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**Ashok S. Guha, Brishti Guha**

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## TARGET SAVING IN AN OVERLAPPING GENERATIONS MODEL

Ashok S. Guha<sup>1</sup>

Indian Statistical Institute

Brishti Guha<sup>2</sup>

Singapore Management University

### Abstract

We examine a model in which the utility function has been engineered so that it is optimal for consumers to aim for a fixed target level of retirement resources. In this case consumption displays excess sensitivity to current income as well as perfect old age insurance. In an overlapping generations model, this leads naturally to multiple and unstable equilibria. Under static expectations, it also leads to a well-defined dynamics, including possible historical traps, implosions involving ever-diminishing capital stock and ever-increasing interest rates, and the feasibility of optimal one-time interventions.

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**Keywords:** Targets, history, excess sensitivity, static expectations, rational expectations, uniqueness.

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<sup>1</sup> Indian Statistical Institute, New Delhi, India. E-mail: ashoksanjayguha@hotmail.com.

<sup>2</sup>Corresponding author. Department of Economics, School of Economics and Social Sciences, Singapore Management University, 90 Stamford Road, Singapore 178903. E-mail : [bguha@smu.edu.sg](mailto:bguha@smu.edu.sg) The authors acknowledge input from Kaushik Basu.

## 1. Introduction

Consumers often aim at target consumption levels of specific goods. Consumer durables (houses, cars) frequently come in indivisible units and the consumer is often satiated with a single unit. He desires just one piece, no more and no less. Where such a good bulks large in the consumer's budget, changes in its price may have unusual repercussions on demand for other goods. The gross substitutability property may not obtain. A rise in the price of the target good may reduce the budget available for other goods and the consumer's demand for them. This affects the existence and uniqueness of market equilibrium.

An important example occurs when the consumer aims at a future consumption target – say a house to retire to or a fixed payment to an old age home that will support him. This affects his saving behavior, particularly his reactions to interest rate changes. Oddly enough, although there is empirical evidence from household surveys that target saving is an important motivation for saving behavior,<sup>1</sup> there has been hardly any theoretical modeling of its implications.<sup>2</sup> Perhaps this is on account of the fact that target saving behavior is not easily generated from standard utility functions, an omission that we attempt to repair in a subsequent section. We show that target saving can be generated for specific and time-varying values of risk aversion parameters which characterize more standard utility and saving functions. Later we also describe in detail and provide examples of how our results differ from those obtainable with more standard utility functions.

If target saving behavior occurs on a large scale, equilibrium in the capital market, the interest rate and the level of output are affected in curious ways. Saving becomes negatively interest-elastic, so that multiple and unstable equilibria become possible, some with *high saving even at very low (possibly zero) interest rates*.

It is however well-known that multiplicity and instability of equilibrium are not easy to handle within the standard rational expectations framework. In the absence of an extraneous coordination mechanism, people may expect and act upon any of the many possible equilibria. If they differ, the outcome will predictably diverge from everyone's expectations, thus violating the basic conceptual structure of rational expectations. We therefore postulate static expectations<sup>3</sup> while exploring multiple and unstable equilibria: people expect current interest rates to persist, and act accordingly – an assumption that yields a well-defined dynamics. The past now determines the future: initial conditions decide which of the possible equilibria the economy will converge to, or if it will con-

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<sup>1</sup>Samwick (1998) cites evidence on this from the Survey of Consumer Finances (1992).

<sup>2</sup>The few studies on this subject that we are aware of are mentioned in the literature review section.

<sup>3</sup>Static expectations imply that people expect the future value of the variable in question to be identical to its value in the present. This is particularly useful when dealing with long time horizons (eg 30 years) – that may reasonably occur within an OLG framework. Since it is very difficult to forecast so far into the future, people may not attempt to do so and may reasonably base future expectations on the past.

verge to one at all. If the multiple equilibria are Pareto-rankable, policies can be prescribed to ensure convergence to a preferred equilibrium.

Excess response of consumption to current income is built into our model. While a violation of the permanent income hypothesis, this is in line with the fact that there *is* an “excess sensitivity” puzzle, particularly for developing countries where consumers often violate the permanent income hypothesis (see Rao [2005], Lavi [2003], Chakrabarty and Schmalenbach [2002]).

In section 2 we discuss some related literature, especially that on target saving and on multiple equilibria in OLG models. In section 3 we show how the target saving motive can be derived from standard CRRA utility functions (much used in the literature on intertemporal choice as summarized, for instance, in Azariadis (1993) and Romer (1995)) for *specific* and *time varying* values of the risk aversion parameters. Section 4 presents the details of our model and derives results, along with a number of examples, both for the rational expectations case and for the static expectations case. Section 5 details further ways in which our model, its equilibria and results differ from those obtainable with the same production functions but with more standard utility functions. Section 6 concludes.

## 2. Some Related Literature

To the best of our knowledge theoretical work on the target saving motive and its implications is extremely limited. Samwick (1998) studies the likely effect of pension-related tax reform on savings, for savers with different types of motivations. Target savers form one of the groups he considers. Berninghaus and Seifert-Vogt (1993) study the return migration decision of target-saving guest workers who have to choose between returning to their home country to invest in a business and staying on in the host country for one more period.

We now come to the other themes incidental to our paper. Multiple equilibria can also be obtained in other ways as in Azariadis and Lambertini (2003) who focus on an imperfect credit market in a deterministic three-generation world. Our focus is on sketching the consequences of a particular type of behavior – saving for a fixed target - and our model accordingly differs substantially from theirs. Moreover, expectations in their model are always rational. While a multitude of papers deal with multiple equilibria in overlapping generations models in various contexts – such as unemployment (Pissarides, 1992), growth and poverty traps (Boldrin, 1992), fertility (Palivos, 2001), education (Futagami and Ishiguro, 2004) and endogenous public policy (Glomm and Ravikumar, 1995) – these models do not deal with the kind of saving behavior which it is our purpose to model, nor do they depart from rational expectations. Evans and Honkapohja (2001) survey the vast literature on multiple equilibria in OLG models, including Shell (1977), Cass and Shell (1983), Azariadis (1981) and Azariadis and Guesnerie (1982). Evans and Honkapohja themselves postulate adaptive learning as a selection device among multiple equilibria. Woodford (1990) studies a monetary economy with an adaptive learning rule, converging to a stationary sunspot equilibrium.

Our model is also related to the strand of literature which deals with the “history versus expectations” debate in economies with externalities. This liter

ature includes Krugman (1991), Matsuyama (1991) and Adsera and Ray (1998) and centers around decisions to move or migrate into a different sector with externalities - decisions based both on current returns and expected future returns. Although our model does not fall in this class, as there are no intersectoral movement decisions involved, there is an externality in our model as individuals make their private saving decisions in light of their interest rate expectations, while the true realizations of interest rates will depend on the decisions of others. The history-versus-expectations strand of literature looks at whether long run equilibria are predictable given history, or whether they depend instead on “self-fulfilling expectations” and are indeterminate in spite of a known history. In our model, we find that for some technologies, indeterminacy can obtain if expectations are rationally formed – however, when people have static expectations, history has the power to predict the future course of the economy. Our model differs in two other important respects from the group of models considered in the history versus expectations debate. First, people in our model have a fixed future goal. Hence their response to possible returns is affected by this. Secondly, our model has dynamics in the static expectations case, but there are no dynamics if expectations are rational.

