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Bootstrapping long memory tests: some Monte Carlo results

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Bootstrapping Long Memory Tests: Some Monte Carlo Results

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Abstract: We investigate the bootstrapped size and power properties of five common long memory tests - the modified R/S, KPSS, V/S, GPH and Robinson's $\widehat{H}$ tests. Even in samples of size 100, the moving block bootstrap controls the empirical size of the tests in the DGPs examined. The $\widehat{H}$ test appears to be the most powerful. Moreover, the bootstrapped tests suffer little loss of power against fractionally integrated processes vis á vis asymptotic tests with samples of 250 or more observations. This is true both for distributions with heavy tails and with stochastic volatility (SV).

Keywords: Moving block bootstrap, long memory, fractional integration.

JEL Code: C10, C12

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1. Introduction

Long memory processes, especially fractionally integrated processes, describe many financial time series as well as some macroeconomic series rather well. It is important to distinguish long memory processes from more common I(0) and I(1) processes as they imply different long run predictions and responses to shocks (Baillie, 1996). A range of tests for long memory are available. Unfortunately, the evidence is that tests based on asymptotic critical values are often badly sized.

In this paper we report the results of a series of Monte Carlo experiments used to examine the size and power properties of five, commonly used, long memory tests using asymptotic and bootstrapped critical values. The five tests are Lo's modified rescaled range or R/S statistic (Lo, 1991), the KPSS statistic (Kwiatkowski *et al*., 1992), the rescaled variance or V/S statistic (Giraitis *et al*., 2003), the GPH statistic (Geweke and Porter-Hudak, 1983) and the $\widehat{H}$ statistic in Robinson (1995) and Robinson and Henry (1999). The set of tests considered is broader than in other papers.

We use the moving block bootstrap (MBB) to mimic the dependence in the data. All the test statistics are asymptotically pivotal. This means that, for dependent stationary data satisfying reasonable regularity conditions, bootstrapped critical values should provide a higher order of accuracy than asymptotic critical values. We found this when we used the post-blackened MBB to examine the size and power of the modified R/S statistic (Izzeldin and Murphy, 2000).

For the data generation processes we consider, we find that we can control the size of all five tests using the moving block bootstrap even in small samples with as few as 100 observations. We also find that bootstrapped tests suffer little loss of power against fractionally integrated (FI) processes vis á vis asymptotic tests with samples of 250 or more observations. This is true both for distributions with heavy tails (log-normal random errors) and with stochastic volatility (SV). We also show that all of the tests lack power against a particular type of fractionally integrated process, the sum of a FI and a SV process as opposed to a FI process with a SV error.

The outline of this paper is as follows. We discuss the five tests of long memory in next section. We briefly review the relevant empirical literature on the size and power of these tests, as well as bootstrapped long memory tests, in Section 3. We discuss the moving block bootstrap in Section 4 and discuss the Monte Carlo experiments and our findings in Sections 5 and 6. We present a financial application in Section 7 and conclude in Section 8.

2. Tests of Long Memory

We consider five tests of long memory – the modified rescaled range or R/S statistic, the KPSS statistic, the rescaled variance or V/S statistic, the GPH statistic and the $\widehat{H}$ statistic. The modified R/S, KPSS and V/S statistics for a time series $\{x_t\}$ may be expressed in term of the partial sum of the standardized series $S_T(t)=\sum_{s=1}^{t}(x_s-\bar{x})/\left(\sqrt{T\hat{\sigma}_\infty}\right)$, where $\bar{x}=\frac{1}{T}\sum_{t=1}^{T}x_t$ is the sample mean, $\hat{\sigma}^2_\infty$ is an estimate of the long run variance of $\{x_t\}$ and T is the sample size. Then:

$$T^{-1/2}R/S=\max_{0\le t\le T}S(t)-\min_{0\le t\le T}S(t) \tag{1}$$

$$KPSS=\frac{1}{T}\sum_{t=1}^{T}S_T(t)^2 \tag{2}$$

$$V/S=\frac{1}{T}\sum_{t=1}^{T}(S_T(t)-\bar{S}_T)^2 \tag{3}$$

When $\{x_t\}$ is stationary and under suitable regularity conditions:

$$T^{-1/2}R/S\Rightarrow\max_{0\le r\le 1}W_1(r)-\min_{0\le r\le 1}W_1(r) \tag{4}$$

$$KPSS \Rightarrow \int_0^1 W_1(r)^2 dr \tag{5}$$

$$V / S \Rightarrow \int_0^1 W_1(r)^2 dr - \left(\int_0^1 W_1(r)^2 dr\right)^2 \tag{6}$$

where $\Rightarrow$ denotes convergence in distribution, $W_1(r) = W(r) - rW(1)$ is a standard first order Brownian bridge process and $W(r)$ is a standard Brownian motion process.

Giraitis *et al*. (2003), *inter alia*, derive the asymptotic distribution of the R/S, KPSS and V/S statistics under short and long memory assumptions. All three tests are consistent against fractionally integrated alternatives. In addition, all three tests are asymptotically pivotal, so appropriate bootstrap critical values should outperform asymptotic critical values in smaller samples.

Geweke and Porter-Hudak (1983) show how to consistently estimate the fractional integration parameter d in an ARFIMA model using a semi-nonparametric, frequency domain procedure and derived its asymptotic distribution. For frequencies near zero, d can be estimated from the least squares regression:

$$\ln(I(w_j)) = c - d \ln\{(4\sin^2(w_j / 2)\} + \eta_j, \quad j = 1,...n \tag{7}$$

where $I(w_j)$ is the periodogram of the $\{x_t\}$ series at the n frequencies $w_j = 2\pi j / T$. Often the setting $n = [\sqrt{T}]$ is chosen, where [ ] denotes the integer part. With a proper choice of *n*, the asymptotic distribution of *d* does not depend on either the order of the ARMA process or on the distribution of the error term in the ARFIMA process $\{x_t\}$. Asymptotically d is normally distributed with variance $\pi^2 / 6$.

Robinson (1995) derives a semi-parametric, frequency domain estimator of the fractional integration parameter d which is closely related to the trimmed Whittle estimator in Kunsch (1987). He refers to it as a Gaussian or local Whittle estimator. The estimator is shown to be consistent and asymptotically normal under relatively weak conditions. Moreover, the asymptotic variance of this estimator is free of unknown parameters. Robinson also shows that it dominates the Geweke and Porter-Hudak (1983) estimator. Robinson and Henry (1999) show that, under weak conditions, these results continue to hold under common forms of conditional heteroscedasticity of both the long and short memory kind.

