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Abstract

This paper introduces endogenous capital income tax rates as in Schmitt-Grohe and Uribe (1997), into the overlapping generations model with endogenous labor and consumption in both periods of life (e.g., Cazzavillan and Pintus, 2004). In contrast with the previous result that the existence of endogenous labor income taxes raises the possibility of local indeterminacy (Chen and Zhang 2009), it shows that increasing the size of capital income taxes can make shrink the range of values of the consumption–to–wage ratio associated with local indeterminacy, because of two conflicting effects on savings that operate through wage and interest rate.

Keywords: Indeterminacy; Endogenous capital income tax rate.

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1. Introduction

In a recent article, Chen and Zhang (2009a) investigates how government expenditure financed by labor income taxes influences local dynamics near a normalized steady state in an overlapping generations model with endogenous labor and consumption in both periods of life. They find that local indeterminacy can easily arise with small distortionary labor income taxes, provided that the elasticity of capital-labor substitution is less than the share of capital in total income and the wage elasticity of the labor supply is large enough. Moreover, they show that increasing the size of labor income taxes enlarges the range of values of the consumption-to-wage ratio associated with local indeterminacy, because of two conflicting effects on savings that operate through wage rate and interest rate. Therefore, (endogenous) labor income taxes are favorable to local indeterminacy.

In this note, we extend their model and investigate how government expenditure financed by capital income taxes influences local dynamics near the normalized steady state in the very same OLG model. First, we show that for a reasonable share of total consumption over the output, local indeterminacy can easily occur when there are small capital income taxes. Second, we find that increasing the size of capital income taxes can make shrink the range of values of the consumption-to-wage ratio associated with local indeterminacy, because of two conflicting effects on savings that operate through wage rate and interest rate. Lastly, we show that for a given technology (θ), adding capital income taxes can make decrease the critical value of the input substitution ($\overline{\sigma}$) associated with multiple equilibria. Therefore, endogenous capital income taxes are *not* favorable to local indeterminacy.

As in Cazzavillan and Pintus (2006), the existence of two conflicting effects that operate through wage and interest rate is essential for generating endogenous fluctuations. Chen and Zhang (2009a) conclude that increasing labor income tax rates makes smaller the share of consumption out of wage income in the first period of life, thus making sunspots more likely to occur.[1] In contrast with their findings, the presence of capital income taxes will have different effects on these conflicting effects. To be more precise, there is one force which tends to dampen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make larger the lower bound of the ratio (between savings and wage income) for indeterminacy, thus making sunspots less likely to occur. In addition, there is another force which tends to strengthen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make the after-tax interest rate more and more negatively sensitive to variations in the capital stock, thus making sunspots more likely to occur. When the former effect is stronger than the latter effect, increasing capital income tax rates will make local indeterminacy hard to arise.

The paper is organized as follows. Section 2 sets up the model. Section 3 establishes the existence of a normalized steady state. In section 4, we use the geometrical method to analyze the local dynamics near the normalized steady state and then deliver the main results on local indeterminacy. Section 5 concludes.

2. The model

This note introduces constant government expenditure financed by capital income taxes in the OLG model studied in Cazzavillan and Pintus (2004). We consider a competitive, non-monetary model with production and consumption in both periods. It involves a unique perishable good, which can be either consumed or saved as investment. Identical competitive firms all face the same technology. Identical households live for two periods. The agent consumes in both periods, supplies labor and saves when young. When old, her saved income is rented as physical capital to the firm.

Assuming additively separable preferences, the household born at time $t \geq 0$ maximizes her

[1] In fact, increasing labor income tax rates can make smaller the lower bound of the ratio (between savings and wage income) for indeterminacy, thus making sunspots more likely to occur.

lifetime utility

$$\max_{c_{1t},\ \lambda_t,\ c_{2t+1}} \left[U_1(c_{1t}/B) - U_3\left(\lambda_t\right) + \beta U_2\left(c_{2t+1}\right)\right]$$

subject to the constraints

$$c_{1t} + z_t = \Omega_t \lambda_t, \tag{1}$$

$$c_{2t+1} = \widetilde{R}_{t+1} z_t, \tag{2}$$

$$c_{1t} \geq 0,\ c_{2t+1} \geq 0,\ \overline{\lambda} \geq \lambda_t \geq 0, \text{ for all } t \geq 0,$$

where λ_t, c_{1t} and z_t are labor, consumption and saving, respectively, of the individual of the young generation, c_{2t+1} is the consumption of the same individual when old, and $\Omega_t > 0$ and $\widetilde{R}_{t+1} > 0$ are the real wage at time t and the after–tax gross interest rate at time $t+1$. Moreover, $\beta \in (0,1)$, $B > 0$ and $\overline{\lambda}$ are the discount factor, a scaling parameter and the maximum amount of labor supply, respectively.

The preferences satisfy the following condition as in Cazzavillan and Pintus (2004).

Assumption 1. The functions $U_1(c/B)$, $U_3\left(\lambda\right)$ and $U_2\left(c\right)$ are defined and continuous on the set R_+. Moreover, they are C^r, for r large enough, on the set R_{++}, with $U_1'(c/B) > 0$, $U_2'\left(c\right) > 0$, $U_3'\left(\lambda\right) > 0$, $U_1''(c/B) < 0$, $U_2''\left(c\right) < 0$, $U_3''\left(\lambda\right) > 0$. $\lim_{\lambda \to \overline{\lambda}} U_3'\left(\lambda\right) = +\infty$, where $\overline{\lambda} > 1$, and $\lim_{\lambda \to 0} U_3'\left(\lambda\right) = 0$. In addition, $0 < R_1(c/B) \equiv -(c/B)U_1''(c/B)/U_1'(c/B) < 1$, $0 < R_2(c) \equiv -cU_2''\left(c\right)/U_2'\left(c\right) < 1$, and $R_3(\lambda) \equiv \lambda U_3''\left(\lambda\right)/U_3'\left(\lambda\right) > 0$.

The conditions $0 < R_1(c/B) < 1$ and $0 < R_2(c) < 1$ are used to ensure that consumption and leisure are gross substitutes, and the saving function is increasing in R.

