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Elisa Pagani, Letizia Pellegrini

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Scalarization and sensitivity analysis in Vector Optimization. The linear case.

Elisa Pagani and Letizia Pellegrini*

Department of Economics, University of Verona

viale Università 3, Palazzina 32

37129 Verona, Italy

Abstract In this paper we consider a vector optimization problem; we present some scalarization techniques for finding all the vector optimal points of this problem and we discuss the relationships between these methods. Moreover, in the linear case, the study of dual variables is carried on by means of sensitivity analysis and also by a parametric approach. We also give an interpretation of the dual variables as marginal rates of substitution of an objective function with respect to another one, and of an objective function with respect to a constraint.

Keywords: Vector Optimization, Image Space, Separation, Scalarization, Shadow Prices.

AMS Classification: 90C, 49K.

JEL Classification: C61.

*Corresponding author. Email: letizia.pellegrini@univr.it

1 Introduction and Preliminaries

This paper deals with a vector optimization problem and its dual variables (Lagrange multipliers or shadow prices) that we want to investigate in details. To this aim, we present some scalarization methods and, subsequently, the interpretation of dual variables is given by means of sensitivity analysis. We propose also a scheme to derive marginal rates of substitution between two different objective functions and also between an objective and a constraint. The setting is the linear case; in fact, it is the simplest to operate in at the beginning of the work, but obviously our intent is to propose in the future similar results in a more general framework and to focus also on duality arguments.

The paper deals mainly with scalarization methods. As it is well known, under convexity assumptions, a weighting scalarization technique permits to find all the optimal solutions of a vector optimization problem.

Otherwise, without any assumption on the given problem, a way of scalarizing consists in the construction of ℓ scalar problems - as many as the objective functions - called ε-constraint problems (see (3.2.1) of [11]).

Another scalarization method, without any assumption, is proposed in [9]. All the optimal points of the vector minimum problem are found by solving a scalar quasi-minimum problem, that, in the linear case, reduces to establish whether or not a parametric system of linear inequalities has a unique solution that is exactly the value of the parameter; a necessary and sufficient condition is given, which allows us to obtain this result. We also will point out some relationships between the two above methods.

The paper is organized as follows. In Sect.2, we present the scalarization techniques and some results about them; then, we describe a way to find all the optimal solutions of a vector optimization problem. Sect.3 is devoted to the study of shadow prices. Starting from a complete vector Lagrangian function, through scalarization and sensitivity analysis, we can derive two matrices of marginal rates of substitution, or rates of change; the former, call it Θ, refers to the rates of change between two objectives; the latter, call it Λ, refers to the rates of change between an objective and a constraint. In Subsect.3.1, we obtain these matrices trough the ε-constraint method and we prove that the matrix Θ is reciprocal and consistent; in Subsect.3.2, we obtain these matrices by means of the scalarization technique proposed in [9] and we point out the relationships between these two methods. The properties and the relations between Θ and Λ

are then outlined. All these aspects are discussed in Sect.4 through examples.

In the remaining part of this section, we recall the notations and basic notions useful in the sequel. O_n denotes the n-tuple, whose entries are zero; when there is no fear of confusion the subfix is omitted; for $n = 1$, the 1-tuple is identified with its element, namely, we set $O_1 = 0$. Let the positive integers ℓ, m, n, the cone $C \subseteq \mathbb{R}^\ell$, the vector-valued functions $f : \mathbb{R}^n \to \mathbb{R}^\ell$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ and the subset $X \subseteq \mathbb{R}^n$ be given. In the sequel, it will be assumed that C be convex, closed and pointed with apex at the origin (so that it identifies a partial order) and with $\operatorname{int} C \neq \emptyset$, namely with nonempty interior. Set $C_0 := C \backslash \{O\}$. We consider the following vector minimization problem, called *generalized Pareto problem*:

$$\min{}_{C_0}\ f(x) \ \text{ subject to } \ x \in K = \{x \in X : g(x) \geq O\}, \tag{1.1}$$

where $\min_{C_0}$ denotes vector minimum with respect to the cone C_0: $x^0 \in K$ is a (global) vector minimum point (for short, v.m.p.) to (1.1), if and only if

$$f(x^0) \not\geq_{C_0} f(x), \ \forall x \in K, \tag{1.2}$$

where the inequality means $f(x^0) - f(x) \notin C_0$. In what follows, we will assume that v.m.p. exist. Obviously, $x^0 \in K$ is a v.m.p. of (1.1), i.e (1.2) is fulfilled, if and only if the system (in the unknown x)

$$f(x^0) - f(x) \in C, \quad f(x^0) - f(x) \neq O, \quad g(x) \geq O, \quad x \in X \tag{1.3}$$

is impossible. System (1.3) is impossible if and only if $\mathcal{H} \cap \mathcal{K}(x^0) = \emptyset$, where $\mathcal{H} := C_0 \times \mathbb{R}^m_+$ and $\mathcal{K}(x^0) := \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u = f(x^0) - f(x), \ v = g(x), \ x \in X\}$. In the sequel, when there is no fear of confusion, $\mathcal{K}(x^0)$ will be denoted merely by $\mathcal{K}$. $\mathcal{H}$ and $\mathcal{K}$ are subsets of $\mathbb{R}^\ell \times \mathbb{R}^m$, that is called *image space*; $\mathcal{K}$ is called *image* of problem (1.1). In general, to prove directly $\mathcal{H} \cap \mathcal{K}(x^0) = \emptyset$ is a difficult task; hence this disjunction can be proved by means of a sufficient condition, that is the existence of a function, such that two of its disjoint level sets contain $\mathcal{H}$ and $\mathcal{K}$, respectively. To this end, let us consider the sets $U = C_0, \ V = \mathbb{R}^m_+$ and $U^*_{C_0} := \{\Theta \in \mathbb{R}^{\ell \times \ell} : \Theta u \geq_{C_0} O, \ \forall u \in U\}, \quad V^*_C := \{\Lambda \in \mathbb{R}^{\ell \times m} : \Lambda v \geq_C O, \ \forall v \in V\}$. Let us introduce the class of functions $w : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^\ell$, defined by:

$$w = w(u, v, \Theta, \Lambda) = \Theta u + \Lambda v, \quad \Theta \in U^*_{C_0}, \quad \Lambda \in V^*_C \tag{1.4}$$

where Θ, Λ play the role of parameters. For every vector-valued function of family (1.4) the positive and nonpositive level sets are given by:

$$W_{C_0}(u,v;\Theta,\Lambda) = \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u,v,\Theta,\Lambda) \geq C_0\};$$

$$\overline{W}_{C_0}(u,v;\Theta,\Lambda) = \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u,v,\Theta,\Lambda) \ngeq C_0\}.$$

Proposition 1.1. (see Proposition 1 of [9]) Let w be given by (1.4). We have $\mathcal{H} \subset W_{C_0}$, $\forall \Theta \in U^*_{C_0}$, $\forall \Lambda \in V^*_C$.

Proposition 1.1 is a first step towards a sufficient condition for the optimality of x^0. It is obvious that, if we can find one of the functions of class (1.4) such that $\mathcal{K}(x^0) \subset \overline{W}_{C_0}$, then the optimality of x^0 is proved. Indeed, we have the following result.

Theorem 1.1. (see Theorem 1 of [9]) Let $x^0 \in K$. If there exist matrices $\Theta \in U^*_{C_0}, \Lambda \in V^*_C$, such that

$$w(f(x^0) - f(x), g(x), \Theta, \Lambda) = \Theta(f(x^0) - f(x)) + \Lambda g(x) \ngeq_{C_0} O, \ \forall x \in X, \tag{1.5}$$

then x^0 is a (global) v.m.p. of (1.1).

If $C = \mathbb{R}^\ell_+$, then (1.1) becomes the classic *Pareto vector problem* and (1.2) is the definition of Pareto optimal point.
At $\ell = 1$ and $C = \mathbb{R}_+$, the above theorem collapses to an existing one for scalar optimization (see Corollary 5.1 of [7]). Observe that the identity matrix of order ℓ, say I_ℓ, belongs to $U^*_{C_0}$ and that, when $\ell = 1$, Θ can be replaced by 1.

2 Scalarization of Vector Problems

There exist many scalarization methods; we recall here those ones exploited in this work to deepen the study of shadow prices.
A) Weighting method (see (3.1.1), Part II of [11]).
Problem (1.1), where $C = \mathbb{R}^\ell_+$, is associated with the following scalar problem:

$$\min \sum_{i=1}^{\ell} w_i f_i(x) \ \text{ subject to } \ x \in K, \tag{2.1}$$

where the w_i are weighting coefficients, such that $w_i \geq 0 \ \forall i, \ \sum_{i=1}^{\ell} w_i = 1$.
Classical results are represented by the following propositions (see [11]).

Proposition 2.1. A minimum point of problem (2.1) is Pareto optimal if the weighting coefficients are positive, that is $w_i > 0, \ \forall i = 1, \dots, \ell$.

