Endogenous Rational Bubbles

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Abstract

Tests on simulated data from an asset pricing model with heterogeneous forecasts show excess variance in the price and ARCH effects in the returns, features not explained by the strong version of the efficient markets hypothesis. An evolutionary game theory dynamic describe how agents switch between a fundamental forecast, a rational bubble forecast and the reflective forecast, which is a weighted average of the former two. Conditions determining the frequency and duration of episodes where a significant fraction of agents adopt the rational bubble forecast leading to large deviations in the price-dividend ratio are discussed.

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1 Introduction

Though the efficient markets hypothesis (EMH) has been a dominant paradigm in asset pricing for decades, the assumption that there is a representative forecast for an asset price contradicts the observed heterogeneity of forecasts and cannot explain important features of the data, such as the volatility of prices and serially dependent variance in the returns. Furthermore, the strong version of the EMH, meaning asset prices are determined solely by expectations of fundamental information such as earnings or dividends, is at odds with the popular perception that bubbles are a common phenomenon in asset markets.

There are models that allow for bubbles. Models of rational bubbles (Blanchard (1979), Evans (1991)) are appealing, since expectations are rational (unbiased), and the model matches the stylized fact that prices and returns are unpredictable in the short run. However, these models do not provide an explanation for ARCH effects in the returns, and it is unclear how agents could coordinate on a forecast based on extraneous information when an alternative forecast based on the strong EMH is available. Models with heterogenous behavioral forecasting strategies (Brock and Hommes (1998), LeBaron (2010) etc.) can also produce large deviations in the asset price and price-dividend ratio from the predicted values of the strong EMH, but such approaches involve strategies do not satisfy rationality in any sense.

The present paper explains how agents with a choice of heterogeneous forecasting strategies could adopt a rational bubble forecast leading to a bubble in the asset price. Bubbles endogenously collapse given the assumption that a small fraction of agents do not abandon the fundamental forecast. Furthermore, the outbreak of such bubbles can explain excess volatility in the price-dividend ratio and ARCH effects in returns, while also producing unpredictable returns.

An evolutionary game theory dynamic describes how agents switch between forecasting strategies based on their past performance, given by payoff based on forecast errors as in Parke and Waters (2007, 2011). Agents choose from a fundamental forecast, which corresponds to the strong EMH prediction, a mystic forecast, which includes an extraneous martingale as in the rational bubble model, and a reflective forecast, which is a weighted average of the former two forecasts. The reflective forecast embodies all the information available to the agents including the other forecasts and their relative popularity and is the unique unbiased forecast in an environment with heterogeneous forecasting strategies.

The behavior of all agent satisfies the *cognitive consistency principle,* described in Evans and Honkapohja (2011), which specifies that agents in a model act are as smart as economists. More precisely, agents should form expectations using reasonable models according to economic theory. In the present approach, agents adopting fundamentalism and mysticism are using models that satisfy rational expectations in the homogeneous case. While they are not fully rational in the heterogeneous case, the vast majority of the asset pricing literature, Cochrane (2001) is a representative example, focuses on the homogeneous case. Furthermore, agents have every opportunity to adopt reflectivism, a forecasting strategy that does satisfy rationality in the heterogeneous case.

The exponentially weighted replicator[1] used is an example of an imitative dynamic, see Sandholm (2011) for a discussion, that allows for a parameterization of how aggressive agents are in switching to strategies with superior performance. If there are no mystics in the model, the fundamental and reflective forecasts coordinate on that of the strong EMH. The key issue is to find conditions under which a small fractions of agents adopting mysticism can gain a following sufficient to cause a significant deviation in the price and price-dividend ratio. The introduction of a small fraction of deviants is related to evolutionary stability commonly studied with imitative dynamics, see Weibull (1998).

There are a number interesting alternative approaches to asset pricing that involve deviations from the strong EMH. Commonly, some type of linear model is used to forecast prices or some other market indicator, which satisfies cognitive consistency, though other restrictions on the forecast are often required. Adam, Marcet and Niccolini (2008) are able to match a number of the features of the U.S. stock market data where expectations of prices are formed using a simple linear model whose parameters are updated each period. As is common in such approaches, they must impose a *projection facility* to limit the possible choices of parameters in the forecasting model. Whether the assumption that all agents know and use such an approach satisfies cognitive consistency is open to interpretation.

A model of bubbles in asset markets that is not subject to this criticism is found in Branch and Evans (2010) where a representative agent updates an estimate of the conditional variance of the

[1] A continuous time version of this dynamic was originally studied in Bjornerstedt and Weibull (1996) and appear prominently in Hofbauer and Weibull (1996).

return using a linear model. The time series implications of this approach have yet to be explored in detail.

Another related approach with a representative agent is that of Lansing (2010), where the forecasting model ("perceived law of motion") includes a geometric random walk, making bubbles a possibility, but agents also update the parameter on the bubble term, so the importance of the bubble term can change over time. This paper goes on to examine the implications of the resulting asset price dynamics within a macroeconomic model.

The model in LeBaron (2010) has multiple forecasting strategies with a variety of linear forecasting models, though some agents use a "buy and hold" strategy. The gain parameter, which parameterizes how quickly agents adjust their parameters in the model used for forecasting, varies for the different strategies. Agents are allowed to switch strategies according to past performance though the mechanism is rather ad hoc.

Section 2 give details about the asset price model with heterogeneous expectation, while section 3 presents the dynamic describing the evolution of the forecasting strategies. Section 4 describes the simulations and the conditions for the formation of bubbles. Section 5 gives the results of formal tests on the simulated data, and Section 6 concludes.

2 Asset Pricing

This section specifies the three forecasts and the resulting realization of the asset price, which thereby determines the forecast errors for each strategy. The underlying model is the standard asset pricing equation

$$p_t = \alpha p^e_{t+1} + d_t, \tag{1}$$

where the asset price is p_t, the dividend is d_t and the parameter α is the discount factor. This model is not fully sufficient for our purpose, since there is a unique representative forecast of the price. Brock and Hommes (1998) develop a model with mean-variance optimization where investors choose between a riskless and risky asset in constant supply. With risk neutral agents and a common belief about the variance of the returns, the model with heterogeneous forecasts can be written as

$$p_t = \alpha \sum_{h=1}^{n} x_{h,t} e_{h,t} + d_t \tag{2}$$

where the vectors $e_t = (e_{1,t}, ..., e_{n,t})$ and $x_t = (x_{i,t}, ..., x_{n,t})$ are the different forecasts of p_{t+1} and the fractions of agents using the forecasts, respectively.

The forecasts considered are motivated by the multiplicity of solution to the model (1) of the homogeneous case. According to the strong efficient markets hypothesis (EMH), the price is given by the discounted expected future dividends as given by the following solution to the model.

$$p_t^* = d_t + \sum_{j=1}^{\infty} \alpha^j E_t(d_{t+j})$$

Agents referred to as *fundamentalists* adopt the forecast $e_{2,t}$ determined by the above solution.

$$e_{2,t} = E_t(p_{t+1}^*) = \sum_{j=1}^{\infty} \alpha^{j-1} E_t(d_{t+j}) \tag{3}$$

However, this solution is not unique. As discussed in the rational bubble literature, see Lansing (2010), there is a continuum of solutions to (1) of the form

$$p_t^m = p_t^* + \alpha^{-t} m_t$$

where the stochastic variable m_t is a martingale such that $m_t = m_{t-1} + \eta_t$, for *iid*, mean zero shocks η_t. Though the information contained in the martingale m_t may be extraneous with respect to the fundamental information in d_t, if agents believe that information is important, it affects the asset price. Agents that adopt the forecast $e_{3,t}$ based on the rational bubble solution above are called *mystics*, and their forecast is as follows.

$$e_{3,t} = E_t\left(p_{t+1}^m\right) = E_t(p_{t+1}^*) + \alpha^{-t-1} m_t \tag{4}$$

Both the mystic and fundamental forecasts satisfy rational expectations in the homogeneous case. However, our goal is to allow for possible heterogeneity in forecasting strategies. The *reflective* forecast $e_{1,t}$ is an average of the alternative forecasts used in the population weighted according to the relative popularity.

$$e_{1,t} = (1 - n_t)\, e_{2,t} + n_t e_{3,t} \tag{5}$$

where

$$n_t = \frac{x_{3,t}}{x_{2,t} + x_{3,t}}$$

The variable n_t is the relative popularity of mysticism among agents using mysticism or reflectivism.