When the solution of the maximization problem is interior, the first order conditions are given by

$$U_1'(c_{1t}/B)/B = \beta \widetilde{R}_{t+1} U_2'\left(c_{2t+1}\right) = U_3'\left(\lambda_t\right)/\Omega_t. \tag{3}$$

Using the first order conditions, the current consumption can be written as follows

$$c_{1t} = B\left(U_1'\right)^{-1}\left(\frac{BU_3'\left(\lambda_t\right)}{\Omega_t}\right), \tag{4}$$

and the savings of the young agent born at time t are[2]

$$z_t = \Omega_t\lambda_t - B\left(U_1'\right)^{-1}\left(\frac{BU_3'\left(\lambda_t\right)}{\Omega_t}\right). \tag{5}$$

Multiplying both terms of the last equality in Eq. (3) by z_t yields

$$\beta U_2'\left(c_{2t+1}\right)c_{2t+1} = \frac{z_tU_3'\left(\lambda_t\right)}{\Omega_t}, \text{ or, } \widetilde{R}_{t+1}z_t = u_2^{-1}(\frac{z_tU_3'\left(\lambda_t\right)}{\beta\Omega_t}), \tag{6}$$

where $u_2(c_{2t+1}) = U_2'\left(c_{2t+1}\right)c_{2t+1}$ is an increasing function of c_{2t+1}.

The perishable output (y) is produced using capital (k) and labor (λ),

$$y = AF\left(k,\lambda\right) = A\lambda f\left(a\right),$$

where $a = k/\lambda$ and $A > 0$ is a scaling factor. The competitive factor market implies that the real wage rate and the real gross rate of return on capital stock are

$$\Omega\left(a\right) \equiv A\left[f\left(a\right) - af'(a)\right] = A\omega\left(a\right), R\left(a\right) \equiv Af'\left(a\right) + 1 - \delta = A\rho\left(a\right) + 1 - \delta, \tag{7}$$

where $0 \leq \delta \leq 1$ is the constant depreciation rate of capital.[3]

[2] $U_1'(\frac{c}{B})$ is decreasing and invertible in view of assumption 1.

[3] The reduced production function $y/\lambda = Af(a)$ is a continuous function of the capital-labor ratio $a = k/\lambda \geq 0$ and has continuous derivatives of all required orders for $a > 0$, with $f'(a) > 0$, $f"(a) < 0$. In particular, the marginal productivity of capital $A\rho\left(a\right) = Af'(a)$ is a decreasing function of a, while the marginal productivity of labor $A\omega\left(a\right) = A[f\left(a\right) - af'(a)]$ is increasing with a.

As in Schmitt-Grohe and Uribe (1997), at each point in time, the government finances its **constant** expenditure through capital income taxes, i.e.,

$$g = \tau_{kt} r_t k_t > 0, \tag{8}$$

where r_t and τ_{kt} are the marginal productivity of capital ($r_t = Af'(a_t)$) and the capital income tax rate. It is easy to show that the after–tax gross interest rate at time t is

$$\widetilde{R}_t = (1 - \tau_{kt})\, r_t + 1 - \delta.$$

Using the fact that at the equilibrium $k_{t+1} = z_t$ holds, we can easily derive the dynamic system characterizing equilibrium paths of $(k_t,\, a_t)$.

$$R(a_{t+1})k_{t+1} = u_2^{-1}\left(\frac{k_{t+1}U_3'\left(k_t/a_t\right)}{\beta\Omega(a_t)}\right) + g, \tag{9}$$

$$k_{t+1} = \Omega(a_t)\frac{k_t}{a_t} - B(U_1')^{-1}\left(\frac{BU_3'(k_t/a_t)}{\Omega(a_t)}\right). \tag{10}$$

3. Steady state existence

A steady state is a pair $(\overline{k},\, \overline{a})$ such that.

$$A\rho\left(\overline{a}\right) + 1 - \delta = \frac{1}{\overline{k}}\left[u_2^{-1}\left(\frac{\overline{k}U_3'\left(\overline{k}/\overline{a}\right)}{\beta A\omega\left(\overline{a}\right)}\right) + g\right], \tag{11}$$

$$\overline{k} = A\omega\left(\overline{a}\right)\frac{\overline{k}}{\overline{a}} - B(U_1')^{-1}(\frac{BU_3'(\overline{k}/\overline{a})}{A\omega\left(\overline{a}\right)}). \tag{12}$$

To simplify the algebra, we follow the procedure described in Cazzavillan and Pintus (2004) and use the parameters A and B to normalize the steady state.

Proposition 1. *Under the assumptions on the utility and production functions,* $(\bar{k}, \bar{a}) = (1,1)$ *is a normalized steady state (NSS) of the dynamic system (9) and (10) if and only if* g *is not too large,* $A^{*}\omega(1) > 1$, $\beta u_2[\frac{\rho(1)}{\omega(1)} + 1 - \delta - g] < U_3'(1)$ *and* $\lim_{c\to 0} cU_1'(c) < \frac{A^{*}\omega(1)-1}{A^{*}\omega(1)} U_3'(1)$, *where* A^{*} *is the unique solution of* $A\rho(1) + 1 - \delta - g = u_2^{-1}\left(\frac{U_3'(1)}{\beta A\omega(1)}\right)$.

Proof. See Appendix A.1. ■

Multiplicity of steady states can arise in our model. For brevity, we just analyze the local dynamics around the NSS.[4]

4. Local dynamics analysis

Let us linearize the dynamic system (9) and (10) around the NSS (1,1). We shall define ε_{Ω} and ε_{R} as the elasticities of the functions $\Omega(a)$ and $R(a)$ evaluated at the NSS. Moreover, let $\theta \equiv \Omega(\bar{a})/\bar{a} = \Omega(1) = A^{*}\omega(1) > 1$, $R_1 \equiv R_1(\frac{\bar{c_1}}{B^{*}})$, $R_2 \equiv R_2(\bar{c_2})$, and $R_3 \equiv R_3(1)$. Then, we have the following proposition.