Proposition 2.2. If there exists a unique minimum point of problem (2.1), then it is Pareto optimal.

Proposition 2.3. Let (1.1) be convex. If $x^0 \in K$ is Pareto optimal, then there exists a weighting vector w ($w_i \geq 0, \ i = 1, \dots, \ell, \ \sum_{i=1}^{\ell} w_i = 1$), such that x^0 is a minimum point of (2.1).

B) ε-constraint method (see (3.2.1), Part II of [11]).
Problem (1.1), when $C = \mathbb{R}^{\ell}_{+}$, is associated with the following ℓ scalar problems:

$$P_k(\varepsilon): \qquad \min f_k(x) \ \text{ s.t. } \ f_j(x) \leq \varepsilon_j \ \ \forall j \neq k, \ \ x \in K; \ \ k = 1, \dots, \ell, \tag{2.2}$$

where ε_j is an upper bound for function f_j with $j = 1, \dots, \ell$. For the proofs of the following results we refer to [11].

Proposition 2.4. A vector $x^0 \in K$ is Pareto optimal if and only if it is a minimum point of all problems $P_k(\varepsilon)$, $\forall k = 1, \dots, \ell$, at $\varepsilon_j = f_j(x^0)$ for $j = 1, \dots, \ell, \ \ j \neq k$.

Therefore it is possible to find every Pareto optimal solution to (1.1) by the ε-constraint method. Further results are the following ones.

Proposition 2.5. A point $x^0 \in K$ is Pareto optimal if it is the unique minimum point of an ε-constraint problem for some k with $\varepsilon_j = f_j(x^0)$ for $j = 1, \dots, \ell, \ \ j \neq k$.

Proposition 2.6. The unique minimum point of the kth ε-constraint problem, i.e. $P_k(\varepsilon)$, is Pareto optimal for any given upper bound vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_{k+1}, \dots, \varepsilon_{\ell})$.

Proposition 2.7. Let $x^0 \in K$ be a minimum point of (2.1) and $w_i \geq 0, \ \forall i = 1, \dots, \ell$; we have:

i) if $w_k > 0$, then x^0 is a solution to $P_k(\varepsilon)$ for $\varepsilon_j = f_j(x^0)$, with $j = 1, \dots, k, \ j \neq k$;

ii) if x^0 is the unique solution to (2.1), then x^0 is a solution to (2.2) for $\varepsilon_j = f_j(x^0)$.

Moreover, if (1.1) is convex and $x^0 \in K$ is a minimum point of (2.2) and $\varepsilon_j = f_j(x^0)$ for $j = 1, \dots, \ell, \ \ j \neq k$, then there exists a weighting vector $w_i \geq 0$, $\forall i = 1, \dots, \ell$, with $\sum_{i=1}^{\ell} w_i = 1$, such that x^0 is also a minimum point of (2.1).

Finally, we will shortly recall the scalarization approach proposed in [9] for solving vector optimization problems.

C) Scalarization by Separation.

In [9] it has been shown how to set up a scalar minimization problem, which leads to detect either all the v.m.p. of (1.1) or merely only one.

For every $y \in X$, let us define the following set:

$$S(y) := \{x \in X : f(x) \in f(y) - C\}.$$

Then, let us consider any fixed $p \in C^* := \{z \in \mathbb{R}^n : \langle z, x\rangle \geq 0, \forall x \in C\}$ and introduce the (scalar) *quasi-minimum problem* (in the unknown x):

$$\min \langle p, f(x)\rangle, \quad \text{s.t.} \quad x \in K \cap S(y), \tag{2.3}$$

whose feasible region depends on the parameter y.

We stress the fact that, in what follows, unlike y, p will not play the role of a parameter and will be considered fixed. Observe that, if $C = \mathbb{R}^\ell_+$, then $S(y) = \{x \in X : f(x) \leq f(y)\}$, so that $S(y)$ is a (vector) level set of f.

As it will be clear in the sequel, it is interesting to find conditions under which the set $K \cap S(y)$ (which obviously contains y) is a singleton.

Proposition 2.8. $K \cap S(x^0) = \{x^0\}$ if and only if x^0 is a v.m.p. of (1.1) and $\nexists x \in K\backslash\{x^0\}$ such that $f(x) = f(x^0)$.

Proof. $K \cap S(x^0) = \{x^0\}$ if and only if

$$f(x^0) - f(x) \notin C, \quad \forall x \in K\backslash\{x^0\}, \tag{2.4a}$$

or, equivalently,

$$f(x^0) - f(x) \notin C \Longleftrightarrow f(x^0) - f(x) \notin C_0 \text{ and } f(x^0) - f(x) \neq O. \tag{2.4b}$$

Then (2.4) is equivalent to claim that x^0 is a v.m.p. of (1.1) and $f(x^0) \neq f(x)$, $\forall x \in K\backslash\{x^0\}$. □

The scalarization approach of [9] is based on the results expressed by the following propositions.

Proposition 2.9. Let any $p \in \operatorname{int} C^*$ be fixed. Then x^0 is a v.m.p. of (1.1), if and only if it is a (scalar) minimum point of (2.3) at $y = x^0$.

Proposition 2.10. If x^0 is a (global) minimum point of (2.3) at $y = y^0$, then x^0 is a (global) minimum point of (2.3) also at $y = x^0$.

Proposition 2.10 suggests a method for finding a v.m.p. of (1.1). Let us choose any $p \in \operatorname{int} C^*$; p will remain fixed in the sequel. Then, we choose any $y^0 \in K$ and solve the (scalar) problem (2.3) at $y = y^0$. We find a solution x^0. According to Proposition 2.9, x^0 is a v.m.p. of (1.1). If we want to find all the v.m.p. of (1.1) – this happens, for instance, when a given function must be optimized over the set of all v.m.p. of (1.1) – then, starting with $y = x^0$, we must parametrically move $y \in K$ and maintain y itself as a solution to (2.3). Propositions 2.9 and 2.10 guarantee that all the solutions to (1.1) will be reached. Note that such a scalarization method does not require any assumption on (1.1).
This method is proposed also in [17] and [3], and problem (2.3) is sometimes called "hybrid problem" (see (3.3.1), Part II of [11]), but in [9] the results are obtained, by means of separation arguments, independently of [17] and [3]. The following example shows the application of the above scalarization approach.

Example 2.1. Let us set $\ell = 3$, $m = 1$, $n = 2$, $X = \mathbb{R}^2$, $C = \mathbb{R}^3_+$ and

$$f_1(x) = x_1 + 2x_2, \quad f_2(x) = 4x_1 + 2x_2, \quad f_3(x) = x_1 + 3x_2, \quad g(x) = -|x_1| + x_2.$$

Choose $p = (1, 1, 1)$ and $y^0 = (0, 1)$. Then (2.3) becomes:

$$\min(6x_1 + 7x_2), \quad \text{s.t.} \quad -|x_1| + x_2 \geq 0, \quad x_1 + 2x_2 \leq 2, \quad 2x_1 + x_2 \leq 1, \quad x_1 + 3x_2 \leq 3. \tag{2.4}$$

The (unique) solution to (2.4) is easily found to be $x^0 = (0, 0)$. Because of Proposition 2.9, x^0 is a v.m.p. of (1.1) in the present case. Furthermore, we have $K \cap S(x^0) = \{x^0\}$, namely the parametric system (in the unknown x):

$$-|x_1| + x_2 \geq 0, \quad x_1 + 2x_2 \leq y_1 + 2y_2, \quad 2x_1 + x_2 \leq 2y_1 + y_2, \quad x_1 + 3x_2 \leq y_1 + 3y_2 \tag{2.5}$$

has the (unique) solution x^0. In order to find all the v.m.p. of (1.1), we have to search for all $y \in K$, such that (2.5) has y itself as the (unique) solution. (2.5) is equivalent to

$$\begin{cases} |x_1| \leq x_2 \leq -\frac{1}{2}x_1 + \frac{1}{2}y_1 + y_2 \\ \quad x_2 \leq -2x_1 + 2y_1 + y_2 \\ \quad x_2 \leq -\frac{1}{3}x_1 + \frac{1}{3}y_1 + y_2 \end{cases} \tag{2.6}$$

Due to the simplicity of the example, it is easy to see, by direct inspection, that $y \in \mathbb{R}^2$ is the unique solution to (2.6), if and only if $y_1 + y_2 = 0, y_1 \leq 0$ or

$$y = (y_1 = -t, y_2 = t), \quad t \in [0, +\infty),$$

which gives us all the v.m.p. of (1.1).

We wish to propose a general method for solving the problem of finding the unique solution to $K \cap S(x^0)$, but this is not a simple matter. Meanwhile let us introduce a way to do this in the linear case. First of all, we suppose to have a non-parametric system.

Theorem 2.1. Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$, a bounded and nonempty polyhedron $P := \{x \in \mathbb{R}^n : Ax \geq b\}$ is a singleton if and only if

$$\max_{x\in P} \left\langle c^i, x\right\rangle = \min_{x\in P} \left\langle c^i, x\right\rangle, \quad \forall i \in J := \{1, \dots, n\}, \tag{2.7}$$

where $c^1, \dots, c^n \in \mathbb{R}^n$ are fixed linearly independent vectors.