### 3. Target Savings and Standard Utility Theory

Can target saving behavior be reconciled with standard utility models ? Consider an individual with a first period budget of  $W$  which he must allocate to consumption over a two-period horizon in the light of an interest rate of  $r$ . Suppose his utility function is the conventional CRRA function except for the fact that the coefficient of RRA differs between the two periods:

$$U = \frac{c_1^{1-\theta_1}}{1-\theta_1} + \frac{c_2^{1-\theta_2}}{(1-\theta_2)(1+\rho)}$$

Here  $\theta_1$ ,  $\theta_2$  and  $\rho$  are all positive. He equates the marginal rate of substitution between the two consumptions to the marginal cost of one in terms of the other to derive

$$\frac{c_1^{\theta_1}}{c_2^{\theta_2}} = \frac{1+\rho}{1+r}.$$

Without more restrictions on the parameters, one cannot draw any meaningful inferences about his saving behaviour. The standard CRRA function in which  $\theta_1 = \theta_2 = \theta$  yields

$$\frac{c_1}{c_2} = \left[ \frac{1+\rho}{1+r} \right]^{1/\theta}$$

The consumption growth rate is positive if the interest rate exceeds the rate of time preference and is an increasing function of the interest rate. If  $\theta = 1$  as with logarithmic felicity functions, it is proportional to the interest rate:

$$\frac{c_1}{c_2} = \frac{1+\rho}{1+r}$$

All this is standard and occasions no surprise. But differences in RRA coefficients change the story. Suppose  $\theta_1 = 0$ , implying linear first period felicity. Then

$$c_2 = \left[ \frac{1+r}{1+\rho} \right]^{1/\theta_2}.$$

If  $\theta_2$  is non-zero,  $c_2$  will be independent of the budget  $W$ , though still related to the interest rate  $r$ . Now, if in addition  $\theta_2 = \infty$ , we have  $c_2 = 1$  – the consumer aims at a fixed second period consumption target. The unitary value of  $c_2$  is not restrictive, given that we are free to choose our own units. Target saving is thus

a consequence of a sharp asymmetry between the consumer's attitudes to risk in the present and the future. It occurs if he is risk-neutral in the present but intensely risk-averse in his view of the future. Alternatively, interpreting  $\theta$  as the elasticity of the marginal utility of consumption, target saving characterizes individuals with insatiable consumption needs today ( $\theta_1 = 0$  – marginal utility constant) but very limited wants tomorrow ( $\theta_2 = \infty$  – marginal utility dropping to zero, implying satiation). Note that the consumption target is independent not only of the budget  $W$  but also of both interest rate and time preference.

A comparison of the asymmetric case with that of symmetric RRA coefficients  $\theta_1 = \theta_2 = \theta = \infty$  is of some interest. The latter implies  $c_1 = c_2$  – stability of consumption between the two periods.<sup>4</sup> However, the level at which consumption will be intertemporally stable is an increasing function of the interest rate (though independent of time-preference). The higher the interest rate, the higher will be the income from positive saving (as in the present case) – and the higher accordingly the constant level of consumption that it can sustain. Since not only future, but present consumption as well is increasing in interest rate, saving will be negatively related to  $r$  – though not as strongly as in the case where the future consumption target is fixed.

#### 4. The Model

Assume a world of overlapping generations in which all individuals are identical. Population is constant with the number in each generation being normalized to one. Each individual lives for two periods. At the outset of the first, he borrows capital from the previous generation, now retired, and promises to repay principal and prenegotiated interest after the production process. He then produces according to a production function  $y_t = f(k_t) + e_t$ , where  $e_t$  is a production shock with zero mean. Thereafter, he makes the contractual repayment and allocates the remaining output between consumption  $c_t$  and saving in the light of his interest rate expectations and preferences. At the beginning of the next period, he lends the savings to the new generation of producers at the interest rate expected in that period and lives in retirement thereafter on his interest and capital.

As in all models in which population is constant and individuals exhaust their income over their life-cycle, there is no aggregate saving: however, the young save to finance their retirement.

Output is a CRS function of capital and labor. The labor force comprises the entire younger generation and is constant through time. We therefore suppress the labor argument in the production function and write it simply as  $f(k)$ . Competition for capital between producers drives the contractual interest rate to the marginal product of this capital  $f'(k_t)$ . Since all capital is borrowed, the young working generation, after repaying its loans, has only its wage income to consume and save with.

The role of expectations in this process implies that different models of expectation formation will have different consequences. We shall highlight in

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<sup>4</sup>The utility function in this case can be shown, by appropriate transformations, to be of the Leontief variety  $U = \text{Min}(c_1, c_2)$ , implying strict complementarity between consumptions in the two periods and L-shaped indifference curves with vertices along the 45 degree line.

particular the implications of rational expectations on the one hand and of static expectations on the other. The latter is of interest because rational expectations *in our case* do not deliver a definite prediction – in the absence of a predictable exogenous co-ordination mechanism - if the underlying model displays multiple equilibria.

#### 4.1.1. Rational Expectations

Consider first the case of rational expectations. Under rational expectations, the marginal product of capital can be fully forecast given the capital stock at the time of contracting (which is just the saving carried over from the previous period by the then working generation). Savers form correct expectations of this contractual interest rate and base their saving decisions on it. Thus the contract is signed before the production shock hits. As savers are guaranteed the interest rate  $f'(k_t)$ , they are fully insured against the production shock – whether positive or negative - which is borne accordingly by the current producing generation.

Suppose now that the second period consumption of each individual is fixed at  $\underline{a}$ . Let the younger generation's consumption in period  $t$  – the first period of its life – be denoted by  $c_{1,t}$  while the older's consumption is  $c_{2,t}$  (the older generation's consumption in its youth was  $c_{2,t-1}$ ). Because individuals are saving to meet fixed targets, we always have

$$(1) \quad c_{2,t} = \underline{a} = (1 + f'(k_t))s_{2,t-1}$$

After paying the older generation,<sup>5</sup> the younger one also wants to save just enough to ensure a second period consumption of  $\underline{a}$ . Accordingly the production shock is entirely absorbed in  $c_{1,t}$ , the current consumption of the working generation. This excess sensitivity of current consumption to transitory income goes against the permanent income hypothesis, but is a characteristic of target-savers. The following equation describes the younger generation's behavior in period  $t + 1$ , when *it* gets old:

$$(2) \quad c_{2,t+1} = \underline{a} = (1 + f'(k_{t+1}))s_{2,t}$$

Moreover, we have

$$(3) \quad k_{t+1} = s_{2,t} \quad \nabla t$$

From (1), (2) and (3) we can easily check that

$$(4) \quad s_{2,t} = s_{2,t-1} = s, \quad k_t = k_{t+1} = k, \quad s = k$$

Thus the game is stationary, despite the uncertain production shock. If in any period, the shock is so adverse as to reduce the total income of the younger generation – after interest payments on the older generation's loan – below the minimum required to reach the second period target, the young will not save, but will instead consume everything left over after making interest payments. No capital will be available next period to sustain production and the economy will disappear. An economy can be wiped out by a sufficiently unfavorable shock  $e_t$  if

$$f(k_t) + e_t - k_t\{1 + f'(k_t)\} < \frac{\underline{a}}{1 + f'(k_t)}.$$

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<sup>5</sup>The young generation can commit to repayment. This is because repayment is verifiable and accordingly default can be deterred by the threat of sufficiently harsh legal penalties. Note that, as we explain later, we focus on a parameter space where true bankruptcy is impossible and its spectre does not therefore deter lenders.