Reflectivism depends on alternative strategies, so to ensure its existence, we make the following key assumption.

Assumption: The fraction of fundamentalists $x_{2,t}$ never falls below some minimum $\delta_2 > 0$.

This assumption is not particularly restrictive, considering that in most asset pricing models, all investors are fundamentalists. Given these three forecasting strategies (3), (4) and (5) and the asset pricing model allowing for heterogeneity (2), the realization of the asset price is

$$p_t = p_t^* + \alpha^{-t} n_t m_t. \tag{6}$$

Agents evaluate the performance of the forecasting strategies by comparing payoffs based on squared forecast errors. Hommes (2001) shows that, the mean-variance optimization underpinning the model (2) is equivalent to minimizing squared forecast errors. Payoffs are defined as follows.

$$\pi_{i,t} = -(p_t - e_{i,t-1})^2 \tag{7}$$

The reflective forecast error U_t plays an important role in the payoffs to all three forecasting strategies, and is comprised of two terms.

$$U_t = (p_t^* - E(p_t^*)) + \alpha^{-t}(n_t m_t - n_{t-1} m_{t-1}) \tag{8}$$

The first term is the current period dividend payment, which is the new fundamental information. The second term is the martingale innovation weighted according to the change in the martingale and can be written as $n_t \eta_t - \Delta n_t m_t$. The representation of U_t shows that the reflective forecast is unbiased, under the assumption that agents are unable to forecast changes in n_t. The innovations to the dividend (d_t) and the martingale (η_t) and the change in n_t are all independent, mean zero, so the forecast error is also mean zero and the reflective forecast is unbiased. The reflective forecast satisfies rational expectations in the presence of heterogeneity.

The fundamental and mystic forecasts satisfy rationality in the homogeneous case, but their forecast errors are affected by the level of the martingale in the presence of heterogeneity. A key term in the payoffs is the weighted martingale $A_{t-1} = \alpha^{-t} m_{t-1}$. The reflective forecast depends only on U_t and, using (7) and (8), has payoff

$$\pi_{1,t} = -U_t^2. \tag{9}$$

Fundamentalism has forecast error $U_t + n_{t-1} A_{t-1}$, so its payoff is

$$\pi_{2,t} = -U_t^2 - 2n_{t-1} U_t A_{t-1} - n_{t-1}^2 A_{t-1}^2. \tag{10}$$

Similarly, the payoff to mysticism is as follows.

$$\pi_{3,t} = -U_t^2 + 2(1 - n_{t-1}) U_t A_{t-1} - (1 - n_{t-1})^2 A_{t-1}^2 \tag{11}$$

Much of the intuition behind the possibility of mysticism gaining a following can be observed in the above three payoffs. The third terms in the payoffs to mysticism (11) and fundamentalism (10) are unambiguously damaging to those payoffs in comparison with the payoff to reflectivism (9). Since the covariance $U_t A_{t-1}$ has mean zero, in expectation, reflectivism outperforms the other two strategies.

However, mysticism can outperform the other strategies in some periods. If the covariance $U_t A_{t-1}$ is positive and sufficiently large, the second term in (11) may outweigh the third term so that $\pi_{3,t} > \pi_{1,t} > \pi_{2,t}$. Such a positive covariance corresponds to a fortunate correlation between the martingale and the innovations in the model. In distribution, dividends are uncorrelated with the martingale, but over a number of periods, such correlations are likely to occur.

3 Evolutionary Dynamics

A generalization of the replicator dynamic, a workhorse in the evolutionary game theory literature, describes the evolution of the vector x_t of the fractions of agents using the different forecasting strategies. Let the weighting function $w(\pi)$ be a positive, increasing function of the payoffs. The general replicator dynamic[2] is

$$x_{i,t+1} - x_{i,t} = x_{i,t} \frac{w(\pi_{i,t}) - \overline{w}_t}{\overline{w}_t}, \tag{12}$$

where the expression $\overline{w}_t$ is the weighted population average $\overline{w}_t = x_{1,t} w(\pi_{1,t}) + \cdots + x_{n,t} w(\pi_{n,t})$. A strategy gains followers if its weighted payoff above the weighted population average, i.e. has positive fitness in evolutionary game theory terminology. Such a dynamic is said to be *imitative* since strategies that are popular today, larger $x_{i,t}$, tend to gain more adherents if they are successful, i.e. the numerator in (12) is positive.

A general form for the dynamic (12) allows for a range of behavior of the agents. For a linear weighting function $w(\pi)$, the adjustment to better performing strategies is sluggish, but for convex $w(\pi)$, agents switch faster. A linear weighting function in the dynamic (12) gives the special case of the replicator dynamic studied in Weibull (1998) and Samuelson (1998). Sandholm (2010) gives a thorough comparison of the features of a number of evolutionary dynamics.

Using a version of the dynamic (12) with an alternate timing, Parke and Waters (2011) demonstrate that, for bounded dividends, the payoff to reflectivism is always above the population average. Therefore, under the replicator (linear $w(\pi)$), mysticism cannot take followers away from reflectivists. While the formal details and implications of this statement are quite involved (see the reference above) the intuition is straightforward and is relevant for the simulations using the present approach. Under linear weighting, the covariance (second) terms in the payoffs to mysticism and fundamentalism, (11) and (10), cancel in the population average payoff, but the third terms with A_{t-1}^2 do not. Since the payoff to reflectivism is unaffected by the martingale, it is larger than the population average, so reflectivism gains followers.

The logic of the superiority of reflectivism does not apply in the case of a convex weighting

[2] Parke and Waters (2011) focus on a dynamic with the same form, but slightly altered timing and perform simulations with the present form as a robustness check.

function. Here, a positive covariance $U_t A_{t-1} > 0$ has greater benefit to mysticism than harm to fundamentalism, so if it is large enough, mysticism can gain a following. The model used for simulations focuses on the exponential weighting function

$$w(\pi) = e^{\theta^2 \pi}, \tag{13}$$

so θ parameterizes the aggressiveness of the agents. An increase in θ means that agents are switching more quickly to the best strategy, but as θ decreases the dynamic approaches the linear weighting case. See Parke and Waters (2011) for formal results and a more detailed discussion.

One drawback to imitative dynamics such as the generalized replicator (12) is their lack of inventiveness. If a strategy has no followers ($x_i = 0$) then it cannot gain any. Hence, game theorists usually focus on equilibria that are evolutionarily stable, meaning they are robust to the introduction of a small fraction of deviating agents. Similarly, the focus of the present class of models is whether the fundamental forecast is robust to the introduction of a small fraction of mystics.

It is possible for mysticism to gain a following given the following conditions. Some agents believe that extraneous information may be important to the value of a asset. In some periods, the extraneous information must be correlated with fundamentals. Lastly, agents must be sufficiently aggressive in switching to superior performing strategies.

