Demographic structure and capital accumulation

Hippolyte d'Albis

GREMAQ, University of Toulouse I, 21 allée de Brienne, 31000 Toulouse, France

Abstract

This paper develops an overlapping generations model to analyze the consequences of demographic structure changes induced by an exogenous shift in the birth rate. We first show that a finite growth rate of the population that maximizes long-run capital per capita exists. Then, we examine the theoretical properties of this growth rate by showing that: (i) it corresponds to the demographic structure such that the average ages of capital holders and workers are equal; (ii) it is associated to an efficient steady state; (iii) it increases with compulsory transfers from younger to older generations. Finally, we explain why standard overlapping generations models do not exhibit such a growth rate.

JEL Classification: D91; E13; J10

Key words: Continuous-time overlapping-generations models; Population aging

1   Introduction

Overlapping generations (OLG) models are the neoclassical literature's common tool for analyzing the economic consequences of demographic structure changes. In this paper, we focus on the impact of exogenous shifts in the birth rate on long-run capital accumulation. This relationship is crucial for analyzing the consequences of population aging on most macro-variables[1], such as growth, assets prices and unemployment. An increase in the population growth rate simultaneously induces a reduction in capital per worker and an increase in savings; the problem is to find out which one has the strongest effect. Standard OLG models with production in discrete or continuous time developed by Diamond [13] and Blanchard [4], respectively, can be used to

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[0]   *E-mail address*: hippolyte.d-albis@univ-tlse1.fr.

[1]   See notably the paper by Abel [1] on demographic growth and stock prices.

show that a birth-rate increase reduces long-run capital per capita. The economic intuition given to this result relies on the absence of intergenerational altruism: the size of future generations does not directly influence the saving choices of current generations. The reduction in aggregate consumption that follows a birth rate increase is therefore insufficient for compensating the capital dilution effect. Following Weil [36], the birth rate measures the degree of disconnection between generations, and newborn individuals are interpreted as "unloved children" or immigrants.

Empirical studies, however, do not show that demographic structure changes have a significant impact of on capital accumulation or, more precisely, on its marginal productivity. In a recent paper, Poterba [30] analyzes the historical relationship between population structure and assets return, including the interest rate. Using time series from the United States, Canada and the United Kingdom for the last seventy years. Poterba finds no robust evidence of any impact of demographic variables on assets prices.

At a first glance, these empirical findings seem to invalidate the OLG models while validating the Ramsey framework. In this framework [2], where the population is composed of one unique, altruistic family, the demographic variables have no impact at the balanced equilibrium. That is because the optimal response to a birth-rate increase is to reduce consumption in order to keep capital per capita constant. However, in this paper, we argue that Poterba's results can be reproduced in an OLG model. We build a continuous time OLG model of individuals with finite life-spans to show there exists a finite population growth rate, or equivalently an age structure of the population, that maximizes capital per capita. The existence of this capital maximizer implies that (i) the sign of the impact of demographic growth on capital per capita is ambiguous; (ii) the two different demographic structures could be associated to the same capital per capita; (iii) and the demographic structure changes have a little impact on capital per capita when these structures are at the neighborhood of the one that maximizes capital per capita. Hence, we conjecture that the demographic structures of developed countries have remained close to the structure which maximizes capital per capita. Of course an estimation is required to prove this statement, which is not within the scope of this paper. Instead, we focus on the theoretical characterization of the demographic growth rate that maximizes steady-state capital per capita.

Our argument hinges on the simple but crucial assumption of a population composed of non-altruistic individuals who accumulate assets following a life-cycle behavior. This is standard in neoclassical literature, but is barely implemented in general equilibrium models, because of the technical difficulties that arise in the aggregation procedure. Thus, the two-period lifetime model

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[2] See Cass [9], Koopmans [24].

by Diamond [13] and the continuous time models by Blanchard [4] and Weil [36], which avoid these difficulties, are therefore extensively used in the literature. We show, however, that the assumptions that keep these models tractable, are responsible for the monotonic relationship they exhibit between population growth and capital per capita.

We argue that it is possible to develop an analytically tractable model that reproduces the stylized fact highlighted by Poterba [30]. We build a continuous time OLG model of individuals with finite life-spans based on the pioneering works of Tobin [33] and of Cass and Yaari [10][3]. We assume a general pattern of individual mortality, which includes such cases as the certain lifetime, the Blanchard's Poisson process and the Weil's infinitely-lived families. This formalization makes the comparative discussion straightforward. Aggregation is made following Lotka's [27] assumption of a stable population structure as well as assumptions of perfect financial and insurance markets.

We then derive the following results. First, we give sufficient conditions for the existence and uniqueness of a steady-state path. These conditions, which are easy to interpret and compare with the existing OLG literature. We then perform a static comparative analysis and show that a finite population growth rate that maximizes capital per capita exists. This result crucially depends on the finiteness of individual life-span. Moreover, the considered growth rate corresponds to the demographic structure, such that the average age of capital holders weighted by their wealth equals the average age of the workers weighted by their earnings. We demonstrate this result using the property of stable populations to exhibit easily averaged ages associated to individual variables[4]. Thus, in the steady-state equilibrium, the impact of demographic growth on capital per capita is positive if the average age of capital holders is lower than the average age of workers. By construction, this case cannot exist in the Diamond [13] and Blanchard [4] models.

Before introducing the model, we want to emphasize that our analysis is positive. Indeed, our intention is different from the issue raised by Samuelson [31] on the optimal population growth rate. This growth rate is the one that maximizes welfare when the equilibrium is golden-rule. Instead, we make a static comparative analysis when the equilibrium is balanced. Nevertheless, we show that the equilibrium associated to the demographic growth rate that maximizes capital per capita, is efficient. Moreover, we complete the analysis by introducing compulsory transfers between generations and show that transfers from younger to older generations increase the demographic growth

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[3] Further developments have been made recently by Burke [8], Malinvaud [28], Bommier and Lee [5] and Demichelis and Polemarchakis [12].
[4] These were first introduced in economic models by Arthur and McNicoll [2] and Lee [26].

rate that maximizes capital per capita.

The basic framework of the model is presented in Section 2 and the analysis of the effect of a shift in the demographic growth rate on the steady-state capital per capita is made in Section 3. A general discussion of the results is proposed in Section 4. Section 5 concludes and the Appendix presents all the proofs of the propositions.

2 The model

This section presents an aggregation of individual consumption behavior under an uncertain lifetime in an overlapping generations model of neoclassical growth. It presents the steady state of such an economy and derives sufficient conditions for its existence and uniqueness. Time is continuous, and all the variables that depend on time are assumed to be continuous and differentiable.

2.1 Individual behavior

We closely follow Yaari [37][5]. Individuals are uncertain about the length α of their life. Let $\pi(\alpha) > 0$ denote the probability density function of the random variable α with support $[0, \Omega]$, where Ω is the finite maximum possible lifetime. Individuals also exhibit a pure discount factor $\theta(\alpha)$ that satisfies $\theta : (0, \Omega] \rightarrow (0, 1)$ and $\theta(0) = 1$; hence, the pure discount rate, denoted $\rho(\alpha) \equiv -\theta'(\alpha)/\theta(\alpha)$ which may vary with age and might even be negative. Individuals only derive satisfaction from their consumption, denoted $c(\alpha) \geq 0$ and have no bequest motives. Instantaneous utility is isoelastic with an elasticity of intertemporal substitution $\sigma \in (0, 1]$. Hence, expected utility at birth of any individual is:

$$\int_0^\Omega \pi(\alpha) \left(\int_0^\alpha \theta(z) \frac{c(z)^{1-1/\sigma} - 1}{1 - 1/\sigma} dz \right) d\alpha \tag{1}$$

During a lifetime, the labor supply is fixed and, $w(\alpha) > 0$, a given stream of labor income is received. There is a single asset and individuals have access to competitive capital and complete insurance markets. Individuals hence hold their entire wealth $a(\alpha)$ in the form of actuarial notes that makes an insurance company their legatee. Since the notes are assumed actuarially fair, they

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[5] Our problem is less general, however, since we are considering a specific instantanous utility function, namely a CRRA with $\sigma \in (0, 1]$. Relaxing this assumption drastically complicates the proof of the existence of an aggregate equilibrium.

yield the regular risk-free interest rate, $r > 0$, plus the hazard rate of death and are therefore the most lucrative asset if the individual wealth is positive. Moreover, the uncertainty about life-span makes the subscription to these notes the only way to borrow. Let $p(\alpha) = \int_\alpha^\Omega \pi(z) \, dz$ denote the probability at birth that an individual will be alive at age α. Hence, $\pi(z | \alpha) = \pi(z) / p(\alpha)$ is the conditional probability density evaluated at age z, given that the individual is alive at age α, and $\pi(\alpha | \alpha)$ is the hazard rate of death at age α. The hazard rate of death is supposed to be non-decreasing with respect to age: $\pi'(\alpha | \alpha) \geq 0$. The individual budget constraint hence writes:

$$a'(\alpha) = [r + \pi(\alpha | \alpha)] \, a(\alpha) + w(\alpha) - c(\alpha) \tag{2}$$