As long as production shocks are bounded, there is always a parameter space where this will never happen. For simplicity, we assume that production shocks are uniformly distributed in the interval  $[-e^*, e^*]$ , and for the rest of this paper concentrate on the case where even the worst shocks leave the young with enough to reach their second period target – so that the economy is in no danger of disappearing.

#### 4.1.2. Equilibrium under rational expectations

The supply of capital from the old is given by

$$(5) \quad s = k = \frac{a}{1+r^*}$$

Here  $r^*$  is the pre-negotiated interest rate that savers will face in the beginning of the next period. The demand of the young for capital, is given by

$$(6) \quad f'(k) = r^*$$

Thus equilibrium capital stock solves the equation

$$(7) \quad k(1 + f'(k)) = a$$

The supply curve is negatively interest-elastic and representable as a rectangular hyperbola in  $(s, r)$  space. It is asymptotic to the vertical axis and to the horizontal through  $r = -1$ ; it intersects the horizontal axis at  $s = a$ . The shape of the demand curve depends on the production function, and determines the number of intersections. Given diminishing marginal productivity of capital, the demand curve is also down-sloping so that multiple intersections are possible. We now prove

**Proposition 1:** If the production function has continuous second derivatives, the demand curve always lies below the supply curve for sufficiently small  $s$ : and if there is no capital saturation ( $f'(k) > 0$  for all  $k$ ), there is at least one intersection.

**Proof:** If, as  $k \rightarrow 0$ , the limiting elasticity of capital-labor substitution is less than one,  $r$  converges to a finite ceiling,<sup>6</sup> so that the demand curve intercepts the vertical axis at a finite  $r$ . In this case clearly the demand curve lies below the supply curve (which is asymptotic to the vertical axis) for sufficiently small  $k$  (or  $s$ ). If the elasticity of substitution in the limit exceeds one, as  $k \rightarrow 0$ , the relative share of capital in total output goes to zero while output tends to a finite limit;<sup>7</sup> if the elasticity equals one, the relative capital share remains constant while output goes to zero as  $k \rightarrow 0$  with  $L$  constant. In either event, the absolute share of capital sinks to zero. This implies that the area below the demand curve would go to zero, while that below the supply curve goes to a positive limit,  $a$ , as  $r \rightarrow \infty$ . Thus, the demand curve must lie below the supply curve as  $r \rightarrow \infty$  (or as  $k \rightarrow 0$ ) whatever the elasticity of substitution. Now suppose there is no capital saturation, so that the demand curve never intercepts the horizontal axis. In this case, clearly the demand curve lies above the supply curve for large enough values of  $k$ , as the supply curve intersects the horizontal axis at  $k = a$ . Since the demand curve lies below the supply curve for small  $k$ , and above it for large  $k$ , then given the continuity of both curves, there must be at least one intersection. Q.E.D.

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<sup>6</sup>For proof see Guha (1963).

<sup>7</sup>Op.cit.

**Corollary:** If there is no capital saturation, there is an odd number of intersections : capital saturation is necessary for an even number of intersections.

**Proposition 2:** A sufficient condition for a unique equilibrium is that  $\text{mod}[f''(k)] < \frac{a}{k^2}$ .

**Proof:**  $f''(k)$  is the slope of the demand curve (6) and  $-\frac{a}{k^2}$  the slope of the supply curve (5). The given condition simply implies that the supply curve is steeper than the demand curve for any  $k$ , so that no more than one intersection is possible. Along with Proposition 1, this guarantees a unique equilibrium.

However, not every intersection between the supply and demand curves constitutes an equilibrium in which the economy is sure to survive unfavorable production shocks. To insure the economy against vanishing due to lack of saving, capital stock in equilibrium must be such that

$$(8) \quad e^* < f(k) - \underline{a} \left( \frac{2+f'(k)}{1+f'(k)} \right) = f(k) - k(2 + f'(k)) \quad (\text{using (7)})$$

Thus equilibria where the capital stock is not large enough to satisfy (8) are not “safe”: there is some danger of the economy disappearing.

*An Example*

Consider the standard Cobb-Douglas production function  $f(k) = k^\alpha$ . Using (7), in equilibrium,

$$(9) \quad \alpha k^\alpha + k = \underline{a}$$

While the right hand side of (9) is invariant with  $k$ , the left hand side is monotonically increasing, its derivative being

$$1 + \alpha^2 k^{-(1-\alpha)} > 1 > 0.$$

Therefore, only one  $k$  satisfies (9). Moreover, there is no capital saturation-  $f'(k) = \alpha k^{-(1-\alpha)} \rightarrow 0$  only as  $k \rightarrow \infty$ . Therefore, by Proposition 1, an equilibrium exists and is in this case unique (Figure 1).<sup>8</sup> Consider parameter values  $\alpha = 1/3$ ,  $\underline{a} = 7/24$ ,  $e^* = 1/13$ . Then the equilibrium  $k$  satisfying (9) is  $k = 1/8$ . Moreover, (8) is satisfied, as its left hand side is  $1/13$  while its right hand side is  $1/12 > 1/13$ . Thus in this equilibrium, there is no danger of not being able to save enough to meet the target.