Proposition 2. *The linearized dynamics generated by the two-dimensional system (9) and (10) around the NSS are determined by the determinant* D *and the trace* T *of the Jacobian matrix associated with Eqs. (9) and (10).*

$$dk_{t+1} = \left[\theta + \frac{\theta - 1}{R_1} R_3\right] dk_t + \left[\theta\left(\varepsilon_{\Omega} - 1\right) - \frac{\theta - 1}{R_1}\left(R_3 + \varepsilon_{\Omega}\right)\right] da_t, \qquad (13)$$

[4] Thanks to Yoichi Gokan for pointing this out to us. By selecting appropriately A and B and imposing some limiting conditions, we can normalize one steady state at $(1,1)$.

$$
\begin{aligned}
& da_{t+1} && (14) \\
= \; & -\frac{1}{|\varepsilon_R|}\{\tfrac{1-\tau_k^{nss}\mu}{1-R_2}\left[\theta+\left(\tfrac{\theta-1}{R_1}+1\right)R_3\right]-\left(\theta+\tfrac{\theta-1}{R_1}R_3\right)\}dk_t \\
& \frac{-1}{|\varepsilon_R|}\{\tfrac{1-\tau_k^{nss}\mu}{1-R_2}\left[\theta\left(\varepsilon_\Omega-1\right)-\left(R_3+\varepsilon_\Omega\right)\left(\tfrac{\theta-1}{R_1}+1\right)\right]-\left[\theta\left(\varepsilon_\Omega-1\right)-\tfrac{(\theta-1)(R_3+\varepsilon_\Omega)}{R_1}\right]\}da_t,
\end{aligned}
$$

where $\mu \equiv \frac{A^*\rho(1)}{A^*\rho(1)+1-\delta} \in (0,1]$ *and* $\tau_k^{nss} \in (0,1)$ *is the steady state capital income tax rate around the NSS. Moreover, the expressions of* D *and* T *are given by*

$$
\begin{aligned}
T \; = \; & \frac{1}{|\varepsilon_R|}\left[\theta\left(\varepsilon_\Omega-1\right)-\tfrac{\theta-1}{R_1}\left(R_3+\varepsilon_\Omega\right)\right]+\theta+\tfrac{\theta-1}{R_1}R_3 && (15) \\
& -\tfrac{\left(1-\tau_k^{nss}\mu\right)}{|\varepsilon_R|(1-R_2)}\left[\theta\left(\varepsilon_\Omega-1\right)-\left(R_3+\varepsilon_\Omega\right)\left(\tfrac{\theta-1}{R_1}+1\right)\right],
\end{aligned}
$$

$$
D = \frac{\theta\varepsilon_\Omega\left(1+R_3\right)}{|\varepsilon_R|\left(1-R_2\right)}\left(1-\tau_k^{nss}\mu\right) > 0. \qquad (16)
$$

Using the same geometrical method as in Cazzavillan and Pintus (2004), we shall analyze the variations of T and D in the (T, D) plane when some parameters are made vary continuously. In particular, we are interested in the two roots of the characteristic polynomial $Q(\pi) = \pi^2 - T\pi + D$. There is a local eigenvalue which is equal to $+1$ when $1 - T + D = 0$. It is represented by the line (AC) in Fig. 1. Moreover, one eigenvalue is -1 when $1 + T + D = 0$. That is to say, in this case, (T, D) lies on the line (AB). Finally, the two roots are complex conjugate of modulus 1, whenever (T, D) belongs to the segment [BC] which is defined by $D = 1$, $|T| \leq 2$. Since both roots are zero when both T and D are 0, then, by continuity, they have both a modulus less than one iff (T, D) lies in the interior of the triangle ABC, which is defined by $|T| < |1 + D|$, $|D| < 1$. The steady state is then locally indeterminate given that there is a unique predeterminate variable k. If $|T| > |1 + D|$, the stationary state is a saddle-point. Finally, in the complementary region $|T| < |1 + D|, |D| > 1$, the steady state is a source.[5]

[5] For details, see Cazzavillan and Pintus (2004, pp. 462, 463).

The diagram below can also be used to study local bifurcations. When the point (T, D) crosses the interior of the segment [BC], a *Hopf bifurcation* is expected to occur. If, instead, the point crosses the line (AB), one root goes through -1. In that case, a *flip bifurcation* is expected to occur. Finally, when the point crosses the line (AC), one root goes through $+1$, one expects an exchange of stability between the NSS and another steady state through a *transcritical bifurcation.*

In our model, we focus on two parameters, the elasticity of capital–labor substitution (σ) and the relative curvature of the second-period utility function R_2. To be more precise, we shall fix the technology, i.e. θ, the elasticities ε_Ω and ε_R, as well as R_1 and R_3, and make R_2 vary continuously in the open interval $(0, 1)$. This means that we shall consider the parametrized curve $(T(R_2),\, D(R_2))$ when R_2 lies in the interval $(0, 1)$. From the expressions of D and T given in proposition 2, one sees that $(T(R_2),\, D(R_2))$ describes a half-line Δ which starts from the point $(T_0(\sigma), D_0(\sigma))$ for $R_2 = 0$, where $T_0(\sigma)$ is the trace in (15) and $D_0(\sigma)$ is the determinant in (16) evaluated at $R_2 = 0$. In addition, the slope of Δ is

$$\frac{D'(R_2)}{T'(R_2)} = \frac{\theta\varepsilon_\Omega\,(1 + R_3)}{R_3\left(\frac{\theta-1}{R_1} + 1\right) + \varepsilon_\Omega\,(\theta - 1)\left(\frac{1}{R_1} - 1\right) + \theta} > 0. \tag{17}$$

and does not depend on R_2.

It is easy to express the elasticities ε_Ω and ε_R as functions of the depreciation rate δ, the share of capital in total income $0 < s(a) = a\rho(a)/f(a) < 1$, and the elasticity of capital–labor substitution $\sigma(a) \geq 0$,

$$\varepsilon_\Omega = \frac{s(a)}{\sigma(a)} \text{ and } |\varepsilon_R| = \mu\,(a)\,\frac{1 - s\,(a)}{\sigma\,(a)}, \tag{18}$$

where $\mu(a) \equiv \frac{s(a)\theta(a)}{s(a)\theta(a)+(1-s(a))(1-\delta)} \in (0,1]$. Moreover, the coordinates of the origin of the half-line $\Delta(\sigma)$ as functions of the elasticity parameter σ are:

$$T_0(\sigma) = \frac{\sigma R_3+s}{\mu(1-s)} + \theta + \frac{(\theta-1)R_3}{R_1} - \frac{\sigma\tau_k^{nss}}{1-s}\left[R_3\left(\frac{\theta-1}{R_1}+1\right) + \frac{s(\theta-1)(1-R_1)}{\sigma R_1} + \theta\right], \quad (19)$$

$$D_0(\sigma) = \frac{s\theta(1+R_3)}{\mu(1-s)}\left(1-\tau_k^{nss}\mu\right) \geq 0, \quad (20)$$

where $s = s(\overline{a})$, $\theta = \theta(\overline{a})$, $\mu = \mu(\overline{a}) = \frac{s(\overline{a})\theta(\overline{a})}{s(\overline{a})\theta(\overline{a})+(1-s(\overline{a}))(1-\delta)}$, and $\sigma = \sigma(\overline{a})$. In addition, the slope of the half-line $\Delta(\sigma)$ can be written as follows $\frac{s\theta(1+R_3)R_1}{s(\theta-1)(1-R_1)+\sigma[R_3(\theta-1+R_1)+\theta R_1]}$.

Assumption 2. $R_3 > \frac{\tau_k^{nss}\frac{s}{1-s}\theta}{\left(\frac{s}{1-s}\theta+1-\delta\right)R_1}\left[R_3(\theta-1+R_1)+\theta R_1\right]$. It corresponds to the case of small distortionary capital income tax rates, that is, τ_k^{nss} not large. This condition can be met for a sufficiently high R_3 (if labor supply elasticity is finite), so that $T_0(\sigma)$ is an increasing function of σ. In our case, $T_0(\sigma)$ increases from $T_0(0)$ to $+\infty$ along the half line Δ_1, as σ increases from zero to $+\infty$.

To understand the main results, it is useful to relate the parameters θ and τ_k^{nss} to the consumption–to–wage ratio. It is easy to show that $c_1/(\Omega\lambda) = \frac{\theta-1}{\theta}$. From this equation, one can recover the results by Cazzavillan and Pintus (2004) by setting $\tau_k^{nss} = 0$.

If s and θ are kept fixed and σ is regarded as an independent parameter, we find that as σ increases from zero to $+\infty$, the point $(T_0(\sigma), D_0(\sigma))$ moves along a flat half-line Δ_1. More precisely, $D_0(\sigma)$ doesn't change, but $T_0(\sigma)$ increases from a finite number to $+\infty$ along the flat line (Δ_1), when τ_k^{nss} is small. In addition, $\Delta(\sigma)$ pivots rightward and it has a positive slope when $\sigma = 0$, and it is horizontal when $\sigma = +\infty$, but the origin $(T_0(\sigma), D_0(\sigma))$ moves to the right along the line Δ_1, when σ varies from zero to $+\infty$.

In order to get local indeterminacy, first, $D_0(\sigma)$ should be less than 1, which requires that s and θ be small enough, i.e., a sufficiently low share of capital in total income and a sufficiently low ratio of consumption while young to saving ($\bar{c_1} = \theta - 1$ in the NSS).[6] As Cazzavillan and Pintus (2004) point out, the latter requirement is crucial to local indeterminacy. Second, we should impose some other restrictions as in Cazzavillan and Pintus (2004, the first and second paragraphs on p. 466).[7]

Following Cazzavillan and Pintus (2004), we consider the case (I) where $D_0(\sigma) < 1$, $T_0(0) < 1 + D_0(\sigma)$, $slope_\Delta(\sigma) > slope_\Delta(\bar{\sigma})$ and the latter slope ($slope_\Delta(\bar{\sigma})$) is bigger than 1. Here $\bar{\sigma}$ is the value of σ such that the line Δ_1 intersects the line *(AC)*. It is easy to know that the half-line $\Delta(\sigma)$ intersects the interior of the segment BC for σ in $(0, \sigma_H)$, where σ_H is the value of σ such that $\Delta(\sigma)$ goes through C. Then we know that, for all σ in $(\bar{\sigma}, \sigma_H)$, the half-line $\Delta(\sigma)$ intersects not only the line *(AC)* at $R_2 = R_{2T}$, but also the segment BC at $R_2 = R_{2H}$. When σ moves beyond σ_H, $\Delta(\sigma)$ will not cross the interior of the segment BC, but it can cross the line AC up to $\sigma = \sigma_T$, where σ_T is the value of σ such that the $slope_\Delta(\sigma)$ is one. When $\sigma > \sigma_T$, the $slope_\Delta(\sigma)$ is less than one. We

[6] In Cazzavillan and Pintus (2004), they show that the relative curvature of the disutility of labor (R_3) has to be small in order to make $D_0(\sigma)$ less than 1. In Cazzavillan and Pintus (2006), R_3 has to be sufficiently high in order to make the slope of Δ_1 positive and small enough. In our model, R_3 has to be sufficiently high in order to ensure that $T_0(\sigma)$ is increasing with σ.

[7] In other words, local indeterminacy requires complementary inputs (σ is not large) and $R_1 \approx 1$ (the relative curvature of the first period consumption is close to the logarithmic specification). R_2 is not too close to 1, since local indeterminacy requires the generic point $(T(R_2), D(R_2))$ to lie in the interior of the stability triangle ABC, provided that $\Delta(\sigma)$ intersects the triangle ABC.

provide these parameters here.[8]