Individuals are born with no financial assets and face a terminal condition that forbids Ponzi games. The two following conditions therefore complete the individual program:

$$a(0) = 0 \tag{3}$$
$$a(\Omega) \geq 0 \tag{4}$$

We, moreover, assume that $\pi(\alpha)$, $\theta(\alpha)$, $w(\alpha)$ and $a(\alpha)$ are of class C^2 on $[0, \Omega]$ and that $c(\alpha)$ is of call C^1 on $[0, \Omega]$. For simplicity's sake, we define $G(.)$ and $H(.)$ such that:

$$G(x) \equiv \int_0^\Omega p(\alpha) \, w(\alpha) \exp(-x\alpha) \, d\alpha \tag{5}$$

$$H(x) \equiv \int_0^\Omega p(\alpha) \, [\theta(\alpha)]^\sigma \exp(-x\alpha) \, d\alpha \tag{6}$$

which are both positive and decreasing. Note that $\int_0^\Omega p(\alpha) \exp(-r\alpha) \, d\alpha$ represents the expected discounted value of a constant flow of one unit of output; this value is weighted with the income distribution in $G(r)$ and with the preference parameters in $H(r)$.

Given prices r and w, each individual chooses $\{c(\alpha)\}$ that maximizes (1) subject to (2), (3) and (4). The optimal behavior is now described by the following proposition:

Proposition 1 *There exists a unique solution to the individual problem. For $\alpha \in [0, \Omega]$, the optimal consumption profile satisfies:*

$$c(\alpha) = \frac{G(r)}{H((1 - \sigma) r)} \, [\theta(\alpha)]^\sigma \exp(\sigma r \alpha) \tag{7}$$

while the optimal wealth satisfies:

$$a\left(\alpha\right) = \int_{\alpha}^{\Omega} \frac{p\left(z\right)}{p\left(\alpha\right)} \exp\left(-r\left(z - \alpha\right)\right) \left[c\left(z\right) - w\left(z\right)\right] dz \qquad (8)$$

with $a\left(\Omega\right) = 0$.

The individual behavior described by equation (7) is the standard neoclassical one. Note that $G\left(r\right)/H\left(\left(1 - \sigma\right)r\right)$ represents the product of the expected discounted flow of earnings at birth and the marginal propensity to consume out of total individual wealth at birth, i.e. the initial consumption $c\left(0\right)$. Deriving equation (7) with respect to α yields:

$$c'\left(\alpha\right) = \sigma\left[r - \rho\left(\alpha\right)\right] c\left(\alpha\right) \qquad (9)$$

Perfect annuity markets imply that the optimal consumption growth rate only depends on the difference between the interest rate and the pure discount rate. Since the latter varies with age, individual consumption may then decrease during the life-span. Moreover, it may also be a concave function of age if the pure rate of discount sufficiently increases with age.

Concerning the optimal asset accumulation, its profile is characterized in the following proposition:

Proposition 2 *For* $\alpha \in \left[0, \Omega\right]$, *the optimal wealth is positive and hump-shaped if:*

$$\sigma\left[r - \rho\left(\alpha\right)\right] > \frac{\pi'\left(\alpha \mid \alpha\right)}{\pi\left(\alpha \mid \alpha\right)} \quad and \quad \frac{\pi'\left(\alpha \mid \alpha\right)}{r + \pi\left(\alpha \mid \alpha\right)} > \frac{w'\left(\alpha\right)}{w\left(\alpha\right)} \qquad (10)$$

A borrowing period during the life-cycle is a possible output of the individual problem. However, under rather plausible assumptions, proposition 2 rules out this possibility. It is indeed sufficient to have a consumption growth rate greater than the hazard rate of death's growth rate and a bounded income growth rate. Remark that the hazard rate of death is usually assumed by demographers to follow a Gompetz' law such that $\pi\left(\alpha \mid \alpha\right) = \exp\left(x\alpha\right)$ with $x > 0$. Alternatively, if the individual consumption is decreasing during the life-span, the optimal wealth may not be hump-shaped. Finally, note that condition (10) remains to be check at the equilibrium since it involves the interest rate that is to be an endogenous variable.

Remark 1. Our rather general specifications ease the comparison with OLG models developed by Blanchard [4] and Weil [36]. Blanchard assumes that an individual lifetime follows a Poisson process and that the pure discount rate and the wage stream are constants. With our notation, $\Omega = \infty$, $\pi\left(\alpha \mid \alpha\right) = p$, $\rho\left(\alpha\right) = \rho$, and $w\left(\alpha\right) = w$ with $p, \rho, w > 0$. Thus, $G\left(x\right) = w/\left(x + p\right)$ and $H\left(x\right) = 1/\left(x + p + \sigma\rho\right)$. It is moreover necessary to replace condition (4) by

the No-Ponzi-Game condition: $\lim\limits_{\alpha \to +\infty} \exp\left(-\left(r+p\right)\alpha\right) a\left(\alpha\right) \geq 0$. Weil interprets the individual as an infinitely lived family, an assumption that can be obtained by setting $p = 0$. Replacing in equation (7), the individual consumption behavior becomes for all $\alpha \geq 0$:

$$c\left(\alpha\right) = w\frac{\left[\left(1-\sigma\right)r + p + \sigma\rho\right]}{\left(r+p\right)} \exp\left(\sigma\left(r-\rho\right)\alpha\right) \qquad (11)$$

Simple manipulations of equation (11) yields the individual asset accumulation:

$$a\left(\alpha\right) = \frac{w}{\left(r+p\right)}\left[\exp\left(\sigma\left(r-\rho\right)\alpha\right) - 1\right] \text{ for } \alpha \geq 0 \qquad (12)$$

In Blanchard and Weil models, the consumption and asset accumulation profiles are hence a monotonic function of age.

2.2 Aggregation and competitive equilibrium

The demographic structure is based on Lotka's [27] stable population theory. Each individual belongs to a large cohort of identical individuals. Therefore even though each individual's life-span is stochastic, there is no aggregate uncertainty. The law of large numbers is supposed to apply[6], and thus, the size of each cohort is decreasing at rate $\pi\left(\alpha | \alpha\right)$. Then, at time t, the size of the age-α cohort is $\beta N\left(t-\alpha\right)p\left(\alpha\right)$ where $\beta > 0$ is the birth rate and $N\left(t-\alpha\right)$ is the size of the population at time $t-\alpha$. Since a new cohort is born at each instant, $N\left(t\right)$ satisfies:

$$N\left(t\right) = \int_0^\Omega \beta N\left(t-\alpha\right)p\left(\alpha\right)d\alpha \qquad (13)$$

Lotka demonstrates that the demographic structure (i.e. the relative size of each cohort in the population) reaches a steady state when the birth and death rates have been constant over a sufficiently long period of time. The stationary distribution, characterized by the demographic growth rate n, is obtained by replacing $N\left(t\right) = N\left(0\right)\exp\left(nt\right)$ in (13) and then solving:

$$\int_0^\Omega p\left(\alpha\right)\exp\left(-n\alpha\right)d\alpha = \frac{1}{\beta} \qquad (14)$$

The unique real solution of equality (14) defines[7] the population growth rate n whose sign depends on the difference between the birth rate and the inverse of the life expectancy at birth. Formally, $n \geq 0$ if and only if $\beta \geq 1/\int_0^\Omega p\left(\alpha\right)d\alpha$.

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[6] See the discussion in Judd [21] and Feldman and Gilles [16].

[7] The left hand side of (14) is a positive, decreasing function of n, whose limits are respectively $+\infty$ and 0 when n goes to $-\infty$ and $+\infty$.

Note that the relative size of each cohort in the population, $\beta p(\alpha) \exp(-n\alpha)$, is then time independent, which is why the population is said to be stable.

Aggregate wage per capita, denoted as w, satisfies:

$$w \equiv \int_0^\Omega \beta p(\alpha) \exp(-n\alpha) w(\alpha) \, d\alpha \tag{15}$$

while consumption per capita, denoted as c, satisfies:

$$c \equiv \int_0^\Omega \beta p(\alpha) \exp(-n\alpha) c(\alpha) \, d\alpha \tag{16}$$

Hence, replace equation (7) in (16) and replace β using (5) and (15), to obtain:

$$c = w \frac{G(r)}{G(n)} \frac{H(n - \sigma r)}{H((1 - \sigma) r)} \tag{17}$$

Moreover, when the individuals consumption profile is optimal, aggregate financial wealth per capita, denoted as a, satisfies[8] :

$$a = \frac{c - w}{r - n} \tag{18}$$

There is a unique material good, whose price is normalized to 1. It can be used for consumption or for adding to the capital stock. This good is produced by many competitive firms whose aggregate activity is described by a constant return-to-scale production function with labor and capital as inputs. Let k be the capital per capita, $\delta \geq 0$ be the depreciation rate and $f(.)$ be the production function in intensive form. Assume that $f : [0, \infty) \to [0, \infty)$, that $f'(k) > 0$ and $f''(k) < 0$ for all $k > 0$, and that $\lim_{k \to 0} f'(k) = \infty$. Factor prices equal marginal products; hence $w = f(k) - kf'(k)$ and $r = f'(k) - \delta$.