For the sort of long memory processes usually estimated using financial time series data, the $\widehat{S}_k$ test of Harris, McCabe and Leybourne (2006) appears to have fairly similar size and power properties to the $\widehat{H}$ test in Robinson (1995), so we have not examined its performance here.

3. A Review of Previous Monte Carlo Studies

In this section we briefly review some of the more recent Monte Carlo results in the literature on testing long memory. Lee and Schmidt (1996) show that the power of the KPSS test against basic fractionally integrated (FI) alternatives in sample sizes ranging from 50 to 500 is comparable to that of the modified R/S test. However, they argue that rather larger sample sizes, such as T = 500 or T = 1000, are required to distinguish reliably between a long memory process and a short memory process with comparable short-term autocorrelation. Their results show that both tests are sensitive to the choice of lag truncation i.e. the number of covariance terms used to calculate the long run variance $\hat{\sigma}_\infty^2$.

Hauser (1997) investigates the size and power properties of the GPH test, the modified R/S test, a semi-parametric frequency domain test due to Robinson (1994) and a test based on the trimmed Whittle likelihood (Kunsch, 1987), *inter alia.* He examines IID, AR(1), MA(1), FI, ARFIMA, GARCH and IGARCH data generation processes (DGPs) but only consider one sample size, namely T = 1000. No single test performs satisfactorily for all of the models considered. He suggests that the R/S statistic is generally robust with the disadvantage of relatively small power. The trimmed Whittle likelihood has high power in general and is robust except for large short run effects.

Teverovsky *et. al*. (1999) also show that the value of Lo's (1991) modified R/S statistic is sensitive to the choice of the truncation lag used to estimate $\hat{\sigma}_{\infty}^{2}$. As the truncation lag increases, the test statistic has a strong bias towards accepting the null of no long run dependence, even when the DGP is a basic FI process.

Giraitis *et. al*. (2003) examined the size and power of the modified R/S, KPSS and V/S statistics using sample sizes of 500 and 1000 using AR(1), FI and long and short memory linear ARCH (Robinson, 1991) DGPs. They find that the V/S statistic achieves a somewhat better balance of size and power than the R/S and KPSS test. They also highlight the sensitivity of the test to the choice of the truncation lag when estimating $\hat{\sigma}_{\infty}^{2}$.

Robinson and Henry (1999) report an extensive range of Monte Carlo results. They consider IID, ARCH, FI, nearly integrated GARCH, EGARCH and long memory linear ARCH models and three sample sizes (T = 64, 128 and 256). Their estimator $\widehat{H} = \hat{d} - 1/2$ appears to perform reasonably well except in the nearly integrated GARCH case.

We now consider Monte Carlo studies using bootstrap methods. Hiemstra and Jones (1997) use the original non-parametric bootstrap of Efron (1979), designed for IID observations, to test for long memory in stock returns using the modified R/S statistic. Anderson and Gredenhoff (1998) use the AR-sieve bootstrap in a Monte Carlo experiment looking at the size and power of the modified R/S and GPH tests, as well as a LM test due to Agiaklogou and Newbold (1993), in detecting fractional integration using sample sizes of 750 and 1000 observations. They use four bootstrap re-sampling procedures. Their basic sieve or residual based bootstrap involves re - sampling (with replacement) the residuals from an estimated AR model, the maximal order of which is selected using the Bayesian information criterion of Schwartz (1978). They extend this procedure to incorporate ARCH(1) dependence in the residuals. They find that the sieve bootstrap works well in controlling the size of the tests.

Izzeldin and Murphy (2000) use the post-blackened moving block bootstrap to examine the size and power of the modified R/S statistic. They consider IID, AR(1), MA(1), ARCH(1), GARCH(1,1), MA(1) plus GARCH(1,1) and fractionally integrated data generation processes with both normal and log-normal random errors. The post-blackened MBB works well. Compared to the asymptotic critical values in Lo (1991), the MBB controls the empirical size of the test well without reducing the power against FI alternatives much.

De Peretti (2003) examines the size and power of the R/S, modified R/S, GPH and two other test statistics using an AR model to pre-whiten the data and various parametric and non-parametric bootstrap procedures. He does not use the MBB. He presents his results using a variety of P value plots and size-power curves using AR(p) and FI DGPs. He suggests that the proposed bootstrap procedure controls the empirical size of the various tests reasonably well without any loss of power.

Finally, Grau-Carles (2005) follows Izzeldin and Murphy (2000) and uses the post-blackened moving block bootstrap to examine the size and power of the R/S, modified R/S, Robinson's $\widehat{H}$ and one other test of long memory. He looks at relatively small samples (T = 100 and 300) and considers a range of DGPs - IID uniform, normal and log-normal; AR(1) and MA(1); ARCH(1) and AR(1) plus ARCH(1) as well as FI and ARFIMA(1,1,0). He finds that the size of the post blackening MBB is generally good although the tests are not very powerful. However, this may because he used a small block length of 5 for the MBB. In our Monte Carlo experiments, the modified R/S test statistic, and to some extent the $\widehat{H}$ statistic, is a good deal more powerful than in Grau-Carles (2005).

4. The Moving Block Bootstrap

The two most common bootstrap procedures for time series are the moving block bootstrap (MBB) and the AR-sieve bootstrap for stationary linear time series (Buhlmann, 2002). Both procedures are easy to implement, at least in principle. However the MBB bootstrap is the more general procedure so we use it in our Monte Carlo experiments. In the most common version of the MBB, introduced by Kunsch (1989) and Liu and Singh (1992), the bootstrap sample is obtained by resampling fixed size blocks of observations rather than the individual observations themselves. The blocks may overlap. We experiment with the post blackening bootstrap suggested by Davison and Hinkley (1997), which combines the MBB and AR-sieve methods, and obtained no better results than the ones reported below.

Of course, there are some practical and other problems with the MBB (Maddala and Kim, 1998, p. 329-330). For example, the pseudo-time series generated by the moving block method is not stationary even if the original series $\{x_t\}$ is stationary. The choice of block length may be problematic, so the cross-validation and plug-in procedures in Hall, Horowitz and Jing (1995) and Lahirir, Furukawa and Lee (2007), as well as the frequency domain bootstrapping procedures in Hidalgo (2003), may be worth investigating. However, in practice, we did not find this to be the case. In addition, there are few theoretical results on bootstrapping long memory data.

5. The Monte Carlo Experiments

We consider a range of data generation processes (DGPs) in our Monte Carlo experiments. Here we present representative results for five DGPs:- (i) the IID case; (ii) the first order autoregressive AR(1) case; (iii) the AR(1) with stochastic volatility (SV) case; (iv) the fractionally integrated (FI) case and (v) the fractionally integrated (FI) with stochastic volatility (SV) case. These five cases seem relevant when considering financial data.