$$
\begin{aligned}
R_{2H} &= 1-\frac{s\theta\left(1+R_3\right)\left(1-\tau_k^{nss}\mu\right)}{\mu\left(1-s\right)},\\
\sigma_H &= \frac{2-\frac{s\left(1-\tau_k^{nss}\mu\right)}{\mu(1-s)(1-R_{2H})}\left(\frac{\theta-1}{R_1}+1-\theta\right)-\frac{s}{\mu(1-s)}\left(\theta-\frac{\theta-1}{R_1}\right)-\theta-\frac{\theta-1}{R_1}R_3}{\frac{1}{\mu(1-s)}\left\{\frac{1-\tau_k^{nss}\mu}{1-R_{2H}}\left[R_3\left(\frac{\theta-1}{R_1}+1\right)+\theta\right]-\left(\theta+\frac{\theta-1}{R_1}R_3\right)\right\}},\\
\sigma_T &= \frac{s\left[\theta\left(1+R_3\right)-\left(\theta-1\right)\left(1/R_1-1\right)\right]}{\theta+R_3\left(1+\frac{\theta-1}{R_1}\right)},\\
\overline{\sigma} &= \frac{\chi(\theta)\theta\left(1+R_3\right)\left(1-\tau_k^{nss}\mu\right)+\frac{\tau_k^{nss}s(\theta-1)}{1-s}\left(\frac{1}{R_1}-1\right)-\left(\theta-1\right)\left(1+\frac{R_3}{R_1}\right)-\chi(\theta)}{\frac{\chi(\theta)R_3}{s}-\frac{\tau_k^{nss}}{1-s}\left[R_3\left(\frac{\theta-1}{R_1}+1\right)+\theta\right]},\\
\text{where } \chi(\theta) &= \frac{s\theta+(1-s)(1-\delta)}{(1-s)\theta}.\\
R_{2T} &= 1-\frac{\left(1-\tau_k^{nss}\mu\right)\left\{\sigma\left[R_3\left(\frac{\theta-1}{R_1}+1\right)+\theta\right]+s\left(\frac{\theta-1}{R_1}+1-\theta\right)-s\theta\left(1+R_3\right)\right\}}{\mu\left(1-s\right)+\left(\theta+\frac{\theta-1}{R_1}R_3\right)\left[\sigma-\mu\left(1-s\right)\right]-s\left(\theta-\frac{\theta-1}{R_1}\right)}.
\end{aligned}
$$

It is easy to find that the introduction of capital income taxes can affect the critical values of these parameters. In contrast with the results of Chen and Zhang (2009a), endogenous capital income taxes are less helpful to local indeterminacy than labor income taxes. We provide the following result.

Proposition 3. *The introduction of endogenous capital income taxes does not affect the critical values σ_H and σ_T.*

Proof. Since the slope of $\Delta(\sigma)$ does not depend on τ_k^{nss} and $slope_\Delta(\sigma_T)=1$, we know that σ_T does not depend on τ_k^{nss}. From the expression of R_{2H}, we know that $\frac{1-\tau_k^{nss}\mu}{1-R_{2H}}=\frac{\mu(1-s)}{s\theta(1+R_3)}=\frac{1}{\chi(\theta)\theta(1+R_3)}$, which does not depend on τ_k^{nss}. Replacing $\frac{1-\tau_k^{nss}\mu}{1-R_{2H}}$ with $\frac{1}{\chi(\theta)\theta(1+R_3)}$ in the formula of σ_H, we know that σ_H does not depend on τ_k^{nss}. ■

Four possible dynamic regimes in the case (I) are the same as in Cazzavillan and Pintus (2004, Fig. 1 on pp. 463, 466) except that the critical values of the independent parameter σ and the

[8] For how to derive these parameters, see the appendix A.2. in Cazzavillan and Pintus (2004). It means that σ_H is the solution of $T(R_{2H})=2$; σ_T is the solution of slope$_\Delta(\sigma)$= 1; R_{2H} is the solution of $D(R_2)=1$; R_{2T} solves $T(R_2)=1+D(R_2)$.

bifurcation parameter R_2 are different from those in their model. We summarize these results in the following theorem.

Theorem 1. *Let* $(\bar{k}, \bar{a}) = (1, 1)$ *be a normalized steady state which is set according to the procedure outlined in proposition 1. Then, under assumptions 1, 2, and those stated in the appendix A.2, the following holds.*

(i) $0 < \sigma < \bar{\sigma}$*: the steady state* $(1, 1)$ *is a sink for* $R_2 < R_{2H}$*, undergoes a Hopf bifurcation at* $R_2 = R_{2H}$*, and becomes a source for* $R_2 > R_{2H}$*;*

(ii) $\bar{\sigma} < \sigma < \sigma_H$*: the steady state* $(1, 1)$ *is a saddle for* $R_2 < R_{2T}$*, undergoes a transcritical bifurcation at* $R_2 = R_{2T}$*, becomes a sink for* $R_{2T} < R_2 < R_{2H}$*, undergoes a Hopf bifurcation at* $R_2 = R_{2H}$*, and becomes a source for* $R_2 > R_{2H}$*;*

(iii) $\sigma_H < \sigma < \sigma_T$*: the steady state* $(1, 1)$ *is a saddle for* $R_2 < R_{2T}$*, undergoes a transcritical bifurcation at* $R_2 = R_{2T}$*, and becomes a source for* $R_2 > R_{2T}$*;*

(iv) $\sigma > \sigma_T$*: the steady state* $(1, 1)$ *is a saddle for all* R_2 *in the open interval* $(0, 1)$*.*

Proof. See Appendix A.2. ■

Insert Figure 1 here.

For brevity, we will not turn to analyze the case (II) where the origin $(T_0(0), D_0(0))$ lies outside the triangle ABC and the slope of the half-line $\Delta(\sigma)$ is steeper than that of the line connecting the origin with the point C. This means that $T_0(0) > 1 + D_0(0)$, $D_0(0) < 1$, $1 < T_0(0) < 2$ and $slope_{\Delta}(0) > \frac{1-D_0(0)}{2-T_0(0)}$. Similar to Cazzavillan and Pintus (2004), we may have the very same theorem 2 except that the critical values of the independent parameter σ and the bifurcation parameter R_2 are different from those in their original model.[9]

[9] Three cases in theorem 2 can appear.

Perhaps the reader is interested in studying the impact of small capital income tax rates on the conditions leading to local indeterminacy, as shown in Figure 1 (or Theorem 1). The lemma 1 in the appendix shows that if $1 < \theta < \theta_1$ holds, indeterminacy can arise. Here θ_1 is a critical value above which local indeterminacy can not arise. The interesting finding is that θ_1 can be *decreasing* in the level of capital income tax rates (τ_k^{nss}) provided that the rates (τ_k^{nss}) are not too large. Therefore, increasing the size of distortionary capital income taxes from zero can *not* enlarge the range of the values of θ that are compatible with local indeterminacy.

Proposition 4. *Under the assumptions of Theorem 1, the critical lower bound θ_1 above which local indeterminacy can not arise is decreasing in the level of capital income tax rates provided that the distortionary tax rates (τ_k^{nss}) are not too large. Moreover, $R_3 > \frac{\tau_k^{nss} \frac{s}{1-s}\theta}{\left(\frac{s}{1-s}\theta+1-\delta\right)R_1}\left[R_3\left(\theta-1+R_1\right)+\theta R_1\right]$ will be met if the utility function in the first period of life is close enough to logarithmic $(R_1 = 1)$ and τ_k^{nss} is not too large.*

Insert Figure 2 here.