Mysticism cannot maintain a following indefinitely given the existence of a minimum fraction of fundamentalists δ_2. If fundamentalism is eliminated from the population, then $n_t = 1$ and the payoff to mysticism (10) is identical to the payoff to reflectivism (9). However, the presence of a minimum fraction of fundamentalists implies that $n_t < 1$ and that the reflective and mystic forecasts are not identical. Since the expected value of the covariance term $U_t A_{t-1}$ in (10) is zero, reflectivism outperforms mysticism in the long run. Further, the magnitude of A_t grows over time, so the third term in the payoff to mysticism (10) dominates and the performance of mysticism deteriorates over time. While mysticism can gain a following temporarily so the martingale affects the asset price, eventually agents abandon mysticism, so bubbles endogenously form and collapse. The goal of the simulations is to determine to quantitative importance of such outbreaks of mysticism.

Since it limits the life of bubbles, the minimum fraction of fundamentalists plays a similar role as the *projection facility* used with least squares learning as in Adam, Marcet and Niccolini (2008). In their approach, a representative agent updates the estimate of the parameters in a forecasting rule, but the projection facility limits the acceptable estimates to those that produce non-explosive behavior. The projection facility places stronger restrictions on agents beliefs than the minimum fraction of fundamentalists. In the present model, a small fraction rejects extraneous information, but under that projection facility, all agents have a sophisticated understanding of the long run dynamics of the forecasting rule.

The model represents a minimal departure from rationality when mystics are introduced into the population. Mysticism appears due to a disagreement about what constitutes fundamental information, but all agents form expectations with a reasonable economic model, i.e. agents meet the *cognitive consistency* principle described in Evans and Honkapohja (2010). Both mysticism and fundamentalism satisfy rationality in the homogeneous case, and reflectivism satisfies rationality when there is heterogeneity in the forecasting strategies, and this forecasting strategy is available to agents at all times. When mystics are eliminated from the population, the reflective and fundamental forecasts coincide and satisfy rationality. Only when mystics are introduced do the mystic and fundamental forecasts deviate from rationality, but mystics believe that the extraneous information in the martingale is relevant to the forecast of the asset price, and that other agents will eventually realize this. Hence, all agents believe that they are making efficient use of the available information.

4 Simulations

Simulations of the model with the three forecasting strategies described above verify that outbreaks of mysticism depend on the aggressiveness of the agents in switching to better performing strategies and the magnitude of the shocks to the dividends and the martingale. Furthermore, for reasonable parameterizations, when a significant portion of the population adopts the mystic forecasting strategy, there can be large bubble-like deviations in the asset price and price-dividend ratio.

Given the dividend d_t and the martingale m_t, the model is determined by the dynamic (12) along with the exponential weighting function (13), the payoffs (9), (10) and (11), and the realization of the asset price (6). The dividend process is specified as a stationary process with parameter choices below.

$$d_t = \overline{d} + \rho \left(d_{t-1} - \overline{d} \right) + v_t$$

$\overline{d}$	ρ	σ_v
0.03	0.465	0.203

The constant $\overline{d}$ is chosen so that for $\alpha = 0.99$, the steady state price-dividend ratio (log difference) is 20, which is close to the long run average for the S&P 500 from the Shiller data. The persistence parameter ρ and shocks $v_t \sim N(0, \sigma_v)$ are chosen to match values from the H-P detrended earnings series. Earnings are used instead of dividends, since not all firms pay dividends and earnings are a more reasonable proxy for firm profitability.

Two other fixed parameters are the minimum fraction of fundamentalists $\delta_2 = 0.01$ and the fraction of mystics introduced into the population 0.001. The minimum fraction of mystics is set much smaller so that the introduction of mystics on its own does not have a quantitatively significant effect on the asset price (6) since n_t is small. If the dynamic used in the simulations is specified so that if the unconstrained dynamic (12) sets one of the fractions below its minimum, that fraction is set to its minimum, and the other two strategies split the remaining followers in the same proportion they would in the unconstrained case. If mysticism falls below its minimum, that level of followers is reintroduced and the martingale is restarted at $m_t = 0$.

The free parameters θ, which measures agent aggressiveness, and σ_η, the standard deviation of the martingale innovations, play are large role in determining the potential for outbreaks of mysticism and bubbles. For those events to occur, agents must be sufficiently aggressive, meaning θ is sufficiently large, and the magnitude of the martingale innovations must be large enough to have a noticeable impact on the payoffs and the asset price, but not so large so that the third term in the payoff to mysticism (11) dominates.

Figures 1-5 demonstrate the role of the parameters θ and σ_η in determining the frequency and duration of bubbles. Figures 1 and 2 show the evolution of the price dividend ratio, the forecast errors of the reflective and fundamental forecasts and the fraction of followers of the three forecasting strategies for two different choices of θ. In figure 1, this parameter is set to $\theta = 5/8$, which indicates sluggish adjustment to strategies with superior performance. The simulations are initiated at a point where the fraction of followers of reflectivism, the potentially dominant strategy, is at its maximum. For the low level of θ, the introduction of a small fraction of mystics does not induce others to adopt the strategy and has no appreciable impact on the evolution of the asset price. Again, for smaller θ's, the dynamic (12) approaches the linear weighting case where reflectivism dominates.

Figure 2 shows the same variables as Figure 1, but for a higher level of θ at $\theta = 5.0$. Here, agents are sufficiently aggressive for mysticism to gain a following for significant stretches of time. There are a number of instances where well over half of the the population is using mysticism and some of these are associated with large fundamentalist forecast errors and large deviations from the steady state value in the price-dividend ratio. Note that the martingale does not damage the reflectivist forecast error, since reflectivists use the martingale and information about its relative popularity in their forecast.

Figures 3 and 4 illustrate the role of the standard deviation of the martingale innovations σ_η in the formation and duration of bubbles. The agent aggression parameter is set to $\theta = 5.0$ as in Figure 2, but the parameter σ_η is lower at $\sigma_\eta = 0.25\sigma_v$. Hence, though mysticism often gains a following, it is more difficult for the martingale to attain a sufficient magnitude to noticeably affect the asset price. However, when they do occur, bubbles in the asset price tend to last longer, since the martingale grows relatively slowly and more time is required for the martingale (third term) in the mystic payoff (11) to overwhelm the covariance term. Conversely, a higher magnitude for martingale innovation, as in Figure 4 with $\sigma_\eta = 2.0\sigma_v$, shows that bubble outbreaks become rare and short-lived as the martingale quickly, if not immediately, grows too large for mysticism to dominate.

Finally, Figure 5 shows the case where the agent aggression parameter is large at $\theta = 10.0$, while the parameter $\sigma_\eta = \sigma_v$ as in Figures 1 and 2. Here, the martingale innovations are at a magnitude where mysticism can dominate and the agents are quickly switching to superior strategies means that the dynamics are dominated by the covariance term in the payoffs to mysticism (11) and fundamentalism (10), and there is little inertia in the evolution of the $x_{i,t}$'s. Hence, mysticism quickly gains a following with a positive covariance, but quickly loses is with the opposite. There are some occurrences of bubble-like behavior in the price-dividend ratio, but the primary impact of the martingale is an increase in the volatility of the asset price. We proceed by examining more formal econometric features of the data to support these qualitative observations.

5 Time Series Tests

The simulated data matches econometric features of asset market data in multiple respects. In the presence of bubbles, the price-dividend ratio has greater persistence than the dividend series. Returns are unpredictable in the short run. Excess variance in the price-dividend ratio and ARCH effects in the returns can arise in the presence of bubbles arising due to outbreaks of mysticism. The stationarity of the price-dividend ratio is difficult to characterize, but this is true of stock market data as well.