In the AR(1) case, we set ρ = 0.5 which is definitely on the high side for financial data. However if the MBB bootstrap works well with ρ = 0.5, it will also work well when the level of autocorrelation is lower. Conditional heteroscedasticity is common in financial data, so we consider a range of GARCH and SV DGPs. The two DGPs generated similar results so we only present the SV results here.

The DGP in (iii) is $x_t = (1-0.5L)^{-1}u_t$ where $u_t = \exp(h_t/2)\varepsilon_t$ with $h_t = 0.95h_{t-1} + \eta_t$. The 0.95 coefficient on h_{t-1} means that the SV conditional heteroscedasticity is slow to decay. The random errors ε_t and η_t are mean zero, independent normal random variables with variances equal to $1/10$. For the fractionally integrated DGPs, we set the FI parameter d equal to $1/3$, a reasonable value given the range of results in many empirical papers. In the case of (i), (ii) and (iv), we look at normal and log normal random errors. We also consider to variants of cases (iii) and (v) involving the sum of an AR(1) or FI process and a SV process

Many of the Monte Carlo results summarized in the previous section are based on either rather large or quite small sample sizes. We use four sample sizes - T = 100, 250, 500 and 1000 - which covers a reasonable range. In practice, sample sizes of 250 or more observations are the norm in most economic applications. Much larger sample sizes are common in financial applications.

The Monte Carlo results are based on 1000 replications. A 100 observation "burn in" period is used. The bootstrap results are based on 999 bootstrap replications using the moving block bootstrap with a block length of 10. In general, the results are not sensitive to the choice of block length, as long as it is not too short.

The long-run variance $\hat{\sigma}_{\infty}^{2}$ in the R/S, KPSS and V/S statistics is calculated using $\left[8\sqrt[4]{T/100}\right]$ estimated covariance terms - the midpoint of the two settings considered by Lee and Schmidt (1999). We use the standard Newey and West (1987) estimator of $\hat{\sigma}_{\infty}^{2}$. When calculating the GPH and $\widehat{H}$ test statistics, we use $\left[\sqrt{T}\right]$ frequency domain terms. All the calculations are carried out in Ox (Doornik, 1999).

6. The Monte Carlo Results

The Monte Carlo results in Table 1 for the IID case show that, in line with other results in the literature, the MBB is reasonably successful in controlling the size of all five tests, especially in small samples (T = 100 or 250). This is true for both the normal and heavy-tailed, lognormal error cases. The empirical and nominal sizes of the asymptotic tests can differ quite a lot, especially for the modified R/S and $\widehat{H}$ test in small samples. Similar results are obtained in Table 2 using the AR(1) DGP.

We report the results for the AR(1) model with a stochastic volatility random error term in Table 3. The SV random error with $h_t = 0.95h_{t-1} + \eta_t$ adds a slowly decaying conditional heteroscedastic error, similar to a GARCH (1,1) error, to the AR(1) model. The sizes of the asymptotic tests can be poor, whereas the nominal and empirical sizes of the bootstrapped tests are reasonably close, even when T = 100. The results are in line with the ARCH and GARCH results in Izzeldin and Murphy (2000) and Grau-Carles (2005).

We report the power of the tests against the fractional integrated FI(d) alternative, with d = 1/3, in Table 4. The power of the tests is higher when the random error is log-normal than when it is normal. Unsurprisingly, the asymptotic tests are generally more powerful than the bootstrapped tests, since we are reporting power as opposed to size adjusted power. However, for moderate samples sizes (T ≥ 250), the difference in power is generally small, the exception being the $\widehat{H}$ test when T = 250. When T ≥ 250, the power ranking of the bootstrapped tests appears to be $\widehat{H}$, GPH, V/S followed jointly by the KPSS and the modified R/S tests. In smaller samples, the power of all of the tests, apart from the asymptotic $\widehat{H}$ test, is low and the $\widehat{H}$ test is not the most powerful one. Similar results are obtained for other values of d in the range 0.1 to 0.4.

The power of the five tests against the FI alternative with a stochastic volatility error term is set out in Table 5. The introduction of the SV error term only results in a small reduction in power. The asymptotic tests are somewhat more powerful when T = 250. The power ranking of the tests is much the same as in Table 4. The $\widehat{H}$ test is the most powerful, followed by either the GPH or V/S test.

Finally we present some Monte Carlo results in Tables 6 for DGPs obtained by summing an AR(1) or FI(d) process and a stochastic volatility process. Unfortunately, in the FI-SV composite error case, none of the bootstrapped or asymptotic tests has much power. In most cases, there is little difference in power between the bootstrapped and asymptotic tests. The low power of the tests continues to hold when, for example, $h_t = 0.5h_{t-1} + \eta_t$ is used to generate the SV component of the DGP.

7. Financial Application

We apply the R/S, KPSS, V/S, GPH and $\widehat{H}$ long memory tests to daily Standard and Poor's (SP500) returns, absolute returns and squared returns. We use the data in Tsay (2005). We select the seven year sample period from January 1993 to December 1999, a total of 1768 trading days. We also use a smaller, two year sample from January 1997 to December 1998, a total of 505 days.

The long memory test results are shown in Table 7. Statistically significant outcomes are shown in bold. In line with the literature, using the larger sample, we cannot reject the null hypothesis of short memory in daily returns and we generally reject the null hypothesis of short memory in absolute and squared daily returns. However, in the smaller sample, we cannot always reject the null hypothesis of short memory in the squared daily returns. These results are consistent with our Monte Carlo results regarding the power of the tests.

In this example, the asymptotic and bootstrapped tests produce similar results. However, even in the case of actual returns, the bootstrapped and asymptotic critical values (and any corresponding P values) can differ quite a lot so it is worthwhile bootstrapping the test statistics.

8. Summary and Conclusion

We use Monte Carlo methods to examine the size and power properties of five widely used long memory tests – the modified R/S statistic, the KPSS statistic, the rescaled variance or V/S statistic, the GPH statistic and Robinson's $\widehat{H}$ statistic. The set of tests considered is broader than in other papers. We use the moving block bootstrap to mimic the dependence in the data. All the test statistics are asymptotically pivotal.

For all of the data generation processes we consider, we find that we can control the size of all five tests using the moving block bootstrap even in small samples with as few as 100 observations. We also find that bootstrapped tests suffer little loss of power against fractionally integrated (FI) processes vis á vis asymptotic tests with samples of 250 or more observations. This is true both for distributions with heavy tails (log-normal random errors) and with stochastic volatility (SV). We also show that all of the tests lack power against a particular type of fractionally integrated process, the sum of a FI and a SV process as opposed to a FI process with a SV error.