The following numerical example shows how the share of wage devoted to savings has to be large for local indeterminacy to arise (θ should be small) and how endogenous capital income tax rates are *not* helpful to local indeterminacy: the latter conclusion is consistent with recent works, for example, Schmitt-Grohe and Uribe (1997). Schmitt-Grohe and Uribe (1997) have shown that, in a standard neoclassical growth model, endogenous labor income tax rates are essential for the existence of stationary sunspot equilibria. Moreover, we illustrate, using numerical examples, our main results

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Case 1: $0 < \sigma < \sigma_H$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC but the half line $\Delta(\sigma)$ crosses both the line (AC) and the interior of the segment BC. The NSS is a saddle-point for $0 < R_2 < R_{2T}$, undergoes a transcritical bifurcation and exchanges stability with another steady state at $R_2 = R_{2T}$, becomes a sink for $R_{2T} < R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$.

Case 2: $\sigma_H < \sigma < \sigma_T$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC and the slope satisfies the condition $slope_\Delta(\sigma) > 1$, i.e. the half-line $\Delta(\sigma)$ crosses the line (AC). The NSS is a saddle for $0 < R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, and becomes a source for $R_2 > R_{2T}$.

Case 3: $\sigma > \sigma_T$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC and the slope satisfies the condition $slope_\Delta(\sigma) < 1$. The NSS is a saddle for all R_2 in the open interval $(0, 1)$.

that increasing steady state capital income tax rates may make shrink the range of parameter values ($\overline{\sigma}$) associated with multiple equilibria.

To fix ideas and ease comparisons with Cazzavillan and Pintus (2004), we set $s = 1/3$ and $\delta = 1$, where full capital depreciation is perfectly consistent with the time period implied by the OLG setting, and the chosen value of the capital share in total income is close to the one that Cazzavillan and Pintus (2004) use. We further assume that τ_k^{nss} can take the values of 0.1, 0.12, 0.14, 0.16, 0.18 and 0.20. These values can imply the bound of θ (i.e., θ_1). The values of R_1 and R_3 must belong to the relevant intervals defined in lemma 1. And we assume that $R_1 = 0.95$ and $R_3 = 0.82$.[10] Considering the elasticity of capital–labor substitution, we find that the condition $\sigma < \sigma_H$, which is necessary to get endogenous fluctuations, places a upper bound on σ. It is easy to find that $\sigma_H < \sigma_T$. Numerical examples show that $\sigma_T < s$ and, therefore, that $\sigma_H < \sigma_T < s$. This suggests that σ_H may be below the capital share. In fact, we illustrate that, irrespective of the values for R_1 and R_3, σ_H and σ_T do not depend on τ_k^{nss}, σ_H decreases when θ increases for a given τ_k^{nss}, and $\overline{\sigma}$ is decreasing with τ_k^{nss} for a given θ.[11] This conclusion shows that endogenous capital income tax rates are not helpful to local indeterminacy. Similar to Cazzavillan and Pintus (2004), we can show that total consumption, including consumption by the old agents, has to be less than 45% of output in the case of Fig. 1.

Insert Table 1 here.

We are now in a position to intuitively explain why endogenous capital income tax rates are less helpful to local indeterminacy. Cazzavillan and Pintus (2004) have already shown that when intertemporal substitution in consumption across periods is introduced, endogenous fluctuations require very low values of the propensity to consume out of wage income of the young generation

[10] For how to select these proper values of R_1 and R_3, see the matlab programs which are available upon request.

[11] Again, R_1 and R_3 must belong to the relevant intervals defined in lemma 1.

(in our model, $1 - \frac{1}{\theta}$). In addition, endogenous fluctuations require elasticities of capital–labor substitution that are well below the share of capital in total income. We find that (1) adding capital income tax rates (τ_k^{nss}) will make larger the lower bound of the ratio (between savings and wage income, $\frac{1}{\theta_1}$) for indeterminacy, thus making sunspots less likely to occur and; (2) for a given technology θ, adding tax rates will make the bound on σ associated with multiple equilibria ($\overline{\sigma}$) smaller (this bound is less than the share of capital in total income). To be more precise, we provide the following intuitive interpretation. Endogenous fluctuations arise due to the interaction of two conflicting effects: when the capital stock increases, it leads to an increase in wage rate and, therefore, an increase in savings which leads the capital stock in the next period to be higher. However, capital accumulation is followed by a decrease in the real interest rate that will depress savings and/or capital accumulation. There is one force which tends to dampen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make larger the lower bound of the ratio (between savings and wage income) for indeterminacy, thus making sunspots less likely to occur. In addition, there is another force which tends to strengthen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make the after-tax interest rate more and more negatively sensitive to variations in the capital stock (the elasticity of $\widetilde{R}$ with respect to k is $\varepsilon_{\widetilde{R},k} = [\frac{1}{1-\tau_k^{nss}\mu(a)}]\frac{\mu(a)[s(a)-1]}{\sigma(a)} < 0$ and decreases with τ_k^{nss} for small values of σ when $\sigma < \sigma_H$), thus making sunspots more likely to occur. When the former effect is stronger than the latter effect, increasing capital income tax rates will make local indeterminacy hard to arise.

5. Conclusion

In this note, we study the dynamic effects of government expenditure financed by capital income taxes in an aggregate OLG model with consumption in both periods of life. Using the same method as in Cazzavillan and Pintus (2004), we investigate how government expenditure influences local indeterminacy around the normalized steady state. In contrast with the previous result that the existence of endogenous labor income taxes raises the possibility of local indeterminacy (Chen and Zhang 2009a), this note shows that increasing the size of capital income taxes can make shrink the range of values of the consumption–to–wage ratio associated with local indeterminacy, thus making local indeterminacy less likely to occur.

Acknowledgements

We would like to thank Yoichi Gokan for stimulating discussion. All remaining errors are our own.

