m = f(y,-) + \frac{\alpha\mu_w}{r} \int_0^{\bar{x}_m} (g(x) - g(\bar{x}_m))\, h_w(x)dx \tag{9}$$

respectively. In equilibrium the arrival rate of offers μ_m and the conditional taste distribution $H_m(x)$ must be consistent with male agents' reservation strategy defined by equation (9). Likewise, $H_w(x)$ and μ_w must be consistent with females' strategy given by equation (8).

frictions in the market. In this case all singles set $\bar{x}$ to the maximum and accept the first single they encounter. On the other hand, a large extent θ implies that singles can easily meet each other or that they are very patient. For $\theta \to \infty$ there are no search frictions, and every agent will "wait" (for a time span of zero) for her perfect match. This corresponds to the perfect assortative mating results in previous models without search frictions, e.g. Becker (1973). However, this "notion of assortative mating" differs from the usual definition of assortative mating, as the taste parameter is not an ordinal measure. In our context we observe perfect correlation between taste characteristics if search frictions disappear. With maximum search fractions there is no correlation between spouses in taste. For search frictions in between the correlation is decreasing with increasing search frictions as the reservation quality $\bar{x}$ is decreasing as Corollary 1 shows.

3.2. Introducing vertical heterogeneity ("Two circles")

In this section we introduce vertical heterogeneity in our model. Therefore we drop the assumption of a single income level and consider two income levels $y_H > y_L$. These two income levels are equally likely in the population. As before, the taste parameter t is distributed uniformly on $T = [0, 2]$ and is drawn independently of an agent's income. By this specification we have two different types of agents: High type agents with high income y_H and low type agents with low income y_L. Whenever it is necessary to distinguish between one particular agent and types of agents, we denote individuals by small letters and types by capital letters.

Contrary to the model without vertical heterogeneity, this time agent i takes into account intra-marital redistribution of income as $f(y_i, y_H) > f(y_i, y_L)$. We will look for the symmetric equilibrium of the extended model. As before, agents balance waiting costs and the foregone utility of marrying a mediocre partner. But concerning the traits of a potential partner both characteristics serve now as substitutes: When deciding on proposals agents may be willing to accept lower income for a better match in taste and vice versa.

The derivation of the expression for lifetime utility analogously to equation (1) is different as there are now two different offer distributions an agent faces: Offers received from high type agents and offers from low type agents. For high type agents who are willing to marry her, let $H^H_{-i}(x_H|y_i)$ denote the cumulative probability that the difference in taste is equal to x_H or less. Analogously, we define $H^L_{-i}(x_L|y_i)$ and x_L for low types.[16] Let α^H_i denote the probability that agent i contacts a high type

[16]To stress that the offer distribution of agent i is fundamentally determined by her income type

single who is willing to marry her. Likewise, let α_i^L denote the probability that she contacts a low type single who is willing to marry her. As in the case without vertical heterogeneity agent i accepts proposals if utility out of marriage exceeds her lifetime utility. Hence, agents apply a reservation utility strategy. Expected discounted lifetime utility is then determined by

$$\begin{aligned} V_i = \frac{1}{1+\Delta r} \Bigg[& \Delta f(y_i, -) \\ & + \Delta(\alpha_i^H + \alpha_i^L) \left(\sum_K \frac{\alpha_i^K}{\alpha_i^H + \alpha_i^L} E_{-i}^K \left(\max \left\{ V_i, \frac{f(y_i, y_K) + g(x_K)}{r} \right\} \right) \right) \\ & + (1 - \Delta(\alpha_i^H + \alpha_i^L)) V_i \Bigg] + o(\Delta) \end{aligned} \tag{11}$$

where $K \in \{H, L\}$.

Rearranging equation (11) and letting $\Delta \to 0$ yields

$$rV_i = f(y_i, -) + \sum_K \frac{\alpha_i^K}{r} E_{-i}^K \left(\max \left\{ 0, f(y_i, y_K) + g(x_K) - rV_i \right\} \right). \tag{12}$$

So far, we established that agents use a reservation utility strategy. Equation (12) now implies that for the two types of agents an individual agent will optimally choose cut-off-values in the taste difference as g is decreasing. For each type of agent there will be a different cut-off-value as f is increasing in the partner type's income. In particular, agent i now calculates two reservation taste levels $\bar{x}_i(\cdot)$ depending on the income type of proposing agents. Generally, as the lifetime utility of staying single V_i is an outside option independent of the proposal received the cut-off-value for high type proposals will be higher. I.e. an agent is less selective among high types. High type agents compensate for greater differences in taste by a higher income. Similarly to Lemma 1 we can give a general characterization of the reservation utility strategies as follows:

Lemma 2 *Reservation utility strategies for two income levels*

Given the expected distribution of offers by other agents $H_{-i}^K(x_K|y_i)$ where $K \in \{H, L\}$, agent i uses a reservation taste strategy and accepts all agents with $x \leq \bar{x}_i(y_K)$ where either

y_i, we write $H_{-i}^K(\cdot|y_i)$ instead of the (rather correct) $H_{-i}^K(\cdot)$ for $K \in \{H, L\}$.

1. $rV_i = f(y_i, y_K) + g(\bar{x}_i(y_K))$ *and*

$$rV_i = f(y_i, -) + \sum_K \frac{\alpha_i^K}{r} \int_0^{\bar{x}_i(y_K)} \left(g\left(x_K\right) - g(\bar{x}_i\left(y_K\right))\right) dH_{-i}^K\left(x_K|y_i\right) \tag{13}$$

or

2. $rV_i < f(y_i, y_K) + g(\bar{x}_i(y_K))$, $H_{-i}^K(\bar{x}_i(y_K)) = 1$ *and*

$$rV_i = f(y_i, -) + \sum_K \frac{\alpha_i^K}{r} E_{-i}^K\left(f(y_i, y_K) + g(x_K) - rV_i\right). \tag{14}$$

Equilibrium

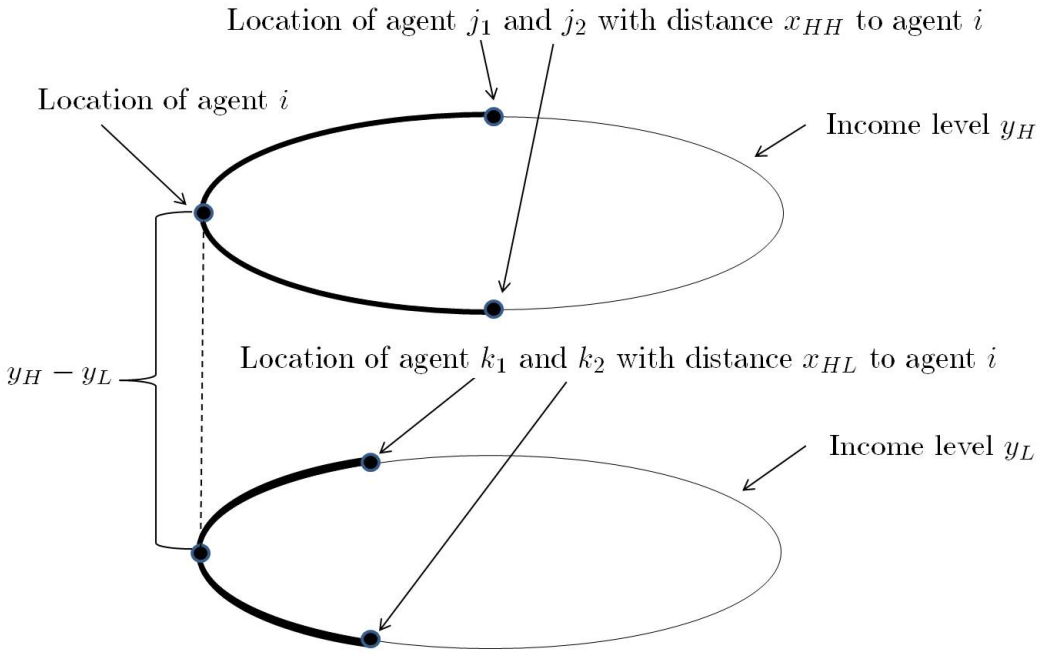
After the introduction of the two-sided matching-model with vertical heterogeneity we now discuss the equilibrium. So far we know, that individuals apply a reservation utility strategy which coincides with a cut-off-value concerning the taste difference for every type of agent. The reservation utility for every agent is determined endogenously.

As in the previous section we again focus on symmetric equilibria. All agents with the same income are ex-ante identical due to our assumptions on the distributions of all individual's traits. But compared to the case without vertical heterogeneity in this part we have two different types of agents, high types and low types. By imposing symmetry on the equilibrium outcome we restrict our considerations to cases where agents of the same type apply the same strategies. By the same reasoning as in the previous section agents of a particular type end up with the same lifetime utilities.