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Table 1: Size of Long Memory Tests for IID Models

Sample Size	Test Statistic	Critical Values	Normal Random Error						Demeaned Log-Normal Error					
			20%	**15%**	**10%**	**5%**	**2½%**	**1%**	**20%**	**15%**	**10%**	**5%**	**2½%**	**1%**
T = 100	R/S	Bootstrapped	20.1	13.9	8.6	3.3	1.4	0.4	19.2	13.4	6.4	2.9	0.9	0.2
		Asymptotic	7.1	2.8	0.4	0.1	0.0	0.0	4.3	1.5	0.2	0.0	0.0	0.0
	KPSS	Bootstrapped	21.2	15.8	10.4	5.0	3.3	1.2	22.1	16.3	10.4	5.1	2.6	1.1
		Asymptotic	21.7	15.3	9.4	3.8	1.0	0.0	23.4	16.7	10.0	2.8	1.10	0.0
	V/S	Bootstrapped	20.5	14.7	8.5	3.4	1.4	0.5	20.3	14.2	8.7	4.9	2.0	0.7
		Asymptotic	19.5	12.0	3.8	0.5	0.0	0.0	20.9	12.0	4.6	0.5	0.0	0.0
	GPH	Bootstrapped	16.3	10.6	5.9	3.2	1.4	0.4	18.0	12.1	7.0	2.9	1.0	0.4
		Asymptotic	6.7	5.5	3.7	1.5	0.7	0.2	8.9	6.8	2.9	1.3	0.3	0.3
	$\widehat{H}$	Bootstrapped	13.5	8.5	5.5	2.6	1.0	0.1	15.6	10.1	5.3	2.1	0.6	0.1
		Asymptotic	16.3	13.1	10.4	6.9	5.0	3.2	16.9	14.8	10.9	7.1	5.0	2.4
T = 250	R/S	Bootstrapped	21.1	15.0	10.3	5.4	3.4	1.2	19.2	13.5	8.5	4.3	2.2	0.5
		Asymptotic	12.8	8.5	5.0	1.4	0.4	0.0	9.4	5.9	2.6	0.2	0.1	0.0
	KPSS	Bootstrapped	20.5	14.8	10.2	3.8	1.7	0.8	21.5	15.8	10.3	5.3	2.3	0.9
		Asymptotic	20.6	15.5	9.6	3.5	1.2	0.4	21.4	16.0	10.3	4.2	1.4	0.3
	V/S	Bootstrapped	19.5	13.9	9.5	5.3	2.6	1.0	19.9	14.7	9.4	4.7	2.8	1.5
		Asymptotic	18.7	13.1	8.0	3.5	1.1	0.4	20.1	14.0	7.9	3.5	1.2	0.2
	GPH	Bootstrapped	19.9	14.6	10.5	5.1	2.4	1.1	18.8	13.7	9.2	4.8	2.2	0.9
		Asymptotic	10.3	7.9	5.1	2.3	0.9	0.3	8.8	6.9	4.3	2.4	1.2	0.4
	$\widehat{H}$	Bootstrapped	19.8	14.6	9.5	4.4	2.0	1.0	18.5	13.2	8.7	4.1	1.5	0.5
		Asymptotic	15.8	13.0	10.7	7.1	4.8	2.9	13.0	11.1	9.0	6.5	4.0	1.8

Notes: The DGP is $x_t = \varepsilon_t$ with $\varepsilon_t \sim n.i.d(0,1)$ or, before demeaning, $\ln \varepsilon_t \sim n.i.d(0,1)$. The Monte Carlo results are based on 1000 replications using a 100 observation "burn-in" period. The bootstrap results are based on 999 bootstrap replications using the moving block bootstrap with a block length of 10. The long run variance in the R/S, KPSS and V/S statistics is calculated using $[8\sqrt[4]{T/100}]$ estimated covariance terms. [√T] frequency domain terms are used to calculate the GPH and H test statistics.

Table 1 (Continued): Size of Long Memory Tests for IID Models

Sample Size	Test Statistic	Nominal Size	Normal Random Error						Demeaned Log-Normal Error					
			20%	**15%**	**10%**	**5%**	**2½%**	**1%**	**20%**	**15%**	**10%**	**5%**	**2½%**	**1%**
T = 500	R/S	Bootstrapped	21.4	15.7	10.2	5.6	2.6	1.3	20.2	14.6	9.8	4.4	2.1	0.9
		Asymptotic	15.7	11.2	7.1	2.6	0.9	0.2	11.9	9.1	4.3	1.4	0.6	0.2
	KPSS	Bootstrapped	19.6	14.8	10.1	5.2	3.2	1.5	21.3	15.6	11.0	5.7	3.0	1.3
		Asymptotic	19.7	14.8	10.2	5.0	2.8	1.1	21.9	15.8	10.5	5.1	2.8	0.7
	V/S	Bootstrapped	20.4	15.9	10.5	5.2	2.9	1.0	21.4	15.1	9.5	4.6	2.6	1.3
		Asymptotic	20.2	16.0	9.3	4.9	1.9	0.5	20.2	14.6	9.5	3.5	1.9	0.6
	GPH	Bootstrapped	19.5	15.3	9.7	4.5	2.2	1.1	18.9	13.7	9.3	5.4	2.8	1.2
		Asymptotic	9.5	6.8	3.9	2.0	0.8	0.3	8.2	7.0	4.9	2.4	0.9	0.6
	$\widehat{H}$	Bootstrapped	21.2	16.4	10.9	4.7	1.8	0.7	18.5	13.9	9.1	4.8	2.8	1.0
		Asymptotic	15.3	11.9	9.0	5.0	2.7	1.0	11.9	9.6	7.2	5.2	2.8	1.7
T = 1000	R/S	Bootstrapped	18.7	13.5	9.3	4.2	2.1	0.9	18.4	13.3	9.0	4.0	1.9	0.6
		Asymptotic	14.6	10.6	6.3	3.1	1.3	0.2	13.1	9.4	5.4	1.9	0.8	0.1
	KPSS	Bootstrapped	19.0	15.2	9.8	4.3	2.0	0.7	19.8	14.9	10.7	5.1	2.9	1.4
		Asymptotic	19.1	15.0	9.5	4.0	1.8	0.7	19.4	14.8	10.6	5.1	2.4	1.0
	V/S	Bootstrapped	18.9	14.5	9.2	3.9	1.8	1.1	19.3	13.9	8.8	4.7	2.8	1.5
		Asymptotic	18.2	14.2	9.0	4.0	1.5	0.7	18.8	13.8	8.6	4.3	2.7	0.8
	GPH	Bootstrapped	20.0	15.5	10.4	4.7	2.2	0.9	21.6	15.9	9.9	4.9	2.0	0.8
		Asymptotic	10.2	7.1	4.2	1.6	0.7	0.2	8.9	6.3	4.3	1.4	0.9	0.4
	$\widehat{H}$	Bootstrapped	18.7	13.9	8.6	4.7	1.8	0.9	20.7	16.2	11.1	4.7	1.7	0.5
		Asymptotic	11.5	8.9	6.5	3.5	1.9	0.9	13.3	10.7	7.7	3.4	1.8	0.8

Notes: See first part of Table.