Proof. The necessity is trivial. The sufficiency is proved *ab absurdo*: if $\exists x^1, x^2 \in P$ with $x^1 \neq x^2$, then two cases are possible for $\hat{x} := x^2 - x^1$: either $\hat{x}$ is orthogonal to every $c^i,\ i \in J$, or $\exists h \in J$ such that $\hat{x}$ is not orthogonal to c^h. In the former case, we have $\hat{x} = O$, because $\hat{x}$ is orthogonal to n linearly independent vectors; hence $x^1 = x^2$ which contradicts the assumption. In the latter case, we contradict (2.8) since $\langle c^h, \hat{x}\rangle \neq 0$ and we have

$$\max_{x\in P} \left\langle c^i, x\right\rangle \neq \min_{x\in P} \left\langle c^i, x\right\rangle$$

for $x = \hat{x}$ and $c^i = c^h$. Finally, we conclude that P is a singleton. $\square$

Remark 1. Let $R(y) := \{x \in \mathbb{R}^n : \bar{A}x \geq \bar{B}y + \bar{b}\}$, where

$$\bar{A} := \begin{pmatrix} A \\ -F \end{pmatrix} \in \mathbb{R}^{(m+\ell)\times n},\ \bar{B} := \begin{pmatrix} O_{m\times n} \\ -F \end{pmatrix} \in \mathbb{R}^{(m+\ell)\times n} \ \text{ and } \ \bar{b} := \begin{pmatrix} b \\ O_\ell \end{pmatrix} \in \mathbb{R}^{(m+\ell)}.$$

The set

$$Y := \{y \in \mathbb{R}^n : \max_{x\in R(y)} \left\langle c^i, x\right\rangle = \min_{x\in R(y)} \left\langle c^i, x\right\rangle, \quad i = 1, \dots, n; \quad y \in \arg\max_{x\in R(y)} \left\langle c^1, x\right\rangle\}$$

is the set of parameters y we are looking for to obtain as unique solution to $K \cap S(x^0)$, $y = x^0$. Note that $Y \neq \emptyset$ if and only if $R(y)$ is a singleton; moreover, the following theorem gives a sufficient condition to have $Y = \emptyset$.

Theorem 2.2. Assume that $R(y) \neq \emptyset$. If $\bar{A}^* := \{x \in \mathbb{R}^n : \langle \bar{a}^i, x \rangle \geq 0 \;\; i = 1, \ldots, m+\ell\} \neq \{O\}$, then $Y = \emptyset$.

Proof. By hypothesis $\exists \hat{x} \neq O$ such that $\langle \bar{a}^i, x \rangle \geq 0 \quad i = 1, \ldots, m+\ell$, i.e., $\bar{A}\hat{x} \geq O$. From $R(y) \neq \emptyset$ follows $\exists \tilde{x} \in R(y)$ such that $\bar{A}\tilde{x} \geq \bar{B}y + \bar{b}$. Let us consider the vectors $\tilde{x} + \alpha\hat{x}$, $\forall \alpha > 0$; we have $\bar{A}(\tilde{x} + \alpha\hat{x}) \geq \bar{B}y + \bar{b}$ and hence, since $\hat{x} \neq O$, $\tilde{x} + \alpha\hat{x}$ is an element of $R(y)$ for any choice of α. We have found that $R(y)$ is not a singleton, and from Theorem 2.1 and Remark 1, follows $Y = \emptyset$. □

In other words we have proved that if the positive polar cone of the cone generated by the rows of $\bar{A}$ does not collapse to the origin $O \in \mathbb{R}^n$, then $\nexists y \in \mathbb{R}^n$ such that it is the unique solution to the system $\bar{A}x \geq \bar{B}y + \bar{b}$.

3 Shadow Prices

This section aims at carrying on the study of the dual problem of (1.1) and the strictly related analysis of the shadow prices. A sufficient optimality condition, given in Theorem 1.1, states that $x^0 \in K$ is a v.m.p. of (1.1) if $\Theta(f(x^0) - f(x)) + \Lambda g(x) \not\geq_{C_0} O$, $\forall x \in X$, for $\Theta \in U^*_{C_0}$, $\Lambda \in V^*_C$. From this condition, and remembering the form for a complete vector Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^{\ell \times \ell} \times \mathbb{R}^{\ell \times m} \to \mathbb{R}^\ell$: $L(x; \Theta, \Lambda) := \Theta f(x) - \Lambda g(x)$, we can observe that there are two kinds of dual variables: $\Lambda = (\lambda_{ij})$, $i = 1, \ldots, \ell$, $j = 1, \ldots, m$ is the matrix of Lagrangian multipliers, where λ_{ij} denotes the change of the constrained optimal value of the ith objective function with respect to the level of the jth constraining function, while $\Theta = (\theta_{ik})$, $i, k = 1, \ldots, \ell$, is the matrix of Lagrangian multipliers, where θ_{ik} denotes the rate of change in the value of f_i when it occurs a change in the value of f_k, sometimes called trade-off. Hence, in Vector Optimization it is easy to understand the interest not only in studying the rate of change of every objective function with respect to the movement of any constraint, but also in evaluating, by means of the dual variables, the rate of change of every objective function with respect to the change of the other objectives. To this aim, we will study problems (2.2) and the scalar quasi-minimum problem (2.3). More precisely, we refer to Proposition 2.4 for the ε-constraint method, and to Proposition 2.9 for the quasi-minimum problem. In both cases, the analysis will be accomplished in the linear case; nevertheless, since the results of Sect.2 do not require any assumption on (1.1), it is possible to apply the same arguments to more general cases. Hence, let us consider the following positions: $C = \mathbb{R}^\ell_+$; $f(x) = Fx$, with $F \in \mathbb{R}^{\ell\times n}$; $g(x) = Ax - b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$; $X = \mathbb{R}^n$. $\forall i \in I$, we will denote by F_i the ith row of the matrix F; problem (1.1) becomes:

$$\min{}_{\mathbb{R}^\ell_+\setminus\{O\}}\ Fx, \quad \text{s.t.} \quad x \in K = \{x \in \mathbb{R}^n : Ax \geq b\}. \tag{3.1}$$

3.1 Shadow prices by means of ε-constraint method

In the linear case, when $\varepsilon = Fx^0$, problems (2.2) become:

$$L_k(x^0): \quad \min\langle F_k, x\rangle, \quad \text{s.t.} \quad x \in \{x \in \mathbb{R}^n : Ax \geq b, \langle F_i, x\rangle \leq \langle F_i, x^0\rangle, i \in I \setminus \{k\}\}, \quad k \in I.$$

In the sensitivity analysis, we will assume that x^0 be a v.m.p. of (3.1) and hence also of all problems $L_k(x^0), \forall k \in I$. Since we are considering the linear case, at least one of the constraints of problem (3.1) is fulfilled as equality at an optimal solution x^0; let $B \in \mathbb{R}^{n_1\times n}$ and $d \in \mathbb{R}^{n_1}$, with $n_1 < m$, be the submatrix of A and the subvector of b corresponding to a subset of the binding constraints at x^0. Now, let us pay attention to $L_k(x^0)$ for a fixed $k \in I$: suppose that B^k be a basis corresponding to the solution x^0 and that the first n_1 rows of B^k are those of B and the remaining $n_2 = n - n_1 > 0$ are from F. Therefore, it results:

$$B^k = \begin{pmatrix} B \\ -F_i^T, i \in I_{n_2^k} \end{pmatrix},$$

where $I_{n_2^k}$ is a subset of $I \setminus \{k\}$ of cardinality n_2. Among all the bases fulfilling the above properties, let us consider only those for which x^0 is also a minimum point of the problem:

$$\min\langle F_k, x\rangle, \quad \text{s.t.} \quad x \in R_k := \{x \in \mathbb{R}^n : Bx \geq d, \langle -F_i, x\rangle \geq \langle -F_i, x^0\rangle, i \in I_{n_2^k}\}. \tag{3.2}$$

Observe that, in general, such a property is not maintained for any basis B^k; in the sequel, it will be understood. Define the perturbed problem:

$$\min\langle F_k, x\rangle, \text{ s.t. } x \in R_k(\eta;\xi) := \{x \in \mathbb{R}^n : Bx \geq d+\eta, \langle -F_i, x\rangle \geq \langle -F_i, x^0\rangle + \xi_i, i \in I_{n_2^k}\}, \tag{3.3}$$

where $\eta \in \mathbb{R}^{n_1}$ and $\xi \in \mathbb{R}^{n_2^k}$. Now, let us consider the Kuhn-Tucker multipliers of the problem (3.2); they are the solution to the following system:

$$B^T\lambda^k - \sum_{i\in I_{n_2^k}} F_i\vartheta_i^k = F_k, \ \tilde{\lambda}^k \in \mathbb{R}^{n_1}_+, \ \tilde{\vartheta}_i^k \in \mathbb{R}_+, \ i \in I_{n_2^k}. \tag{3.4}$$

It is well known that they are the derivatives of the optimal value of the function $\langle F_k, x\rangle$ with respect to the level of the constraint function (when it is binding at x^0, as previously supposed):

$$\tilde{\lambda}_i^k = \frac{\partial}{\partial \eta_i}\left(\min_{x\in R_k(\eta;\xi)} \langle F_k, x\rangle\right), \quad i = 1,\dots,n_1, \tag{3.5}$$

$$\tilde{\vartheta}_i^k = \frac{\partial}{\partial \xi_i}\left(\min_{x\in R_k(\eta;\xi)} \langle F_k, x\rangle\right), \quad i \in I_{n_2^k}. \tag{3.6}$$

Recall the following definitions [13].