While the solution for the model without vertical heterogeneity is characterized by two equations for the two endogenous variables V and $\bar{x}$, there are now in principle 6 endogenous variables. These are the expected discounted lifetime utility for both high and low type agents (V_H and V_L) and the reservation qualities for type K_1 proposing to type K_2 which we denote by $x_{K_1K_2}$, where $K_1, K_2 \in \{H, L\}$. For example the *least acceptable agent* for high types proposing to low types is denoted by x_{HL}. Figure 2 shows the basic geometry for the model with vertical heterogeneity.

There are two income levels $y_H > y_L$ and hence agents are located on two circles. Due to the assumptions on the taste distribution they are distributed uniformly on the two circles (while the fraction of high and low type agents is equal). Exemplified, agent i is assumed to have high income y_H and is located to the very left. For any given offer distributions from the high types and the low types her best response is a

Figure 2: Geometry of the Two-Circles-Model



cut-off-strategy. The figure describes the two reservation qualities x_{HH} and x_{HL} for one particular high-type agent i. As partners with high income are more attractive compared to their low income equivalents with identical taste agent i is less selective among rich types, i.e. we have $x_{HH} \geq x_{HL}$.

As a first step, the set of initially six relevant unknown variables can be reduced. A simple observation shows that in equilibrium high types will have a higher reservation utility than low types due to their higher utility when being single. Suppose to the contrary they would have a lower reservation utility than their low type counterparts. Then, a high type could deviate to the strategy of a low type. As she is even more attractive than the low type due to her income she would get at least a low type's utility out of marriage events (as all agents use reservation utility strategies). In addition her utility out of staying single is higher. This shows that in equilibrium we must have $V_H > V_L$ as $y_H > y_L$. As a direct consequence of this high types will set a higher minimal taste quality for low types than the low types do when facing high types. Thus, we have $x_{HL} \leq x_{LH}$, i.e. for equilibrium considerations x_{LH} does not play a role as poor types are un-decisive concerning mixed marriages.[17]

After these preliminary steps we come to the more thorough discussion of the equilibrium. For the beginning of the analysis let us assume that the extent of

[17] Consequently, we do not even characterize x_{LH} in the solution which follows. Clearly, the least acceptable partner for low types among the rich is the one which gives the low type agent exactly her lifetime utility out of the equilibrium result described in the following.

the market is very large ($\theta \to \infty$). Intuitively, in this case a high type agent would even reject his poor type taste-equivalent, i.e. an agent with characteristics (y_H, t_i) will reject another agent with characteristics (y_L, t_i) because of intra-marital redistribution of income. Hence, high type agents generally reject low type agents and an income segregation equilibrium emerges. Only below a certain threshold $\tilde{\theta}$, high type agents will consider marrying low type agents. For all extents θ above this threshold the equilibrium solution exhibits $x_{HH} > 0$ and $x_{HL} = 0$. We will discuss the characteristics of $\tilde{\theta}$ at the end of the derivation of equilibrium.

We start with the case $\theta < \tilde{\theta}$, where mixed marriages occur in equilibrium. First, we look at the equilibrium strategy of a high type agent i. As all agents apply reservation utility strategies and by the symmetry argument it is clear that the distribution of outgoing and received proposals do not have holes. Furthermore, as $x_{HL} \leq x_{LH}$ we know that a high type agent i is decisive. Let $H^K_{-i}(x_K|y_H)$ denote the offer distributions of agents of type $K \in \{H, L\}$, where μ^K_{-i} denotes the fraction of singles of type K who are willing to marry agent i on contact. As both high and low types are equally likely, this implies that agent i faces an arrival rate of $0.5\alpha\mu^H_{-i}$ from high types and respectively $0.5\alpha\mu^L_{-i}$ from low types. By the decisiveness of agent i we can use the first part of Lemma 2 which yields

$$rV_i = f(y_i, -) + \sum_K \frac{0.5\alpha\mu^K_{-i}}{r} \int_0^{\bar{x}_i(y_K)} \left(g\left(x_K\right) - g(\bar{x}_i\left(y_K\right))\right) dH^K_{-i}\left(x_K|y_i\right). \quad (15)$$

As the offer distributions do not have holes and individuals are uniformly located on the two circles the relevant densities are $h^K_{-i}(x_K|y_i) = 1/\mu^K_{-i}$. Using the type-based notation this allows to rewrite equation (15) for high types as follows

$$rV_H = f(y_H, -) + \frac{1}{2}\theta \left[\int_0^{x_{HH}} g(x)dx + \int_0^{x_{HL}} g(x)dx - g(x_{HL})x_{HL} - g(x_{HL})x_{HL} \right]. \quad (16)$$

Using once more agent i's decisiveness the first part of Lemma 2 yields:

$$rV_H = f(y_H, y_H) + g(x_{HH}) \quad (17)$$

$$rV_H = f(y_H, y_L) + g(x_{HL}) \quad (18)$$

Altogether, the three equations (16), (17) and (18) give a full characterization of the first three variables of the model V_H, x_{HH} and x_{HL}. Equations (17) and (18) give a

further intuitive insight as

$$g(x_{HL}) - g(x_{HH}) = f(y_H, y_H) - f(y_H, y_L).$$

This result shows that under any specification of search frictions a high type agent "requires" a higher reservation quality from low type agents. More specifically, this compensation is a constant defined through the utility specification. The difference between the reservation qualities measured in g equals the income loss due to redistribution.

Now, we turn to the solution for a low type agent i. Recall that $x_{LH} \geq x_{HL}$, as high types are decisive for mixed marriages. From this we know that low type agents are only decisive among other poor agents; the decisiveness again holds because of the symmetry. Regarding Lemma 2 this indicates that one can use the integral part for the marriage among low types and one has to use the additive term for mixed marriages.

Hence, for a low type agent i Lemma 2 yields

$$\begin{aligned} rV_i = f(y_i, -) &+ \frac{\alpha_i^H}{r} E_{-i}^H \left(f(y_i, y_H) + g(\bar{x}_i(y_H)) - rV_i \right) \\ &+ \frac{\alpha_i^L}{r} \int_0^{\bar{x}_i(y_L)} \left(g\left(x_L\right) - g(\bar{x}_i\left(y_L\right)) \right) dH_{-i}^L \left(x_L | y_i\right). \end{aligned} \tag{19}$$

By the successive procedure of our equilibrium derivation the reservation quality for high types concerning low types x_{HL} is already determined. The mass of high types which proposes to agent i among the high types is $2x_{HL}/2 = x_{HL}$ due to individuals' uniform distribution on the circle. Alternatively put, the fraction of high type agents proposing to low type agents μ_{-i}^H is equal to x_{HL}. This simplifies the arrival rate of offers from the high types to $\alpha_i^H = 0.5\alpha x_{HL}$. Furthermore, the density of $H_{-i}^L\left(x_L | y_i\right)$ is $h_{-i}^L = 1/\mu_{-i}^L$.

The decisiveness of the poor types among themselves by Lemma 2 yields

$$rV_L = \quad f(y_L, y_L) + g(x_{LL}). \tag{20}$$

Using these simplifications altogether and rewriting equation (19) in the type-based notation then yields

$$rV_L = f(y_L, -) + \frac{1}{2}\frac{\alpha}{r} x_{HL} \left[f(y_L, y_H) + E\left(g(x) \mid x \leq x_{HL}\right) - f(y_L, y_L) - g(x_{LL}) \right] + \frac{1}{2}\frac{\alpha}{r} \left[\int_0^{x_{LL}} g(x)dx - x_{LL} g(x_{LL}) \right]. \tag{21}$$

We summarize the results for $\theta < \tilde{\theta}$ in the following proposition.

