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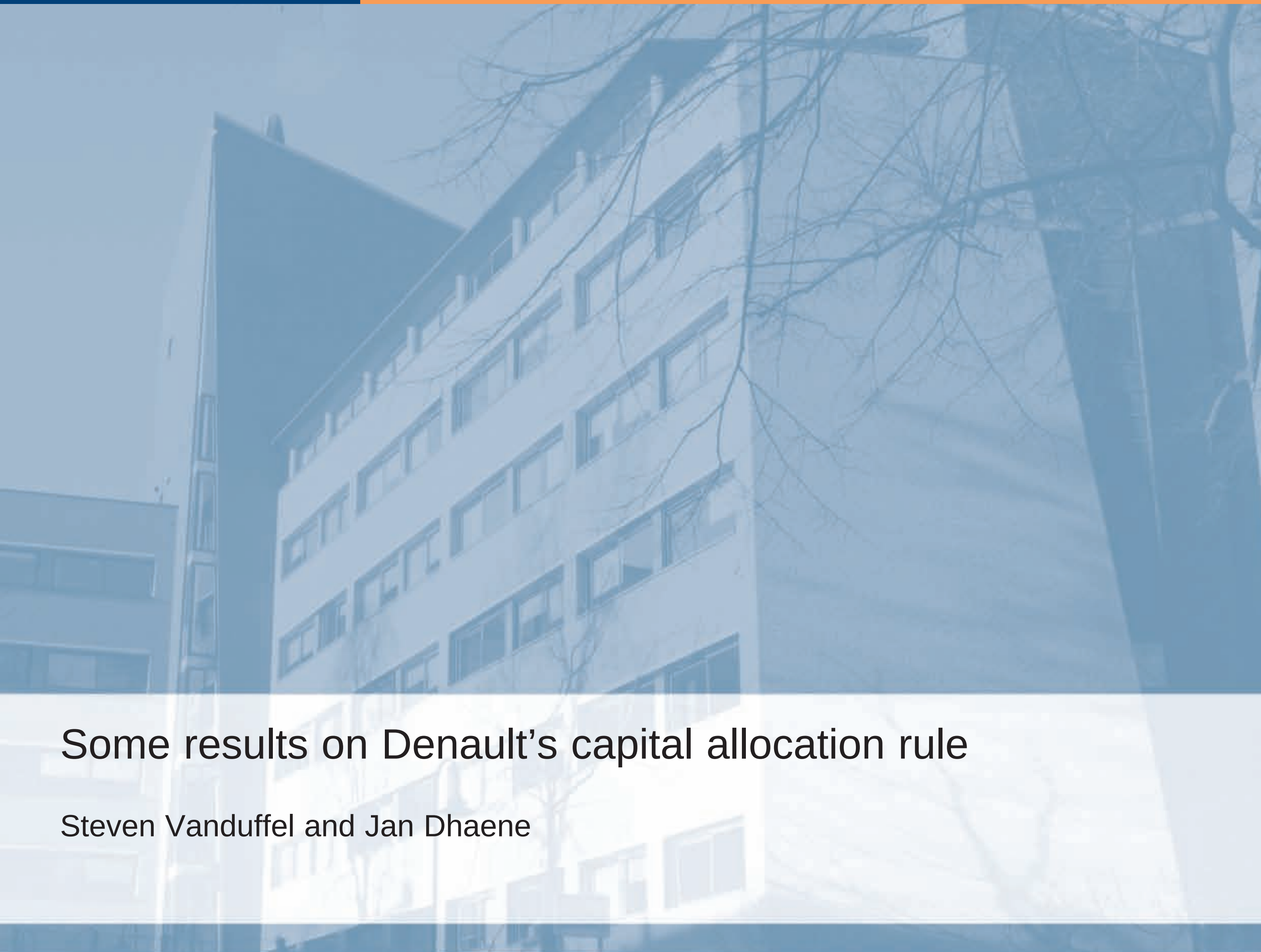

Some results on Denault's capital allocation rule

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Abstract

Denault (2001) introduces a capital allocation principle where the capital allocated to any risk unit is expressed in terms of the contribution of that risk to the aggregate conditional tail expectation. Panjer (2002) derives a closed-form expression for this allocation rule in the multivariate normal case. Landsman & Valdez (2003) generalize Panjer's result to the class of multivariate elliptical distributions.