As in Diamond [13] and Blanchard [4], the equilibrium condition is obtained by identifying asset holdings with capital stock: $a = k$. The analysis of steady-state equilibrium turns out to be a fixed-point problem, $k = \phi(k)$, where function ϕ is obtained by first replacing equation (17) in (18) and then replacing factor prices such that:

$$\phi(k) \equiv \frac{f(k) - kf'(k)}{f'(k) - (n + \delta)} \left[\frac{G(f'(k) - \delta)}{G(n)} \frac{H(n - \sigma(f'(k) - \delta))}{H((1 - \sigma)(f'(k) - \delta))} - 1 \right] \tag{19}$$

Let us introduce $s(k) \equiv kf'(k)/f(k)$, the share of capital in output and $\varepsilon(k) \equiv -f'(k)[1 - s(k)]/[kf''(k)]$, the elasticity of substitution between capital and labor. The concavity of f implies that $s \in (0, 1)$ and $\varepsilon > 0$ for all $k > 0$.

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[8] See the appendix for the details.

Proposition 3 *There exists $\hat{k} > 0$ such that $\hat{k} = \phi\left(\hat{k}\right)$ if*

$$\lim_{k \to 0} \left[-kf''(k)\right] > \frac{1}{\sigma\Omega} \tag{20}$$

$\hat{k}$ is unique if

$$s(k) \le \varepsilon(k) \quad and \quad \frac{d}{dk}\left(\frac{s(k)}{\varepsilon(k)}\right) \ge 0 \tag{21}$$

Proposition 3 gives existence and uniqueness conditions for a non-trivial equilibrium. Technically, condition (20) implies $\lim_{k \to 0}\phi(k) = +\infty$ and conditions (21) ensure $\phi'\left(\hat{k}\right) < 1$. Let us now turn to the interpretation.

Our existence condition (20) is close to the condition proposed by Bommier and Lee [5] who extend to a continuous-time framework the findings of Konishi and Perera-Tallo [23] for the Diamond model [13]. Their condition, named the "non-vanishing labor share", has the advantage of relying only on the production function. Formally it can be written as:

$$\lim_{k \to 0} s(k) \in [0, 1) \tag{22}$$

However, (22) is more restrictive than (20). To demonstrate that point, let us observe that

$$\lim_{k \to 0} \left[\frac{1 - s(k)}{s(k)} f'(k)\right] \equiv \lim_{k \to 0}\left[-kf''(k)\right] \tag{23}$$

Let us now recall the Inada condition on $f'(k)$ and conclude that (22) implies $\lim_{k \to 0}\left[-kf''(k)\right] = +\infty$.

In condition (20), the key elements are the maximum possible lifetime and the elasticity of inter-temporal substitution. Strong income effects yielded by a small σ, may hence be compensated by a longer lifespan. Moreover, in Blanchard [4] and Weil's [36] models, where $\Omega = \infty$, a competitive equilibrium always exists.

Multiple equilibria may occur in such a framework: see notably Kehoe [22] for production economies and Ghiglino and Tvede [18] for economies with heterogeneous endowments. Further restrictions on the production function are hence needed to exclude this possibility. Our conditions (21) state that inputs should be rather substitutable, especially if capital per capita is low at equilibrium. These conditions are notably satisfied by production functions with a constant elasticity of substitution that is greater than one ($\varepsilon \ge 1$); that is the functions that verify the Inada condition for $k = 0$. Let us also note that relaxing the assumption of $\sigma \in (0, 1]$ may produce multiple equilibria.

Remark 2. The complete analysis of the inter-temporal equilibrium is beyond the scope of this paper. Note nevertheless, that recent contributions have pointed out the importance of the initial condition of the economy on the existence of the equilibrium. Burke [8] notably extend to continuous time the result that the equilibrium may fail to exist if time has a finite starting point (See Geanakoplos and Polemarchakis [14]). As a consequence, the particular solution that is given by the steady-state equilibrium may also fail to exist. Moreover, the dynamics of the equilibrium path is not necessarily monotoneous: in a pure exchange economy, Demichelis and Polemarchakis [12] show the existence of exponnentially decreasing fluctuations along the transition while Ghiglino and Tvede [19] show that a cycle may exist.

A graphical illustration. We propose a graphical example of the equilibrium in the (k, c) space when conditions (20) and (21) are respected. Note first that equation (18) may be written as $c = (r - n) a + w$; at the equilibrium, this defines $c_1 (.)$ such that:

$$c_1 (k) = f (k) - (n + \delta) k \qquad (24)$$

The $c_1 (k)$ line represents the resource constraint of the economy. If $n + \delta > 0$, $c_1 (k)$ is hump-shaped with golden-rule consumption that occurs at the point where $c_1 (k)$ reaches its maximum. Then, we use the definition of the wages given above and replace it in (17); at the equilibrium, this defines $c_2 (.)$ such that:

$$c_2 (k) = \frac{k \left[f' (k) - (n + \delta) \right]}{1 - \frac{G(n)}{G(f'(k) - \delta)} \frac{H((1 - \sigma)(f'(k) - \delta))}{H(n - \sigma(f'(k) - \delta))}} \qquad (25)$$

The $c_2 (k)$ line stands for the aggregation of individual consumption decisions for each level of capital. Here, for example, figure 1 represents [9] the situation where $n + \delta > 0$, $f (0) = 0$ and there is dynamic efficiency.

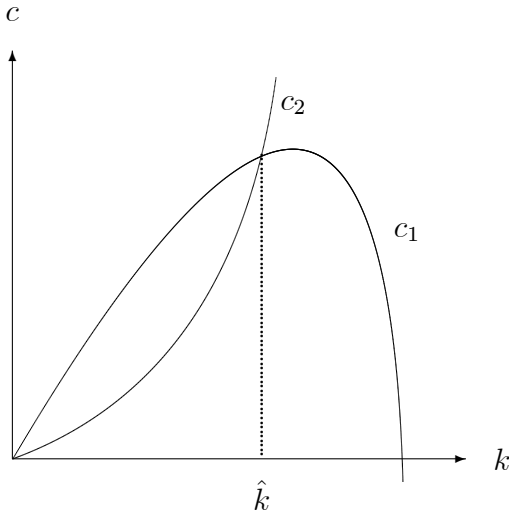[9] Details of the derivation of this figure are given in the Appendix.

Figure 1

This figure is the extension, for any stationary demographic structure, of the one presented in Blanchard [4].

3 Demographic growth and capital per capita

This section analyses the impact of a shift in the population growth rate on the steady-state capital per capita and shows that a finite growth rate that maximizes capital per capita exists. It then characterizes this capital maximizer.

3.1 Main results

It is useful to define $\bar{\alpha}_x$, the average age calculated cross-sectionally such that:

$$\bar{\alpha}_x \equiv \frac{\int_0^\Omega \alpha p(\alpha) \exp(-n\alpha) x(\alpha) d\alpha}{\int_0^\Omega p(\alpha) \exp(-n\alpha) x(\alpha) d\alpha} \tag{26}$$

where $x(\alpha)$ is a relevant characteristic. In the case $x(\alpha) = 1$, $\bar{\alpha}_1$ is the standard definition of the average age of the population. If $x(\alpha) = w(\alpha)$, $\bar{\alpha}_w$ is the average age of workers weighted by their earnings and equivalently, if $x(\alpha) = a(\alpha)$, $\bar{\alpha}_a$ is the average age of capital holders weighted by their wealth.

We now implicitly consider that the change in the demographic growth is induced by a change in the birth rate. It is, however, not worth representing changes in the birth rate since its long-run relationship with demographic

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growth is monotonously increasing. Indeed, implicit differentiation of condition (14) yields:

$$\frac{dn}{d\beta} = \frac{1}{\beta\bar{\alpha}_1} > 0 \qquad (27)$$

Hence, a static comparative of demographic growth on steady-state capital per capita yields the following result:

Proposition 4 *The impact of an increase in population growth on capital per capita depends on the difference between the average age of capital holders and the average age of workers; one indeed has:*

$$\frac{d\hat{k}}{dn} = -\frac{\hat{k}}{1 - \phi'\left(\hat{k}\right)} \left[\bar{\alpha}_a - \bar{\alpha}_w\right] \qquad (28)$$

Proposition 4 states that an increase in population growth does not necessarily reduce the steady-state capital per capita but depends on income and population distributions. Observe moreover that $\bar{\alpha}_a$ is an endogenous variable while $\bar{\alpha}_w$ is exogenous. The interpretation of this result will be given after the next proposition.