5.1 Mystic dominance and bubbles

Simple measures to detect mysticism and bubbles allow a demonstration of the correspondence between the impact of the martingale and formal econometric features of the data such as excess variance. We run 10,000 trials of 100 periods, roughly the size of the sample in the Shiller data, with 50 periods for initiation. Table 1 reports the fraction of periods (across all trials) where mysticism dominated, i.e. when the fraction of followers of mysticism is greater than 0.5. Table 2 reports the fraction of trials with an occurrence of a bubble in the asset price, defined as a price-dividend ratio that 50% greater than its steady state value. This is a necessarily arbitrary but rather strict interpretation of a bubble. Observing the major U.S. stock market averages and using a steady state ratio of 20, the price-earnings ratio in the Shiller data only exceeded 30 after the start of the "technology bubble" of 2000. In the present model of a bubble, a negative bubble, when prices fall below their fundamental value, are just as likely as positive bubbles. If both classes of bubbles are included, the values in Table 2 should be doubled.

Tables 1 and 2 verify that outbreaks of mysticism and bubbles require sufficiently large choices for the parameter θ, the measure of agent aggressiveness, and the parameter σ_η the standard deviation of the shocks to the martingale. For low values of these parameters, there are no occurrences of bubbles or mystic dominance.

As the choices of the parameters θ and σ_η become very large, the occurrences of bubbles and mystic dominance fall from their maximum values. For example in Table 1 given $\theta = 3/4$, the fraction of mystic dominance initially rises with σ_η to a maximum of 0.188 at $\sigma_\eta = 1.0$, corresponding to Figure 2, but falls for larger magnitudes of the shock to the martingale for two reasons. For large σ_η, the martingale (third) term in the payoff to mysticism (11) dominates, diminishing the payoff and making the emergence of mysticism more difficult, as shown in Figure 4. Second, for large θ and σ_η, bubble rise and collapse faster, lowering the number of periods satisfying the criteria for mystic dominance and bubbles, as in Figure 5.

If agents are sufficiently aggressive about switching to superior strategies, $\theta \geq 3/4$, the role of the martingale becomes significant. In these cases, the fraction of periods showing mystic dominance is always greater than the fraction with bubbles. Even if mysticism has a large following, the magnitude of the martingale may not be large enough to have a dramatic effect on the asset price, pointing up the difficulty identifying bubbles. It is possible that agents are always using extraneous information to value assets, but that information only drives asset prices away from their fundamental values on rare occasions.

5.2 Persistence and volatility

For parameter settings that produce outbreaks of mysticism and bubbles, the simulated price-dividend series displays greater persistence than the dividend series and matches the volatility observed in the Shiller data. Tables 3 reports the average autocorrelation coefficient across the 10,000 trials and demonstrates higher persistence for values of θ and σ_η where bubbles can arise. While the highest value in the table of 0.62 does not show the persistence in the annual data of 0.8, Table 4 reports the standard deviation of the autocorrelation coefficients over the trials and shows that such levels of persistence do occur in a number of trials. Interestingly, for high values of θ and σ_η, the persistence falls to low levels as mysticism is adopted and abandoned very quickly as in Figure 5. Table 5 reports the standard deviation of the price-dividend series, and for sufficiently large θ and σ_η the volatility matches the standard deviation in the Shiller data of 0.38.

5.3 Return Predictability

For a large majority of the simulated series, returns are not predictable in the short run, an implication of the weak version of the EMH. We examine whether the price-dividend ratio is informative about per share excess returns

$$Z_t = d_t + y_t - \alpha^{-1} y_{t-1}, \tag{14}$$

which is the part of the optimization problem underlying the asset pricing model (2), see Brock and Hommes (1998). Furthermore, per share excess returns are the same as the reflective forecast up to a constant such that $U_t = Z_t + C$, where C is the constant risk premium in (2), see Parke and Waters (2007) for a discussion.

To test predictability, the following equation to test whether lagged prices dividend ratios contain information about current returns, similar to those used in Fama and French (1988), is estimated on simulated data with 100 observations.

$$U_t = \beta_0 + \beta_1 \left(p_{t-k} - d_{t-k} \right),$$

where k is the lead time for the prediction. If the R^2 from the estimation is over 0.1, returns are defined to be predictable. Table 6 reports the fraction of runs with predictable returns, and, for a lead time of two years $k = 2$, less than two percent of the series had predictable returns. Returns at longer horizons are also unpredictable, which is unsurprising give the stationarity of the dividend process.

5.4 Price-dividend stationarity

The the potential presence of the martingale has an ambiguous impact on the stationarity of the price-dividend ratio. Given the stationary process for dividends above, the price-dividend ratio should be stationary according to the strong EMH, i.e. the martingale-free solution. Formerly, a stationary price-dividend ratio was considered to be a stylized fact of financial markets data (Cochrane 2001), but inclusion of data from the past decade shows a non-stationary price-dividend ratio, though not for the price-earnings ratio in the Shiller series. A Dickey-Fuller test at a significance level of 5% is performed on the simulated series with 100 observations. Table 7 reports the fraction of trials where the stationarity of $p_t - d_t$ is rejected. For the cases where mysticism does not arise, stationarity of the price-dividend ratios cannot be rejected, which is not surprising, given the moderate amount of persistence in the dividend process ($\rho = 0.5$). However, there are simulations with a non-stationary price-dividend ratio, but the presence of bubbles is a necessary but not sufficient condition for non-stationarity. Comparing Tables 2 and 4, the fraction of trials with a bubble in the price-dividend ratio is always larger than the fraction where that ratio is non-stationary. The presence of bubbles is a necessary but insufficient condition for non-stationarity of the price-dividend ratio.

The presence of mysticism can have econometrically detectable effects on the data similar to those found in asset markets. In particular, for parameter choices where mysticism can gain a significant following, there is excess variance in the asset price, and ARCH effects in the returns.

5.5 Excess Variance

Studies such as Shiller (1981) demonstrate that asset prices fluctuate more than predicted by the EMH, and endogenous rational bubbles can explain such excess variance . Simulations determine a ratio of the realized variance and the predicted variance based on the variance of the dividends and the EMH, though the statistical significance is difficult to assess. A statistical test of the variance of the price-dividend ratio provides more definitive evidence.

In the absence of mysticism ($n_t = 0$), the asset price behaves according to the strong version of the EMH and depends only on the dividend process.

$$y_t^* = \overline{d}\left(\frac{\alpha}{1-\alpha} - \frac{\alpha\rho}{1-\alpha\rho}\right) + d_t\left(1 + \frac{\alpha\rho}{1-\alpha\rho}\right)$$

Hence, the variance of the asset price should be $\sigma_{y^*}^2 = (1-\alpha\rho)^{-2}\sigma_d^2$. Table 8 reports the ratio $\sigma_y^2/\sigma_{y^*}^2$ of the variance of the simulated asset prices and the predicted variance using the variance of the simulated dividends. Under the strong EMH, the ratio is unity, which occurs for very low levels of θ and σ_η. For higher levels, the ratio rises above one, and, for one pair of parameter values well over three. This level is much smaller than Shiller's initial estimate of 20, but other research[3] has found smaller estimated values.

To examine the statistical significance of the observed excess volatility in the asset prices, we conduct a test on a transformation of the price-dividend ratio. Let the notation $\widehat{x}$ denote the deviation of x from its steady state value. The series $\widehat{pd}_t = \frac{1-\alpha\rho}{\alpha\rho\sigma_d}\left(\widehat{y}_t - \widehat{d}_t\right)$ has the standard normal distribution so the variance of $\widehat{pd}_t$ is distributed $\chi^2(n)/n$ where n is the number of periods. Table 9 reports the fraction of runs such that the variance of the realized $\widehat{pd}_t$ is excessive at a significance level of 0.05. The results demonstrate that the excess variance shown in Table 8 is statistically significant and corresponds to outbreaks of bubbles. The pattern of the excess return probabilities in Table 9 follows that of the probability an occurrence of a bubble in Table 9 with higher probabilities in every case. For example, for a sufficiently large choice of θ such that $\theta \geq 3/2$, both probabilities rise with the magnitude of the martingale innovation σ_η for all values reported, but for smaller choice of θ, the probabilities both peak at a choice of σ_η less than 16.