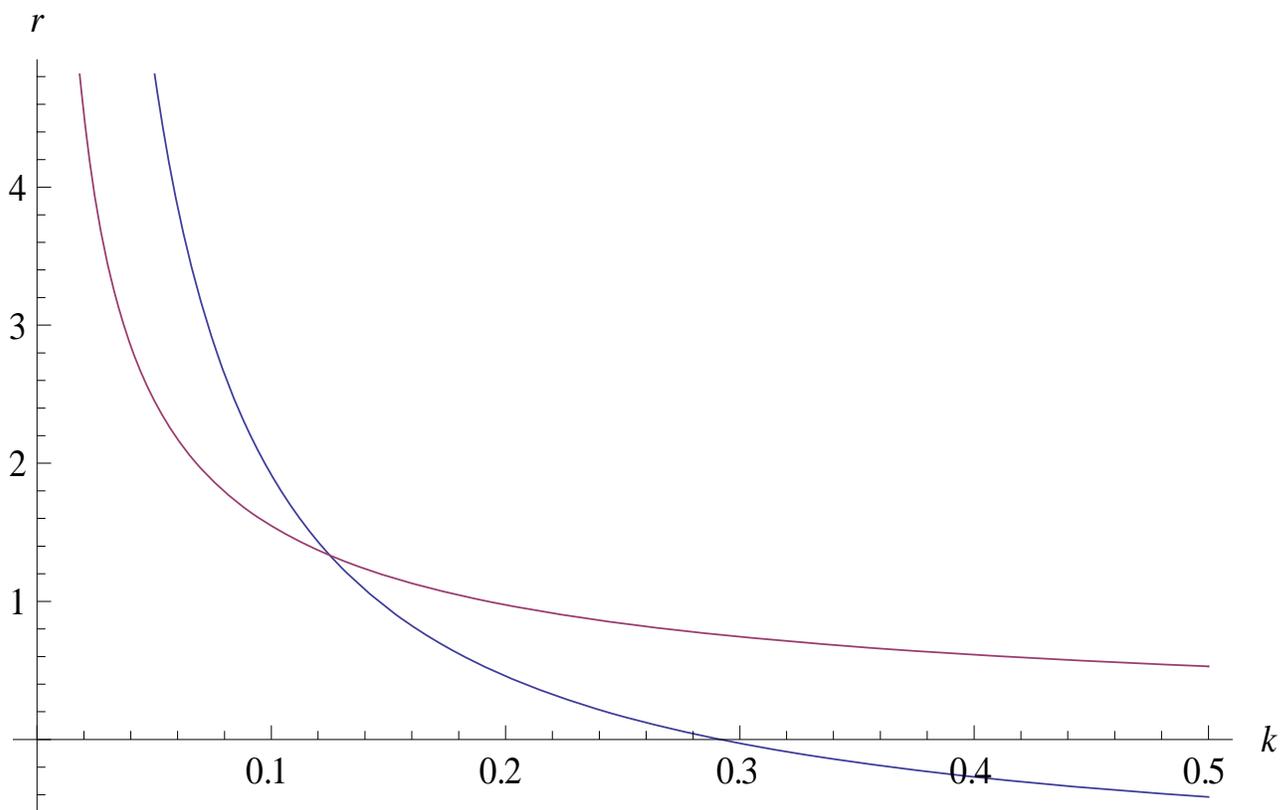
One could also provide examples of multiple equilibria, but since rational expectations would then lead to indeterminacy except in the presence of some exogenous coordination mechanism, we now turn to a different model of expectation formation.

#### 4.2. Static Expectations

Under static expectations, every one expects the current period’s interest rate to persist into the next period. In period  $t$ , the young observe the marginal product of capital in production  $r_t = f'(k_t)$ , expect this interest rate to continue and save accordingly, so that  $s_t = k_{t+1} = \frac{\underline{a}}{1+r_t}$ . In the beginning of period  $t+1$ , they loan out their savings for production to the then young generation, charging an interest rate of precisely  $r_t$ , thus ensuring their target consumption of  $\underline{a}$ . However, the marginal product of capital that emerges after absorption of the wealth inherited from the previous period is  $f'(k_{t+1})$ : it will be different from

<sup>8</sup>All our figures are plotted using Mathematica. Interested readers can refer to our mathematica code at <http://www.economics.smu.edu.sg/faculty/economics/bguha.asp>

**Figure 1**



$r_t$  unless  $k_t = k_{t+1}$ , and the currently young who base their saving decisions on it will save differently from the previous generation. Dynamic equilibrium will then obtain when

$$r_{t+1} = f' \left( \frac{a}{1+r_t} \right) = g(r_t) = r_t$$

This is subject to the initial condition

$$r_0 = f'(k_0)$$

Here  $g(\cdot) : R^+ \rightarrow R^+$  is a function mapping  $r_t$  into  $r_{t+1}$ . The dynamic equilibrium is a fixed point of  $g(\cdot)$ . The dynamic adjustment path comprises of a series  $\{r_t\}_{t=0}^{\infty}$ , given the initial capital stock and technology. The sequence of interest rates also determines the sequence of future capital stocks via the relationship  $k_{t+1} = \frac{a}{1+r_t}$ .

In Figure 2<sup>9</sup> we plot the function  $g(\cdot)$  on the vertical axis against  $r_t$  on the horizontal axis. Intersections of the  $g(\cdot)$  function with the 45 degree line represent dynamic equilibria. Now  $g(0) = f'(\underline{a}) > 0$  is the vertical intercept of the  $g$  function, so it starts above the 45 degree line *provided* there is no capital saturation (the upper curve in Fig. 2). If there is capital saturation at  $\bar{k}$ , on the other hand, so that  $\bar{k} = \frac{a}{k^s}$ , then the  $g$  function lies along the horizontal axis until we reach  $r_t = r^s = \frac{a}{k^s} - 1$  where  $k^s$  is the level of capital at which saturation sets in (this is where the lower curve in Fig. 2 cuts the x axis). We also note that the slope of the  $g$  function is

$$g'(r_t) = -\frac{a}{(1+r_t)^2} f'' \left( \frac{a}{1+r_t} \right) > 0 \text{ for } f'' < 0$$

So the  $g$  function is upward sloping, and unless the production function exhibits capital saturation at  $k = \underline{a}$ , starts above the 45 degree line.

If there is capital saturation at  $k = \underline{a}$ , there is trivially at least one intersection between the  $g$  function and the 45 degree line (at  $r = 0$ ). Now restricting our attention to cases with no capital saturation at  $k = \underline{a}$ , a necessary condition for at least one crossing is that for some  $r_t$ , we have  $f' \left( \frac{a}{1+r_t} \right) < r_t$ .

Now provided the existence condition holds, and again restricting our attention to cases without capital saturation at  $k = \underline{a}$ , a *sufficient* condition for uniqueness is  $g''(r_t) < 0$ . If the  $g$  function is concave to the origin throughout its domain, once it crosses the 45 degree line (from above), it will not cross back again. This condition can be rewritten as

$$\frac{2a}{(1+r_t)^3} f'' \left( \frac{a}{1+r_t} \right) + \frac{a^2}{(1+r_t)^4} f''' \left( \frac{a}{1+r_t} \right) < 0 \text{ for all } r_t$$