Proposition 5 *There exists a finite value of the population growth rate, denoted as n^*, that maximizes steady-state capital per capita.*

Propositions 4 and 5 show that the relationship between the population growth rate and steady-state capital per capita is non monotonic and that capital per capita reaches a maximum when the average age of capital holders equals the average age of workers. The intuition of this result can be obtained by comparing the impact of demographic growth on the age structure of the working population and the pattern of individual assets. Let us first observe that the average age of the workers is decreasing with n; at the limit, it equals 0 for $n = +\infty$ and Ω for $n = -\infty$. Next, let us recall that the optimal asset accumulation pattern during lifetime is such that $a(0) = a(\Omega) = 0$. Therefore, when n goes to plus or minus infinity, (i.e. when $\bar{\alpha}_1$ goes to 0 or Ω) the capital accumulated in the economy is very low. Conversely, the steady-state capital per capita is maximal when the distribution of population concentrates a large mass of individuals around the maximum wealth in the life-cycle accumulation profile. Remark that Proposition 5 does not rule out multiples n^*. Observe nevertheless, that if n^* is unique, the relationship between n and k is hump-shaped.

Remark 3. An n-maximizer has also been recently presented in Boucekkine, de la Croix and Licandro [6]. In an endogenous growth model with schooling and retirement choices, they find a hump-shaped relationship between the population growth rate and the per-capita growth rate of human capital. Their

result is based on a functional but realistic survival law and an instantaneous utility that is linear with respect to consumption. There is no capital in their model, but the existence of an n-maximizer relies on the same reasoning: the vintage structure of human capital.

3.2 Optimality

The population growth rate that maximizes capital per capita is not optimal since it does not maximize individual welfare or aggregate consumption; it can be only stated that the equilibrium to which it is associated with is efficient. To see that point, let us first consider the following proposition:

Proposition 6 *Let $\bar{k}$ be such that $\phi\left(\bar{k}\right) = 0$ and μ such that $f'\left(\bar{k}\right) = \mu + \delta$. Then,*

$$n^* < \mu \tag{29}$$

Proposition 6 introduces μ, the lower bound on the interest rate, associated to $\bar{k}$, the maximal value for capital per capita at equilibrium. This bound corresponds to the interest rate above which outstanding debt in the economy outweighs positive assets. If the individual pure discount rate is constant, such that $\rho\left(\alpha\right) = \rho$, then $\mu = \rho$. Otherwise, μ depends on the model parameters, including demographic ones. Thus, proposition 5 states that the population growth rate that maximizes capital per capita is lower than the lower bound on the interest rate. This result is quite intuitive if the individual consumption profile is supposed to be increasing: then, the age at which individuals have a maximal wealth takes place during the second half of their lives. Therefore, capital per capita is maximized by a demographic structure with a relatively high average population age, which means a rather low population growth rate.

Corollary 1. *At the steady-state equilibrium associated with n^*, there is under-accumulation of capital.*

The definition of μ, given in proposition 6, implies that any steady-state $\hat{k}$ satisfies $f'\left(\hat{k}\right) - \delta > \mu$. Consequently, the equilibrium associated with n^* exhibits an interest rate strictly greater that the population growth rate. Consequently, the equilibrium (i) cannot be a golden-rule equilibrium and (ii) is efficient as proved by de La Croix and Michel [11] [10]. This does not implies that the equilibrium associated with n^* is Pareto optimal. This problem has been first discussed in Cass and Yaari [10] and carefully studied by Grandmont [20],

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[10] Proposition 2.4 page 83.

Wang [34], and Duc and Ghiglino [15]. The latter notably propose a general characterization of the endowment distributions leading to an optimal barter steady-state.

Remark 4. The aggregate consumption per capita is not maximized when the population growth rate equals n^*. Our work then differ from those of Samuelson [31] and Michel and Pestieau [29] who have studied the existence of an "optimal" population growth rate that maximize individual's welfare when the economy is at the golden-rule equilibrium. In a continuous-time OLG framework, Arthur and McNicoll [3], Lee [26] and Willis [35] interpret this "optimal" population growth rate in terms of average ages of consumption and earnings.

3.3 Intergenerational transfers

Real world economies include many kinds of non-competitive intergenerational transfers such as child rearing, Pay-As-You-Go pension systems and intra-familial transfers. In this subsection, we will introduce a simple intergenerational transfer scheme in our model and analyze its impact on n^*.

Following Blanchard [4], we introduce a simple intergenerational transfers scheme assuming that disposable incomes are growing with age at the constant rate of γ: a $\gamma > 0$ (respectively, a $\gamma < 0$) implies a transfer scheme from younger to older (from older to younger) individuals. Let us assume that the disposable income, denoted $\tilde{w}(\alpha)$, is such that:

$$\tilde{w}(\alpha) = w(\alpha)\lambda \exp(\gamma\alpha) \tag{30}$$

where λ is a constant whose value is determined to ensure that the following equality is satisfied:

$$\int_0^\Omega \beta p(\alpha) w(\alpha) \exp(-n\alpha) \, d\alpha = \int_0^\Omega \beta p(\alpha) \tilde{w}(\alpha) \exp(-n\alpha) \, d\alpha \tag{31}$$

Replacing (30) yields $\lambda = G(n)/G(n-\gamma)$ and therefore:

$$\tilde{w}(\alpha) = w(\alpha) \frac{G(n)}{G(n-\gamma)} \exp(\gamma\alpha) \tag{32}$$

Then, equation (7) can be rewritten as:

$$c(\alpha) = \frac{G(n)}{G(n-\gamma)} \frac{G(r-\gamma)}{H((1-\sigma)r)} [\theta(\alpha)]^\sigma \exp(\sigma r\alpha) \tag{33}$$

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and equation (19) becomes:

$$\phi\left(k\right) \equiv \frac{f\left(k\right) - kf'\left(k\right)}{f'\left(k\right) - \left(n + \delta\right)}\left[\frac{G\left(f'\left(k\right) - \delta - \gamma\right)}{G\left(n - \gamma\right)}\frac{H\left(n - \sigma\left(f'\left(k\right) - \delta\right)\right)}{H\left(\left(1 - \sigma\right)\left(f'\left(k\right) - \delta\right)\right)} - 1\right]$$
(34)

It is easy to show that, upon conditions stated in proposition 3, the existence and uniqueness of a steady-state equilibrium are guaranteed for any finite γ. The impact of γ on $\hat{k}$ is negative, which is standard: increasing transfers to older individuals reduces private savings and therefore steady-state capital per capita. We will discuss the impact of γ on n^* in the following proposition.

Proposition 7 *Let n^* be the demographic growth rate that maximizes $\hat{k}$, given by $\hat{k} = \phi\left(\hat{k}; n, \gamma\right)$, and suppose n^* is unique; thus:*

$$\frac{dn^*}{d\gamma} > 0 \tag{35}$$

Proposition 7 states that transfers from younger to older (respectively, from older to younger) individuals increase (reduce) the demographic growth rate that maximizes capital per capita.

To interpret proposition 7, it is necessary to recall proposition 4 and observe that the impact of γ on n^* depends on two opposite effects. On the one hand, $\bar{\alpha}_w$ increases with γ: equation (32) implies that transfers to older individuals reduce the wage of newborn individuals and increase the wage growth rate. On the other hand, $\bar{\alpha}_a$ also increases with γ. This is the consequence of the induced increase in the interest rate that postpones the age at which individuals stop saving. Hence, proposition 7 states that the first effect dominates the second one, and therefore that the introduction of transfers to older individuals increases n^*. This result is also supported by analyzing the impact of γ on μ. Simple calculus shows that the impact is positive, which enhances the idea of proposition 7.

4 Comparison with the literature

We will now explain why commonly used OLG models developed by Diamond [13] and Blanchard [4] imply a negative relationship between population growth and capital per capita.

Diamond [13] supposes each individual lifetime to be composed of two discrete periods. During the first period, individuals work and earn a wage that allows them to save for retirement. To do so, they buy the assets accumulated by those in the second period of their life. This assumption eases aggregation

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because the savings of the younger individuals correspond to the total capital of the economy. However, by construction, elder individuals hold all the assets. Thus, the average age of capital holders is always greater than the average age of workers. It is true that in our model, individuals work their entire lives. Such an assumption would not change the implication of the two-period model: if individuals were allowed to work during both periods, the average age of earnings would be equal to the average age of the population, but would still be lower than the average age of capital holders[11]. The monotonic result of the Diamond model therefore comes from the exogenous length of time that the individual saves. Conversely, if a lifetime covers $N \geq 3$ discrete periods as in Gale [17] or a finite interval as in our framework, the "economic youth", defined as the saving period, is endogenous and the result of proposition 5 applies.