Proposition 2 *Integration equilibrium*

The solution for the five unknown endogenous variables is characterized by the following system of equations:

$$(I)\; rV_H = f(y_H, y_H) + g(x_{HH})$$

$$(II)\; rV_H = f(y_H, y_L) + g(x_{HL})$$

$$(III)\; rV_H = f(y_H, y_H) + \frac{1}{2}\frac{\alpha}{r} \left[\int_0^{x_{HH}} g(x)dx + \int_0^{x_{HL}} g(x)dx - x_{HH} g(x_{HH}) - x_{HL} g(x_{HL}) \right]$$

$$(IV)\; rV_L = f(y_L, y_L) + g(x_{LL})$$

$$(V)\; rV_L = f(y_L, y_L) + \frac{1}{2}\frac{\alpha}{r} x_{HL} \left[f(y_L, y_H) + E\left(g(x) \mid x \leq x_{HL}\right) - f(y_L, y_L) - g(x_{LL}) \right] + \frac{1}{2}\frac{\alpha}{r} \left[\int_0^{x_{LL}} g(x)dx - x_{LL} g(x_{LL}) \right]$$

Equation (III) of the solution captures the standard reservation equation for high type agents. It implies that the flow value of search rV_H is equal to the current payoff $f(y_H, -) = f(y_H, y_H)$ plus the expected surplus generated by the optimal search strategy. This optimal strategy is given by x_{HH} and x_{HL}. Equations (I) and (II) show another standard reservation strategy result: High agents must be indifferent between searching further and marrying the (optimally chosen) least acceptable partner among high types (in taste distance x_{HH}) and among low types (in taste distance x_{HL}). In addition the two equations capture the constraints for this strategy discussed above; that is high type agents require compensation along the horizontal dimension from low type agents. Equation (V) describes the standard reservation equation for low type agents. Since low type agents are not decisive for mixed marriages, they treat x_{HL} essentially as given. Therefore, the first bracket of the right hand side can be interpreted as the current payoff including a stochastic

component generated by possible mixed marriages. The second bracket equals the expected surplus from choosing the optimal reservation quality within their own types. Low types have only one decision variable and hence equation (IV) completes the solution. This equation describes the usual indifference condition and is comparable to equations (I) and (II).

Worth mentioning is the procedure through which we derived Proposition 2. The concept of decisiveness we used is generalizable to all (discrete) numbers of agents as we will show in the section about possible extensions of the model. For the case of two agents the high types are decisive and *decide* whom to marry. The poor types take the high types' decisions in equilibrium as given and optimize only among their own type actively.

We now turn to the case $\theta > \tilde{\theta}$. By the definition of $\tilde{\theta}$ no (decisive) high type will ever accept a proposal of a low type agent. Hence, in this case there are no mixed marriages and from the perspective of an agent the only considerable offers she faces are from agents of her own type. Therefore, for these values of θ, the two types of agents partition themselves into the two classes in equilibrium and we can immediately apply the results from the case without vertical heterogeneity.

To solve for the symmetric equilibrium we use Proposition 1 for the two types of agents separately. The only difference to our discussion of the model without vertical heterogeneity in section 2 is, that the adjusted arrival rate is now 0.5α. This is because of the random matching assumption and the fact that half of the individuals are of either type. The result for the segregation equilibrium is shown in the following Proposition:

Proposition 3 *Segregation equilibrium*

The solution for the three endogenous variables is characterized by the following system of equations.

$$\begin{aligned} g(\bar{x}) &= \frac{\theta}{2}\left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x})\right] \\ rV_H &= f(y_H, y_H) + g(\bar{x}) \\ rV_L &= f(y_L, y_L) + g(\bar{x}) \end{aligned}$$

As the first equation holds independently of the income type, both types of agents use the same reservation quality $x_{LL} = x_{HH} = \bar{x}$. This is due to the fact that agents do not face intra-marital redistribution of income because of segregation. But clearly,

the utilities of either type differ according to the different income levels. This can be seen in Proposition 3 as $f(y_H, y_H) > f(y_L, y_L)$.

As a final remark we give a characterization of the critical level of search frictions $\tilde{\theta}$ which separates the integration equilibria from the segregation equilibria. The threshold extent $\tilde{\theta}$ is implicitly characterized by equation (I) and (II) from Proposition 2 which gives

$$f(y_H, y_H) + g(x_{HH}) = f(y_H, y_L) + g(0). \tag{22}$$

I.e., at $\tilde{\theta}$ a high type agent is indifferent between accepting the offer from a high type with taste distance x_{HH} or accepting the offer of a low type with taste distance 0. As equation (22) shows, the threshold extent directly translates into a reservation quality among high types $\tilde{x}_{HH}$. This threshold reservation quality $\tilde{x}_{HH}$ can be determined by equation (III) from Proposition 2 by using $x_{HL} = 0$. It is easy to show that this equation and the first equation from Proposition 3 coincide when $x_{HL} = 0$. Hence, the threshold extent is well-defined. This completes the solution.

Properties of the general solution

In the following we discuss some properties of the general solution of the model with vertical heterogeneity and give some intuition for the results developed so far. As the exogenous parameter in the model relevant for equilibrium behavior is the extent θ we discuss the properties of the solution regarding changes in θ. We therefore write particular endogenous variables of the model as functions of the extent θ.[18]

First, we want to discuss some continuity properties of the general solution:

Proposition 4 *Continuity Properties*

The reservation qualities $x_{HH}(\theta), x_{HL}(\theta)$ and $x_{LL}(\theta)$ are continuous in $\theta \in \mathbb{R}_+$.

The continuity for $\theta \neq \tilde{\theta}$ is straightforward as the optimal cut-off strategies are adjusted smoothly by the agents. When search frictions increase (θ decreases) starting at $\tilde{\theta}$, high type agents start sending offers to low types. High type agents apply a reservation utility strategy and as the utilities out of marriage are distributed in a continuous way on the two circles there will be continuity in their reservation

[18] Note the small difference of the two components of extent θ, which are arrival rate α and discount factor r. As long as an agent stays single both of them are relevant for the optimal decision. From the point in time onwards when an agent gets married α becomes irrelevant for her as the search stops through marriage.

qualities. For low types these first proposals from high types slightly below $\tilde{\theta}$ start out as probability zero events and hence their reservation utility is not influenced in a discontinuous manner. Therefore, there is no discontinuity in their proposal behavior among themselves. For the reservation quality which high types require from low types we already established that $x_{HL} = 0$ and hence continuity is obvious.

The continuity property is worth to be stressed from another angle of view. In the search and matching literature so far the focus is on (continuous) vertical heterogeneity only. For example the approaches of Burdett and Coles (1997), Eeckhout (1999) and Smith (2006) share the common feature that agents partition themselves into a number classes of agents with identical (reservation) utilities. Search frictions mainly influence the size of these classes. The intuition behind the partitioning result is that agents can pool with better types which are willing to marry them. I.e. as long as the *best* type in a particular class is proposing to another agent the latter agent in effect has the same reservation utility as the best agent in her class and therefore applies the same reservation utility strategy as the best agent. From an applied perspective the result may seem questionable as the reservation utility of each agent (but the highest) drops sharply if search frictions decrease in a way that the agent falls out of the class. This discontinuity emerges because at certain levels of search frictions the agent is not decisive for any marriage he will ever be engaged in. Our approach results in a continuous adjusting of reservation utility strategies. This is because the agents are always decisive for some marriages at least among their own vertical class. In effect, they adjust strategies continuously whenever search frictions decrease.[19]

We proceed by analyzing the limiting behavior of the different reservation qualities:

Proposition 5 *Limiting Properties*

The limiting behavior of the reservation qualities is as follows:

a) $\lim_{\theta\to 0} x_{HH}(\theta) = 1, \lim_{\theta\to 0} x_{LL}(\theta) = 1$ *and* $\lim_{\theta\to 0} x_{HL}(\theta)$ *is given by* $g(x_{HL}) = f(y_H, y_H) - f(y_H, y_L)$.

b) $\lim_{\theta\to\infty} x_{HH}(\theta) = 0, \lim_{\theta\to\infty} x_{LL}(\theta) = 0$

The first part of Proposition 5 covers the case of maximum search frictions. Due to our normalization of function $g(x) \geq 0$ for all possible taste differences x agents always (weakly) prefer marrying within their own class rather than staying single.

[19] This feature of our model is *not* driven by reducing the number of (vertical) types to two. We will come back to this point when we speak about the generalization to the N-type case.

As a high type faces re-distributional losses when marrying a low type, she is only willing to marry low type agents who compensate her for this loss. This holds as well for maximum search frictions and explains the limiting behavior of $x_{HL}(\theta)$. The second part of Proposition 5 is straightforward. In a market without search frictions complete segregation emerges. Every agent sets her reservation quality to the maximum as she meets all agents instantaneously.

Proposition 6 *Differentiability Properties*

a) Reservation quality $x_{LL}(\theta)$ is differentiable for all $\theta \in \mathbb{R}_+ \setminus \tilde{\theta}$ and is not differentiable in $\tilde{\theta}$.

b) Reservation quality $x_{HH}(\theta)$ is differentiable for all $\theta \in \mathbb{R}_+$ and is decreasing in θ.

c) Reservation quality $x_{HL}(\theta)$ is differentiable for all $\theta \in \mathbb{R}_+ \setminus \tilde{\theta}$ and is non-increasing in θ.

Once again, the differentiability properties outside $\tilde{\theta}$ are straightforward. Agents smoothly adjust their reservation utility strategies in response to a small change in search frictions. In contrast, around $\tilde{\theta}$ only x_{HH} is differentiable.