In this paper we provide an alternative and much simpler proof for the allocation formula in the elliptical case. Further, we show how to derive accurate closed-form approximations for Denault's allocation formula in case of lognormal risks.

1 Introduction

Evaluating the total capital requirement of a financial conglomerate as well as the allocation of this capital to its various business units is an important risk management issue. Various capital allocation techniques have been proposed in literature. Dhaene, Valdez, Tsanakas & Vanduffel (2005) introduce a general capital allocation rule which is the solution of a distance optimisation problem. Several well-known allocation principles in literature turn out to be special cases of this general allocation rule, and hence can be seen as solutions of a particular optimisation problem. These authors also argue that no allocation formula is universally optimal. The choice of the appropriate allocation rule depends on problem specifications such as the purpose of the allocation and the manner in which the different risks under consideration interact.

Denault (2001) considers a capital allocation principle that is based on Tail Value-at-Risk. Starting from a total capital that is determined as the Conditional Tail Expectation at a given probability level, the capital allocated to any of the risk units involved is expressed in terms of the contribution of that risk to the aggregate Conditional Tail Expectation. Panjer (2002) provides a closed-form expression for this allocation rule in case the risks involved are multivariate normal.

Landsman & Valdez (2003) show how Panjer's result can be extended to the case where the risks are multivariate elliptically distributed. The proof of their result is rather technical and certainly not straightforward. In this paper we give an elegant and short proof for the Landsman & Valdez formula.

Moreover, using the main idea of our proof, we derive accurate closed-form approximations for Panjer's allocation formula in case the risks of the different units have a multivariate lognormal distribution.

2 Elliptical distributions

2.1 Definitions and some properties

In this section we recall some results concerning multivariate elliptical distributions. An extended reference to this class of distributions is Fang, Kotz & Ng (1990).

Definition 2.1 (Multivariate elliptical distributions). *The random vector* $\mathbf{X} = (X_1, \ldots, X_n)^T$ *is said to have an elliptical distribution with parameters the vector* $\mu(n \times 1)$ *and the matrix* $\mathbf{\Sigma}(n \times n)$ *if its characteristic function can be expressed as*

$$E\left[\exp\left(i\mathbf{t}^T\mathbf{X}\right)\right] = \exp\left(i\mathbf{t}^T\mu\right)\ \phi\left(\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\right), \qquad \mathbf{t}^T = (t_1, t_2, \ldots, t_n), \tag{1}$$

for some scalar function ϕ *and where* $\mathbf{\Sigma}$ *is given by* $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$ *for some matrix* $\mathbf{A}(n \times m)$.

The function ϕ is called the *characteristic generator* of $\mathbf{X}$. In case $\mathbf{X}$ has the elliptical distribution as defined above, we write $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$.

We denote the elements of μ and $\mathbf{\Sigma}$ by $\mu = (\mu_1, ..., \mu_n)^T$ and $\mathbf{\Sigma} = (\sigma_{kl})$ for $k, l = 1, 2, ..., n$. Since $\mathbf{\Sigma}$ is symmetric and positive definite, for any k and l, one has that $\sigma_{kl} = \sigma_{lk}$, whereas $\sigma_{kk} \geq 0$.

The moments of $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$ do not necessarily exist. However, in case of existence, they will be given by

$$\mathrm{E}\left[X_k\right] = \mu_k \tag{2}$$

and

$$\mathrm{Cov}\left[X_k, X_l\right] = -2\ \phi'(0)\, \sigma_{kl}. \tag{3}$$

In this paper we will always consider random variables with finite mean.