Table 2: Size of Long Memory Tests for AR(1) Model with ρ = 0.5

Sample Size	Test Statistic	Critical Values	Normal Random Error						Demeaned Log-Normal Error					
			20%	15%	10%	5%	2½%	1%	20%	15%	10%	5%	2½%	1%
T = 100	R/S	Bootstrapped	17.5	12.6	8.3	3.2	1.1	0.5	17.6	11.8	5.7	2.7	0.6	0.2
		Asymptotic	4.0	1.1	0.2	0.0	0.0	0.0	2.9	0.6	0.2	0.0	0.0	0.0
	KPSS	Bootstrapped	21.4	14.7	10.7	5.5	2.9	0.8	22.4	16.3	10.3	5.1	2.7	1.1
		Asymptotic	26.4	19.6	12.6	5.0	2.0	0.0	27.7	21.2	13.1	4.5	1.6	0.2
	V/S	Bootstrapped	21.4	15.1	9.4	3.4	1.6	0.6	21.5	15.4	9.4	5.0	2.1	0.6
		Asymptotic	25.5	17.2	7.2	1.0	0.0	0.0	26.4	17.4	7.7	1.1	0.1	0.0
	GPH	Bootstrapped	21.1	14.5	7.6	4.0	1.4	0.7	22.1	15.1	9.2	3.6	1.5	0.5
		Asymptotic	17.6	12.3	7.3	4.1	1.9	1.1	18.6	13.7	9.9	4.6	1.6	0.5
	$\widehat{H}$	Bootstrapped	18.5	12.0	6.9	3.3	0.7	0.4	19.6	13.5	7.3	2.9	0.8	0.0
		Asymptotic	34.5	30.4	25.2	17.8	13.5	8.3	31.5	27.2	23.7	17.8	13.3	8.6
T = 250	R/S	Bootstrapped	20.7	15.4	10.3	5.6	3.3	1.4	19.0	13.4	8.7	4.7	2.0	0.7
		Asymptotic	13.7	8.7	5.1	1.7	0.4	0.1	11.2	6.9	3.4	0.7	0.2	0.0
	KPSS	Bootstrapped	21.3	15.7	10.0	4.4	1.9	0.6	22.2	16.3	11.1	5.2	2.2	0.8
		Asymptotic	24.5	19.2	12.0	5.4	1.9	0.6	26.1	19.9	13.4	6.1	2.5	0.8
	V/S	Bootstrapped	20.7	14.4	10.1	5.6	3.2	1.2	20.7	15.7	10.6	5.2	2.9	1.6
		Asymptotic	26.4	17.5	11.4	5.5	2.5	0.6	25.4	19.5	12.2	5.3	2.5	0.6
	GPH	Bootstrapped	22.7	16.0	11.1	5.8	2.3	1.3	21.2	15.4	10.6	4.9	2.6	1.0
		Asymptotic	15.3	11.8	8.1	3.9	1.9	0.7	12.5	9.8	7.1	3.4	2.1	0.9
	$\widehat{H}$	Bootstrapped	23.5	16.4	10.6	5.3	2.5	1.1	20.8	14.7	9.7	4.0	1.6	0.6
		Asymptotic	25.0	21.5	17.4	12.1	8.3	5.4	21.4	17.3	14.3	9.8	7.7	4.9

Notes: See Table 1. The DGP is $x_t = 0.5x_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim n.i.d.(0,1)$ or, before demeaning, $\ln \varepsilon_t \sim n.i.d.(0,1)$

Table 2 (Continued): Size of Long Memory Tests for AR(1) Model with ρ = 0.5

Sample Size	Test Statistic	Critical Values	Normal Random Error						Demeaned Log-Normal Error					
			20%	15%	10%	5%	2½%	1%	20%	15%	10%	5%	2½%	1%
T = 500	R/S	Bootstrapped	21.1	16.2	11.3	6.2	2.6	1.2	21.3	15.2	9.9	5.0	2.5	0.8
		Asymptotic	19.0	13.7	9.2	3.6	1.7	0.3	16.1	10.9	6.4	2.8	1.0	0.3
	KPSS	Bootstrapped	20.3	15.2	10.5	5.4	3.3	1.4	22.0	16.3	11.3	6.2	3.3	1.5
		Asymptotic	24.4	18.6	12.2	6.8	3.8	1.8	24.7	20.3	14.2	7.5	4.1	1.5
	V/S	Bootstrapped	22.0	16.7	11.9	6.3	3.3	1.2	22.2	17.0	10.7	4.8	3.0	1.6
		Asymptotic	26.0	20.6	14.4	7.1	3.7	1.1	26.4	20.4	13.4	6.3	3.2	1.1
	GPH	Bootstrapped	20.9	16.0	10.3	5.2	2.3	0.9	19.9	15.3	10.6	6.2	3.1	1.3
		Asymptotic	11.3	9.0	5.5	2.5	1.2	0.4	10.6	8.4	6.1	3.2	1.7	0.7
	$\widehat{H}$	Bootstrapped	21.9	18.0	12.1	5.8	2.0	0.8	20.3	15.1	10.2	5.1	2.9	1.3
		Asymptotic	19.6	16.2	12.1	7.9	4.9	2.1	15.6	12.6	9.9	6.3	4.8	2.4
T = 1000	R/S	Bootstrapped	20.2	14.2	9.9	4.6	2.5	1.1	19.4	14.0	9.5	4.5	2.1	0.8
		Asymptotic	20.3	13.9	9.6	4.4	2.1	0.6	18.2	12.5	8.8	3.8	1.3	0.5
	KPSS	Bootstrapped	19.5	15.3	10.4	5.1	1.9	0.8	20.0	15.4	11.1	6.0	3.1	1.3
		Asymptotic	23.3	17.7	13.1	6.6	2.9	1.1	23.6	18.6	12.7	6.7	3.5	1.6
	V/S	Bootstrapped	20.3	15.1	10.0	4.4	2.1	1.1	20.5	14.4	9.7	4.9	3.2	1.5
		Asymptotic	25.8	18.5	13.3	5.7	3.0	1.1	24.3	19.5	12.5	6.7	3.5	2.0
	GPH	Bootstrapped	20.9	16.1	11.0	5.8	2.7	1.3	21.5	16.4	10.6	5.4	2.6	0.7
		Asymptotic	11.3	8.5	6.0	2.3	1.2	0.2	10.6	7.6	5.2	2.2	0.9	0.5
	$\widehat{H}$	Bootstrapped	19.6	14.2	9.7	4.9	2.3	1.0	21.8	16.7	11.9	5.3	2.0	0.7
		Asymptotic	13.5	11.0	7.8	4.4	2.6	1.3	15.8	12.7	9.9	5.0	2.2	1.0