When search frictions increase (θ decreases) beyond $\tilde{\theta}$, high types start to propose to low type agents. Contrary to low types, high types are decisive. As marriages with low types start out as a probability zero event, high types do not need to adjust the change rate of their proposals to other high types when search frictions increase. Hence, the solution for x_{HH} exhibits differentiability in $\tilde{\theta}$.

In general, reservation qualities x_{HL} and x_{LL} are not differentiable in $\tilde{\theta}$. For an extent slightly below $\tilde{\theta}$, low types receive offers from high types with a positive probability. Low type agents receive a larger utility gain from these offers than from the least acceptable partner within their own class. Since the low types are not decisive for these mixed marriages, one can interpret the possibility of marrying a high type as a new opportunity which happens by chance. Since low types correct their reservation quality for these lucky events, their reservation quality is not differentiable in $\tilde{\theta}$. An example which illustrates this is discussed in section 4. We then show that the low types may even profit from higher search frictions because of these lucky events.

3.3. Comparative Statics

In section 2 we defined an agent's utility as the sum of two sources: (i) Utility out of an household's income and (ii) utility out of the taste characteristics of the two spouses (if married). By defining a range for income-related payoff f and for taste-related payoff g one can weigh the two sources of utility; e.g. if income-related utility is relatively high compared to the possible gain through taste, agents will be more concerned about the income of their partner than on her taste characteristic. To allow for some comparative statics in these different specifications of our model we now assign weights to the two parts of utility. Subsequently, we discuss the quantitative changes associated with variations in these weights.

In particular, let $\beta \in (0, 1)$ denote the weight for the payoff related income and let $1 - \beta$ be the weight on taste-related utility. Agent i's utility if married to agent j is then defined as

$$U(y_i, t_i, y_j, t_j) = \beta f(y_i, y_j) + (1 - \beta)g(x),$$

where x is the taste difference as before.[20] The limiting cases $\beta \to 0$ and $\beta \to 1$ describe agents who are "romantically minded" or "greedy for money" respectively.[21]

First, we want to highlight that these weights do not change the general analysis at all. As our concept of utility is ordinal, we can simply transform the utility function by dividing with weight β. This transformed utility function is

$$\tilde{U}(y_i, t_i, y_j, t_j) = f(y_i, y_j) + \tilde{g}(x) \text{ with } \tilde{g}(x) = \frac{(1 - \beta)}{\beta} g(x).$$

As $\tilde{g}$ is decreasing and $\tilde{g}(1) = 0$ the transformed utility $\tilde{U}$ meets all requirements on the utility specification imposed in section 2. Hence, the analysis does not change and replacing g by $\tilde{g}$ in Proposition 1 yields the solution for the case without vertical heterogeneity. Likewise, Proposition 2 and Proposition 3 implicitly characterize the solution for the integration and segregation equilibria respectively.

In the following, we discuss the comparative statics for the reservation qualities associated with the limiting cases $\beta \to 0$ and $\beta \to 1$. By equations (I) and (II) of Proposition 2 the critical market extent $\tilde{\theta}$ which divides the integration equilibrium

[20] The only thing which therefore remains normalized as before is $g(1) = 0$. Again, this just simplifies in a sense, that from a taste-perspective everyone could be acceptable to anyone else.

[21] We stick to the assumption that all agents have the same utility function. Weakening this requires the model more fundamentally.

and the segregation equilibrium (where $x_{HL} = 0$) is implicitly defined by

$$\beta(f(y_H, y_H) - f(y_H, y_L)) = (1 - \beta)(g(0) - g(x_{HH})). \tag{23}$$

This can also be seen directly in equation (22) when using the modified utility specification. As before, this equation yields an expression for x_{HH} at the critical market extent $\tilde{\theta}$. It is obvious from equation (23) that an increase of β requires a higher threshold x_{HH} at $\tilde{\theta}$ as g is decreasing in x. What are the implications for $\tilde{\theta}$? From the previous analysis we know that the high types' reservation qualities are increasing in θ, i.e. the lower the search frictions (increasing θ) the lower the taste difference for the least acceptable partners (e.g. $\bar{x}$ is decreasing). Hence, this shows that higher values of β are associated with smaller values of the critical market extent $\tilde{\theta}$.

Using this result, it is straightforward to show that the integration equilibrium does not exist for values of β beyond an upper bound $\tilde{\beta}$. Note that the right-hand-side of equation (23) captures the maximum possible gain along the horizontal dimension relative to the least acceptable high type partner. It is bounded by $(1 - \beta)g(0)$ as $g(1) = 0$. Recall that the left-hand-side of this equation describes the income loss for the high type agent due to intra-marital redistribution when marrying a low type. Imposing the constraint that high type agents never consider marrying low types requires that the redistributional loss is larger than the maximum gain along the taste dimension. Hence, we have to solve the respective inequality

$$\beta(f(y_H, y_H) - f(y_H, y_L)) > (1 - \beta)g(0)$$

for β. It follows that for

$$\beta > \tilde{\beta} := \frac{g(0)}{f(y_H, y_H) - f(y_H, y_L) + g(0)}$$

complete segregation emerges for all values of θ. Clearly, we have $0 < \tilde{\beta} < 1$. As before, both low and high type agents apply the same cut-off-value concerning the taste difference within their own class in this equilibrium. It is fully characterized by Proposition 3 (by using $\tilde{g}$) and $x_{HL} = 0$.

Intuitively, if $\beta > \tilde{\beta}$ a high type agent is potentially proposing to every other high type agent and still prefers the worst high type agent to the most preferred low type agent. As marrying the least acceptable high type for her is equivalent to staying single (due to $g(1) = 0$ and $f(y_H, -) = f(y_H, y_H)$) we have complete segregation. Therefore, for

high weights on the income function f the model with vertical heterogeneity can be viewed as individuals living in two separate worlds or castes. With this interpretation the arrival rate of potential mates with identical income is $\alpha/2$.

Obviously, decreasing β is associated with larger values of $\tilde{\theta}$. In the limiting case, income becomes irrelevant for the agents' decision problem as in the case without vertical heterogeneity. Hence, the functions x_{HH}, x_{HL} and x_{LL} converge pointwisely to the same function which is the one-circle solution for an arrival rate of α. Individuals live in a world where nobody cares about vertical heterogeneity in the limit.

In sum, we have established two results. First, from a purely technical point of view assigning weights to the two parts of the utility function does not change the analysis. More interestingly, in both limiting cases the equilibrium can be derived from the solutions for the model without vertical heterogeneity. However, the critical threshold $\tilde{\theta}$ differs across the two limiting cases.

The reasoning for the two limiting cases is also quite different. Intuitively, for large values of β only high types are acceptable for a high type. Hence, low types are stuck with each other and this leads to complete segregation. Since the markets of low and high type agents are divided into two separate parts, the arrival rate of offers is equal to 0.5α. Hence, the reservation qualities for this case can be obtained by inserting this modified arrival rate into the one-circle-solution. By contrast, for low values of β income is not very important and in the limiting case ($\beta \to 0$) income does not affect agents' decisions. In this case both low and high type agents are basically identical. Hence, the result converges to the one-circle solution with the original arrival rate α.

4. An Example

In the previous section we characterized the general solution of the two-sided matching problem with vertical and horizontal heterogeneity. We illustrate the results in this section by the means of an example. In order for doing so, we have to choose a specification for the redistribution of intra-marital household income $f(\cdot, \cdot)$. Furthermore, we specify the intra-marital utility by taste $g(\cdot)$ which is defined as a function of the distances in taste for married couples.

We assume that singles and spouses receive the average household income:

$$\begin{aligned} f(y_H, y_H) = f(y_H, -) &= y_H \\ f(y_H, y_L) = f(y_L, y_H) &= \frac{y_H + y_L}{2} \\ f(y_L, y_L) = f(y_L, -) &= y_L \end{aligned}$$

By this symmetric choice, both partners within a marriage receive the same utility, although partners may have been differently endowed ex-ante. Let intra-marital utility through taste be linearly related to the distance:

$$g(x) = 1 - x.$$

Clearly, this specification fulfills all the assumptions from the previous section.