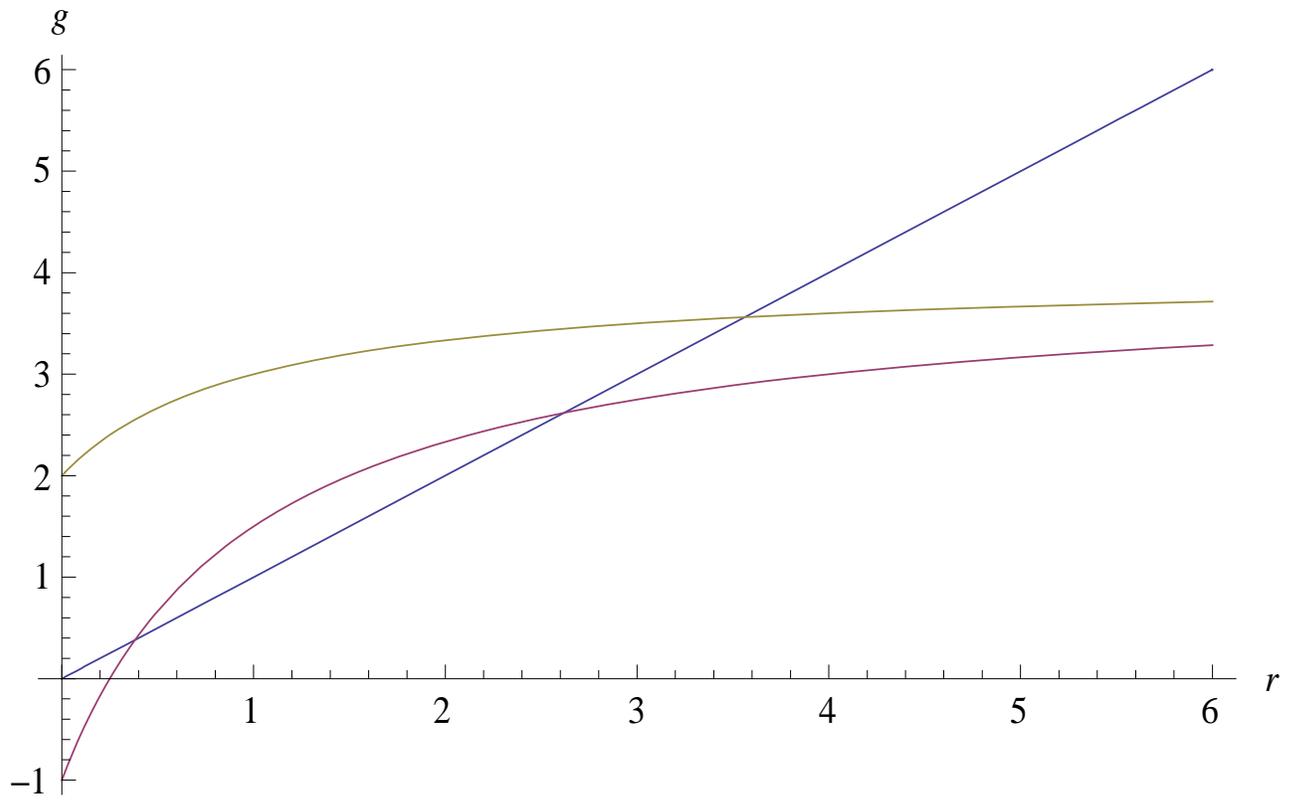
As  $f'' < 0$ , we may immediately infer that if we have  $f''' \leq 0$ , we will have  $g''(r_t) < 0$  (although this is not *necessary* to have  $g''(r_t) < 0$ ). In this case, we will have a unique equilibrium, provided there is no capital saturation at  $k = \underline{a}$ .

#### Examples

1. As in the rational expectations case, consider the Cobb Douglas production function with  $f(k) = k^\alpha$ . Here  $f'(\underline{a}) = \alpha(\underline{a})^{\alpha-1} > 0$ , so that the  $g$  function has a positive vertical intercept. Further,  $g'(r_t) = -\frac{a}{(1+r_t)^2} f'' \left( \frac{a}{1+r_t} \right) = \frac{a}{(1+r_t)^2} \alpha(1-\alpha) \left( \frac{a}{1+r_t} \right)^{\alpha-2} = \frac{\alpha(1-\alpha)}{(1+r_t)^\alpha \underline{a}^{1-\alpha}}$  and hence  $g''(r_t) = -\frac{\alpha^2(1-\alpha)}{(1+r_t)^{1+\alpha} \underline{a}^{1-\alpha}} < 0$ . Therefore, the criterion for a unique equilibrium is met. We can now check that

<sup>9</sup>The curves in Figure 2 are plotted for the specific example of a piecewise production function under varying assumptions (our example 2).

**Figure 2**



Notes : The lower curve is plotted for parameter values  $a=4, \underline{a}=2.5/b$ , the upper one for  $a=4, \underline{a}=1/b$

the function  $g$  has a unique fixed point in the positive domain. At this fixed point we have  $g(r) = r = \alpha \left[ \frac{1+r}{\underline{a}} \right]^{1-\alpha}$ . Consider  $\alpha=1/2$ ,  $\underline{a}=1/4$ . Then the fixed point solves  $r = (1+r)^{1/2}$  or squaring,  $r^2 - r - 1 = 0$ . This quadratic has one positive and one negative root. We can rule out the negative root, the positive root is  $r = \frac{1+\sqrt{5}}{2}$  which is the unique intersection of the  $g$  function with the 45 degree line (Figure 3).

2. Consider the piecewise quadratic production function with capital saturation

$$\begin{aligned} f(k) &= ak - bk^2 \text{ for } k < a/2b \\ f(k) &= \frac{a^2}{4b} \text{ for } k \geq a/2b. \end{aligned}$$

We have  $f'(k) = a - 2bk$ ,  $k < \frac{a}{2b}$  and  $f'(k) = 0$ ,  $k \geq \frac{a}{2b}$ . Let us assume first that  $f'(a) > 0$ , or equivalently that  $\underline{a} < \frac{a}{2b}$ , so that there is no capital saturation at  $k = \underline{a}$  and the  $g$  function starts above the 45 degree line (the upper curve in Figure 2). We may check that  $f''(k) = -2b$ ,  $f''' = 0$ . Therefore, we will always have  $g''(r_t) < 0$ . This indicates that there will be a unique intersection of the  $g$  function with the 45 degree line provided  $\underline{a} < \frac{a}{2b}$ . We now check this below. The fixed point solves  $g(r) = a - 2b \left( \frac{a}{1+r} \right) = r$ , or  $r^2 - (a-1)r - (a-2b\underline{a}) = 0$ . We assume that  $a > 1$ . Now given our assumption that  $\underline{a} < \frac{a}{2b}$ , we can see that though the sum of the roots is positive, their product is negative and hence we have only one root in the positive domain, which is given by  $r = \frac{a-1 + \sqrt{(a-1)^2 + 4(a-2b\underline{a})}}{2}$ .

However, suppose that there is capital saturation at  $k = \underline{a}$ . This implies that  $\underline{a} > \frac{a}{2b}$ . In this case, the  $g$  function corresponds to the x-axis until we reach  $r_t = \frac{2b\underline{a}}{a} - 1$ : at this point the  $g$  function becomes upward sloping. We still have  $g'' < 0$ . However, because the  $g$  function now starts below the 45 degree line, this no longer implies that there must be a unique intersection. In fact, we can check that the fixed points of  $g$  will solve the same quadratic as in the no-saturation case, but the difference will be that as  $\underline{a} > \frac{a}{2b}$ , the product of the two roots is positive so that both roots are admissible fixed points of the  $g$  function. We then have  $r_1, r_2 = \frac{a-1 \pm \sqrt{(a-1)^2 - 4(2b\underline{a}-a)}}{2}$ . We denote the smaller of these two roots by  $r_1$ . In this case,  $r_1$  is an unstable root (the  $g$  function cuts the 45 degree line from below here), and is bounded by two stable roots,  $r = 0$  and  $r = r_2$ . Thus if the initial capital stock is such that  $f'(k_0) < r_1$  (a relatively high capital stock) – the economy will converge to the stable equilibrium with  $r = 0$ ,  $k = \underline{a} > \frac{a}{2b}$ . If the initial capital stock is lower, so that  $f'(k_0) > r_1$ , we converge to the other stable equilibrium with  $r = r_2$ ,  $k = f'^{-1}(r_2) = \frac{a}{1+r_2}$ . This equilibrium has a higher interest rate and lower savings.