Definition 3.1. Let A be an $n \times n$ matrix with non-zero elements. A is said to be *reciprocal*, if and only if $a_{ij} = \frac{1}{a_{ji}} \quad \forall i, j = 1,\dots,n$.

Definition 3.2. Let A be an $n \times n$ reciprocal matrix. A is said to be *consistent*, if and only if $a_{ij}a_{jk} = a_{ik} \quad \forall i, j, k = 1,\dots,n$.

We have the following results.

Proposition 3.1 (Reciprocity). Let $j, k \in I$ be such that for the corresponding problems $L_j(x^0)$ and $L_k(x^0)$ there exist two bases B^j and B^k, such that $I_{n_2^k} \cup \{k\} = I_{n_2^j} \cup \{j\}$. Then, it results that $\tilde{\vartheta}_j^k = 1/\tilde{\vartheta}_k^j$.

Proof. $\tilde{\vartheta}_j^k$ is the jth component of the solution $\tilde{\vartheta}^k$ of (3.4), while $\tilde{\vartheta}_k^j$ is the kth component of the solution $\tilde{\vartheta}^j$ of the following system:

$$B^T\lambda^j - \sum_{i\in I_{n_2^j}} F_i\vartheta_i^j = F_j.$$

$\tilde{\vartheta}_j^k$ and $\tilde{\vartheta}_k^j$ can be determined by applying the Cramer rule. Hence:

$$\tilde{\vartheta}_j^k = \frac{det\left(B^T \,\middle|\, - F_i, i \in \left(I_{n_2^k} \setminus \{j\}\right) \cup \{k\}\right)}{det\left(B^T \,\middle|\, - F_i, i \in I_{n_2^k}\right)},$$

where the order of the indexes $i \in \left(I_{n_2^k} \setminus \{j\}\right) \cup \{k\}$ is rearranged in such a way that j is replaced by k. Likewise and with the same remark about the indexes, it results:

$$\tilde{\vartheta}_j^k = \frac{det\left(B^T \,\middle|\, - F_i, i \in \left(I_{n_2^j} \setminus \{k\}\right) \cup \{j\}\right)}{det\left(B^T \,\middle|\, - F_i, i \in I_{n_2^j}\right)}.$$

Since, by assumption, we have $I_{n_2^k} \cup \{k\} = I_{n_2^j} \cup \{j\}$, it results $\big(I_{n_2^k} \cup \{k\}\big) \setminus \{j\} = I_{n_2^j}$ and $\big(I_{n_2^j} \cup \{j\}\big) \setminus \{k\} = I_{n_2^k}$. We can conclude that

$$\tilde{\vartheta}_j^k = \frac{det\Big(B^T \Big| - F_i, i \in I_{n_2^j}\Big)}{det\Big(B^T \Big| - F_i, i \in I_{n_2^k}\Big)}, \qquad \tilde{\vartheta}_k^j = \frac{det\Big(B^T \Big| - F_i, i \in I_{n_2^k}\Big)}{det\Big(B^T \Big| - F_i, i \in I_{n_2^j}\Big)};$$

i.e., that $\tilde{\vartheta}_j^k = 1/\tilde{\vartheta}_k^j$. □

Proposition 3.2 (Consistency). Let $j, k, s \in I$. Consider the problems $L_j(x^0)$ and $L_k(x^0)$ having $\langle F_j, x\rangle$ and $\langle F_k, x\rangle$ as the objective functions, respectively. Suppose that:

(i) for $L_j(x^0)$ there exist two bases

$$B^{j_k} = \begin{pmatrix} B \\ -F_i^T, i \in I_{n_2^{j_k}} \end{pmatrix} \quad \text{and} \quad B^{j_s} = \begin{pmatrix} B \\ -F_i^T, i \in I_{n_2^{j_s}} \end{pmatrix}$$

with $k \in I_{n_2^{j_k}}$, $s \in I_{n_2^{j_s}}$ and $I_{n_2^{j_k}} \cup \{s\} = I_{n_2^{j_s}} \cup \{k\}$;

(ii) the matrix B^{j_s} maintains the optimality of x^0 also for problem $L_k(x^0)$; i.e.,

$$B^k = \begin{pmatrix} B \\ -F_i^T, i \in I_{n_2^k} \end{pmatrix}, \quad \text{with} \quad I_{n_2^k} = I_{n_2^{j_s}}.$$

Then, it results that $\tilde{\vartheta}_k^j \tilde{\vartheta}_s^k = \tilde{\vartheta}_s^j$.

Proof. $\tilde{\vartheta}_k^j$ is the kth component of the solution $\tilde{\vartheta}^j$ of the following system:

$$B^T \lambda^j - \sum_{i \in I_{n_2^{j_k}}} F_i \vartheta_i^j = F_j.$$

Therefore, by the Cramer rule, it turns out:

$$\tilde{\vartheta}_k^j = \frac{det\Big(B^T \Big| - F_i, i \in \big(I_{n_2^{j_k}} \setminus \{k\}\big) \cup \{j\}\Big)}{det\Big(B^T \Big| - F_i, i \in I_{n_2^{j_k}}\Big)}.$$

By assumption (i), we have that

$$I_{n_2^{j_k}} \setminus \{k\} = I_{n_2^{j_s}} \setminus \{s\} \tag{3.7}$$

and hence:

$$\tilde{\vartheta}^j_k = \frac{det\left(B^T \,\middle|\, -F_i, i \in \left(I_{n_2^{j_s}} \setminus \{s\}\right) \cup \{j\}\right)}{det\left(B^T \,\middle|\, -F_i, i \in I_{n_2^{j_k}}\right)}.$$

Now, calculate $\tilde{\vartheta}^k_s$, that is the sth component of the solution $\tilde{\vartheta}^k$ of (3.4):

$$\tilde{\vartheta}^k_s = \frac{det\left(B^T \,\middle|\, -F_i, i \in \left(I_{n_2^k} \setminus \{s\}\right) \cup \{k\}\right)}{det\left(B^T \,\middle|\, -F_i, i \in I_{n_2^k}\right)}.$$

By (3.7) and by assumption (ii), it results $I_{n_2^{j_k}} \setminus \{k\} = I_{n_2^k} \setminus \{s\}$ or, equivalently, $I_{n_2^{j_k}} = \left(I_{n_2^k} \setminus \{s\}\right) \cup \{k\}$. It follows that:

$$\tilde{\vartheta}^j_k \tilde{\vartheta}^k_s = \frac{det\left(B^T \,\middle|\, -F_i, i \in \left(I_{n_2^{j_s}} \setminus \{s\}\right) \cup \{j\}\right)}{det\left(B^T \,\middle|\, -F_i, i \in I_{n_2^k}\right)} = \frac{det\left(B^T \,\middle|\, -F_i, i \in \left(I_{n_2^{j_s}} \setminus \{s\}\right) \cup \{j\}\right)}{det\left(B^T \,\middle|\, -F_i, i \in I_{n_2^{j_s}}\right)},$$

where the second equality is again because of (ii). In conclusion, by the Cramer rule, we have that $\tilde{\vartheta}^j_k \tilde{\vartheta}^k_s$ is the sth component of the solution $\tilde{\vartheta}^j$ of the system

$$B^T \lambda^j - \sum_{i \in I_{n_2^{j_s}}} F_i \vartheta^j_i = F_j,$$

and hence $\tilde{\vartheta}^j_k \tilde{\vartheta}^k_s = \tilde{\vartheta}^j_s$. □

From statement (3.6) and the thesis of Proposition 3.1 follows the interpretation of the reciprocity as rule for deriving the inverse function, while from (3.6) and the thesis of Proposition 3.2 the consistency property can be interpreted as rule for deriving the composed function.