The continuous time model of Blanchard [4] is also widely used in macro-economics. We will now discuss Buiter [7] and Weil's [36] models; these two extensions of Blanchard allow for changes in the population growth rate. Buiter assumes a non-stationary population that grows at rate $n = \beta - p$, where β is the birth rate and p is the death rate defined in remark 1. The identification of the aggregate with the private death rate implies that the average age of the population, and consequently the average age of workers when the wage is age-independent, depends only on the birth rate: $\bar{\alpha}_w = 1/\beta$. To obtain Weil specifications, we only need to assume that $p = 0$. In that case, the population growth rate is equal to the birth rate. Because cohorts have homogenous horizons whatever their age, human wealth and propensity to consume out total wealth are age-independent. Moreover, as mention in remark 1, the consumption and asset accumulations profiles are monotonic functions of age. Now let us compute (12) in (26) to obtain the average age of capital holders:

$$\bar{\alpha}_a = \frac{(2\beta - \sigma(r - \rho))}{\beta(\beta - \sigma(r - \rho))} \text{ for } r \in \left(\rho, \rho + \frac{\beta}{\sigma}\right) \tag{36}$$

Hence, $\bar{\alpha}_a > \bar{\alpha}_w$. This result hinges on the assumption that individuals have age-independent horizons. Every agent therefore behaves as a newborn and accumulates assets forever: $a(\alpha)$ is not hump-shaped but increasing and convex. On the other hand the relative size of a cohort, given by $\beta \exp(-\beta\alpha)$, is always decreasing even if the demographic growth rate is negative. Therefore, Blanchard-Buiter and Weil's models systematically exhibit an average age of capital ownership greater than the average age of the population.

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[11] It is only at the limit (i.e., in a discrete time model, when the population growth rate goes to -1) that both ages would be equal.

5 Conclusion

In this paper, we have developed a continuous time, overlapping generations model to analyze the impact of demographic changes on steady-state capital per capita. First, we presented new conditions for the existence and the uniqueness of a steady state with a general pattern of individual mortality and income distribution. We then performed a comparative static analysis of the impact of the population growth rate on capital per capita, and showed that the functional relationship between those two variables is non-monotonic. Models by Diamond [13] and Blanchard [4], therefore, constitute particular exceptions by exhibiting a strictly decreasing relationship. We next defined an age structure of the population that maximizes capital per capita. This structure made the average ages of capital holders and workers equal. Finally, we showed that the growth rate of the population that corresponds to the considered structure is lower than the equilibrium interest rate and increases with non-competitive transfers to older individuals.

Our model constitutes a rather simple framework for the analysis of age-dependant behaviors. Notably, consequences of fertility and labor-market decisions can be directly investigated in such a general equilibrium setting. Moreover, a study of transition dynamics should be an interesting extension of this paper since the dynamics are not likely to be monotonic. Lotka [27] demonstrates that any population with constant birth and death rates converges toward a stable age structure with exponentially decreasing oscillations. The problem is similar in economic models, although they deal with a forward variable. Numerical methods are proposed in Laitner [25] and Boucekkine *et al.* [6] to analyze the behavior of non-stationary paths. The latter shows that the vintage structure of the model creates discrete delays in the dynamic behavior of aggregate variables which are then governed by echo effects.

Appendix

PROOF. [Proof of proposition 1] The individual program can be solve using classical calculus of variation[12]; change first the order of integration of function (1) to rewrite the objective as

$$\int_0^\Omega p(\alpha)\,\theta(\alpha)\,u(c(\alpha))\,d\alpha \tag{37}$$

with

$$u(c(\alpha)) = \frac{c(\alpha)^{1-1/\sigma} - 1}{1 - 1/\sigma} \tag{38}$$

Using (2), (3) and (4), the program becomes

$$\max \int_0^\Omega L(\alpha, a, a')\,d\alpha$$

$$s.t.\ \begin{vmatrix} a(0) = 0 \\ a(\Omega) \geq 0 \\ \left[r - \frac{p'(\alpha)}{p(\alpha)}\right]a(\alpha) + w(\alpha) - a'(\alpha) \geq 0 \end{vmatrix} \tag{39}$$

with

$$L(\alpha, a, a') = p(\alpha)\,\theta(\alpha)\,u\left(\left[r - \frac{p'(\alpha)}{p(\alpha)}\right]a(\alpha) + w(\alpha) - a'(\alpha)\right) \tag{40}$$

Since L is a C^2-function of (α, a, a'), necessary conditions for a C^2-function $a(\alpha)$ to be a solution of program (39) are the Euler equation:

$$\frac{d}{d\alpha}\left[-p(\alpha)\,\theta(\alpha)\,u'\left(\left[r - \frac{p'(\alpha)}{p(\alpha)}\right]a(\alpha) + w(\alpha) - a'(\alpha)\right)\right]$$

$$= \left[r - \frac{p'(\alpha)}{p(\alpha)}\right]p(\alpha)\,\theta(\alpha)\,u'\left(\left[r - \frac{p'(\alpha)}{p(\alpha)}\right]a(\alpha) + w(\alpha) - a'(\alpha)\right) \tag{41}$$

and the terminal condition:

$$\left(\frac{\partial L(\alpha, a, a')}{\partial a'}\right)_{\alpha=\Omega} \leq 0 \tag{42}$$

Equation (41) rewrites using (38) as follows

$$c'(\alpha) = \sigma\left[r + \frac{\theta'(\alpha)}{\theta(\alpha)}\right]c(\alpha) \tag{43}$$

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[12] See e.g. Seierstad and Sydsæter [32].

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which yields

$$c(\alpha) = c(0) [\theta(\alpha)]^{\sigma} \exp(\sigma r \alpha) \tag{44}$$

Now observe that

$$\frac{\partial L(\alpha, a, a')}{\partial a'} = -\theta(\alpha) [p(\alpha)]^{1+\frac{1}{\sigma}} \tag{45}$$

$$* ([rp(\alpha) - p'(\alpha)] a(\alpha) + p(\alpha) w(\alpha) - p(\alpha) a'(\alpha))^{-\frac{1}{\sigma}}$$

Hence for all $a(\Omega) \geq 0$, the terminal condition (42) is satisfied. We, nevertheless, now show that there exists a unique candidate for optimality. To do so, integrate forward in time condition (2) to obtain

$$a(\alpha) = \int_{\alpha}^{\Omega} \frac{p(z)}{p(\alpha)} \exp(-r(z - \alpha)) [c(z) - w(z)] dz \tag{46}$$

When $\alpha = 0$ use condition (3) to derive the intertemporal budget constraint at birth

$$\int_{0}^{\Omega} p(\alpha) \exp(-r\alpha) c(\alpha) d\alpha = \int_{0}^{\Omega} p(\alpha) w(\alpha) \exp(-r\alpha) d\alpha \tag{47}$$

and replace (44) to obtain the individual initial consumption

$$c(0) = \frac{\int_{0}^{\Omega} p(\alpha) w(\alpha) \exp(-r\alpha) d\alpha}{\int_{0}^{\Omega} p(\alpha) [\theta(\alpha)]^{\sigma} \exp(-(1-\sigma) r\alpha) d\alpha} \tag{48}$$

Consequently, there exists a unique consumption profile that satisfies the necessary condition. Replacing this profile in (46) yields a unique wealth profile. Since for all $z \in (\alpha, \Omega)$ one has $p(z) < p(\alpha)$ observe finally with (46) that $a(\Omega) = 0$.