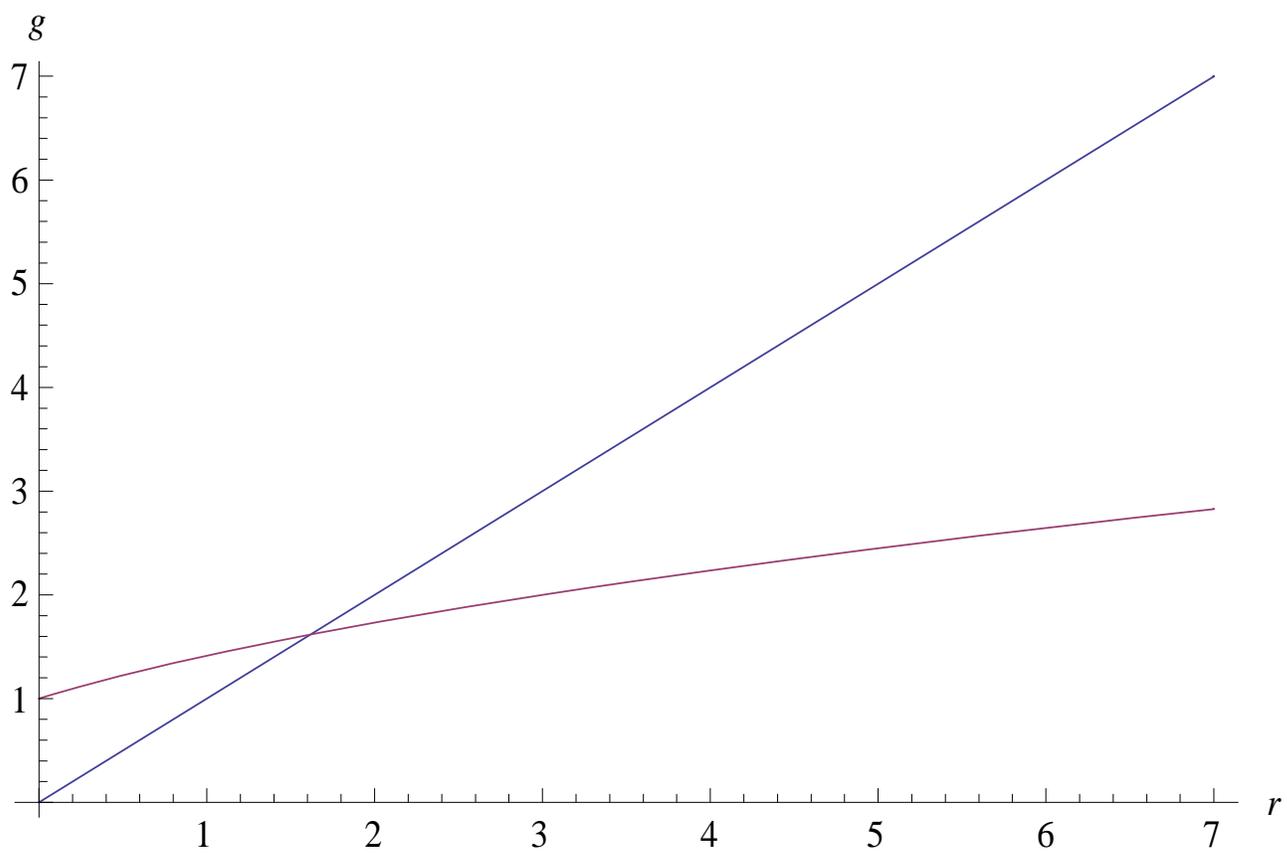
Capital saturation is not however necessary for multiple equilibria, as the following examples will show.

3. Consider the negative rectangular hyperbola

$$\begin{aligned} f(k) &= A - \frac{B}{k} \text{ for } k \geq B/A \\ f(k) &= 0 \text{ for } k < B/A. \end{aligned}$$

Now  $f'(a) = \frac{B}{a^2} > 0$  so again there is a positive vertical intercept. The fixed

**Figure 3**



points of the  $g$  function must solve the equation  $\frac{B}{a^2}(1+r)^2 = r$ , or  $r^2 - (\frac{a^2}{B} - 2)r + 1 = 0$ . Provided  $\underline{a}^2 > 2B$ , the equation has two real positive roots. In this case, at the smaller root the  $g$  function cuts the 45 degree line from above, and in the second it cuts it from below (Fig. 4).

4. Consider

$$f(k) = A + \frac{B}{k^2} - \frac{C}{k}, k > \frac{2B}{C}$$

$$f(k) = A - \frac{C}{4B}, k \leq \frac{2B}{C}$$

This technology reflects minimum size requirements and a marginal product of capital that initially increases and then diminishes. A minimum level of capital is needed to raise output beyond a certain level – only then does the marginal product of additional capital become positive. We assume that parameters are such that  $\underline{a} > \frac{2B}{C}$  so that  $f'(\underline{a}) = g(0) > 0$ . The fixed points of the  $g$  function must solve the equation

$$\frac{C}{\underline{a}^2}(1+r)^2 - \frac{2B}{\underline{a}^3}(1+r)^3 = r,$$

or

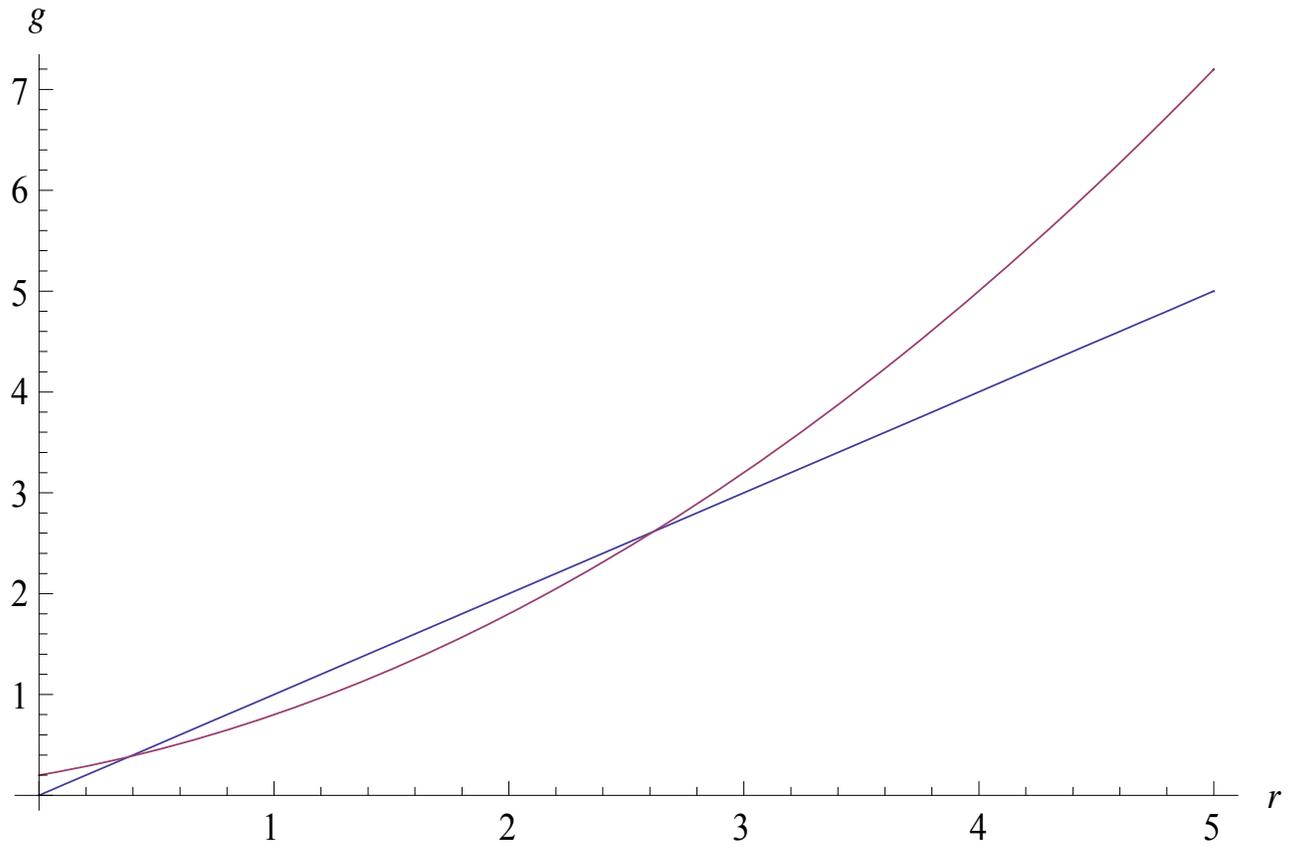
$$-2Br^3 + (\underline{a}C - 6B)r^2 - (\underline{a}^3 - 2\underline{a}C + 6B)r + \underline{a}C - 2B = 0$$