Now, we are in the position to define the shadow prices matrices Θ and Λ. $\forall k \in I$, let us consider the problem $L_k(x^0)$, and $\forall i \in I_{n_2^k}$ solve the corresponding system (3.4). Define Θ as the $\ell \times \ell$ matrix whose diagonal elements are equal to 1 and $\vartheta_{ki} = -\tilde{\vartheta}^k_i$, $i \in I_{n_2^k}$; observe that the change of sign of $\tilde{\vartheta}^k_i$ is because of the change of sign in the corresponding constraint in the calculus of $\tilde{\vartheta}^k_i$ by means of (3.4). The result expressed by (3.6) justifies the definition of "shadow prices matrix" for Θ and gives the following meaning of ϑ_{ki}: how the kth objective function depends on the level of the ith one at the optimal solution x^0. In general, not all the elements of the matrix Θ are obtained by means of the above procedure, it is possible to complete the matrix Θ by imposing that Θ be reciprocal and consistent. From Proposition 3.1

and Proposition 3.2, it turns out that the elements of Θ defined by the above procedure satisfy the property of reciprocity and consistency, except for the sign. From these properties follows also that Θ has null determinant and hence its inverse does not exists. Moreover, starting from the solution of system (3.4), define the matrix $\Lambda \in \mathbb{R}^{\ell \times n_1}$ of shadow prices corresponding to the binding constraints $Bx \geq d$ as the matrix whose kth row is the vector $\tilde{\lambda}^{k^T}$, $\forall k \in I$. In this case too, due to (3.5), the kth row of Λ shows as the kth objective function depends on the levels of the binding constraints at the optimal solution. Observe that, in general, the matrices Λ and Θ are not univocally determined.

The following example shows how to calculate the shadow prices matrices Θ and Λ by means of the procedure just before outlined.

Example 2.1 (continuation). Same data as in Example 2.1, with the sole exception of g, which is now split into $g_1(x) = x_1 + x_2$ and $g_2(x) = -x_1 + x_2$; so that in the present case (1.1) becomes a linear problem; i.e. (3.1). Choose $x^0 = (-2, 2)$ as optimal solution to (3.1) and consider the three problems $L_k(x^0)$, $k = 1, 2, 3$. Observe that in (3.1) only the constraint $g_1(x)$ is binding at $x^0 = (-2, 2)$, and hence, in any basis that we will consider, there is the sub-matrix $B = \begin{pmatrix} 1 & 1 \end{pmatrix}$, while $g_2(x)$ can be disregarded in the problems $L_k(x^0)$, $k = 1, 2, 3$. It results:

$$L_1(x^0): \qquad \min(x_1 + 2x_2), \quad \text{s.t.} \quad x_1 + x_2 \geq 0, \quad 4x_1 + 2x_2 \leq -4, \quad x_1 + 3x_2 \leq 4.$$

There are two bases containing the original constraint $x_1 + x_2 \geq 0$ and one between $4x_1 + 2x_2 \leq -4$ and $x_1 + 3x_2 \leq 4$. If we consider the former of them, that is $B^1 = \begin{pmatrix} 1 & 1 \\ -4 & -2 \end{pmatrix}$, then x^0 is again an optimal solution to the sub-problem:

$$\min(x_1 + 2x_2), \quad \text{s.t.} \quad x_1 + x_2 \geq 0, \quad 4x_1 + 2x_2 \leq -4.$$

The Lagrangian multipliers $\tilde{\lambda}_1^1$, $\tilde{\theta}_2^1$ are the solutions of the system $\begin{cases} \lambda_1^1 - 4\theta_2^1 = 1 \\ \lambda_1^1 - 2\theta_2^1 = 2 \end{cases}$. Therefore, it turns out $\tilde{\lambda}_1^1 = 3$, $\tilde{\theta}_2^1 = \frac{1}{2}$; this implies $\theta_{12} = -\frac{1}{2}$. If we consider the latter basis; i.e., $\begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$, then x^0 it is no more an optimal solution to the sub-problem:

$$\min(x_1 + 2x_2), \quad \text{s.t.} \quad x_1 + x_2 \geq 0, \quad x_1 + 3x_2 \leq 4,$$

and hence such a basis is not considered in order to define the shadow prices matrices Θ and Λ. Now, consider the problem:

$$L_2(x^0): \qquad \min(4x_1 + 2x_2), \quad \text{s.t.} \quad x_1 + x_2 \geq 0, \quad x_1 + 2x_2 \leq 2, \quad x_1 + 3x_2 \leq 4.$$

In this case, there are two bases maintaining the optimality of x^0:

$$B^{2_1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \quad \text{and} \quad B^{2_3} = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}.$$

The Lagrangian multipliers corresponding to B^{2_1} are the solution to the system $\begin{cases} \lambda_1^2 - \theta_1^2 = 4 \\ \lambda_1^2 - 2\theta_1^2 = 2 \end{cases}$.

Therefore, it results $\tilde{\lambda}_1^2 = 6$, $\tilde{\theta}_1^2 = 2$; this implies $\theta_{21} = -2$. Observe that $\theta_{12} = 1/\theta_{21}$.

The multipliers corresponding to B^{2_3} are the solution to the system $\begin{cases} \lambda_1^2 - \theta_3^2 = 4 \\ \lambda_1^2 - 3\theta_3^2 = 2 \end{cases}$. Therefore, it results $\tilde{\lambda}_1^2 = 5$, $\tilde{\theta}_3^2 = 1$; this implies $\theta_{23} = -1$. Lastly, let us consider the problem:

$$L_3(x^0): \qquad \min(x_1 + 3x_2), \quad \text{s.t.} \quad x_1 + x_2 \geq 0, \quad x_1 + 2x_2 \leq 2, \quad 4x_1 + 2x_2 \leq -4.$$

Only the basis $B^3 = \begin{pmatrix} 1 & 1 \\ -4 & -2 \end{pmatrix}$ maintains the optimality of x^0; for such a matrix the corresponding Lagrangian multipliers are the solution to the system $\begin{cases} \lambda_1^3 - 4\theta_2^3 = 1 \\ \lambda_1^3 - 2\theta_2^3 = 3 \end{cases}$.

Therefore, it results $\tilde{\lambda}_1^3 = 5$, $\tilde{\theta}_2^3 = 1$; this implies $\theta_{32} = -1$. Observe that $\theta_{23} = 1/\theta_{32}$. The above procedure allows us to define the following elements of the matrix Θ:

$$\Theta = \begin{pmatrix} * & -\frac{1}{2} & * \\ -2 & * & -1 \\ * & -1 & * \end{pmatrix},$$

where * denotes the missing elements. The complete matrix Θ is obtained by putting the diagonal elements equal to 1 and by applying the properties of reciprocity and consistency. Therefore, the shadow prices matrix associated to the objective functions for the given problem is the following:

$$\Theta = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}.$$

The matrix $\Lambda \in \mathbb{R}^{3\times 1}$ is not univocally determined because of the two bases associated with $L_2(x^0)$; in fact, they give two choices for the second row of Λ. Hence, there are two shadow prices matrices associated with the binding constraint of the given problem:

$$\Lambda^1 = \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix} \quad \text{and} \quad \Lambda^2 = \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}.$$

3.2 Shadow prices by means of the Scalarization by Separation

Also in this second approach, we start by the assumption that x^0 is a v.m.p. of (3.1) and hence (see Proposition 2.9) also of the following problem

$$\min\langle p^T F, x\rangle, \text{ s.t. } x \in \{x \in \mathbb{R}^n : Ax \geq b, Fx \leq Fx^0\}.$$

Since we are considering the linear case, x^0 is an optimal solution to:

$$\min\langle p^T F, x\rangle, \quad \text{s.t.} \quad x \in \{x \in \mathbb{R}^n : Bx \geq d, Fx \leq Fx^0\}, \tag{3.8}$$

where $B \in \mathbb{R}^{n_1\times n}$ and $d \in \mathbb{R}^{n_1}$, with $n_1 \leq m$, are respectively the submatrix of A and the subvector of b corresponding to a subset of the binding constraints at x^0; let us consider a basis corresponding to the optimal solution x^0:

$$\begin{pmatrix} B \\ -F_i^T, \quad i \in I_{n_2} \end{pmatrix},$$

where I_{n_2} is a subset of I of cardinality n_2, with $n_1 + n_2 = n$. Among all the bases fulfilling the above properties, let us consider only those for which x^0 is again an optimal solution to the problem:

$$\min\langle p^T F, x\rangle \text{ s.t. } x \in \{x \in \mathbb{R}^n : Bx \geq d, \langle -F_i, x\rangle \geq \left\langle -F_i, x^0\right\rangle, i \in I_{n_2}\}.$$

Let us consider the problem:

$$\min\langle p^T F, x\rangle \text{ s.t. } x \in R(\eta,\xi) := \{Bx \geq d + \eta, \langle -F_i, x\rangle \geq \left\langle -F_i, x^0\right\rangle + \xi, i \in I_{n_2}\}, \tag{3.9}$$

where $\eta \in \mathbb{R}^{n_1}$ and $\xi \in \mathbb{R}^{n_2}$. Consider the Kuhn-Tucker multipliers $\tilde{\lambda}^S \in \mathbb{R}^{n_1}_+$, $\tilde{\vartheta}_i^S \in \mathbb{R}_+, i \in I_{n_2}$, of the problem (3.9); they are the solution to the system $B^T\lambda^S - \sum_{i\in I_{n_2}} F_i\vartheta_i^S = p^T F$; then, as previously noted for problem (3.3), it turns out that:

$$\tilde{\lambda}_i^S = \frac{\partial\Big(\min_{x\in R(\eta,\xi)} \langle p^T F, x\rangle\Big)}{\partial \eta_i}, \ i = 1, \dots, n_1; \ \ \tilde{\vartheta}_i^S = \frac{\partial\Big(\min_{x\in R(\eta,\xi)} \langle p^T F, x\rangle\Big)}{\partial \xi_i}, \ i \in I_{n_2}.$$

The following proposition provides an alternative procedure to obtain the same shadow prices matrix Θ obtained by the ε-constraint method; recall that I_{n_2} and $\tilde{\vartheta}_j^k$ are those of Subsect.3.1.