Inserting these functions into Propositions 2 and 3 yields the explicit solution as follows:

Corollary 2 *Explicit solution for the example*

a) For $\theta \leq \tilde{\theta}$ an integration equilibrium emerges, where the five unknown endogenous variables V_H, V_L, x_{HH}, x_{HL} and x_{LL} are characterized by the following system of equations:

$$\begin{aligned} (Ia) \quad rV_H &= y_H + 1 - x_{HH} \\ (IIa) \quad rV_H &= \frac{y_H + y_L}{2} + 1 - x_{HL} \\ (IIIa) \quad rV_H &= y_H + \frac{1}{4}\theta \left[x_{HH}^2 + x_{HL}^2\right] \\ (IVa) \quad rV_L &= y_L + 1 - x_{LL} \\ (Va) \quad rV_L &= y_L + \frac{1}{2}\theta x_{HL} \left[\frac{y_H - y_L}{2} + x_{LL} - \frac{x_{HL}}{2}\right] + \frac{1}{4}\theta x_{LL}^2 \end{aligned}$$

b) For $\theta > \tilde{\theta}$ a segregation equilibrium emerges, where the three unknown endogenous variables V_H, V_L and $\bar{x}$ are characterized by the following system of equations:

$$\begin{aligned} (Ib) \quad \bar{x} &= \frac{2}{\theta} \cdot (\sqrt{1+\theta} - 1) \\ (IIb) \quad rV_H &= y_H + 1 - \bar{x} \\ (IIIb) \quad rV_L &= y_L + 1 - \bar{x} \end{aligned}$$

c) The threshold extent $\tilde{\theta}$ is given by

$$\tilde{\theta} = 8\frac{2-(y_H-y_L)}{(y_H-y_L)^2}. \tag{24}$$

Figure 3 shows the solution for the reservation qualities $x_{HH}(\theta), x_{HL}(\theta)$ and $x_{LL}(\theta)$ for the specification given above. As parameters we use $y_H = 1$ and $y_L = 0.25$. Agents discount future utility with $r = 0.05$.[22] By equation (24) the threshold extent $\tilde{\theta}$ is equal to $\frac{160}{9}$.

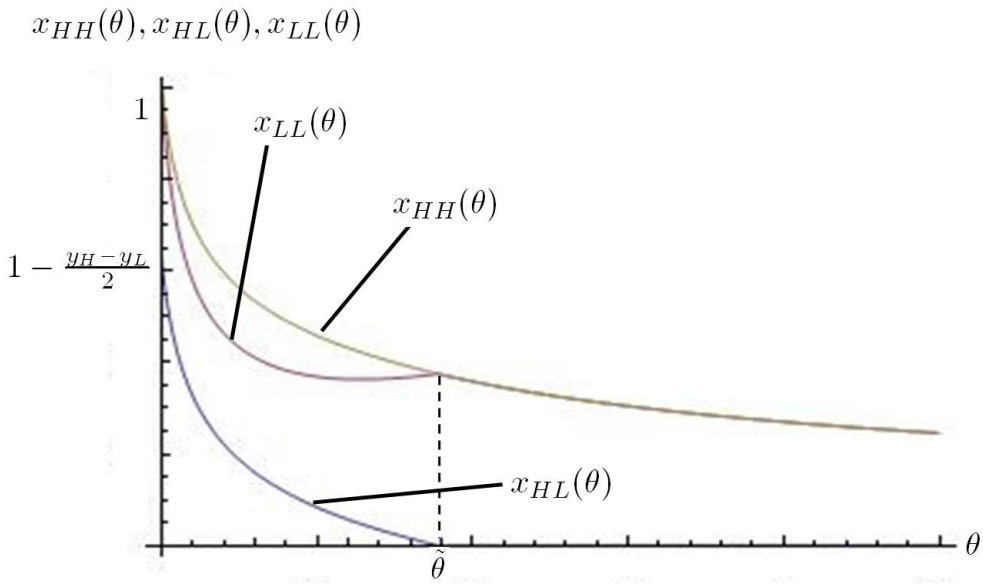
For a market extent above $\tilde{\theta}$ the solution exhibits complete segregation. In this case there are no mixed couples and $x_{HL}(\theta) = 0$. The two reservation qualities for high types within themselves and for low types within themselves coincide as established by Proposition 3. Again, as there are no mixed couples intra-marital redistribution does not play a role. Therefore, agents condition their proposals among their own type only on taste and $\bar{x}$ is identical for both types. In the limit ($\theta \to \infty$) the two reservation qualities $x_{HH}(\theta)$ and $x_{LL}(\theta)$ converge to zero. This is because an infinite market extent implies that all agents meet each other instantaneously.

For $\theta \leq \tilde{\theta}$ integration equilibria emerge. For values around $\tilde{\theta}$, the three reservation qualities clearly exhibit the continuity and differentiability properties discussed in the previous section. Equations (IIa) and (IIIa) of Corollary 2 imply $x_{HH} - x_{HL} = \frac{y_H - y_L}{2}$ which equals $\frac{3}{8}$ in our specific example. This distance is the "compensation" in taste required by high type agents from low type agents for the income loss due to redistribution. Our assumption for $g(\cdot)$ implies that this distance in reservation quality is constant in the example. Hence, in the figure $x_{HL}(\theta)$ and $x_{HH}(\theta)$ are parallel functions for $\theta \leq \tilde{\theta}$. For maximum search frictions (θ close to zero) the reservation qualities x_{HH} and x_{LL} converge to one. In this case all agents propose to all agents within their own class. This is due to the normalization $g(1) = 0$ which assures that marriages among identical income types are always preferred to staying single forever. As discussed, high type agents do not accept all low type agents, but only those who yield at least the value of staying single. The value for $x_{HL}(0)$ can be calculated by using part a) of Proposition 5. In our specific example $x_{HL}(0) = 1 - \frac{y_H - y_L}{2} = \frac{5}{8}$.

A notable feature is that for a certain range of θ the reservation quality within low type agents is *decreasing* in θ. That is, although the probability of meeting other low

[22] We set incomes and the discount rate r to specific values to simplify the example as much as possible. This implies that the figure shows the reservation qualities for different values of α. Since the qualitative findings are independent of specific values of r, we denote all functions as functions depending on θ.

Figure 3: Reservation qualities

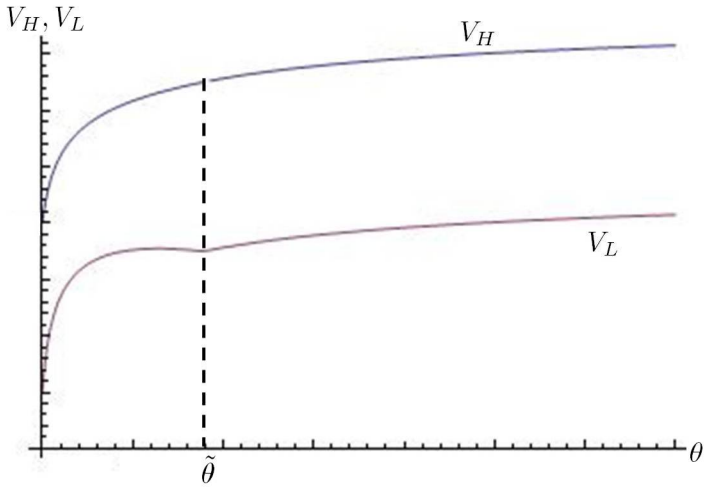


type agents increases, low agents are less selective within their own class. Intuitively, the decisive high type agents reduce the set of acceptable low type partners if the market extent increases. As a consequence low type agents realize that the *lucky* event of being matched to a high type becomes more unlikely. Hence, in this range they get less picky regarding low types although market conditions improve. Figure 4 highlights the consequences of this feature on agents' lifetime utilities.

For the aforementioned range of $\theta < \tilde{\theta}$ low type agents would be better off if the extent of the market shrinks slightly. This is driven by the positive probability of being matched with a high type. Low type agents prefer these mixed matches, because they marry a partner with similar taste and a high income. By contrast, high type agents dislike the income reduction through redistribution and would always prefer lower search frictions.

This result from the example is in harsh contrast to the previous literature on search and matching markets in models with vertical heterogeneity only like Burdett and Coles (1997). There, the agents partition themselves into classes. Within each class all agents have the same lifetime-utility. An agent only suffers from decreased search frictions if she drops out due to a marginal change in the search frictions at discrete values of the frictions. This is a purely mathematical result. In our model agents somehow adjust their strategies when search frictions lead to reduced offering behavior from higher types. By the horizontal heterogeneity among types high types continuously reduce their offers to lower types when search frictions decrease. Depending on the mass of high type agents this already leads to a reduction in the

Figure 4: Lifetime utilities



expected lifetime-utilities of lower type agents. Incorporating this they suffer from lower search frictions and adjust their behavior in a continuous manner. Differently put, decreasing search frictions has two effects: (i) The direct effect leads to meeting potential partners more often. (ii) The indirect effect leads to agents possibly being more selective when deciding on whom to propose to. The example shows that the second, indirect effect may outweigh the first, direct effect of decreasing search frictions.[23]

5. Extensions

In this section we want to show possible extensions of the framework we discussed above.