Appendix

A.1. Proof of Proposition 1

If $(\overline{k}, \overline{a}) = (1, 1)$ is a normalized steady state of the dynamic system (9) and (10), we have the following: ($\bar{c_1}$ is the steady state of the first period consumption.)

$$A\rho\left(1\right) + 1 - \delta - g = u_2^{-1}\left(\frac{U_3'\left(1\right)}{\beta A\omega\left(1\right)}\right), \tag{D-1}$$

$$A\omega\left(1\right) - 1 = B(U_1')^{-1}\left(\frac{BU_3'(1)}{A\omega\left(1\right)}\right) = \bar{c}_1. \tag{D-2}$$

If $g = \tau_k^{nss} A\rho\left(1\right)$ is not too large ($0 \leq \tau_k^{nss} < 1$), $A\rho\left(1\right) + 1 - \delta - g > 0$ can hold. Since the LHS term of (D–2) is positive, it implies that $A > 1/\omega\left(1\right)$. We rewrite (D–1) as follows: $\beta A\omega\left(1\right) u_2[A\rho\left(1\right) + 1 - \delta - g] = U_3'\left(1\right)$ and we find that the LHS term is an increasing function of A. In order to have a unique A^* satisfying (D-1), we require that $\beta A\omega\left(1\right) u_2[A\rho\left(1\right) + 1 - \delta - g]|_{A=1/\omega(1)} < U_3'\left(1\right)$. It is equivalent to $\beta u_2[\frac{\rho(1)}{\omega(1)} + 1 - \delta - g] < U_3'(1)$. We can easily get B^* from (D-2) after we pin down the unique A^* from (D-1). In particular, we can rewrite (D-2) as follows: $\frac{A\omega(1)-1}{B}U_1'(\frac{A\omega(1)-1}{B}) = \frac{A\omega(1)-1}{A\omega(1)}U_3'(1)$. It is easy to see that $\frac{A\omega(1)-1}{B}U_1'(\frac{A\omega(1)-1}{B})$ is a decreasing function of B. In order to have the unique B^*, we should impose the restriction: $\lim_{c\to 0}cU_1'(c) < \frac{A^*\omega(1)-1}{A^*\omega(1)}U_3'(1)$.

A.2. Proof of Theorem 1

Lemma 1. *Let* $1 < \theta < \theta_1 = \frac{\Upsilon+[\Upsilon^2-4\phi(1-\delta)^2]^{1/2}}{2\phi}$, *where* $\Upsilon \equiv \frac{1-s+(1-\delta)(1-2s)-s(1-\delta)\left(1-\tau_k^{nss}\right)}{1-s}$ *and* $\phi \equiv \frac{(1-s)^2-s(1-2s)\left(1-\tau_k^{nss}\right)}{(1-s)^2}$. *Moreover, we assume that* $R_1 > \overline{R}_1$ *and* $\overline{R}_3 < R_3 < \overline{\overline{R}}_3$, *where* $\overline{R}_3 = \frac{\chi(\theta)+(\theta-1)-\chi(\theta)\theta\left(1-\tau_k^{nss}\mu\right)}{1+\chi(\theta)\theta\left(1-\tau_k^{nss}\mu\right)-\theta}$, $\overline{\overline{R}}_3 = \frac{1-\left(1-\tau_k^{nss}\mu\right)\chi(\theta)\theta}{\left(1-\tau_k^{nss}\mu\right)\chi(\theta)\theta}$ *and* $\overline{R}_1 = \frac{(\theta-1)[R_3-s\tau_k^{nss}/(1-s)]}{\chi(\theta)\theta(1+R_3)\left(1-\tau_k^{nss}\mu\right)-\chi(\theta)-(\theta-1)[1+s\tau_k^{nss}/(1-s)]}$, *with* $\frac{s}{\mu(1-s)} = \chi(\theta)$. *Then we have the following results: the origin* $(T_0(0), D_0(0))$ *lies inside the ABC triangle and the half line* $\Delta(\sigma)$ *intersects the interior of the segment BC at* $\sigma = 0$ *(*$T_0(0) < 1+D_0(0)$, $D_0(0) < 1$*). Moreover, we have* $slope_\Delta(0) > slope_\Delta(\overline{\sigma}) > slope_\Delta(\sigma_H) > slope_\Delta(\sigma_T) = 1$.

Proof. *Similar to Cazzavillan and Pintus (2004),* $D_0(0) < 1$ *is satisfied iff* $0 < R_3 < \frac{1-\left(1-\tau_k^{nss}\mu\right)\chi(\theta)\theta}{\left(1-\tau_k^{nss}\mu\right)\chi(\theta)\theta} \equiv \overline{\overline{R}}_3$, *where* $\chi(\theta) = \frac{s\theta+(1-s)(1-\delta)}{(1-s)\theta}$ *and* $\mu = \frac{s}{(1-s)\chi(\theta)}$. *This (*$\overline{\overline{R}}_3 > 0$*) requires that* $\theta < \overline{\theta} \equiv \frac{(1-s)\delta}{\left(1-\tau_k^{nss}\right)s}$. *Since* $\theta > 1$, *we know that* $s < \frac{\delta}{1-\tau_k^{nss}+\delta} < 1$.

$1 + D_0(0) - T_0(0) > 0$ *is satisfied iff* $R_1 > \overline{R}_1 \equiv \frac{(\theta-1)[R_3-s\tau_k^{nss}/(1-s)]}{\chi(\theta)\theta(1+R_3)\left(1-\tau_k^{nss}\mu\right)-\chi(\theta)-(\theta-1)[1+s\tau_k^{nss}/(1-s)]}$ *with* $R_3 > \widetilde{R}_3 \equiv \frac{\chi(\theta)+(\theta-1)[1+s\tau_k^{nss}/(1-s)]}{\chi(\theta)\theta\left(1-\tau_k^{nss}\mu\right)} - 1$. *Since* $R_1 < 1$, *we need that* $\overline{R}_1 < 1$, *which is equivalent to*

$$R_3 > \overline{R}_3 = \frac{\chi(\theta) + (\theta - 1) - \chi(\theta)\,\theta\,(1 - \tau_k^{nss}\mu)}{1 + \chi(\theta)\,\theta\left(1 - \tau_k^{nss}\mu\right) - \theta},$$