Notes: See Table 1. The DGP is $x_t = 0.5x_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim n.i.d.(0,1)$ or, before demeaning, $\ln \varepsilon_t \sim n.i.d.(0,1)$

Table 3: Size of Long Memory Tests for AR(1) Model with Stochastic Volatility Error (ρ = 0.5 and γ = 0.95)

Sample Size	Test Statistic	Critical Values	Nominal Size of Test					
			20%	15%	10%	5%	2½%	1%
T = 250	R/S	Bootstrapped	20.1	15.2	9.2	3.9	1.2	0.3
		Asymptotic	7.0	3.0	0.8	0.0	0.0	0.0
	KPSS	Bootstrapped	22.4	17.3	11.2	6.4	3.4	1.9
		Asymptotic	24.3	17.9	11.3	4.5	2.4	0.8
	V/S	Bootstrapped	22.8	17.3	11.5	5.0	2.1	0.7
		Asymptotic	22.4	16.2	8.3	1.7	0.3	0.0
	GPH	Bootstrapped	19.4	13.1	7.4	3.5	1.2	0.3
		Asymptotic	12.6	9.4	6.5	2.7	1.5	0.7
	$\widehat{H}$	Bootstrapped	17.3	11.0	6.3	1.8	0.4	0.1
		Asymptotic	21.0	18.7	15.9	11.8	8.3	5.8
T = 500	R/S	Bootstrapped	20.1	16.2	9.8	4.2	1.9	0.4
		Asymptotic	14.9	9.5	4.8	1.4	0.2	0.0
	KPSS	Bootstrapped	19.0	15.5	10.6	5.0	3.6	1.4
		Asymptotic	20.6	15.8	11.3	4.8	3.3	1.1
	V/S	Bootstrapped	20.9	15.8	11.3	5.4	2.6	1.0
		Asymptotic	21.5	16.4	11.1	4.2	1.7	0.6
	GPH	Bootstrapped	20.8	16.1	12.3	5.7	3.4	1.6
		Asymptotic	12.0	7.8	5.2	2.9	1.4	0.4
	$\widehat{H}$	Bootstrapped	20.5	15.6	12.4	6.7	3.6	1.5
		Asymptotic	14.2	12.5	10.1	6.6	4.5	3.0
T = 1000	R/S	Bootstrapped	20.7	15.5	10.4	5.3	2.5	1.1
		Asymptotic	17.0	11.6	7.4	2.9	1.1	0.3
	KPSS	Bootstrapped	20.9	16.2	11.4	6.2	3.6	1.6
		Asymptotic	21.7	16.7	11.8	6.1	3.4	1.2
	V/S	Bootstrapped	18.8	15.2	9.6	4.4	2.3	1.4
		Asymptotic	19.7	15.5	10.3	4.0	2.4	1.0
	GPH	Bootstrapped	22.2	18.1	12.5	7.2	4.1	1.7
		Asymptotic	11.8	9.0	6.3	3.3	1.4	0.7
	$\widehat{H}$	Bootstrapped	24.2	18.9	13.9	7.8	4.1	2.1
		Asymptotic	16.7	13.3	10.0	6.8	4.2	2.3

Notes: See Table 1. The DGP is $x_t = (1-0.5L)^{-1}u_t$, where $u_t = \exp\left(\frac{1}{2}h_t\right)\varepsilon_t$ with $h_t = 0.95h_{t-1} + \eta_t$. ε_t and η_t are mean zero, independent normal random variables with variances equal to 0.1

Table 4: Power of Long Memory Tests for Fractionally Integrated Model (d = 1/3)

Sample Size	Test Statistic	Critical Values	Normal Random Error						Demeaned Log-Normal Error					
			20%	15%	10%	5%	2½%	1%	20%	15%	10%	5%	2½%	1%
T = 100	R/S	Bootstrapped	27.9	19.9	11.4	5.0	2.5	1.1	30.2	20.5	12.7	6.2	1.8	0.4
		Asymptotic	7.1	3.0	0.5	0.0	0.0	0.0	7.3	2.1	0.2	0.0	0.0	0.0
	KPSS	Bootstrapped	38.6	33.3	27.0	16.7	10.7	6.0	41.8	35.7	28.9	17.9	11.2	6.1
		Asymptotic	46.9	39.1	32.0	19.0	10.3	2.3	49.6	43.1	34.0	20.5	10.7	1.9
	V/S	Bootstrapped	40.5	33.0	24.0	12.9	6.7	2.3	44.0	36.3	26.2	14.2	7.2	2.8
		Asymptotic	48.1	39.5	24.3	6.7	0.7	0.0	53.3	42.2	26.1	6.4	0.5	0.0
	GPH	Bootstrapped	41.0	31.0	20.0	8.9	4.2	1.6	40.6	30.7	18.7	8.3	3.7	1.3
		Asymptotic	47.8	40.4	31.1	20.3	11.3	5.5	46.7	39.9	30.9	20.5	12.7	6.7
	$\widehat{H}$	Bootstrapped	42.6	30.6	18.2	8.3	3.9	1.1	43.0	29.2	17.2	7.5	3.2	0.9
		Asymptotic	67.6	63.1	58.6	51.0	44.6	36.0	69.9	66.4	62.8	51.6	44.6	35.1
T = 250	R/S	Bootstrapped	53.4	47.4	39.6	27.6	18.2	11.4	55.1	47.8	40.1	28.4	19.7	11.5
		Asymptotic	48.5	41.9	30.8	16.3	7.9	2.4	47.2	39.8	31.0	16.6	8.0	2.8
	KPSS	Bootstrapped	52.2	44.8	36.8	25.7	18.9	12.5	53.9	47.1	37.9	27.4	20.9	13.4
		Asymptotic	60.9	53.4	43.8	30.7	23.1	14.4	58.1	52.4	44.1	32.8	24.2	16.5
	V/S	Bootstrapped	57.2	51.2	41.7	30.3	22.8	15.4	62.2	56.2	47.1	33.6	23.5	15.9
		Asymptotic	66.2	59.4	50.3	37.0	25.0	14.8	58.1	52.4	44.1	32.8	24.2	16.5
	GPH	Bootstrapped	66.1	58.8	48.4	32.4	21.1	11.9	67.4	60.3	51.5	37.7	27.2	15.9
		Asymptotic	65.4	58.3	50.8	39.4	27.1	17.5	67.5	61.3	47.9	30.5	18.6	9.8
	$\widehat{H}$	Bootstrapped	75.8	68.4	56.4	37.6	24.8	11.8	77.8	69.5	57.5	35.3	20.5	8.7
		Asymptotic	83.0	79.9	75.2	68.2	61.1	51.1	84.1	81.5	77.9	69.9	61.3	52.6