The first extension concerns the restriction to two types of agents only. We show that the method for solving the model works for any discrete number for the types of agents.

In the second part we see that it is dispensable that the taste related payoff is related to the distance in location. One can skip w.l.o.g. the intuitive, but very specific interpretation of the agents being located on a circle. This does not change the general results.

In a third section we conclude with a discussion of the assortativeness of the results from the general model from section 3. In general, there is a higher degree of positive

[23] Clearly, only agents which are not decisive concerning potentially formed marriages may suffer from decreased search frictions.

assortativeness concerning the vertical income attribute of the individuals when search frictions become lower. But, for intermediate ranges of market extent θ there is a range where this only partially holds true.

5.1. More than two types of income

In the previous sections we have analyzed the case of at most two discrete levels of income and horizontal heterogeneity. A central assumption regarding the horizontal dimension is symmetry in payoffs, that is both spouses receive exactly the same noneconomic gain when married (cf. Konrad and Lommerud 2010). This symmetry allows us to partially apply results known from the model involving only a vertical dimension (e.g. Burdett and Coles 1997). In particular, this symmetry implies that the high type agent is decisive for mixed marriages. Hence, in equilibrium all low-type agents who receive offers from a particular high type agent also propose to this agent. This observation is crucial to our model as it allows to analyze equilibrium stepwise. Starting from the decisive high type the low types take the high type's equilibrium behavior essentially as given and perform a conditional maximization. In the following, we will exploit this idea to establish an integration equilibrium for N discrete income levels $y_1 < y_2 < ... < y_N$. In generalization to the model discussed so far, now each type except the lowest type will be decisive concerning agents with lower vertical traits. Again, we denote individuals by small letters and income classes with capital letters if a distinction seems helpful. For the following let $J = \{1, 2, ..., N\}$.

Let α_i^J denote the probability that agent i contacts a single of type J who is willing to marry her. Generally, we denote the cumulative probability that the distance of an agent of type J who is willing to marry individual i as $H_{-i}^J(x_J|y_i)$. In comparison to the cases discussed so far there are now in general N different types of agents proposing to individual i and hence the lifetime utility is determined by

$$\begin{aligned} V_i = \frac{1}{1+\Delta r}\Bigg[&\Delta f(y_i, -) \\ &+ \Delta\left(\sum_J \alpha_i^J\right)\left(\sum_K \frac{\alpha_i^K}{\sum_J \alpha_i^J} E_{-i}^K\left(\max\left\{V_i, \frac{f(y_i, y_K) + g(x_K)}{r}\right\}\right)\right) \\ &+ (1 - \Delta(\sum_J \alpha_i^J))V_i\Bigg] + o(\Delta). \end{aligned} \tag{25}$$

For given offer distributions, we could now reformulate Lemma 2 to describe the

case of N different income levels. However, as the reasoning is identical we proceed directly to equilibrium behavior.

For each type of agent J there is one reservation utility level V_J. Furthermore, each type of agent J has N different reservation qualities concerning partners from every potential type of agent. Hence, there are $N+1$ unknowns for each type of agent which sum up to $N \cdot (N+1)$ unknowns to describe individuals' equilibrium strategies completely.

As before matches are formed only by mutual agreements. Exactly as with the case of only two income levels, higher types have higher reservation utilities. Hence they are the decisive players when agents of different income levels propose to each other. It follows that the amount of unknowns relevant for equilibrium behavior can be reduced to those relevant for the formation of matches. As Type N agents are the highest type in the market, they are always decisive. Therefore, there are N binding reservation quality equations associated with type N agents. We call these "decisiveness equations" in the following. An agent of type $N-1$ is decisive concerning all lower types and her own type which yields $N-1$ relevant decisiveness equations. In general, an agent of type J applies J relevant strategies for equilibrium behavior. In addition there are N unknowns concerning the reservation utilities. Thus, the equilibrium is fully described by this set of collectively

$$N + \sum_J J = N + \frac{N \cdot (N+1)}{2}$$

unknowns.

Since equilibrium requires that agents have correct beliefs about the offers they receive, there is a unique reservation utility for each agent of type J. The expression for this reservation utility is a straightforward generalization of Lemma 2. This step provides N equations in total. Furthermore, each type combination between two individuals of types K and respectively L where $K \geq L$ yields an additional equation of the form

$$rV_K - f(y_K, y_L) = g(\overline{x}_K(y_L)).$$

There are N equations concerning the reservation utilities. For each $K \geq L$ there are K additional equations. Adding up yields

$$N + \sum_J J = N + \frac{N \cdot (N+1)}{2}$$

equations in total which equals the number of unknowns. Hence, this system of equations characterizes the equilibrium solution. Once again, without specifying the distance and payoff functions, no explicit derivations can be offered.

Depending on the difference between the income levels of the different types of agents the solution exhibits segregation as well as integration equilibrium features between different types. As in the previous sections let x_{KL} denote the least acceptable L-type-agent from the perspective of K-type-agents. Hence, whenever a K-type-agent meets an L-type agent of quality x_{KL}, she is indifferent between staying single or marrying this agent. The following Proposition summarizes the system of equations for high search frictions in the sense that even the highest (N) type is willing to marry some agents of the lowest type[24]:

Proposition 7 *Complete integration equilibrium*

The solution for the $N + \frac{N\cdot(N+1)}{2}$ unknown endogenous variables is characterized by a system of two types of equations.

Decisiveness equations:
$rV_K = f(y_K, y_L) + g(x_{KL})$ *for all* $K \geq L$
Reservation utility equations:

$$\begin{aligned} rV_K = f(y_K, -) & \\ &+ \frac{1}{N}\frac{\alpha}{r}\left[\sum_{L=K+1}^{N} x_{LK}[f(y_K, y_L) + E\left(g(x) \mid x \leq x_{LK}\right) - f(y_K, y_K) - g(x_{KK})\right] \\ &+ \frac{1}{N}\frac{\alpha}{r}\sum_{L=1}^{K}\left[\int_0^{x_{KL}} g(x) - g(x_{KL})dx)\right] \end{aligned}$$

In general, depending on the extent of the market, there emerge all intermediate equilibria between the perfect integration equilibria discussed in Proposition 7 and the perfect segregation equilibria. We do not provide solutions for all these intermediate cases. The method to construct these cases is clear from the discussion above, while writing down all possible cases is lengthy and involves bulky notation.

We conclude this section with the remark that the continuity and differentiability properties are passed on to the reservation utilities in the same manner as in section 3.2. All the reservation qualities are continuous in the market extent everywhere. Furthermore the reservation qualities are not differentiable for market extents where

[24] Clearly, by the fact that reservation utility is increasing with type and all agents apply reservation utility strategies this implies that every two types of agents are possibly matched in equilibrium.

a type receives the first proposal from an upper class. This missing differentiability for a certain type is inherited to all lower classes as agents of this particular class adjust their behavior concerning all types they make proposals to so far. Both of these properties hold also true for the reservation utilities of all types.[25]

5.2. Different distribution for taste-related payoff

A second possible avenue for altering the payoff structure is to change directly the payoffs. So far, we interpret the utility gain along the horizontal dimension as a result from two steps. First, individuals are randomly and uniformly located on a circle. Second, agents are randomly matched to each other. In general, there are many ways to define utility via the two locations of the spouses; e.g. in a marriage market model with men and women the utility induced by taste could be determined solely through the location of the woman. The central idea of our model is that agents prefer to be matched with agents who are horizontally close to themselves. Therefore, an obvious choice for linking location and the resulting taste-related payoff is the distance x between the two locations. In the model we consider a very general measure of distance, namely $g(x)$.

Since agents are randomly matched to each other and the taste (location) parameter t is uniformly distributed on $[0, 2]$, the distance X of two randomly matched agents is uniformly distributed on $[0, 1]$. However, the solution method and the general form of g allow for a wide range of random variables capturing the taste-related payoff in the horizontal dimension as the following Proposition shows:

Proposition 8 *Possible distributions for match-specific values*

Let $X \sim U[0, 1]$. Let $H_Z(z)$ denote the cumulative distribution function of a random variable Z where Z has finite support $[a, b] \subset \mathbb{R}$. Choosing a utility specification $g(x) := b - H_Z^{-1}(x)$ for the taste-related utility gain is equivalent to solving the model with $Z \sim H_Z(z)$ and $g(z) = b - z$.