We conclude this proof showing that the necessary conditions are sufficient. To do so, we have to prove that for all $\alpha \in [0, \Omega]$, $L(\alpha, a, a')$ is concave as a function of (a, a'). The Hessian matrix writes

$$H = \begin{bmatrix} p(\alpha) \theta(\alpha) \left[r - \frac{p'(\alpha)}{p(\alpha)}\right]^2 u''(c(\alpha)) & -p(\alpha) \theta(\alpha) \left[r - \frac{p'(\alpha)}{p(\alpha)}\right] u''(c(\alpha)) \\ -p(\alpha) \theta(\alpha) \left[r - \frac{p'(\alpha)}{p(\alpha)}\right] u''(c(\alpha)) & p(\alpha) \theta(\alpha) u''(c(\alpha)) \end{bmatrix} \tag{49}$$

Then, L is concave and the unique solution of the Euler equation is globally maximal. $\square$

PROOF. [Proof of proposition 2] Observe first that since the optimal wealth profiles satisfies $a(0) = a(\Omega) = 0$, there exists at least one age denoted $\hat{\alpha} \in$

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$(0, \Omega)$ such that $a'(\hat{\alpha}) = 0$. We proceed by showing that under (10) one has $a''(\hat{\alpha}) < 0$. Deriving equation (2) with respect to α, yields

$$a''(\hat{\alpha}) = \pi'(\hat{\alpha}|\hat{\alpha}) a(\hat{\alpha}) + w'(\hat{\alpha}) - c'(\hat{\alpha}) \tag{50}$$

Then, $a''(\hat{\alpha}) < 0$ is equivalent to

$$c'(\hat{\alpha}) > \frac{\pi'(\hat{\alpha}|\hat{\alpha})}{r + \pi(\hat{\alpha}|\hat{\alpha})} c(\hat{\alpha}) - w(\hat{\alpha}) \left[\frac{\pi'(\hat{\alpha}|\hat{\alpha})}{r + \pi(\hat{\alpha}|\hat{\alpha})} - \frac{w'(\hat{\alpha})}{w(\hat{\alpha})} \right] \tag{51}$$

Then if one assumes $\pi'(\alpha|\alpha) / [r + \pi(\alpha|\alpha)] > w'(\alpha)/w(\alpha)$, it is sufficient that $c'(\alpha)/c(\alpha) > \pi'(\alpha|\alpha)/\pi(\alpha|\alpha)$. Use (9) to obtain (10). Consequently, (i) wealth is positive for $\alpha \in (0, \Omega)$ (ii) there is a unique $\hat{\alpha}$. $\square$

Derivation of equation (18) Recall that the optimal wealth accumulation at age α is such that

$$a(\alpha) = \int_\alpha^\Omega \exp\left(-\int_\alpha^z r + \pi(u|u) du\right) [c(z) - w(z)] dz \tag{52}$$

The definition of aggregate wealth per capita is

$$a \equiv \int_0^\Omega \beta p(\alpha) \exp(-n\alpha) a(\alpha) d\alpha \tag{53}$$

where $p(\alpha) = \exp\left(-\int_0^\alpha \pi(u|u) du\right)$. Replacing (52) in a and rearranging yields

$$a = \int_0^\Omega \beta \exp((r-n)\alpha) \left\{ \int_\alpha^\Omega p(z) \exp(-rz) [c(z) - w(z)] dz \right\} d\alpha \tag{54}$$

which, changing the order of integration, turns to be equal to

$$a = \int_0^\Omega \beta p(\alpha) \exp(-r\alpha) [c(\alpha) - w(\alpha)] \left\{ \int_0^\alpha \exp((r-n)z) dz \right\} d\alpha \tag{55}$$

which yields

$$a = \frac{\int_0^\Omega \beta p(\alpha) \exp(-n\alpha) [c(\alpha) - w(\alpha)] d\alpha}{r - n}$$
$$\quad - \frac{\int_0^\Omega \beta p(\alpha) \exp(-r\alpha) [c(\alpha) - w(\alpha)] d\alpha}{r - n} \tag{56}$$

Using (47), and then (16) and (15), equation (18) follows.

Now it is useful for the next proofs to consider the following lemma:

Lemma 8 *Let function $J : \mathbb{R} \to \mathbb{R}^{++}$ be such that*

$$J(x) \equiv \frac{G(x)}{G(n)} \frac{H(n - \sigma x)}{H((1 - \sigma) x)} \tag{57}$$

(i) J is strictly convex, (ii) $\lim\limits_{x \to +\infty} J(x) = \lim\limits_{x \to -\infty} J(x) = +\infty$.

PROOF. As a preliminary, consider the function J first derivative

$$J'(x) = [\sigma h(n - \sigma x) + (1 - \sigma) h((1 - \sigma) x) - g(x)] J(x) \tag{58}$$

where

$$g(x) \equiv \frac{-G'(x)}{G(x)} = \frac{\int_0^\Omega \alpha p(\alpha) \exp(-x\alpha) \, d\alpha}{\int_0^\Omega p(\alpha) \exp(-x\alpha) \, d\alpha} > 0 \tag{59}$$

and

$$h(x) \equiv \frac{-H'(x)}{H(x)} = \frac{\int_0^\Omega \alpha p(\alpha) [\theta(\alpha)]^\sigma \exp(-x\alpha) \, d\alpha}{\int_0^\Omega p(\alpha) [\theta(\alpha)]^\sigma \exp(-x\alpha) \, d\alpha} > 0 \tag{60}$$

The function J second derivative is

$$J''(x) = \left[-\sigma^2 h'(n - \sigma x) + (1 - \sigma)^2 h'((1 - \sigma) x) - g'(x) \right] J(x)$$
$$+ [\sigma h(n - \sigma x) + (1 - \sigma) h((1 - \sigma) x) - g(x)]^2 J(x) \tag{61}$$

(i) To prove that $J''(x) > 0$ we show that $g' < 0$ and $h' < 0$ and that

$(1 - \sigma)^2 h'((1 - \sigma) x) > g'(x)$. Observe that $g'(x)$ has the sign of $[G'(x)]^2 - G''(x) G(x)$. Let functions u and v be such that $u(\alpha, x) = [\exp(-x\alpha) p(\alpha)]^{1/2}$ and $v(\alpha, x) = \alpha u(\alpha, x)$. Then $G'(x) = -\int_0^\Omega u(\alpha, x) v(\alpha, x) \, d\alpha$; using the Cauchy-Schwartz inequality, $g'(x) < 0$. Similarly $h'(x) < 0$.

Now, since $\lim\limits_{\sigma \to 1} (1 - \sigma)^2 h'((1 - \sigma) x) = 0$ and since $(1 - \sigma)^2 h'((1 - \sigma) x)$ is increasing in σ, for all $\sigma \in (0, 1]$, part (i) of the lemma follows.

(ii) Observe that $\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to +\infty} h(x) = 0$ and $\lim\limits_{x \to -\infty} g(x) = \lim\limits_{x \to -\infty} h(x) = \Omega$. Therefore,

$$\lim_{x \to +\infty} \frac{J'(x)}{J(x)} = -\lim_{x \to -\infty} \frac{J'(x)}{J(x)} = \sigma \Omega \tag{62}$$

and part (ii) of the lemma follows. $\square$

PROOF. [Proof of proposition 3] We derive conditions for (i) existence and (ii) uniqueness of a balanced steady state $\hat{k}$ that solves $\hat{k} = \phi(\hat{k})$, where function ϕ, which is given by (19), can be rewritten for convenience as follows

$$\phi(k) \equiv \frac{f(k) - k f'(k)}{f'(k) - (n + \delta)} [J(f'(k) - \delta) - 1] \tag{63}$$

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with J given by (57).

(i) Existence. We proceed in two steps: we first show it exists a unique $\bar{k} > 0$ satisfying $\phi\left(\bar{k}\right) = 0$, and that $\phi\left(k\right)$ is continuous and positive for all $k \in \left(0, \bar{k}\right)$; then, we propose a condition such that $\lim_{k\to 0}\phi\left(k\right) > 0$.

Step 1. To prove the existence of $\bar{k}$, we study the solutions of equation $\phi\left(k\right) = 0$ for all $k > 0$. Given the concavity of f for all $k > 0$, one has $f\left(k\right) - kf'\left(k\right) > 0$. Hence, it is sufficient to look at the solutions of

$$\frac{J\left(f'\left(k\right) - \delta\right) - 1}{f'\left(k\right) - \left(n + \delta\right)} = 0 \tag{64}$$

Let us define the 'golden rule' level of capital per capita, k^{gr} such that

$$f'\left(k^{gr}\right) = n + \delta \tag{65}$$

Observe with (57) that $J\left(f'\left(k^{gr}\right) - \delta\right) = 1$. Now, applying l'Hôpital's rule, using (58) when $x = n$, yields

$$\lim_{k\to k^{gr}}\frac{J\left(f'\left(k\right) - \delta\right) - 1}{f'\left(k\right) - \left(n + \delta\right)} = h\left(\left(1 - \sigma\right)n\right) - g\left(n\right) \tag{66}$$

Then, $\phi\left(k^{gr}\right)$ does not exist, but $\lim_{k\to k^{gr}}\phi\left(k\right) = 0$ only if $h\left(\left(1 - \sigma\right)n\right) = g\left(n\right)$. If $h\left(\left(1 - \sigma\right)n\right) \neq g\left(n\right)$, there is a unique other solution that solves $J\left(f'\left(k\right) - \delta\right) = 1$ because J is U-shaped (see lemma 1). Let us denote $\bar{k}$ this solution, which is greater than k^{gr} if and only if the slope of J in k^{gr} is negative; formally one has

$$\bar{k} \geq k^{gr} \Leftrightarrow h\left(\left(1 - \sigma\right)n\right) - g\left(n\right) \geq 0 \tag{67}$$

Hence, whatever the relative position of $\bar{k}$ and k^{gr}, it can be stated that for all $k \in \left(0, \bar{k}\right)$ then $J \in \left(0, 1\right)$ if $k > k^{gr}$ and $J > 1$ if $k < k^{gr}$. Conversely, for all $k > \bar{k}$ then $J > 1$ if $k > k^{gr}$ and $J \in \left(0, 1\right)$ if $k < k^{gr}$. Therefore, conclude with (64) that $\phi\left(k\right) > 0$ for all $k \in \left(0, \bar{k}\right)$.