5.6 ARCH

Tests for ARCH effects similarly show a correspondence with bubbles and excess volatility. To test for ARCH, we regress the squared excess returns[4] (14) on four lags of itself and a constant using least squares. Table 10 reports the fraction of runs where the F-test rejects the restriction that all the coefficients on the lagged squared returns are zero at a significance level of 0.05. For low levels of the parameters θ and σ_η, there are no ARCH effects above those produced at random, but, for higher levels, ARCH effects are evident. Thought the magnitude of values in the table are not as high as one might expect, this is primarily due to the short sample in the trials. Much greater incidence of ARCH and GARCH is found for longer samples and a different calibration in Parke and Waters (2007).

For a given standard deviation of the martingale innovations σ_η, the relationship between the frequency of significant ARCH effects and the aggressiveness of the agents (θ) is similar to the pattern with mystic dominance and bubbles in Tables (1) and (2). ARCH, mystic dominance and bubbles all rise with θ but fall for higher levels. However, the correspondence is not perfect, since the peak levels of ARCH are typically at $\theta = 10$, but the peak levels for mystic dominance

[3] Some examples areLeRoy and Porter (1979), Campbell and Shiller (1989) and LeRoy and Parke (1992).

[4] ARCH effects are even more evident using the alternative definition for returns $(P_t + D_t - P_{t-1})/P_{t-1}$, but in this cases there are ARCH effect in simulations with no bubbles, so the focus on excess returns is more revealing.

typically occur at lower levels of θ. For a given level of θ, there is positive relationship between the magnitude of the shocks to the martingale σ_η and the frequency of ARCH. This relationship corresponds to the relationship between the magnitude σ_η and mystic dominance and bubbles for low levels of σ_η but not for high levels. Even small fractions of agents adopting mysticism can affect the data when the magnitude of the martingale is large.

The behavior of the simulated time series of the endogenous rational bubbles model matches multiple features found with stock price and dividend data. As expected given stationary dividends, returns are not predictable. It is possible though not necessarily true that bubbles lead to non-stationarity in the price-dividend ratio. The presence of statistically significant levels of excess variance, and ARCH effects corresponds to outbreaks of mysticism and bubbles.

5.7 Extensions

There are a number of avenues for further research involving alternative approaches to the underlying dividend process, the inclusion of parameter learning and the evolutionary dynamic. In the present paper, the dividend process is stationary, but allowing for trend growth, leading to growth in the asset price as well, is more realistic. Modeling dividends as a random walk with drift as in Adam, Marcet and Niccolini (2008) and LeBaron (2010), should produce predictability of returns in the long run as has been reported in the data, for example Cochrane (2001). The proper method for modeling the martingale in the mystic forecast in such an environment is a non-trivial question requiring further research. Once one includes growth in dividends and prices, introducing parameter learning becomes appealing. Of course, in an environment with heterogenous agents, there are a number of possible ways to model such updating, but that fact does not make the issue less worthy of exploration. Lastly, there are a number of alternatives, such as multinomial logit (Brock and Hommes 1998) and the Brown, von Neumann and Nash dynamic (see Waters 2009), to the weighted replicator dynamic in the present work, see Sandholm (2010) for a theoretical overview of the alternatives.

6 Conclusion

Models of asset pricing where a representative agent forms expectations according to the strong efficient markets hypothesis violates both the heterogeneity of forecasts and reality of bubble-like behavior, as well as some formal econometric features of asset market data. The model with mysticism includes heterogeneous forecasting strategies in a way that satisfies the cognitive consistency principle, describes the necessary conditions for bubbles, and explains excess volatility in asset prices, ARCH effects in returns and other aspects of the data. This model is one of a number of approaches to asset pricing involving minimal deviations from rationality that make it possible to explain the complex behavior of these markets in a disciplined way.

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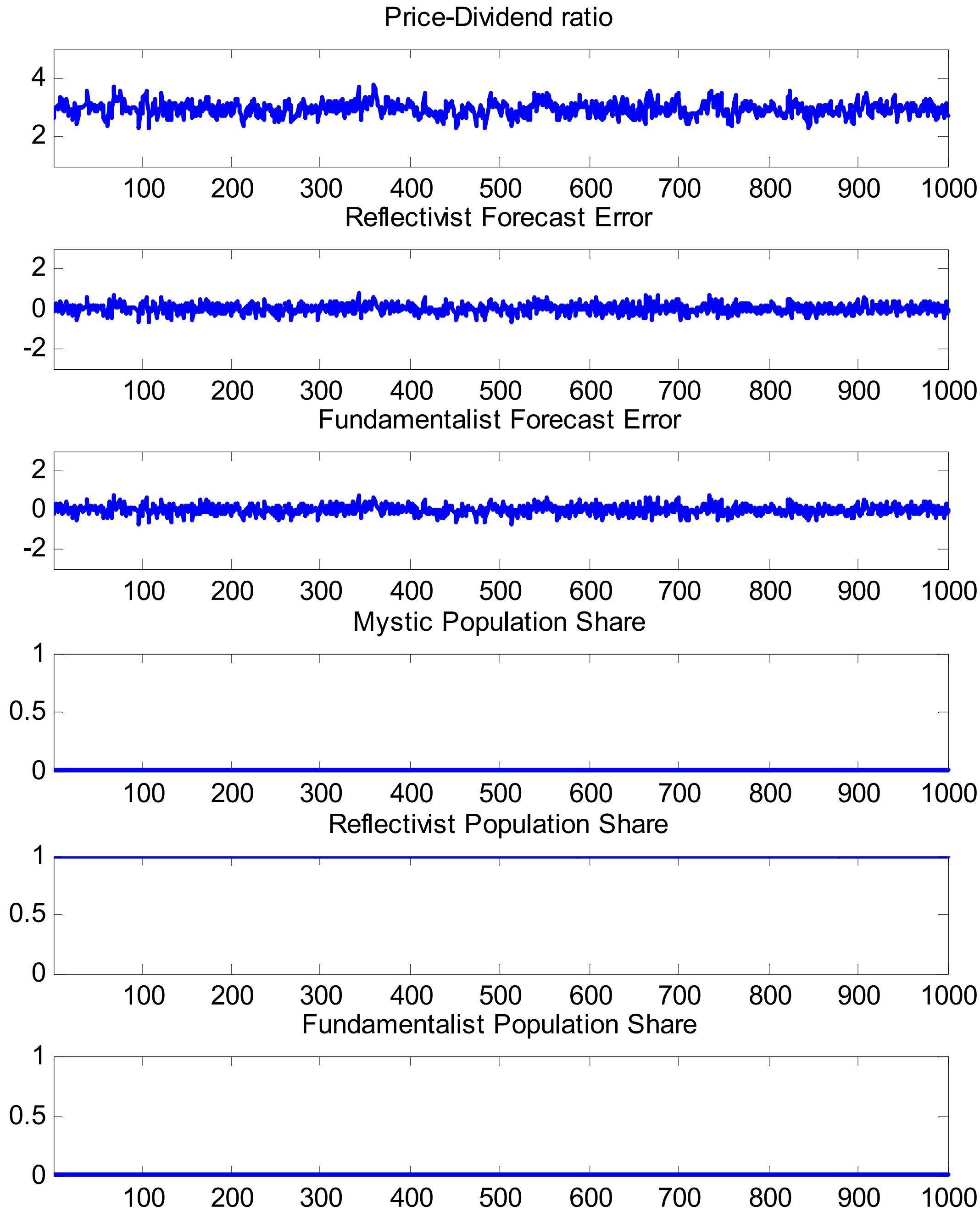