Proposition 3.3. Given problem (3.9), suppose that there are two bases maintaining the optimality of x^0 for problem (3.9):

$$B^j = \begin{pmatrix} B \\ -F_i^T, \ \ i \in I_{n_2^j} \end{pmatrix} \quad B^k = \begin{pmatrix} B \\ -F_i^T, \ \ i \in I_{n_2^k} \end{pmatrix},$$

where $I_{n_2^j} \subset I\backslash\{j\}$ and $I_{n_2^k} \subset I\backslash\{k\}$ are of cardinality n_2 and such that $I_{n_2^j} \cup \{j\} = I_{n_2^k} \cup \{k\}$. Then we have $\tilde{\vartheta}_j^k = \tilde{\vartheta}_j^S / \tilde{\vartheta}_k^S$ with $k, j \in I_{n_2}$.

Proof. Starting from the above positions we get

$$\tilde{\vartheta}_k^S = \frac{det\big(B^T \,\big|\, -F_i, \ i \in I_{n_2^j}\backslash\{k\} \,\big|\, p^T F\big)}{det\big(B^T \,\big|\, -F_i, \ i \in I_{n_2^j}\big)} \ \text{ and } \ \tilde{\vartheta}_j^S = \frac{det\big(B^T \,\big|\, -F_i, \ i \in I_{n_2^k}\backslash\{j\} \,\big|\, p^T F\big)}{det\big(B^T \,\big|\, -F_i, \ i \in I_{n_2^k}\big)},$$

where $p^T F$ substitutes the column vector $-F_k$ in the first equality and the column vector $-F_j$ in the second equality. By observing that the assumption $I_{n_2^j} \cup \{j\} = I_{n_2^k} \cup \{k\}$ is equivalent to $I_{n_2^j}\backslash\{k\} = I_{n_2^k}\backslash\{j\}$, we obtain

$$\tilde{\vartheta}_j^S / \tilde{\vartheta}_k^S = \frac{det\big(B^T \,\big|\, -F_i, \ i \in I_{n_2^j}\big)}{det\big(B^T \,\big|\, -F_i, \ i \in I_{n_2^k}\big)} = \tilde{\vartheta}_j^k$$

where we recall that that $\tilde{\vartheta}_j^k$ is the jth component of the solution to system (3.4). $\square$

We note that changing the set of indexes of cardinality n_2 we get all the elements ϑ_i^S with $i = 1, \dots, \ell$.

Moreover, we want to stress that the quantity $\tilde{\vartheta}_j^S / \tilde{\vartheta}_k^S$ is the quotient between the derivative of the scalarized function with respect to the jth objective and the derivative of the scalarized function with respect to the kth objective, it results that it can be thought as the derivative of the kth objective function with respect to the change of the jth objective, i.e., $\tilde{\vartheta}_j^k$.

Applying the same procedure to the multipliers corresponding to the original binding constraints, the quantity $\tilde{\lambda}_t^S/\tilde{\theta}_k^S$, with $k \in I_{n_2},\ t = 1, \dots, n_1$, is the quotient between the derivative of the scalarized function with respect to the tth constraint and the derivative of the scalarized function with respect to the kth objective; unfortunately, it depends on the parameter p, in fact, we have

$$\frac{\tilde{\lambda}_t^S}{\tilde{\theta}_k^S} = \frac{det\big(B^{-tT}\ \big|p^T F\ \big| - F_i,\ i \in I_{n_2^j}\big)}{det\big(B^T\ \big| - F_i,\ i \in I_{n_2^j}\backslash\{k\}\ \big|p^T F\big)},$$

where $p^T F$ substitutes the tth column of B^T, and B^{-tT} denotes the matrix B^T without its tth column.

Now we want to emphasize some interesting similarities between solving a vector linear problem by the ε-constraint method and solving the same problem by the scalarization method proposed in [9]. In the former case, we have ℓ scalar primal problems, each of them with n variables, $m + (\ell - 1)$ constraints and hence ℓ dual problems with $m + (\ell - 1)$ variables and n constraints; in the latter case, we have only one scalar primal problem with n variables, $(m + \ell)$ constraints and a dual problem with $(m + \ell)$ variables and n constraints. Therefore, in the former case, we have $\Theta \in \mathbb{R}^{\ell\times\ell}$ and $\Lambda \in \mathbb{R}^{\ell\times m}$ that collects all the information given by the ℓ problems and the ith row of the matrices refers to the ith problem, $i \in I$; while, in the latter case, we have the vectors $\theta \in \mathbb{R}^{\ell}$ and $\lambda \in \mathbb{R}^m$ which evidently resume the previous results. The continuation of Example 2.1 elucidates the result of Proposition 3.3 and gives some other hints about the relationships between these two methods.

Example 2.1 (continuation). Our aim is to highlight different aspects of the same problem. We have a scalarized primal and dual problem and three primal and dual problems by the ε-constraint method. We set the parameters $a = \theta_{12},\ b = \theta_{13}$. From Propositions 3.1 and 3.2, we know that Θ is reciprocal and, if positive, also consistent, then we get:

$$\Theta = \begin{pmatrix} 1 & a & b \\ 1/a & 1 & b/a \\ 1/b & a/b & 1 \end{pmatrix}.$$

This kind of matrices $\Theta \in \mathbb{R}^{\ell\times\ell}$ is defined by $\ell - 1$ parameters, has a maximum eigenvalue $\gamma = \ell$ with the other eigenvalues equal to zero, and eigenvector $\bar{\theta}^T = (\frac{1}{k}, \frac{1}{ka}, \frac{1}{kb}),\ \forall k \in \mathbb{R}\backslash\{0\}$, (see e.g.

[10]). We can express the elements of Λ depending on a, b in this way:

$$\Lambda = \begin{pmatrix} 1+4a+b \\ (1+4a+b)/a \\ (1+4a+b)/b \end{pmatrix} \quad \text{or} \quad \Lambda = \begin{pmatrix} 2+2a+3b \\ (2+2a+3b)/a \\ (2+2a+3b)/b \end{pmatrix}$$

Since the dual problems of the ε-constraint method have a feasible region described by 2 equations and 3 unknowns, thus the parameter to set is only one and we may impose $a = 1/2 + b$. One can easily see that the matrix Λ is obtained as follows $\Lambda = \bar{\theta}\lambda^{*T}$ where $\lambda_i^* = c_i^T\theta, \;\; i = 1, 2,$ and $c_1 = (1, 4, 1)$, $c_2 = (2, 2, 3)$, while $\Theta = \bar{\theta}\theta^T$.
We observe that these matrices satisfy $\forall a, b \geq 0$ the sufficient optimality condition:

$$\Theta(f(x^0) - f(x)) + \Lambda g(x) \not\geq_{C_0} O, \; \forall x \in X.$$

The scalarized primal problem is:

$$\min((p_1 + 4p_2 + p_3)x_1 + (2p_1 + 2p_2 + 3p_3)x_2)$$
$$\text{s.t. } x_1 + x_2 \geq 0, \; -x_1 - 2x_2 \geq -2, \; -4x_1 - 2x_2 \geq 4, \; -x_1 - 3x_2 \geq -4$$

and its dual is:

$$\max(-2\theta_1 + 4\theta_2 - 4\theta_3)$$
$$\text{s.t. } \lambda_1 - \theta_1 - 4\theta_2 - \theta_3 = (p_1 + 4p_2 + p_3), \; \lambda_1 - 2\theta_1 - 2\theta_2 - 3\theta_3 = (2p_1 + 2p_2 + 3p_3).$$

The dual feasible region is described by 2 equations and 4 unknowns, so we have 2 parameters to choose. Call λ_1 the unknown related to the original constraint and let $\theta_1, \theta_2, \theta_3$ be the unknowns corresponding to the constraints of the feasible region $S(x^0)$. Finally, let $\theta^T = (k, ka, kb)$ be the vector of parameters, that is $(\theta_1, \theta_2, \theta_3)$. From these positions, we can express a feasible λ_1 as $\lambda_1 = (1, 4, 1)(\theta + p)$ or $\lambda_1 = (2, 2, 3)(\theta + p)$.