Using Descartes' rule of signs, the above cubic equation will have 3 positive roots if  $B > 0, \underline{a}C - 6B > 0, \underline{a}^3 - 2\underline{a}C + 6B > 0, \underline{a}C - 2B > 0$ . We have already assumed the first and last of these inequalities. Consider parameter values of  $B = 1/2, C = \frac{11}{(36)^{1/3}}, \underline{a} = (36)^{1/3}$ . We can check that these values satisfy all of the above conditions and that they yield 3 positive real roots,  $r = 1, r = 2$  and  $r = 5$ . We can also check that the equilibrium capital stocks corresponding to these three interest rates are, respectively,  $k = \frac{(36)^{1/3}}{2}, k = \frac{(36)^{1/3}}{3}$  and  $k = \frac{(36)^{1/3}}{6}$ , all of which are greater than  $k = \frac{2B}{C} = k = \frac{(36)^{1/3}}{11}$ , which is the minimum capital stock at which the marginal product of capital becomes positive. Of the interest rates, the middle one –  $r = 2$  – represents an unstable root while the other two – at  $r = 1$  and  $r = 5$  – are stable (Fig.5).

Thus, multiple equilibria can obtain even without capital saturation – we may have either an odd or an even number of intersections. Generalizing, we can say that in such cases, an odd number of intersections involves unstable equilibria bounded by stable equilibria – the intersections where the  $g$  function cuts the 45 degree line from above are stable, while those where it cuts the 45 degree line from below are unstable. This implies the existence of “traps” – initial historical capital stock fully determines which dynamic stable equilibrium the economy converges to. If the number of equilibria is odd, then starting from any initial capital stock, the economy always converges to some stable equilibrium. However, if the number of intersections is even, then if the initial capital stock is too small such that  $f'(k_0) > \bar{r}$ , where  $\bar{r}$  is the interest rate associated with the even-numbered fixed point with the largest interest rate, then we have an explosive situation of ever-growing interest rates and ever contracting capital stock. We will show below that in this situation it becomes increasingly likely that a bad shock wipes out the economy.

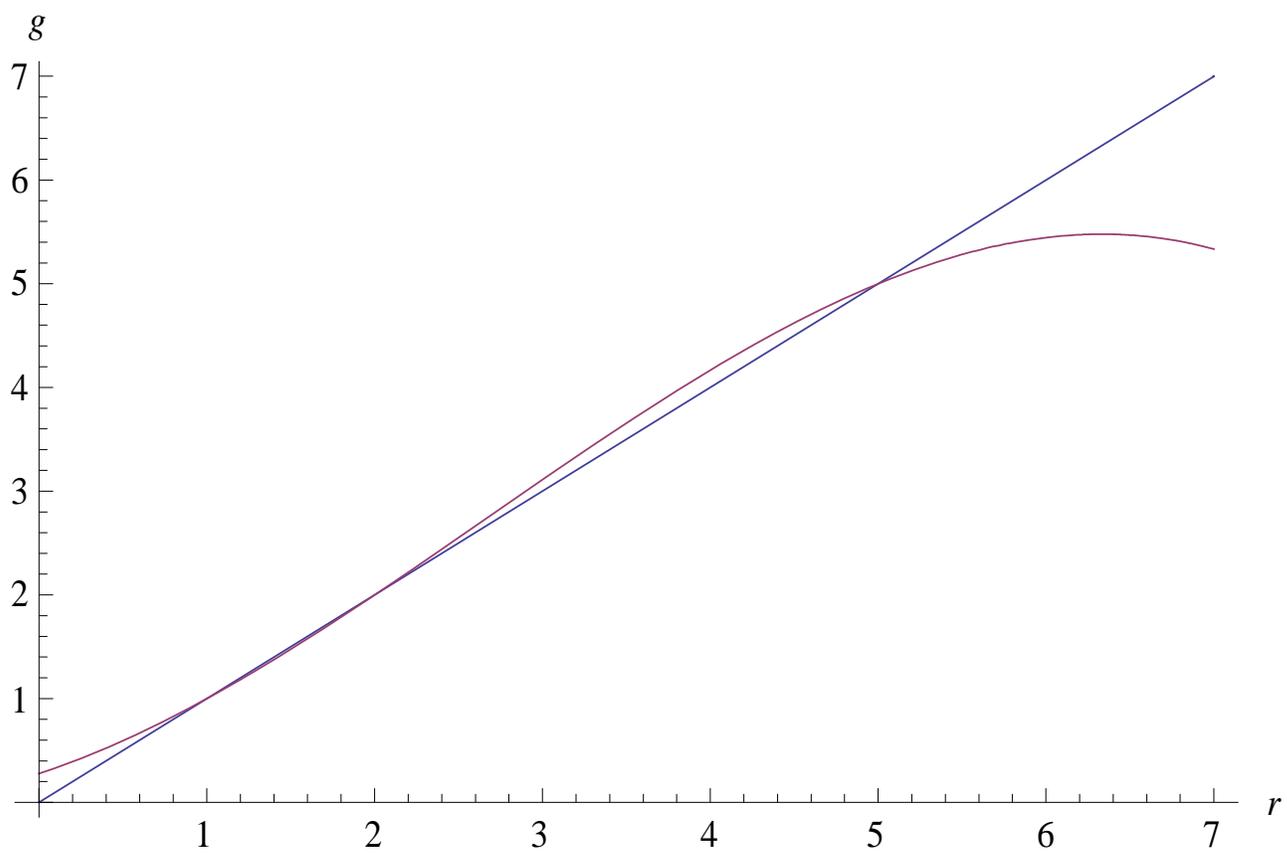
So far we have ignored the possibility that a bad shock may wipe out the economy by making it impossible for young people to have non-negative first

**Figure 4**



Note : This curve is plotted for parameter values of  $\underline{a}=\sqrt{5B}$ .