Figure 1

$\theta = 5/8$, $\sigma_\eta = \sigma_v \; x \; 1.0$

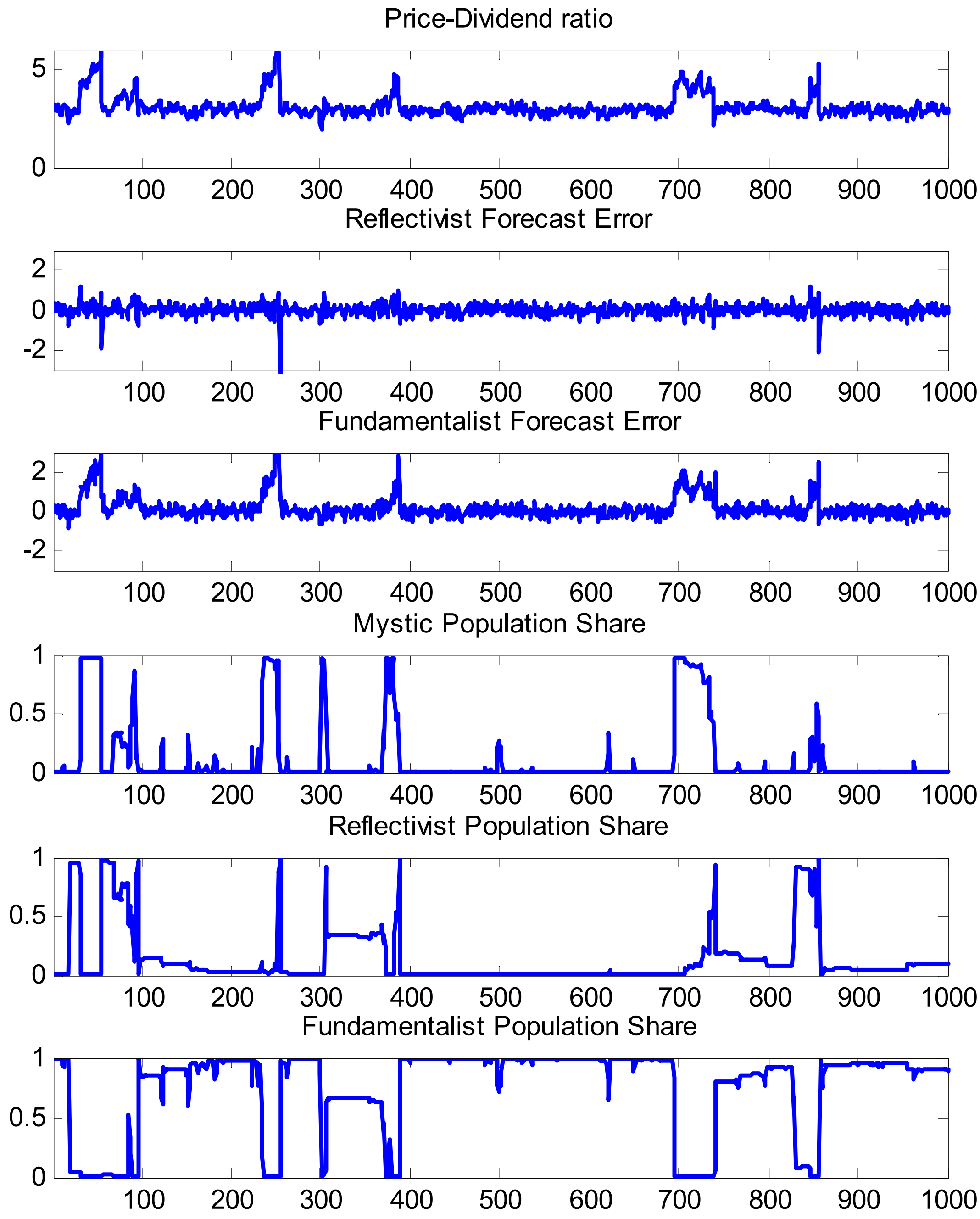

Figure 2

$\theta = 5.0$, $\sigma_\eta = \sigma_v$ x 1.0

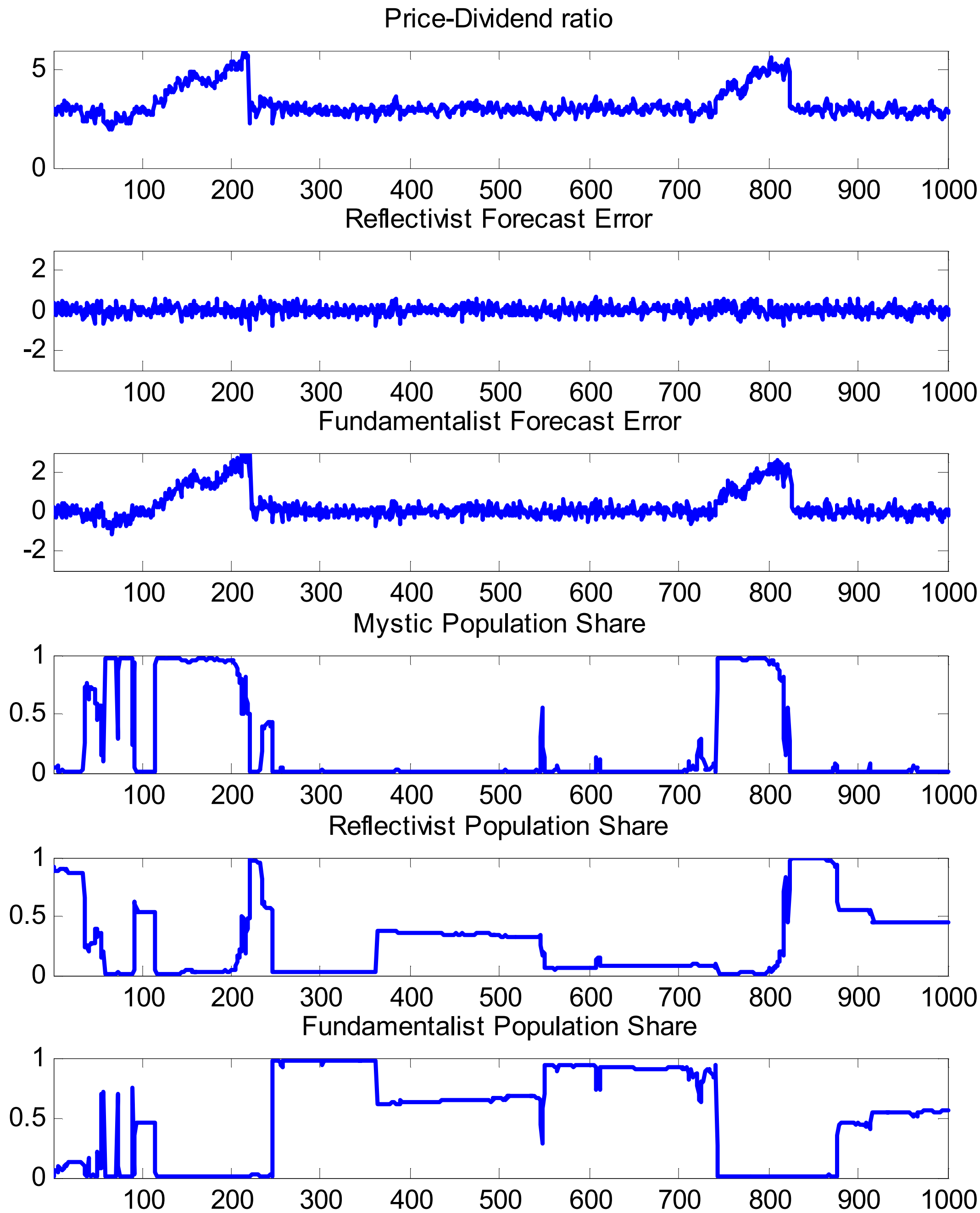

Figure 3

$\theta = 5.0$, $\sigma_\eta = \sigma_v$ x 0.25

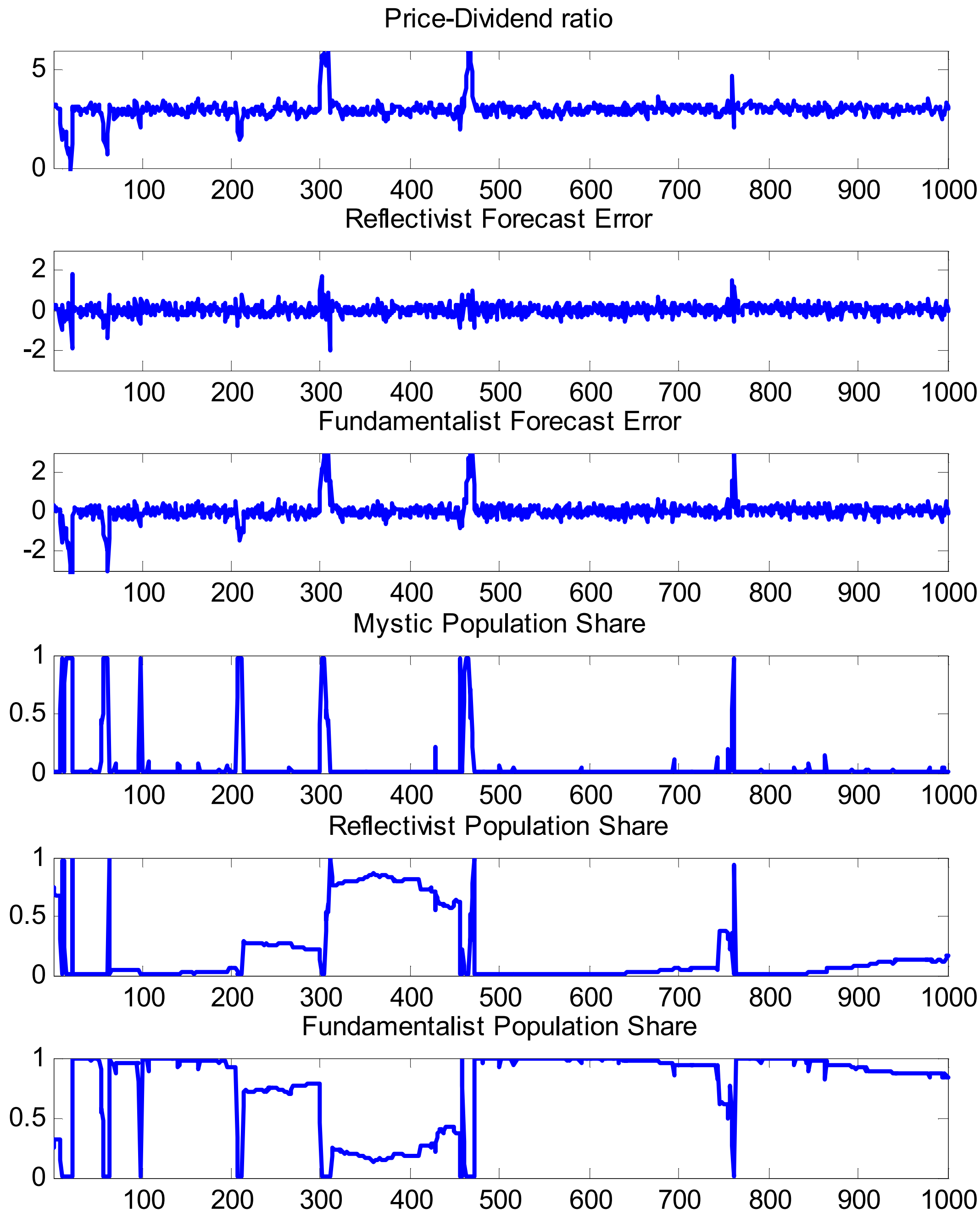

Figure 4

$\theta = 5.0, \sigma_\eta = \sigma_v$ x 2.0

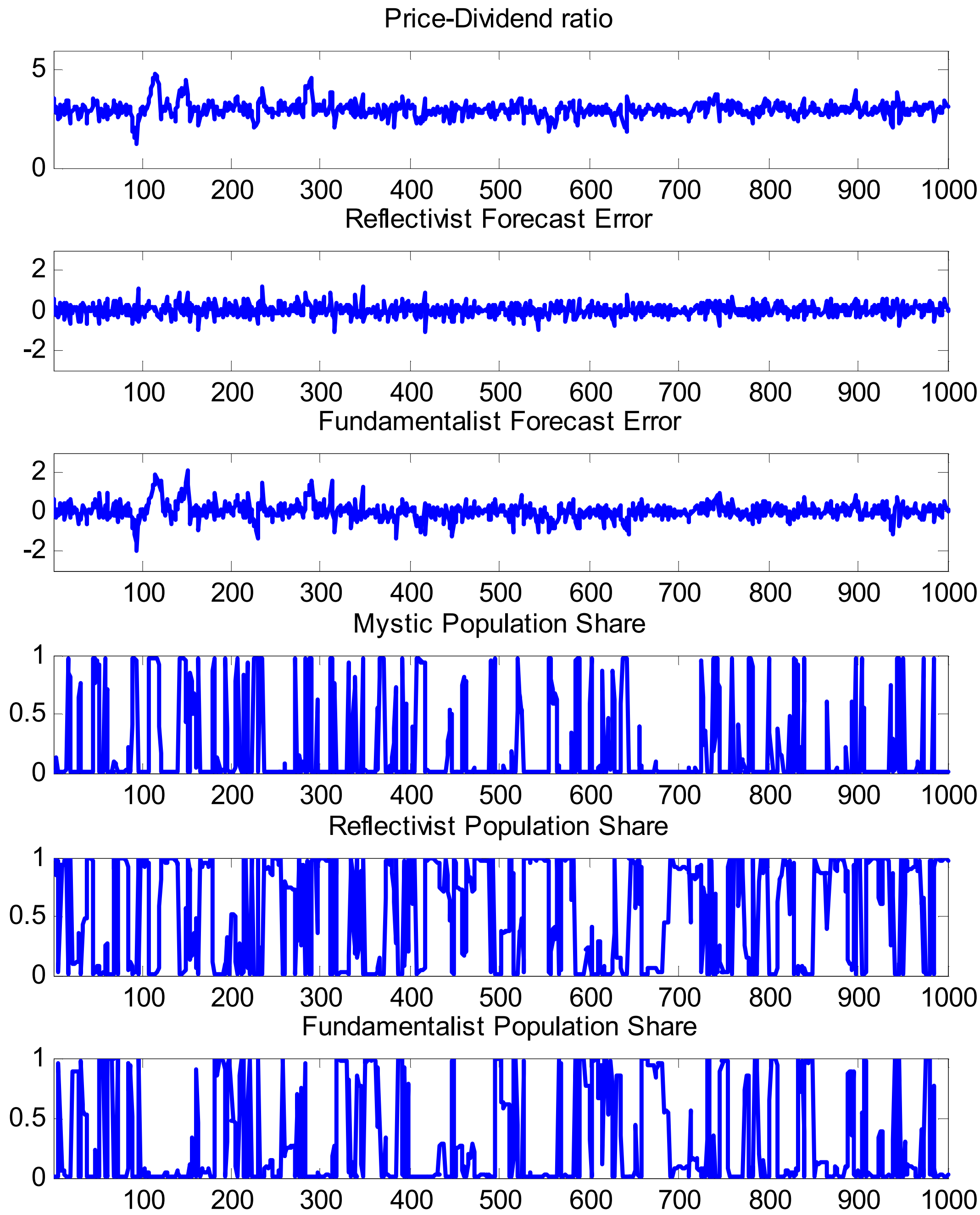

Figure 5
$\theta = 10.0$, $\sigma_\eta = \sigma_v$ x 1.0

				$\sigma_\eta = 0.23$ x			
		1/8	1/4	1/2	1	2	4	8
	5/8	0.000	0.000	0.000	0.000	0.000	0.000	0
	5/4	0.000	0.000	0.000	0.000	0.000	0.000	0.0001
	5/2	0.000	0.000	0.004	0.020	0.025	0.006	0.0014
θ	5	0.044	0.107	0.154	0.120	0.050	0.016	0.0054
	10	0.037	0.094	0.177	0.181	0.117	0.059	0.0298
	20	0.021	0.070	0.132	0.150	0.109	0.062	0.0334
	40	0.025	0.067	0.105	0.107	0.075	0.044	0.0236

Table 1

The fraction of periods over all runs where the mysticism exceeds 50% ($x_2 > 0.5$)

				$\sigma_\eta = 0.23$ x			
		1/8	1/4	1/2	1	2	4	8
	5/8	0.122	0.138	0.124	0.165	0.186	0.261	0.550
	5/4	0.130	0.149	0.135	0.154	0.197	0.230	0.480
	5/2	0.116	0.133	0.161	0.234	0.271	0.287	0.275
θ	5	0.177	0.257	0.362	0.464	0.493	0.468	0.361
	10	0.114	0.221	0.548	0.830	0.938	0.933	0.882
	20	0.122	0.122	0.192	0.567	0.879	0.928	0.909
	40	0.100	0.118	0.141	0.337	0.716	0.830	0.841

Table 2

The fraction of runs with one period where $p_t - d_t > \ln 2 + (\bar{p} - \bar{d})$

		$\sigma_\eta = 0.23$ x						
		1/8	1/4	1/2	1	2	4	8
	5/8	0.432	0.440	0.437	0.432	0.412	0.374	0.275
	5/4	0.436	0.434	0.442	0.433	0.415	0.381	0.302
	5/2	0.441	0.451	0.467	0.493	0.487	0.427	0.384
θ	5	0.479	0.535	0.610	0.620	0.519	0.397	0.355