Notes: See Table 1. The DGP is $x_t = (1-L)^{-1/3}\varepsilon_t$ with $\varepsilon_t \sim n.i.d.(0,1)$ or, before demeaning $\ln\varepsilon_t \sim n.i.d.(0,1)$

Table 4 (Continued): Power of Long Memory Tests for a Fractionally Integrated Model (d = 1/3)

Sample Size	Test Statistic	Critical Values	Normal Random Error						Demeaned Log-Normal Error					
			20%	15%	10%	5%	2½%	1%	20%	15%	10%	5%	2½%	1%
T = 500	R/S	Bootstrapped	72.1	67.0	59.9	49.7	40.0	29.2	74.6	69.0	61.7	50.0	40.3	32.1
		Asymptotic	73.1	67.9	60.3	49.3	38.5	25.1	73.9	67.6	60.4	46.8	37.7	26.7
	KPSS	Bootstrapped	66.6	59.9	51.3	37.9	30.9	23.6	66.5	60.5	52.4	40.3	31.2	24.1
		Asymptotic	74.6	68.6	59.5	46.2	36.6	27.8	74.5	68.6	60.5	47.1	38.1	28.0
	V/S	Bootstrapped	73.7	68.1	60.1	49.9	40.7	31.0	74.6	68.7	61.2	49.5	40.7	31.8
		Asymptotic	82.0	76.0	70.4	58.1	47.8	37.0	82.6	76.9	69.4	56.6	47.3	36.1
	GPH	Bootstrapped	81.5	77.0	69.9	55.3	43.4	31.0	84.2	78.6	68.9	56.0	42.8	28.6
		Asymptotic	77.1	72.6	64.4	51.2	40.0	27.5	77.9	71.6	64.7	51.8	39.3	26.1
	$\widehat{H}$	Bootstrapped	89.0	85.2	80.3	67.8	55.9	40.5	90.9	86.9	81.0	68.5	53.9	39.1
		Asymptotic	89.8	87.7	84.4	78.2	71.6	63.0	91.7	89.7	85.0	79.1	70.7	62.3
T = 1000	R/S	Bootstrapped	87.0	82.9	77.1	67.5	60.5	50.9	88.6	86.0	80.0	71.2	63.3	53.8
		Asymptotic	90.0	86.6	81.7	72.6	64.1	55.2	90.8	87.7	83.2	74.0	66.7	55.7
	KPSS	Bootstrapped	78.5	72.7	65.4	53.5	44.6	34.7	79.8	72.6	64.9	53.5	43.3	34.3
		Asymptotic	85.3	80.1	72.8	61.6	53.0	42.7	86.4	81.1	73.3	61.6	52.5	40.9
	V/S	Bootstrapped	85.2	81.0	75.5	66.1	56.6	46.3	86.3	83.3	78.0	68.3	59.6	51.0
		Asymptotic	91.4	87.8	82.7	75.1	66.7	55.9	92.0	89.1	84.0	76.5	68.8	58.8
	GPH	Bootstrapped	92.9	90.0	85.8	78.1	69.1	55.4	93.4	91.2	87.0	79.4	70.1	56.4
		Asymptotic	88.4	85.1	80.6	72.6	61.7	48.3	89.9	87.2	81.8	72.9	63.2	48.1
	$\widehat{H}$	Bootstrapped	97.0	95.9	93.8	89.8	83.1	75.0	97.7	96.4	94.7	89.4	83.7	74.0
		Asymptotic	96.4	95.3	94.0	91.2	87.9	81.7	97.2	96.5	94.7	92.2	87.8	81.7

Notes: See Table 1. The DGP is $x_t = (1-L)^{-1/3}\varepsilon_t$ with $\varepsilon_t \sim n.i.d.(0,1)$ or, before demeaning $\ln \varepsilon_t \sim n.i.d.(0,1)$

Table 5: Power of Long Memory Tests for a Fractionally Integrated Model with Stochastic Volatility Error (d = 1/3 and γ = 0.95)

Sample Size	Test Statistic	Critical Values	Nominal Size of Test					
			20%	**15%**	**10%**	**5%**	**2½%**	**1%**
T = 250	R/S	Bootstrapped	53.4	47.5	38.9	25.9	17.5	10.5
		Asymptotic	47.5	37.3	26.9	14.6	7.5	1.2
	KPSS	Bootstrapped	53.2	45.9	38.6	26.4	18.8	11.5
		Asymptotic	59.4	53.1	44.6	32.4	21.9	12.8
	V/S	Bootstrapped	54.2	48.9	41.3	29.8	21.3	13.0
		Asymptotic	61.8	55.7	47.2	33.0	22.9	11.4
	GPH	Bootstrapped	57.9	50.7	41.8	29.1	17.1	9.5
		Asymptotic	54.5	49.4	42.4	32.1	23.8	13.5
	$\widehat{H}$	Bootstrapped	65.0	57.7	48.1	32.3	20.8	10.0
		Asymptotic	71.9	69.3	64.1	56.1	49.0	41.2
T = 500	R/S	Bootstrapped	68.5	63.2	56.2	45.5	36.3	25.7
		Asymptotic	69.1	63.7	55.1	44.7	32.7	21.6
	KPSS	Bootstrapped	63.6	56.7	48.4	36.8	29.5	22.2
		Asymptotic	71.0	64.8	56.1	42.9	34.4	25.9
	V/S	Bootstrapped	70.8	65.6	56.4	45.2	36.5	27.4
		Asymptotic	76.6	72.7	65.7	52.5	43.3	31.8
	GPH	Bootstrapped	78.8	73.6	65.3	52.5	41.2	25.6
		Asymptotic	72.6	66.6	60.1	48.5	36.0	21.8
	$\widehat{H}$	Bootstrapped	85.1	81.2	75.9	64.5	52.9	37.0
		Asymptotic	85.3	83.2	79.7	74.9	68.5	59.5
T = 1000	R/S	Bootstrapped	81.6	76.0	70.5	61.1	52.8	43.2
		Asymptotic	80.7	75.6	69.4	59.3	50.2	37.7
	KPSS	Bootstrapped	72.9	65.6	58.0	46.8	37.4	28.2
		Asymptotic	76.7	71.9	62.7	51.8	42.7	31.4
	V/S	Bootstrapped	79.3	74.4	68.2	58.9	48.7	38.5
		Asymptotic	83.5	79.9	73.2	63.5	54.7	42.8
	GPH	Bootstrapped	88.0	84.5	79.1	69.2	58.2	46.5
		Asymptotic	81.8	78.2	72.3	60.6	50.6	39.1
	$\widehat{H}$	Bootstrapped	94.0	91.6	87.8	82.1	73.1	61.9
		Asymptotic	92.5	91.0	87.9	83.7	77.9	70.2