Proposition 8 shows that the assumption of the uniform distribution for the taste-related payoff (distance) is not restrictive. The payoff can follow any continuous

[25] In contrast to the continuity result concerning the reservation utility strategies the missing differentiability is a consequence from the discreteness of the number of vertical types. The differentiability is affected by the strictly positive mass of agents assigned to a certain vertical type. If all types would be atomless in a continuous distribution of types this feature is unclear so far. However, our approach uses the discrete number of types and the continuous case is postponed to further research.

probability distribution which has finite support. Hence, there are virtually no restrictions concerning the distribution of match-specific values. As before, the distance $b - z$ is just a normalization which implies that agents are never matched to individuals who would yield a negative payoff along the taste dimension. As in the untransformed model we could easily allow negative payoffs.

5.3. The assortativeness of the model with both types of heterogeneity

The notion of assortativeness used in the literature on search and matching with heterogenous agents is well-known. For an introduction of the concept we refer to Burdett and Coles (1999). In addition to the theoretical literature there is a lot of empirical work on the assortativeness of spouses concerning particular traits. Becker (1973, 1974) finds a positive correlation between partners' education levels and partners' height. Recent examples in the sociology literature include Kalmijn and van Tubergen (2006) who find a correlation between ethnicity, while Kalmijn (2006) and Mare (1991) document assortative mating for religion and education. Much less empirical literature exists on the similarity of horizontal traits such as e.g. religion or location. However, this literature indicates that such traits play a major role in marriage decisions (e.g. Lundberg and Pollak 2007, Hitsch et al. 2010).

Alas, there is no unique answer how to generalize the term of assortativeness from a model of vertical heterogeneity only to our model of vertical *and* horizontal heterogeneity. What seems clear is, that by construction the two kinds of traits in our model serve as substitutes for each other: If an agent is willing to propose to a low type agent she has to be compensated for this by a higher similarity in the horizontal component. Similarly, to form a match with a partner of the high type an agent is willing to incur compromises concerning the *emotional quality* of the match reflected by a potentially relatively bad match in the horizontal trait.

A first possibility to define assortativeness is as *similarity* in traits between matched agents. This is in line with the common meaning of the term *assortativeness*. Referring to such a definition clearly decreasing search frictions increase the assortativeness among formed matches. Better matching institutions lead to high types reducing their reservation quality among high and low types. In effect, for high types an assortativeness result holds in the following sense: If search frictions decrease (i) the reservation quality for every potential partner decreases and (ii) the probability of agreeing to match with a partner of a lower type relatively decreases. The second statement of this interpretation follows as the difference $f(y_H, y_H) - f(y_H, y_L) = g(x_{HH}) - g(0)$ is a constant and hence, the relative probability $x_{HH}/(x_{HH} + x_{HL})$ of marrying a

high type in the case of forming a match generally increases with decreasing search frictions.

Similarly, the low type agents adjust their behavior when facing decreasing search frictions. Two things seem worth mentioning in this context. When search frictions are maximal any match which assures the parties a utility of at least their single utility is formed. When search frictions decrease, low type agents become far more selective among their own type then among the rich type. I.e. for low values of the extent θ the reservation quality increases faster among their own type. Hence, for a certain range of values the assortativeness in the vertical trait decreases, whereas the assortativeness in the horizontal dimension increases. This feature reverses in the range of extents close to the extent where the complete segregation equilibria emerge. However, this notion does not allow for a natural definition of "positive" or "negative" assortativeness as there is per se no room for this in the horizontal dimension.

Contrary to this first notion of assortativeness one could identify this term with the similarity of formed matches within one pair of matched agents concerning one of the two traits. Naturally, one could restrict attention to the positive or negative assortativeness along the vertical dimension. For this interpretation our model offers perfect correlation whenever the extent is above $\tilde{\theta}$. For extents below this value the positive assortativeness is negatively related with the search frictions for the (decisive) high types. For low types there may be a range (whenever $x_{LL}/(x_{HL}+x_{LL})$ is decreasing in θ) where the correlation in the vertical trait in matches with low types reduces with decreasing search frictions.

6. Conclusion

This article analyzes how matching and sorting takes place when individuals are characterized by both ex-ante heterogeneity and horizontal preferences. Our dynamic model assumes that search frictions are present in the market and that utility is nontransferable. Along the vertical dimension all individuals agree on the ranking of agents and prefer to be matched with the highest agent. This vertical trait can therefore be interpreted as income, wealth or even beauty. Unlike in most of the search-theoretic literature, the horizontal dimension considered here is not ordered. It captures inputs which are not equally ranked across agents. Possible examples are regional preferences concerning a job or sharing hobbies with the spouse. The

analysis for our definition of horizontal heterogeneity[26] is quite different from the more common assumption of preferring similar characteristics along an ordered trait like height (e.g. Clark 2007). [27] It is also possible to define positive or negative assortative mating.

One appealing result is that the strategy of the highest agent is continuous in the market extent. By contrast, in the typical vertical model (e.g. Burdett and Coles 1997) high type agents change their offers to lower types discontinuously. Our result is driven by the fact that agents are able to condition their offers on both dimensions. If the search frictions are below some threshold, high-type agents start sending offers to low-type agents. When the extent of the market is exactly at this threshold, high-type agents propose only to the low-type agent who yields the largest horizontal payoff. When search frictions increase further, high-type agents expand the set of acceptable low-type agents.

Driven by this continuity property, we can identify ranges of search frictions where the direct (positive) effect of decreasing search frictions is outweighed by the indirect (negative) effect for every type of agent (except the highest). In most cases, agents reduce their set of acceptable partners within their own class when search frictions decrease and hence are better off. However, for low-type agents this relation reverses for certain ranges of search frictions. For these particular ranges the probability for mixed matches is elevated when search frictions increase. Since utility is non-transferable, low type agents benefit more than high types from the mixed matches. Whether this effect leads to an increase of aggregate welfare depends on the welfare criterion. However, space constraints do not permit a thorough welfare analysis.

Finally, the approach to tackle the issue of horizontal heterogeneity in the presented way seems appropriate for a handful of reasons. The model is generalizable in many kind of ways. The weights assigned to the utilities from the two traits are not a restriction as long as utility remains additively separable. The match-specific utility derived from the horizontal trait can be distributed quite generally. The number of types in the vertical dimension can be increased to an arbitrary natural number. The prediction of the model depends solely on the level of search frictions and allows for equilibria with mixed marriages as well as for complete segregation in an intuitive way.

[26] Our definition follows Konrad and Lommerud (2010) who analyze the implications of redistributive taxation in a static one-period model.

[27] In these models the probability of being matched is usually not independent of the location. For example, if agents prefer partners of similar height, individuals in the middle of the distribution have higher chances of meeting an appropriate partner.

There are several routes for future research. For simplicity, it is assumed that individuals who leave the market are immediately replaced by clones. Introducing different inflow and outflow specifications should lead to multiple equilibria for a given extent of the market. Another important assumption is the symmetry of the horizontal dimension. This ensures that both agents in a match always receive the same gain along the horizontal dimension. This in turn implies that high-type agents are decisive for mixed matches and considerably simplifies the extension to N discrete income levels.

A. Appendix

Proof of Lemma 1

From equation (2) we know that an agent accepts all proposals which yield her a higher utility than staying single, i.e. agent i accepts if and only if

$$rV_i \leq f(y,y) + g(x). \tag{26}$$

In general, there are two cases to distinguish: In the first case agent i does not accept all proposals which she faces. Thereby, she accepts all proposals up to the taste difference where she is indifferent between marriage and staying single. This determines her reservation quality $\bar{x}_i$ which is then given by

$$rV_i = f(y,y) + g(\bar{x}_i). \tag{27}$$

Using equation (27) we can reformulate equation (2). This yields

$$rV_i = f(y,-) + \frac{\alpha_i}{r} \int_0^{\bar{x}_i} \left(g\left(x\right) - g\left(\bar{x}_i\right)\right) dH_{-i}(x).$$

In the second case the agent accepts every proposal she faces. The agent would be even willing to accept agents whose location is beyond the taste difference of $\bar{x}_i$, but these individuals do not propose to her. Formally, we have

$$rV_i < f(y,y) + g(\bar{x}_i) \text{ and } H_{-i}(\bar{x}_i) = 1$$

as every offer is accepted. Furthermore, as the agent always marries when she faces a proposal the maximum operator from equation (2) simplifies and one gets

$$rV_i = f(y,-) + \frac{\alpha_i}{r} E_{-i}\left(f(y,y) + g(x) - rV_i\right).$$

■

Proof of Proposition 1

The derivations of the equations are shown in the text. For the equilibrium property of this characterization we still have to show that the strategies of the agents generate a consistent offer distribution for a particular agent. As every agent decides to propose to all other agents within taste distance $\bar{x}$ by the symmetry of agent's distance to each other all proposals are accepted. If agents calculate this best response on the

belief on the offer distributions that all other agents play the strategy described in the Proposition, then choices and expectations are consistent with each other. ■

Proof of Corollary 1

The first part simply follows by equations (I) and (II) from Proposition 1 and recalling the utility definition $f(y,-) = f(y,y)$.