	10	0.468	0.503	0.573	0.562	0.403	0.254	0.149
	20	0.432	0.435	0.453	0.417	0.308	0.193	0.118
	40	0.431	0.429	0.428	0.380	0.303	0.231	0.150

Table 3

The average across all trials of the one lag autocorrelation coefficient

		$\sigma_\eta = 0.23$ x						
		1/8	1/4	1/2	1	2	4	8
	5/8	0.092	0.091	0.094	0.089	0.092	0.095	0.097
	5/4	0.092	0.087	0.091	0.093	0.097	0.096	0.099
	5/2	0.094	0.103	0.125	0.169	0.188	0.154	0.142
θ	5	0.123	0.170	0.209	0.211	0.189	0.167	0.164
	10	0.093	0.105	0.128	0.135	0.151	0.167	0.169
	20	0.087	0.090	0.093	0.104	0.121	0.131	0.131
	40	0.089	0.089	0.092	0.098	0.111	0.123	0.120

Table 4

The standard deviation across all trials of the one lag autocorrelation coefficient.

		$\sigma_\eta = 0.23$ x						
		1/8	1/4	1/2	1	2	4	8
	5/8	0.230	0.232	0.234	0.237	0.239	0.250	0.286
	5/4	0.230	0.230	0.235	0.236	0.240	0.247	0.303
	5/2	0.231	0.235	0.289	0.544	0.662	0.405	0.314
θ	5	0.250	0.321	0.523	0.575	0.428	0.319	0.305
	10	0.237	0.255	0.316	0.373	0.387	0.414	0.485
	20	0.230	0.235	0.254	0.289	0.328	0.386	0.499
	40	0.229	0.231	0.240	0.256	0.285	0.333	0.430

Table 5

The standard deviation of the price-dividend ratio.

	$\sigma_\eta = 0.23$ x						
	1/8	1/4	1/2	1	2	4	8
5/8	0.000	0.002	0.001	0.000	0.002	0.000	0.001
5/4	0.002	0.002	0.001	0.001	0.002	0.001	0.002
5/2	0.002	0.001	0.002	0.003	0.007	0.009	0.013
θ 5	0.001	0.004	0.009	0.012	0.004	0.007	0.012
10	0.002	0.001	0.001	0.000	0.000	0.002	0.002
20	0.000	0.001	0.001	0.001	0.000	0.001	0.003
40	0.001	0.001	0.001	0.001	0.000	0.000	0.000

Table 6

The fraction of runs with predictable returns.

	$\sigma_\eta = 0.23$ x						
	1/8	1/4	1/2	1	2	4	8