Notes: See Table 1. The DGP is $x_t = (1-L)^{-1/3} u_t$, where $u_t = \exp\left(\frac{1}{2} h_t\right)\varepsilon_t$ with $h_t = 0.95 h_{t-1} + \eta_t$. ε_t and η_t are mean zero, independent normal random variables with variances equal to 1/10.

Table 6: Size or Power of Long Memory Tests for Various Models with Stochastic Volatility Errors

Data Generation Process	Test Statistic	Critical Values	Nominal Size					
			20%	**15%**	**10%**	**5%**	**2½%**	**1%**
AR(1) with SV Error	V/S	Bootstrapped	20.9	15.8	11.3	5.4	2.6	1.0
		Asymptotic	21.5	16.4	11.1	4.2	1.7	0.6
	$\widehat{H}$	Bootstrapped	20.5	15.6	12.4	6.7	3.6	1.5
		Asymptotic	14.2	12.5	10.1	6.6	4.5	3.0
Sum of AR(1) and SV Errors	V/S	Bootstrapped	20.5	15.8	10.4	5.5	2.6	1.0
		Asymptotic	20.7	15.8	9.9	4.6	1.9	0.6
	$\widehat{H}$	Bootstrapped	23.9	19.0	11.6	6.6	3.6	1.9
		Asymptotic	16.8	13.4	10.3	7.2	4.3	2.5
FI with SV Error	V/S	Bootstrapped	70.8	65.6	56.4	45.2	36.5	27.4
		Asymptotic	76.6	72.7	65.7	52.5	43.3	31.8
	$\widehat{H}$	Bootstrapped	85.1	81.2	75.9	64.5	52.9	37.0
		Asymptotic	85.3	83.2	79.7	74.9	68.5	59.5
Sum of FI and SV Errors	V/S	Bootstrapped	35.8	28.7	22.7	15.2	9.2	4.1
		Asymptotic	36.7	29.6	23.2	14.4	7.8	3.7
	$\widehat{H}$	Bootstrapped	38.8	31.8	24.0	15.5	8.7	3.8
		Asymptotic	31.4	27.4	23.4	17.1	12.0	7.1

Notes: Sample size T = 500. DGPs (i) and (iii) are the same as in Tables 3 and 5. DGP (ii) is $x_t = (1-0.5L)^{-1}u_t + \exp(\frac{1}{2}h_t)\varepsilon_t$ with $h_t = 0.95h_{t-1} + \eta_t$. DGP (iv) is $x_t = (1-L)^{-1/3}u_t + \exp(\frac{1}{2}h_t)\varepsilon_t$ with $h_t = 0.95h_{t-1} + \eta_t$. The random errors u_t, ε_t and η_t are mean zero, independent normal random variables with variances 0.1, 1 and 0.1 respectively.

Table 7: Tests of Long Memory – Standard and Poor's 500 (SP 500) Returns, Absolute Returns and Squared Returns

Period	Test		r_t			$\lvert r_t \rvert$			r_t^2		
			Test Statistic	Critical Values 95%	Critical Values 99%	Test Statistic	Critical Values 95%	Critical Values 99%	Test Statistic	Critical Values 95%	Critical Values 99%
Jan 1993 to Dec 1999 (T = 1768)	R/S	Bootstrapped	1.128	1.650	1.893	3.745	**1.755**	**2.008**	2.795	**1.630**	**1.841**
		Asymptotic		1.747	2.001		**1.747**	**2.001**		**1.747**	**2.001**
	KPSS	Bootstrapped	0.229	0.421	0.620	5.423	**0.557**	**0.789**	2.899	**0.537**	**0.808**
		Asymptotic		0.463	0.739		**0.463**	**0.739**		**0.463**	**0.739**
	V/S	Bootstrapped	0.079	0.170	0.253	1.335	**0.205**	**0.309**	0.792	**0.200**	**0.267**
		Asymptotic		0.187	0.266		**0.187**	**0.266**		**0.187**	**0.266**
	GPH	Bootstrapped	-0.925	1.755	2.241	5.765	**5.031**	**5.534**	2.406	**1.406**	**1.797**
		Asymptotic		1.960	2.575		**1.960**	**2.575**		**1.960**	2.575
	$\widehat{H}$	Bootstrapped	-0.017	2.014	2.873	8.521	**6.473**	**7.109**	5.025	**3.818**	**4.765**
		Asymptotic		1.960	2.575		**1.960**	**2.575**		**1.960**	**2.575**
Jan 1997 to Dec 1998 (T = 505)	R/S	Bootstrapped	1.652	1.651	1.862	1.856	**1.686**	**1.846**	1.604	**1.591**	1.764
		Asymptotic		1.747	2.001		**1.747**	2.001		1.747	2.001
	KPSS	Bootstrapped	0.154	0.484	0.729	1.089	**0.522**	**0.800**	0.775	**0.479**	0.803
		Asymptotic		0.463	0.739		**0.436**	**0.739**		**0.463**	**0.739**
	V/S	Bootstrapped	0.116	0.199	0.248	0.276	**0.172**	**0.270**	0.209	**0.192**	0.262
		Asymptotic		0.187	0.266		**0.187**	**0.266**		**0.187**	0.266
	GPH	Bootstrapped	0.298	1.757	2.403	2.752	**1.561**	**2.740**	1.655	1.889	2.500
		Asymptotic		1.960	2.575		**1.960**	**2.575**		1.960	2.575
	$\widehat{H}$	Bootstrapped	0.805	2.345	2.957	3.658	**2.084**	**3.498**	2.309	2.379	3.606
		Asymptotic		1.960	2.575		**1.960**	**2.575**		**1.960**	2.575

Notes: Statistically significant outcomes are shown in bold. The MBB block length is 10 and the number of bootstrap replications is 999. Other settings are the same as in Section 2.