Continuity of $\phi\left(k\right)$ for all $k \in \left(0, \bar{k}\right)$ is guaranteed because, using (66), one has

$$\lim_{k\to k^{gr}}\phi\left(k\right) = \left[f\left(k^{gr}\right) - k^{gr}f'\left(k^{gr}\right)\right]\left[h\left(\left(1 - \sigma\right)n\right) - g\left(n\right)\right] \ll \infty \tag{68}$$

Step 2. Since $\lim_{k\to 0}\left[f\left(k\right) - kf'\left(k\right)\right] = 0$ and $\lim_{k\to 0}f'\left(k\right) = +\infty$, observe that

$\lim_{k\to 0}\phi\left(k\right)$ can be 0. But since $\phi\left(k\right) > 0$ for all $k \in \left(0, \bar{k}\right)$, a sufficient condition to have $\lim_{k\to 0}\phi\left(k\right) > 0$ is $\lim_{k\to 0}\phi'\left(k\right) < 0$.

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Function ϕ first derivative is

$$\phi'(k) = -f''(k)\,\phi(k)$$
$$* \left[\frac{k}{f(k) - kf'(k)} - \frac{J'(f'(k) - \delta)}{J(f'(k) - \delta) - 1} + \frac{1}{f'(k) - (\delta + n)} \right] \quad (69)$$

Then $\lim\limits_{k \to 0}\phi'(k) < 0$ if and only if

$$\lim_{k \to 0} \left[\frac{k}{f(k) - kf'(k)} - \frac{J'(f'(k) - \delta)}{J(f'(k) - \delta)} \right] < 0 \quad (70)$$

Using l'Hôpital's rule and (62), yields condition (20).

(ii) Uniqueness. We give sufficient conditions that ensure $\phi'\left(\hat{k}\right) < 1$. We first study the case given by $\bar{k} > k^{gr}$ then the one given by $\bar{k} < k^{gr}$ (see condition (67)).

Case 1: $\bar{k} > k^{gr}$. In this case the equilibrium may be dynamically efficient ($\hat{k} < k^{gr}$ or equivalently $J > 1$) or inefficient ($\hat{k} > k^{gr}$ or $J \in (0,1)$). Computing (69) for $k = \hat{k}$ yields

$$\phi'\left(\hat{k}\right) = -\hat{k}f''\left(\hat{k}\right) \frac{J\left(f'\left(\hat{k}\right) - \delta\right)}{J\left(f'\left(\hat{k}\right) - \delta\right) - 1}$$
$$* \left[\frac{\hat{k}}{f\left(\hat{k}\right) - \hat{k}f'\left(\hat{k}\right)} - \frac{J'\left(f'\left(\hat{k}\right) - \delta\right)}{J\left(f'\left(\hat{k}\right) - \delta\right)} \right] \quad (71)$$

Therefore $\phi'\left(\hat{k}\right) < 0$ if and only if

$$\frac{J'\left(f'\left(\hat{k}\right) - \delta\right)}{J\left(f'\left(\hat{k}\right) - \delta\right)} \geq \frac{\hat{k}}{f\left(\hat{k}\right) - \hat{k}f'\left(\hat{k}\right)} \Leftrightarrow \hat{k} \leq k^{gr} \quad (72)$$

Recall (lemma 1) that J'/J is increasing in $f'\left(\hat{k}\right) - \delta$ and therefore decreasing in $\hat{k}$. Moreover, observe, using (68), that condition (72) is always verified for $\hat{k} = k^{gr}$. Then, conclude that a sufficient condition for $\phi'\left(\hat{k}\right) < 0$ is

$$\frac{d}{dk}\left(\frac{k}{f(k) - kf'(k)} \right) \geq 0 \Leftrightarrow \varepsilon(k) \geq s(k) \quad (73)$$

since condition (20) is assumed.

Case 2: $\bar{k} < k^{gr}$. In this case the equilibrium can only be dynamically efficient ($\hat{k} < k^{gr}$ or $J > 1$) and J'/J is necessarily positive. It is easy to see that condi-

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tion (73) is not sufficient in this case because J'/J and $k/\left[f\left(k\right) - kf'\left(k\right)\right]$ may cross each other before $\bar{k}$, opening then the possibility for multiple equilibria. Hence, use (71) to state that $\phi'\left(\hat{k}\right) < 1$ if and only if

$$\frac{J'\left(f'\left(\hat{k}\right) - \delta\right)}{J\left(f'\left(\hat{k}\right) - \delta\right)} > \psi\left(\hat{k}\right) \ \text{ for } \hat{k} < k^{gr} \tag{74}$$

with

$$\psi\left(\hat{k}\right) \equiv \frac{\hat{k}}{f\left(\hat{k}\right) - \hat{k}f'\left(\hat{k}\right)} - \left[1 - \frac{1}{J\left(f'\left(\hat{k}\right) - \delta\right)}\right]\frac{1}{-\hat{k}f''\left(\hat{k}\right)} \tag{75}$$

A sufficient condition for uniqueness is then $\psi'\left(\hat{k}\right) > 0$ because (using l'Hôpital's rule)

$$\lim_{\hat{k}\to 0}\psi\left(\hat{k}\right) = \lim_{k\to 0}\left[\frac{k}{f\left(k\right) - kf'\left(k\right)} - \frac{1}{-kf''\left(k\right)}\right] = 0 \tag{76}$$

This implies that if there exists a $\hat{k}_i$ such that $\phi'\left(\hat{k}_i\right) > 1$, then all $\hat{k}_j$, such that $\hat{k}_j > \hat{k}_i$, will verify $\phi'\left(\hat{k}_j\right) > 1$. Since there exists a finite $\bar{k}$ and since $\phi\left(k\right)$ is continuous over $\left(0, \bar{k}\right)$, this is impossible. To obtain a nice condition implying $\psi'\left(\hat{k}\right) > 0$, rewrite (75) such that

$$\psi\left(k\right) \equiv \frac{k}{f\left(k\right) - kf'\left(k\right)}\left\{1 - \left[1 - \frac{1}{J\left(f'\left(k\right) - \delta\right)}\right]\frac{\varepsilon\left(k\right)}{s\left(k\right)}\right\} \tag{77}$$

where ε and s are defined in the core of the text. Then it is sufficient for $\psi\left(k\right)$ to increase in k that (73) be satisfied and that

$$\frac{d}{dk}\left(\frac{s\left(k\right)}{\varepsilon\left(k\right)}\right) \geq 0 \tag{78}$$

Hence, (73) and (78) are sufficient for uniqueness. $\square$

The following figures give a graphical intuition of this proof: if $\phi\left(k\right)$ is continuous and positive for all $k \in \left(0, \bar{k}\right)$, it is sufficient that $\lim_{k\to 0}\phi\left(k\right) > 0$ to have at least one equilibrium. Moreover, if there are multiple equilibria, the one with the greater $\hat{k}$ must verify $\phi'\left(\hat{k}\right) < 1$. Examples of unique equilibrium

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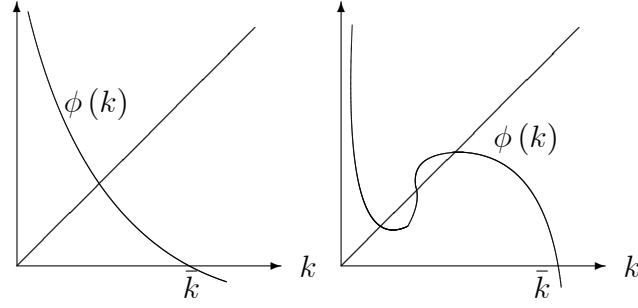and multiple equilibria are respectively presented in Figure (2-a) and (2-b).

Figure 2-a   Figure 2-b

Derivation of figure 1

The function $c_1 : [0, \infty) \to \mathbb{R}$ with $c_1(0) = f(0)$. If $n + \delta > 0$, c_1 is first increasing until it reaches $c_1(k^{gr})$ and then strictly decreasing to $-\infty$. If $n + \delta \leq 0$, c_1 is strictly increasing to $+\infty$. Consider now function $c_2 : [0, \infty) \to [0, \infty)$. Its first derivative with respect of k is given by

$$\frac{dc_2(k)}{dk} = \frac{f'(k) - (n + \delta)}{1 - 1/J(f'(k) - \delta)} + \frac{k f''(k)}{1 - 1/J(f'(k) - \delta)} - f''(k) \frac{J'(f'(k) - \delta)}{[J(f'(k) - \delta)]^2} \frac{k[f'(k) - (n + \delta)]}{[1 - 1/J(f'(k) - \delta)]^2} \tag{79}$$

Condition (20) ensures $c_2(0) = 0$. Then, let, $\hat{k}$ be such that $c_1(\hat{k}) = c_2(\hat{k})$; simple manipulations of (79) show that

$$\frac{dc_1(\hat{k})}{dk} - \frac{dc_2(\hat{k})}{dk} < 0 \tag{80}$$

is equivalent to $\phi'(\hat{k}) < 1$. Hence, if (21) is verified, $c_1(k)$ and $c_2(k)$ cross each other only once and figure 1 follows.