To understand the relationships between the scalarized problem and the ε-constraint problems we may proceed by applying Proposition 3.3. We get $\theta_1^S = p_1 - 2p_2 + p_3, \quad \theta_2^S = -\frac{1}{2}p_1 + p_2 - p_3, \quad \theta_3^S = \frac{1}{2}p_1 - p_2 + p_3$. By the quotients of these elements we obtain the matrix Θ by setting $\theta_j^k = \theta_j^S/\theta_k^S$ and hence we may see that the matrix Θ does not depend on the choice of the parameter p. Alternatively, one can find the derivatives of the scalarized objective function with respect to every constraint. In particular, from the derivatives of the scalarized function with respect to every objective now converted in constraint, we get the vector $\theta \in \mathbb{R}^3$ from which we obtain also the matrix $\Theta \in \mathbb{R}^{3\times 3}$ previously found. We suppose that when we move from the solution $x^0 = (-2, 2)$ we stay on the direction given by the constraint.

4 Numerical Examples

In this section we propose some examples to explain different cases that point out some particular situations. The first one treats a feasible region with 2 binding constraints at x^0.

Example 4.1. Let us consider problem (3.1) and the positions: $\ell = 3,\ m = 2,\ n = 2$ $F_1 = (1, -2),\ F_2 = (-1, 1),\ F_3 = (-1, -1)$,

$$A = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

We now apply the ε-constraint method starting from the optimal point $x^0 = (\frac{2}{3}, \frac{2}{3})$; we get 3 sub-problems:

$$L_i(x^0) : \min \langle F_i, x\rangle\,,\ \text{s.t. } Ax \geq b,\ \langle -F_j, x\rangle \geq \left\langle -F_j, x^0\right\rangle\,,\ j \neq i;\ i = 1, 2, 3.$$

For every $L_i(x^0), i = 1, 2, 3$, we consider the Kuhn-Tucker multipliers corresponding to the basis composed by one (of two) original constraints and one (of two) objective functions now transformed in constraints; hence, we have 4 bases for every single problem. By applying the procedure of Subsect.3.1, we do not have uniqueness in determining the shadow prices matrix Θ; in fact, we obtain the following two matrices:

$$\Theta^1 = \begin{pmatrix} 1 & -4/3 & -4 \\ -3/4 & 1 & 3 \\ -1/4 & 1/3 & 1 \end{pmatrix} \quad \Theta^2 = \begin{pmatrix} 1 & -5/3 & 5 \\ -3/5 & 1 & -3 \\ 1/5 & -1/3 & 1 \end{pmatrix}$$

Alternatively, we can consider the dual problems of $L_i(x^0), i = 1, 2, 3$. Every dual problems has 2 equations and 4 unknowns; if we set $a = \theta_{12}, b = \theta_{13}$, we have the matrix

$$\Theta = \begin{pmatrix} 1 & a & b \\ 1/a & 1 & b/a \\ 1/b & a/b & 1 \end{pmatrix}$$

depending on the 2 parameters a and b. Consequently, we obtain:

$$\Lambda = \begin{pmatrix} (-3a+b+5)/3 & (3a+b-4)/3 \\ (-3a+b+5)/3a & (3a+b-4)/3a \\ (-3a+b+5)/3b & (3a+b-4)/3b \end{pmatrix}.$$

As in Example 2.1, we introduce the scalarized problem:

$$P^S : \begin{cases} \min((p_1 - p_2 - p_3)x_1 + (-2p_1 + p_2 - p_3)x_2) \\ -x_1 - 2x_2 \geq -2, \ -2x_1 - x_2 \geq -2, \\ -x_1 + 2x_2 \geq 2/3, \ x_1 - x_2 \geq 0, \ x_1 + x_2 \geq 4/3 \end{cases}$$

and its dual

$$D^S : \begin{cases} \max(-2\lambda_1 - 2\lambda_2 + \frac{2}{3}\theta_1 + \frac{4}{3}\theta_3) \\ -\lambda_1 - 2\lambda_2 - \theta_1 + \theta_2 + \theta_3 = (p_1 - p_2 - p_3), \\ -2\lambda_1 - \lambda_2 + 2\theta_1 - \theta_2 + \theta_3 = (-2p_1 + p_2 - p_3) \end{cases}$$

The feasible region of the dual is described by 2 equations and 5 unknowns, so we have 3 parameters to characterize the shadow prices matrices. From this problem we get the feasible pair:

$$\lambda_1 = \tfrac{1}{3}(-4,3,1)(\theta + p) = c_1(\theta + p) \text{ and } \lambda_2 = \tfrac{1}{3}(5,-3,1)(\theta + p) = c_2(\theta + p).$$

From these results we can conclude that $\Lambda = \bar{\theta}\lambda^{*T}$ where $\lambda_i^* = c_i^T\theta, \;\; i = 1,2$ and $\Theta = \bar{\theta}\theta^T$. When we move from the solution $x^0 = (\frac{2}{3}, \frac{2}{3})$, we can choose to remain in the direction given either by the first constraint of K or the second one. Thus, we obtain two vectors $\theta^{Si}, \ i = 1,2$. In the former case, we have $\theta_1^{S1} = -p_1 + \frac{3}{4}p_2 + \frac{1}{4}p_3, \quad \theta_2^{S1} = \frac{4}{3}p_1 - p_2 + \frac{1}{3}p_3, \quad \theta_3^{S1} = 4p_1 - 3p_2 - p_3$, while in the latter we have $\theta_1^{S2} = -p_1 + \frac{3}{5}p_2 - \frac{1}{5}p_3, \quad \theta_2^{S2} = \frac{5}{3}p_1 - p_2 + \frac{1}{3}p_3, \quad \theta_3^{S2} = -5p_1 + 3p_2 - p_3$. Then, by the quotient of these elements we obtain the matrices Θ^i by setting $\theta_{kj}^i = \theta_j^{Si}/\theta_k^{Si}$ and again we may observe that these elements do not depend on vector p.

The aim of the next example is to illustrate the results stated in Proposition 3.3.

Example 4.2. Let us consider (3.1) with the following positions: $\ell = 3, \ m = 3, \ n = 3,$ $F_1 = (1,2,-1), \ F_2 = (4,2,1), \ F_3 = (1,3,1),$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As starting optimal point, choose $x^0 = (-2, 2, 2)$. Since only the first constraint is binding at x^0, then the scalarized problem we consider is:

$$P^S : \begin{cases} \min((p_1 + 4p_2 + p_3)x_1 + (2p_1 + 2p_2 + 3p_3)x_2 + (-p_1 + p_2 + p_3)x_3) \\ x_1 + x_2 \geq 0,\ -x_1 - 2x_2 + x_3 \geq 0, \\ -4x_1 - 2x_2 - x_3 \geq 2,\ -x_1 - 3x_2 - x_3 \geq -6 \end{cases}$$

We get 3 bases to choose:

$$B^1 = \begin{pmatrix} x_1 + x_2 \\ -4x_1 - 2x_2 - x_3 \\ -x_1 - 3x_2 - x_3 \end{pmatrix}, B^2 = \begin{pmatrix} x_1 + x_2 \\ -x_1 - 2x_2 + x_3 \\ -x_1 - 3x_2 - x_3 \end{pmatrix}, B^3 = \begin{pmatrix} x_1 + x_2 \\ -x_1 - 2x_2 + x_3 \\ -4x_1 - 2x_2 - x_3 \end{pmatrix}.$$

The corresponding vectors θ have the following components: $\theta_2^{S1} = \frac{4p_2 - 3p_1}{-4}$, $\theta_3^{S1} = \frac{-p_1 + 4p_3}{-4}$, $\theta_1^{S2} = \frac{4p_2 - 3p_1}{3}$, $\theta_3^{S2} = \frac{-3p_3 + p_2}{3}$, $\theta_1^{S3} = \frac{-4p_3 + p_1}{-1}$, $\theta_2^{S3} = \frac{-3p_3 + p_2}{-1}$. From the relation $\theta_j^i = \theta_j^{Si}/\theta_i^{Sj}$, we get:

$$\theta_2^1 = -\frac{3}{4},\ \theta_3^1 = -\frac{1}{4},\ \theta_1^2 = -\frac{4}{3},\ \theta_3^2 = -\frac{1}{3},\ \theta_1^3 = -4,\ \theta_2^3 = -3.$$

While, from ε-constraint method, verifying the optimality condition $F_i^T(B^i)^{-1} \geq 0$, and following the procedure of Subsect.3.1, we have:

$$\Theta = \begin{pmatrix} 1 & -3/4 & -1/4 \\ -4/3 & 1 & -1/3 \\ -4 & -3 & 1 \end{pmatrix}.$$

We observe that Θ is still reciprocal, but it is no longer consistent. In the end we note that, from the dual of P^S we obtain $\lambda = (1, 4, 1)(\theta + p)$ or $\lambda = (2, 2, 3)(\theta + p)$ and if we consider $\theta^T = (k, ka, kb)$ and

$$\Theta = \begin{pmatrix} 1 & a & b \\ 1/a & 1 & b/a \\ 1/b & a/b & 1 \end{pmatrix},$$

then

$$\Lambda = \bar{\theta}\lambda^{*T} = \begin{pmatrix} 1 + 4a + b \\ (1 + 4a + b)/a \\ (1 + 4a + b)/b \end{pmatrix} = \begin{pmatrix} 2 + 2a + 3b \\ (2 + 2a + 3b)/a \\ (2 + 2a + 3b)/b \end{pmatrix}.$$

If we substitute $a = 3/4$ and $b = 1/4$ we get the matrix

$$\Lambda = \begin{pmatrix} 17/4 \\ 17/3 \\ 17 \end{pmatrix}$$

which is the same that we obtain following the other approaches. In fact, in this case, every subproblem $L_i(x^0)$ has an optimal basis, that is the optimality condition $F_i^T(B^i)^{-1} \geq 0$ is verified for all basis $B^i, i = 1, \ldots, 3$.