where $1 + \chi(\theta)\,\theta\,(1 - \tau_k^{nss}\mu) - \theta > 0$. $1 + \chi(\theta)\,\theta\,(1 - \tau_k^{nss}\mu) - \theta > 0$ *holds iff* $\theta < \overline{\overline{\theta}} \equiv \frac{(2-\delta)(1-s)}{1-s-s\left(1-\tau_k^{nss}\right)}$. *It is easy to verify that if* $\delta > \frac{2s\left(1-\tau_k^{nss}\right)}{1-s}$, *the binding upper bound on* θ *is* $\overline{\overline{\theta}}$, *as* $\overline{\overline{\theta}} < \overline{\theta}$. *Otherwise, if* $\delta < \frac{2s\left(1-\tau_k^{nss}\right)}{1-s}$, *the binding upper bound on* θ *is* $\overline{\theta}$, *as* $\overline{\overline{\theta}} > \overline{\theta}$. *In addition,* $\overline{R}_3 > \widetilde{R}_3$ *when* $D_0(0) < 1$ *is satisfied. Then we have that* $D_0(0) < 1$ *and* $T_0(0) < 1 + D_0(0)$ *iff* $R_1 > \overline{R}_1$ *and* $\overline{R}_3 < R_3 < \overline{\overline{R}}_3$, *provided that either* $\theta < \overline{\theta}$, *when* $\delta < \frac{2s\left(1-\tau_k^{nss}\right)}{1-s}$, *or* $\theta < \overline{\overline{\theta}}$, *when* $\delta > \frac{2s\left(1-\tau_k^{nss}\right)}{1-s}$. *The inequality*

$\overline{R}_3 < R_3 < \overline{\overline{R}}_3$ holds iff the polynomial holds.

$$P_1(\theta) = \phi\theta^2 - \Upsilon\theta + (1-\delta)^2 < 0,$$

with $\phi = \frac{(1-s)^2 - s(1-2s)\left(1-\tau_k^{nss}\right)}{(1-s)^2}$ and $\Upsilon = \frac{1-s+(1-\delta)(1-2s)-s(1-\delta)\left(1-\tau_k^{nss}\right)}{1-s}$. In addition, $P_1(\theta)$ has a root in $(1, \overline{\theta})$, which is $\theta_1 = \frac{\Upsilon + [\Upsilon^2 - 4\phi(1-\delta)^2]^{1/2}}{2\phi}$. And $P_1(\theta) < 0$ holds for all $\theta \in (1, \theta_1)$. When $\delta > \frac{2s\left(1-\tau_k^{nss}\right)}{1-s}$, $1 < \theta_1 < \overline{\overline{\theta}} < \overline{\theta}$ can hold for a set of properly chosen parameters. A numerical example is $\tau_k^{nss} = 0.2$, $\delta = 1$ and $s = 1/3$.

Following Cazzavillan and Pintus (2004), it is easy to show that $slope_\Delta(0) > slope_\Delta(\overline{\sigma}) > slope_\Delta(\sigma_H) > slope_\Delta(\sigma_T) = 1$. ■

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Tables and Figures

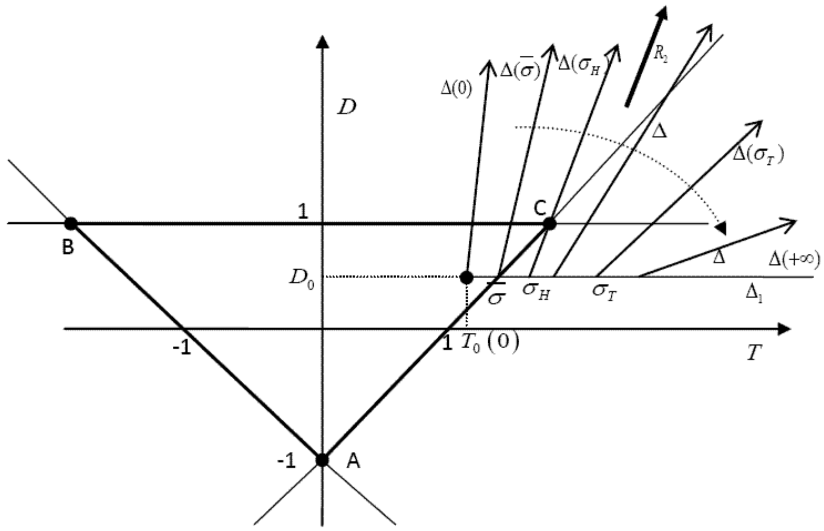

Figure 1.

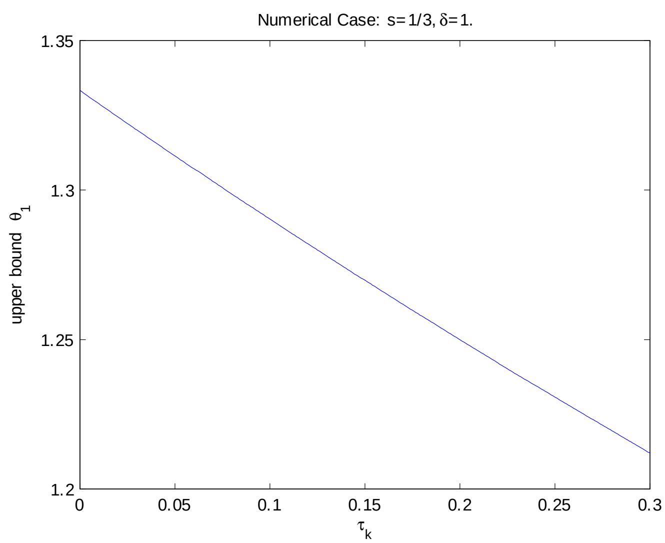

Figure 2.

$\theta\backslash\tau_k^{nss}$	0.10	0.12	0.14	0.16	0.18	0.20
1.05	0.2831	0.2798	0.2762	0.2720	0.2671	0.2614
1.10	0.2313	0.2243	0.2163	0.2070	0.1961	0.1831
1.15	0.1779	0.1666	0.1534	0.1380	0.1197	0.0975
1.20	0.1228	0.1065	0.0874	0.0646	0.0372	0.0035

Table 1. Numerical exercise: $\overline{\sigma}$ ($s = 1/3$, $\delta = 1$, $R_1 = 0.95$, and $R_3 = 0.82$).