**Figure 5**



period consumption after paying their interest obligations and saving to meet their target. However, suppose we have

$$c_t^* < f\left(\frac{a}{1+r_{t-1}}\right) - \frac{a}{1+f'\left(\frac{a}{1+r_{t-1}}\right)} - a \text{ for all } t,$$

Since shocks are bounded, this implies that the economy will not disappear even with the most unfavorable shocks. Here, the first term on the right hand side represents output in period  $t$ , the second term represents the saving that needs to be done to meet next period's target, and the third term represents the repayments (inclusive of both principal and interest) to be made to the older generation. We can check that the right hand side is decreasing in  $r$ : the derivative is

$$-\frac{a}{(1+r_{t-1})^2} f'\left(\frac{a}{1+r_{t-1}}\right) + \frac{a}{(1+f'\left(\frac{a}{1+r_{t-1}}\right))^2} f''\left(\frac{a}{1+r_{t-1}}\right) < 0$$

So the inequality is more difficult to satisfy for large  $r$ . Suppose there are an odd number of fixed points - let  $r'$  be the interest rate at the stable equilibrium with the largest interest rate of all the fixed points. Then, if the inequality is satisfied at  $r = r'$ , it will be satisfied at all smaller interest rates. Then, all the dynamic equilibria will be robust to bad shocks. However, if the initial capital stock is very small such that  $f'(k_0) > r'$ , then although there will be a tendency to converge towards the equilibrium with  $r = r'$ , we cannot guarantee that a sufficiently bad shock will not wipe out the economy during its transition path to this equilibrium. If there are an even number of fixed points, then even if the inequality is satisfied at the fixed point with the largest interest rate, this is an unstable fixed point: if the initial capital stock was such that the initial interest rate was larger than this, we would have an explosive situation during which capital stock would progressively contract and  $r$  would become larger. Therefore the likelihood increases of a bad shock wiping out the economy in this case.

Thus static expectations introduce dynamics into our model. Where there is a unique equilibrium, rational and static expectations are similar in yielding a definite prediction. However, there is a difference when the production technology is such as to admit multiple equilibria. In the rational expectations model, the lack of dynamics and of an explicit co-ordination device results in indeterminacy. In the static expectations model, however, history determines which (stable) equilibrium the economy converges to in the long run. For some production functions (those yielding an even number of fixed points), a sufficiently small initial capital stock can result in long run instability with ever contracting capital stock and ever increasing interest rates. Of course, during this process it is likely that a bad shock may cause the economy to disappear.

The existence of "traps" in the case of multiple equilibria with static expectations means that the role of intervention becomes significant. An economy where people exhibit target saving behavior may converge to a dynamic steady state with high interest rate and low capital stock - and therefore low output - if the initial capital stock is low enough. However if the initial capital stock is a bit higher (enough to push the initial interest rate below that associated with the next unstable root), the economy will converge to a dynamic steady state with low interest rates and high capital stock (high savings and output). The

latter clearly Pareto dominates the former steady state because of higher overall output. This dependence of the ultimate fate of the economy on the initial capital stock implies that a one-time intervention – say, in the form of foreign aid – that increases the initial capital stock would have permanent effects on the long run fate of the economy. The possibility of a permanent Pareto improvement through a one-time intervention, such as acquiring foreign aid, increases the attractiveness of such measures from the perspective both of the country’s government and of the donor.

### 5. Further differences with models with traditional utility functions

In a previous section we have indicated how target saving behavior can be thought of as a special case of the more standard CRRA utility functions with specific – and different – values of the risk aversion parameters. Now we focus on other ways in which the target saving model differs in its results from models with traditional utility and saving functions.<sup>10</sup> We focus on two major distinctions.

First, conventional saving models are derived from utility functions like the symmetric CRRA form:

$$u = \frac{c_1^{1-\theta}}{1-\theta} + \frac{c_2^{1-\theta}}{(1+\rho)(1-\theta)}.$$

This yields the savings ratio

$$s(r) = \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r)^{(1-\theta)/\theta}}$$

The equation of motion becomes

$$k_{t+1} = s[f'(k_{t+1})][f(k_t) - k_t f'(k_t)].$$

With a Cobb-Douglas production function, the steady state value of  $k$  is thus the solution of the equation

$$k[(1+\rho)^{1/\theta} + (1+\alpha k^{\alpha-1})^{(1-\theta)/\theta}] = (1-\alpha)k^\alpha(1+\alpha k^{\alpha-1})^{(1-\theta)/\theta}$$

Evidently, the roots of the equation are functions of  $\rho$  and  $\theta$ , the rate of time preference and the coefficient of risk aversion. In contrast, though our saving model can be related to a specific parametrization of the asymmetric CRRA model, all our results have been established without using this specific form. They are independent of both the parameters  $\theta$  and  $\rho$ , and depend only on the characteristics of the production function and on the value of the target  $\underline{a}$ .

The results of the two models differ, not just in their quantitative dimensions, but qualitatively as well, in the number and characteristics of their equilibria. To illustrate, with our piece-wise quadratic production function, if  $\underline{a} > \frac{a}{2b}$  the target saving model yields an unstable equilibrium bounded by two stable equilibria. With the popular log-linear utility function  $u = \log c_1 + \log c_2/(1+\rho)$ , the same production function yields a unique root  $k = (a - 2 - \rho)/b$ .

The second major difference with traditional models lies in the fact that with rational expectations/perfect foresight, target saving only admits of stationary

<sup>10</sup>For examples of these see Azariadis (1993), Romer (1995). The latter summarizes the Diamond (1965) model. There has been a vast subsequent literature on OLG models of which a few instances are cited in our literature review section.

states, as pointed out in equation (4). Any deviation from stationary values (say through a gift of capital from abroad) is instantly eliminated. This also implies that if there are multiple stationary equilibria, indeterminacy and indeed breakdown will occur in the absence of a coordination mechanism. With more traditional utility functions, perfect foresight is compatible with a variety of time-paths and with multiple equilibria between which choice is determined by history. If we replace the assumption of perfect foresight by an adaptive model of expectation formation (illustrated here by static expectations), target saving too yields the full range of dynamic processes and equilibria for different kinds of production functions – as with conventional utility functions.

## 6. Conclusion

We have analyzed an overlapping generations model of “target savers” . “Target saving” is a form of behavior of which there has been surprisingly little prior theoretical modeling. We have attempted to provide an intuitive justification for this saving motive showing that within the framework of traditional CRRA utility functions target saving can be regarded as a special case where the coefficients of risk aversion take on specific and different values in the two periods of life in an OLG model. We have also compared our model more generally with more standard models and given examples of how its predictions differ from those of standard models for the same set of production functions. In our model, the older generation can insure themselves against all production shocks by setting a pre-negotiated interest rate on their loans to the productive younger generation. We have identified some conditions governing the existence of equilibrium, and the number of equilibria. We have given examples of a unique equilibrium and have also shown that multiple Pareto-rankable equilibria are possible. In the latter case we may have high-interest low-saving equilibria as well as low-interest high-saving equilibria. With rational expectations, indeterminacy arises when the technology admits multiple equilibria, due to the lack of dynamics. But when expectations are static, history enables us to pin down the long run fate of the economy, regardless of the technology. When the technology admits of multiple equilibria, there is the possibility that an economy may become trapped in a sub-optimal equilibrium due to unfavorable initial conditions, highlighting the long-run effects of a one-time intervention.

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