PROOF. [Proof of proposition 4] We have defined $\hat{k}$ such that $\hat{k} = \phi(\hat{k}; n)$. This proof details the simple computing that leads to (28). Application of implicit functions theorem yields

$$\frac{d\hat{k}}{dn} = \frac{\phi'_n(\hat{k}; n)}{1 - \phi'_k(\hat{k}; n)} \tag{81}$$

Deriving equation (19) with respect to n yields

$$\phi'_n(k;n) = \frac{1}{f'(k) - (n+\delta)} \phi(k;n) \tag{82}$$
$$+ \frac{f(k) - kf'(k)}{f'(k) - (n+\delta)} \left[g(n) - h(n - \sigma(f'(k) - \delta)) \right] J(f'(k) - \delta)$$

where J, g and h are defined in lemma 1. At the equilibrium such that $\hat{k} = \phi(\hat{k};n)$ one may, using equations (17) and (19), rewrite (82) such that

$$\frac{\phi'_n(\hat{k};n)}{\hat{k}} = \frac{J(f'(\hat{k}) - \delta)}{J(f'(\hat{k}) - \delta) - 1} \left[g(n) - h\left(n - \sigma\left(f'(\hat{k}) - \delta\right)\right) + \frac{\hat{k}}{\hat{c}} \right] \tag{83}$$

where $\hat{c}$ stands for the aggregate consumption.

Now turn to the definition of the average age of capital holders using (26). Integrating by parts the numerator, using $a(0) = p(\Omega) = 0$, yields

$$\bar{\alpha}_a = \frac{1}{n} - \frac{\int_0^\Omega \alpha \pi(\alpha | \alpha) p(\alpha) \exp(-n\alpha) a(\alpha) d\alpha}{n \int_0^\Omega p(\alpha) \exp(-n\alpha) a(\alpha) d\alpha}$$
$$+ \frac{\int_0^\Omega \alpha p(\alpha) \exp(-n\alpha) a'(\alpha) d\alpha}{n \int_0^\Omega p(\alpha) \exp(-n\alpha) a(\alpha) d\alpha} \tag{84}$$

Replace condition (2) and rearrange using (59), (60) and (7) to obtain

$$\bar{\alpha}_a = \frac{1}{n} - \frac{1}{n} \left(r\bar{\alpha}_a + g(n) \frac{w}{a} - h(n - \sigma r) \frac{c}{a} \right) \tag{85}$$

Then, since $a = k$ and using factor prices equations, it yields

$$h(n - \sigma r) = (f'(k) - (n+\delta))\bar{\alpha}_a \frac{k}{c} + g(n) \frac{f(k) - kf'(k)}{c} + \frac{k}{c} \tag{86}$$

Replace (86) in (83) and arrange, using (57) and recalling that $g(n) = \bar{\alpha}_w$, to obtain

$$\phi'_n(\hat{k};n) = -\hat{k}[\bar{\alpha}_a - \bar{\alpha}_w] \tag{87}$$

replace in (81); this yields (28). $\square$

PROOF. [Proof of proposition 5] We demonstrate it exists a n^* that solve

$$\phi'_n(\hat{k};n^*) = 0 \tag{88}$$

with ϕ'_n defined by (83). Let μ be such that $f'\left(\bar{k}\right) = \delta + \mu$. Since there is no restrictions on n at the equilibrium, it is sufficient for the existence of n^* to show that: (i) ϕ'_n is continuous in n; (ii) $\phi'_n\left(\hat{k}; n\right) < 0$ for all $n > \mu$; (iii)

$$\lim_{n \to -\infty} \phi'_n\left(\hat{k}; n\right) > 0 \tag{89}$$

(i) To show the continuity of ϕ'_n in n, we simply have to prove that $\phi'_n\left(k^{gr}; n\right) \ll$

∞ when $k^{gr} < \bar{k}$. Indeed, observe with (65) and (68) that both numerator and denominator of (83) equal zero, for $\hat{k} = k^{gr}$. Applying l'Hôpital's rule yields

$$\lim_{\hat{k} \to k^{gr}} \phi'_n\left(\hat{k}; n\right) = \frac{k^{gr}}{h\left((1-\sigma)n\right) - g\left(n\right)} *$$

$$\left\{ \begin{array}{c} \sigma h'\left((1-\sigma)n\right) - \left[g\left(n\right) - h\left((1-\sigma)n\right)\right]^2 \\ + \frac{1}{f''(k^{gr})} \frac{d}{dk}\left(\frac{k^{gr}}{f(k^{gr}) - k^{gr} f'(k^{gr})}\right) \end{array} \right\} \tag{90}$$

which, using (67) and (73) and recalling that $h' < 0$, is negative and finite.

(ii) Inequality $n > \mu$ is equivalent to $k^{gr} < \bar{k}$. Hence note with (83) that since

$$\hat{c} = f\left(\hat{k}\right) - (n + \delta)\hat{k} \tag{91}$$

we have to prove that

$$h\left(n - \sigma\left(f'\left(\hat{k}\right) - \delta\right)\right) - g\left(n\right) \leq \frac{\hat{k}}{f\left(\hat{k}\right) - (n + \delta)\hat{k}} \Leftrightarrow \hat{k} \geq k^{gr} \tag{92}$$

Recall with (68), that the equality between the two terms of (92) is guaranteed when $\hat{k} = k^{gr}$. Then, observing that the left hand side is decreasing in $\hat{k}$ (because $h' < 0$) and that the right hand side is increasing in $\hat{k}$, is sufficient to prove that (92) always holds.

(iii) Recall first (lemma 1) that

$$\lim_{n \to -\infty} g\left(n\right) = \lim_{n \to -\infty} h\left(n - \sigma\left(f'\left(\hat{k}\right) - \delta\right)\right) = \Omega \tag{93}$$

Hence, with (83)

$$\lim_{n \to -\infty} \phi'_n\left(\hat{k}; n\right) = \lim_{n \to -\infty} \frac{\hat{k}}{f'\left(\hat{k}\right) - (n + \delta)} \tag{94}$$

which is necessarily positive. $\square$

PROOF. [Proof of proposition 6] This is proven by the point (ii) of the proof of proposition 5. □

PROOF. [Proof of proposition 7] As a preliminary, let us denote $\phi'_n\left(\hat{k}; n, \gamma\right)$ the first derivative of (34) with respect to n; the same kind of computations as those done in the proof of proposition 4 yield that

$$\frac{\phi'_n\left(\hat{k}; n, \gamma\right)}{\hat{k}} = \frac{J\left(f'\left(\hat{k}\right) - \delta\right)}{J\left(f'\left(\hat{k}\right) - \delta\right) - 1}\left[g\left(n - \gamma\right) - h\left(n - \sigma\left(f'\left(\hat{k}\right) - \delta\right)\right) + \frac{\hat{k}}{\hat{c}}\right] \tag{95}$$

where $\hat{c} = f\left(\hat{k}\right) - (n + \delta)\hat{k}$. Hence, we define n^* such that

$$\phi'_n\left(\hat{k}; n^*, \gamma\right) = 0 \tag{96}$$

To show that $dn^*/d\gamma > 0$, we demonstrate that

$$\frac{\partial}{\partial n}\phi'_n\left(\hat{k}; n^*, \gamma\right) < 0 \tag{97}$$

and that

$$\frac{\partial}{\partial \gamma}\phi'_n\left(\hat{k}; n^*, \gamma\right) > 0 \tag{98}$$

Condition (97) is trivial since ϕ'_n is continuous in n and since it is assumed there exists a unique n^*. If there are multiple solutions to (96), one should consider the one that maximizes $\hat{k}$ and the result would be the same.

To prove condition (98), derive (95) when $n = n^*$ (i.e. when $\phi'_n\left(\hat{k}; n^*, \gamma\right) = 0$) to obtain

$$\frac{\partial}{\partial \gamma}\phi'_n\left(\hat{k}; n^*, \gamma\right) = -\hat{k}\frac{J\left(f'\left(\hat{k}\right) - \delta\right)}{J\left(f'\left(\hat{k}\right) - \delta\right) - 1}g'\left(n^* - \gamma\right) \tag{99}$$

which is positive because $g' < 0$ (lemma 1) and $J > 1$ for all $n < \mu$. This establishes the proof. □

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