5/8	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5/4	1.000	1.000	1.000	1.000	1.000	1.000	0.999
5/2	1.000	0.998	0.976	0.923	0.906	0.969	0.996
θ 5	0.989	0.927	0.794	0.784	0.931	0.995	0.995
10	1.000	0.998	0.974	0.992	0.999	1.000	1.000
20	1.000	1.000	1.000	1.000	1.000	1.000	1.000
40	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 7

The fraction of runs with a stationary price-dividend ratio.

		$\sigma_\eta = 0.23$ x						
		1/8	1/4	1/2	1	2	4	8
	5/8	1.006	1.012	1.020	1.030	1.038	1.053	1.127
	5/4	1.007	1.012	1.021	1.031	1.037	1.046	1.162
	5/2	1.010	1.020	1.137	1.946	2.478	1.442	1.186
θ	5	1.050	1.208	1.888	2.129	1.519	1.198	1.159
	10	1.027	1.073	1.237	1.389	1.416	1.486	1.734
	20	1.007	1.022	1.063	1.149	1.231	1.401	1.784
	40	1.002	1.007	1.024	1.065	1.123	1.240	1.529

Table 8
The ratio $Var(y_t^*)/Var(y_t)$ for each run.

		$\sigma_\eta = 0.23$ x						
		1/8	1/4	1/2	1	2	4	8
	5/8	0.360	0.422	0.418	0.481	0.508	0.699	0.986
	5/4	0.354	0.350	0.447	0.483	0.553	0.670	0.955
	5/2	0.348	0.376	0.480	0.559	0.557	0.486	0.529
θ	5	0.456	0.593	0.682	0.720	0.702	0.626	0.542
	10	0.478	0.721	0.921	0.992	0.991	0.994	0.988
	20	0.332	0.459	0.804	0.966	0.994	0.997	0.999
	40	0.341	0.373	0.555	0.801	0.929	0.983	0.996

Table 9
The fraction of runs with excess variance

				$\sigma_\eta = 0.23$ x			
		1/8	1/4	1/2	1	2	4	8
	5/8	0.028	0.040	0.028	0.037	0.026	0.05	0.043
	5/4	0.022	0.026	0.040	0.051	0.032	0.049	0.074
	5/2	0.033	0.035	0.046	0.053	0.158	0.172	0.136
θ	5	0.039	0.059	0.079	0.177	0.410	0.411	0.264
	10	0.035	0.026	0.046	0.129	0.320	0.435	0.39
	20	0.038	0.048	0.036	0.063	0.193	0.323	0.349
	40	0.040	0.027	0.037	0.051	0.180	0.291	0.294

Table 10
The fraction of runs with significant ARCH effects.