In the next example, we want to show what may happen when we start the analysis from an optimal point that binds both constraints.

Example 4.3. Let us consider (3.1) with the following positions: $\ell = 3$, $m = 2$, $n = 3$, $F_1 = (-1, 0, 0)$, $F_2 = (0, -1, 0)$, $F_3 = (0, 0, -1)$,

$$A = \begin{pmatrix} -2 & -3 & -4 \\ -4 & -1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -12 \\ -8 \end{pmatrix}.$$

We want to outline three different cases of optimal points to start with:

i) the first constraint is binding at $x^1 = (0, 0, 3)$;

ii) the second constraint is binding at $x^2 = (2, 0, 0)$;

iii) both constraints are binding at $x^3 = (6/5, 16/5, 0)$.

Case i). The scalarized problem we obtain is:

$$P^S : \begin{cases} \min(-p_1x_1 - p_2x_2 - p_3x_3) \\ -2x_1 - 3x_2 - 4x_3 \geq -12, \; x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 3 \end{cases}$$

We get 3 bases:

$$B^1 = \begin{pmatrix} -2 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}, \quad B^2 = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 0 \\ -4 & 0 & 1 \end{pmatrix}, \quad B^3 = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 0 \end{pmatrix}.$$

The corresponding vectors θ have the following components: $\theta_2^{S1} = \frac{2p_2-3p_1}{2}$, $\theta_3^{S1} = \frac{-4p_1+2p_3}{2}$, $\theta_1^{S2} = \frac{2p_2-3p_1}{-3}$, $\theta_3^{S2} = \frac{3p_3-4p_2}{3}$, $\theta_1^{S3} = \frac{2p_3-4p_1}{-4}$, $\theta_2^{S3} = \frac{3p_3-4p_2}{-4}$. From the relation $\theta_j^i = \theta_j^{Si}/\theta_i^{Sj}$, we get:

$$\theta_2^1 = -\frac{3}{2}, \quad \theta_3^1 = -2, \quad \theta_1^2 = -\frac{2}{3}, \quad \theta_3^2 = -\frac{4}{3}, \quad \theta_1^3 = -\frac{1}{2}, \quad \theta_2^3 = -\frac{3}{4}.$$

While, from ε-constraint method, verifying the optimality condition $F_i^T(B^i)^{-1} \geq 0$, we have:

$$\Theta^1 = \begin{pmatrix} 1 & -3/2 & -2 \\ -2/3 & 1 & -4/3 \\ -1/2 & -3/4 & 1 \end{pmatrix}, \quad \Lambda^1 = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/4 \end{pmatrix}.$$

We observe that from the direct calculus of the derivatives one gets the same result.

Case ii). The scalarized problem we obtain is:

$$P^S : \begin{cases} \min(-p_1x_1 - p_2x_2 - p_3x_3) \\ -4x_1 - x_2 - x_3 \geq -8,\ x_1 \geq 2,\ x_2 \geq 0,\ x_3 \geq 0 \end{cases}$$

We get 3 bases:

$$B^1 = \begin{pmatrix} -4 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},\ B^2 = \begin{pmatrix} -4 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},\ B^3 = \begin{pmatrix} -4 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

We may find the corresponding vectors θ from the ratios $\theta_j^{Si}/\theta_i^{Sj} = \theta_j^i$ or from ε-constraint method, verifying the optimality condition $F_i^T(B^i)^{-1} \geq 0$, or from the calculus of the derivatives; in any case, we get the matrices:

$$\Theta^2 = \begin{pmatrix} 1 & -1/4 & -1/4 \\ -4 & 1 & -1 \\ -4 & -1 & 1 \end{pmatrix},\ \Lambda^2 = \begin{pmatrix} 1/4 \\ 1 \\ 1 \end{pmatrix}.$$

Case iii). The scalarized problem is:

$$P^S : \begin{cases} \min(-p_1x_1 - p_2x_2 - p_3x_3) \\ -4x_1 - x_2 - x_3 \geq -8,\ -2x_1 - 3x_2 - 4x_3 \geq -12, \\ x_1 \geq 6/5,\ x_2 \geq 16/5,\ x_3 \geq 0 \end{cases}$$

Both constraints are binding at the point $x^3 = (6/5, 16/5, 0)$, then, if we want to consider all the constraints of K we can add only one constraint of $S(x^3)$ to the basis. All the bases we obtain are:

$$B^{1_2} = B^{3_2} = \begin{pmatrix} -2 & -4 & 0 \\ -3 & -1 & 1 \\ -4 & -1 & 0 \end{pmatrix},\ B^{1_3} = B^{2_3} = \begin{pmatrix} -2 & -4 & 0 \\ -3 & -1 & 0 \\ -4 & -1 & 1 \end{pmatrix},\ B^{2_1} = B^{3_1} = \begin{pmatrix} -2 & -4 & 1 \\ -3 & -1 & 0 \\ -4 & -1 & 0 \end{pmatrix};$$

$$B_1^1 = \begin{pmatrix} -2 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix},\ B_1^2 = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 0 \\ -4 & 0 & 1 \end{pmatrix},\ B_1^3 = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 0 \end{pmatrix};$$

$$B_2^1 = \begin{pmatrix} -4 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad B_2^2 = \begin{pmatrix} -4 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad B_2^3 = \begin{pmatrix} -4 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

From the first group of bases, we cannot construct the matrices Θ and Λ; the other groups are the same of case i) and ii), hence, even for this point, the pairs (Θ^1, Λ^1) and (Θ^2, Λ^2) are the shadow prices matrices. From the calculus of the derivatives, or from Proposition 3.3, we get the matrix:

$$\Theta = \begin{pmatrix} 1 & -1/14 & 1/10 \\ -14 & 1 & -7/5 \\ 10 & -5/7 & 1 \end{pmatrix}$$

that collects the information about the variation of one objective function with respect to another, but does not exist a corresponding matrix Λ with all positive elements; in fact, the two possible matrices Λ corresponding to Θ are:

$$\begin{pmatrix} -1/14 & 2/7 \\ 1 & 4 \\ 2/7 & -1/7 \end{pmatrix} \text{ or } \begin{pmatrix} -1/10 & 3/10 \\ 2/5 & -1/5 \\ 1 & 3 \end{pmatrix}.$$

5 Concluding Remarks

In this section we want to underscore the main topics of the preceding sections with some observations and remarks about some results, and open questions.

In the linear case, when only one constraint of K is binding at the starting optimal solution, we are obliged to move on it when we perturb the original objective functions now converted in constraints. If the binding constraints are more then one, we have the possibility to choose which one enters in the basis to consider; this is the case of Example 4.1, where we get therefore two matrices Θ, while in Example 4.3 we cannot choose the basis with both original constraints, because it is not optimal.

In connection with this fact, another observation is about the sign of Θ elements. If we are, for instance, in $\mathbb{R}^2$ and we have only one binding constraint, we are bound to move on a line, then if we take other lines moving on it, the possible directions are only two; we want to say that if f_1 moves in the opposite direction of f_2 and of f_3, then necessarily f_2 and f_3 moves in the same

direction; consequently, if we have $\theta_{12} < 0,\ \theta_{13} < 0$, then $\theta_{23} > 0$. Obviously, if we are in $\mathbb{R}^3$ a binding constraint individuates a plane, and the fact that f_1 moves in an opposite direction of f_2 and of f_3, does not imply that f_2 and f_3 move in the same direction, thus the elements of Θ could be: $\theta_{12} < 0,\ \theta_{13} < 0$ and $\theta_{23} < 0$; this fact renders Θ reciprocal, but not consistent.
Another thorny problem is about the relationship between the matrices of Kuhn-Tucker multipliers and the separating hyperplane. In the scalar case, there exists a correspondence between the vector of multipliers and the parameters of the separating function. In this context, instead, the matrices Θ and Λ that we get from the examples do not satisfy the sufficient condition (1.5), moreover they do not define a separation function as in (1.4). Only if we take the elements of the matrix Θ in their absolute value, then we can try to verify condition (1.5), since now the pair (Θ, Λ) is such that $\Theta \in U^*_{C\setminus\{O\}},\ \Lambda \in V^*_C$.
From these considerations the formalization of a necessary optimality condition is required, especially given by means of vector separation; this could be matter of future studies.

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