For the second part let

$$F(\theta, x(\theta)) := \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right] - g(\bar{x}).$$

Using the implicit function theorem we establish that $\bar{x}$ is decreasing in extent θ as

$$\begin{aligned} \frac{d\bar{x}}{d\theta} &= -\left(\frac{dF}{d\bar{x}}\right)^{-1} \cdot \frac{dF}{d\theta} \\ &= -\frac{\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x})}{\theta(g(\bar{x}) - g(\bar{x}) - \bar{x}g'(\bar{x})) - g'(x)} \\ &= \frac{\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x})}{g'(x)(1+\theta\bar{x})} < 0. \end{aligned}$$

The last sign holds as the numerator is positive due to $g' < 0$ and the denominator is negative for the same reason. Hence, we get $d\bar{x}/d\theta < 0$, i.e. the reservation quality required by the agents is increasing in the extent of the market.

It remains to show the limiting behavior of $\bar{x}$. Letting $\theta \to 0$ and finding a solution to equation (10) is equivalent to solving $g(\bar{x}) = 0$ which yields $\bar{x} = 1$. The considerations for $\theta \to \infty$ are similar. We have to find $\bar{x}$ which fulfills

$$g(\bar{x}) = \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right].$$

For given θ we can choose $\bar{x}$ arbitrarily close to 0 such that the right-hand-side is arbitrarily close to zero, whereas the left-hand-side still yields a finite value. Hence, this equation has always a solution. Furthermore, as the right-hand-side is increasing in θ, we have that $\lim_{\theta\to\infty} \bar{x}(\theta) = 0$. ■

Proof of Lemma 2

This is a direct generalization of the proof from Lemma 1. The rest follows directly from the text. ■

Proof of Proposition 2

The proof follows directly from the text. ■

Proof of Proposition 3

The proof follows directly from the text and from Proposition 1. ■

Proof of Proposition 4

First note that $x_{HL}(\tilde{\theta}) = 0$, which we will use at some points throughout the proof.

First, we consider the continuity of $x_{HH}(\theta)$. The continuity for $\theta \in \mathbb{R}_+ \setminus \{\tilde{\theta}\}$ can be seen directly from the equation system (I)-(V) of the integration equilibrium given in Proposition 2 and the expression $g(\bar{x})$ from the income segregation equilibrium given in Proposition 3. For the case of $\theta = \tilde{\theta}$ equating (I) and (III) together with $x_{HL}(\tilde{\theta}) = 0$ yields

$$g(x_{HH}) = \frac{\theta}{2}\left[\int_0^{x_{HH}} g(x)dx - x_{HH}g(x_{HH})\right].$$

Hence, this solution for x_{HH} at $\tilde{\theta}$ coincides with the solution of $g(\bar{x})$ of the segregation equilibrium. Since the solution is the same on both sides of $\tilde{\theta}$ we have continuity for x_{HH} for all $\theta \in \mathbb{R}_+$.

Now we analyze $x_{LL}(\theta)$. The continuity for $\theta \in \mathbb{R}_+ \setminus \{\tilde{\theta}\}$ can be seen directly from the equation system (I)-(V) of the integration equilibrium given in Proposition 2 and the expression $g(\bar{x})$ from the income segregation equilibrium given in Proposition 3. For the case of $\theta = \tilde{\theta}$ equating (IV) and (V) together with $x_{HL}(\tilde{\theta}) = 0$ yields

$$g(x_{LL}) = \frac{\theta}{2}\left[\int_0^{x_{LL}} g(x)dx - x_{LL}g(x_{LL})\right].$$

Hence, this solution for x_{HL} at $\tilde{\theta}$ coincides with the solution of $g(\bar{x})$ of the segregation equilibrium. Since the solution is the same on both sides of $\tilde{\theta}$ we have continuity for x_{LL} for all $\theta \in \mathbb{R}_+$.

Finally, we analyze $x_{HL}(\theta)$. The continuity of x_{HL} for all $\theta \in (0, \tilde{\theta})$ follows directly from the equation system (I)-(V). For $\theta > \tilde{\theta}$ we have $x_{HL} \equiv 0$. Hence, as x_{HL} is zero per definition at $\tilde{\theta}$ we have continuity for all $\theta \in \mathbb{R}_+$. ■

Proof of Proposition 5

a) Let $\theta = 0$. By equations (I) and (III) from Proposition 2 we get $g(x_{HH}) = 0$. By the definition of g this yields $\lim_{\theta \to 0} x_{HH}(\theta) = 1$. The same argument for equations (IV) and (V) from Proposition 2 shows $\lim_{\theta \to 0} x_{LL}(\theta) = 1$. Furthermore, by equating (I) and (II) from Proposition 2 one gets that the difference between least acceptable partners (measured in g) is a constant:

$$g(x_{HL}) - g(x_{HH}) = f(y_H, y_H) - f(y_H, y_L) \tag{28}$$

We already know for $\theta \to 0$ we have $g(x_{HH}) = 0$. Using this in equation (28) shows the last claim from part (a).

b) For $\theta \to \infty$ we have complete segregation and hence Corollary 1 applies. This completes the proof. ■

Proof of Proposition 6

By Proposition 2 and Proposition 3 the differentiability of reservation qualities x_{LL}, x_{HH} and x_{HL} for all $\theta \in \mathbb{R}_+ \setminus \tilde{\theta}$ is obvious. The only thing which remains to be shown is the differentiability of x_{HH} in $\tilde{\theta}$.

For reasons of simplicity we analyze the system of equations given in Proposition 2 and Proposition 3 as functions of the market extent α instead of θ; the latter definition of market extent is just a transformation by the constant discount factor r. By the implicit functions theorem the left- and the right-hand-side derivatives of the reservation quality of the high type among her own class determine as

$$\lim_{\Theta \nearrow \tilde{\Theta}} \frac{\partial x_{HH}}{\partial \alpha} = \lim_{\Theta \searrow \tilde{\Theta}} \frac{\partial x_{HH}}{\partial \alpha} = \frac{\int_0^{x_{HH}} g(x)dx - x_{HH} g(x_{HH})}{g'(x_{HH})(2r + x_{HH}\alpha)}.$$

Hence, x_{HH} is differentiable in $\mathbb{R}_+$. Furthermore, the analysis of the derivatives shows easily, that the reservation qualities x_{HH} and x_{HL} are decreasing resp. non-increasing everywhere. ■

Proof of Corollary 2

All stated equations are the results of calculations using the formula given in Proposition 2 for part a) and Proposition 3 for part b).

It only remains to calculate threshold extent $\tilde{\theta}$. From equation (22) it is clear that at $\tilde{\theta}$ we have

$$y_H + 1 - x_{HH} = \frac{y_H + y_L}{2} + 1 \Leftrightarrow x_{HH} = \frac{y_H - y_L}{2}.$$

By Proposition 4 we know that $x_{HH}(\theta)$ is continuous at $\tilde{\theta}$ and hence, using equation (Ib) at $\tilde{\theta}$ it must hold that

$$\frac{y_H - y_L}{2} = \frac{2}{\tilde{\theta}}(\sqrt{1+\tilde{\theta}} - 1).$$

Solving this for $\tilde{\theta}$ yields

$$\tilde{\theta} = 8\frac{2-(y_H - y_L)}{(y_H - y_L)^2}.$$

This completes the proof. ■

Proof of Proposition 8

So far, distance X is uniformly distributed on $[0,1]$. Therefore, we have $F(x) = P(X \leq x) = x$. Let Z be another random variable with finite support $[a,b] \subset \mathbb{R}$ with $H_Z(z) = P(Z \leq z)$ denoting the corresponding cumulative distribution function and associated quantile function $H_Z^{-1}(q)$. Let Y denote the random variable obtained from transforming X according to $Y := H_Z^{-1}(X)$.

For the distribution of Y we then have:

$$\begin{aligned} P(Y \leq y) &= P(H_Z^{-1}(X) \leq y) = P(H_Z(H_Z^{-1}(X)) \leq H_Z(y)) \\ &= P(X \leq H_Z(y)) = H_Z(y). \end{aligned}$$

Hence, the associated cumulative distribution to random variable Y is function H_Z.

We transform the distance as follows with function $g(x) = b - H^{-1}(x)$. Solving this model is the same as solving the previous untransformed model with taste payoff $g(z) = b - z$ and $Z \sim H_Z(z)$. ■

Proof of Proposition 7

The proof follows directly from the text. ■

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".