Consider the random vector $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$. For each k, one obviously has that X_k is also elliptically distributed, with the same characteristic generator:

$$X_k \sim E_1\left(\mu_k, \sigma_k^2, \phi\right) \tag{4}$$

with $\sigma_k^2 = \sigma_{kk}$. Further, let S be defined by

$$S = \sum_{j=1}^{n} X_j. \tag{5}$$

One can prove that also S is elliptically distributed with the same characteristic generator:

$$S \sim E_1\left(\mu_S, \sigma_S^2, \phi\right), \tag{6}$$

and with μ_S and σ_S^2 are given by

$$\mu_S = \sum_{j=1}^{n} \mu_j \text{ and } \sigma_S^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \sigma_{jk}, \tag{7}$$

respectively.

For the random vector $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$, one can prove that the following regression result holds:

$$E[X_k|S=s] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S^2}(s-\mu_S), \tag{8}$$

with $\sigma_{k,S}$ given by

$$\sigma_{k,S} = \sum_{j=1}^{n} \sigma_{kj}. \tag{9}$$

An elliptically distributed random vector $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$ does not necessarily possess a probability density function (pdf). However, in this paper we will only consider the subclass of elliptical distributions with pdf. This restricted class of elliptical distributions can be characterized as follows.

Definition 2.2 (Multivariate elliptical distributions with pdf). *The random vector* $\mathbf{X} = (X_1, ..., X_n)^T$ *with pdf* $f_{\mathbf{X}}(\mathbf{x})$ *is said to have an elliptical distribution with parameters* μ $(n \times 1)$ *and* $\mathbf{\Sigma}$ $(n \times n)$ *if its pdf can be expressed as*

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c}{\sqrt{\mathbf{\Sigma}}} g\left[(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right], \qquad \mathbf{x}^T = (x_1, x_2, \ldots, x_n) \tag{10}$$

for some $n \times m$ *matrix* $\mathbf{A}$ *such that* $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$. *Further, the normalizing constant* c *is given by*

$$c = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[\int_0^\infty x^{n/2-1} g(x)\, dx\right]^{-1} \tag{11}$$

and g *is a non-negative scalar function satisfying the condition*

$$0 < \int_0^\infty z^{n/2-1} g(x)\, dx < \infty. \tag{12}$$

The function g is called the *density generator*. In case $\mathbf{X}$ has the elliptical distribution as defined above, we write $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, g)$.

In the following example, we consider the multivariate normal distribution which is a member of the class of elliptical distributions.

Example 2.1 (Multivariate normal distributions). *Let* $\mathbf{X} \sim \mathbf{N}_n(\mu, \mathbf{\Sigma})$ *be a normal random vector with mean vector* μ *and variance-covariance matrix* $\mathbf{\Sigma}$. *Its pdf is given by*

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{\Sigma}|}} \exp\left[-\tfrac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right]. \tag{13}$$

Comparing (10) and (13), we immediately find that $\mathbf{N}_n(\mu, \mathbf{\Sigma})$ *is an elliptical distributions with density generator* g *and normalizing constant* c *given by*

$$g(u) = e^{-\frac{u}{2}} \tag{14}$$

and

$$c = \frac{1}{\sqrt{(2\pi)^n}}, \tag{15}$$

respectively.

For actuarial applications of elliptical distributions, see e.g. Landsman & Valdez (2003), Valdez & Dhaene (2005) or Valdez &Chernih (2005).

2.2 Conditional tail expectations

For a given probability level p, the Quantile $Q_p[X]$ and the Conditional Tail Expectation $CTE_p[X]$ of a random variable X are defined by

$$Q_p[X] = \inf\{x \mid F_X(x) \geq p\}, \qquad 0 < p < 1, \tag{16}$$

and

$$CTE_p[X] = \mathrm{E}[X \mid X > Q_p[X]],, \qquad 0 < p < 1, \tag{17}$$

respectively.

Let $X \sim E_1(\mu, \sigma^2, g)$. Landsman & Valdez (2003) prove that the Conditional Tail Expectation $\mathrm{CTE}_p[X]$ is given by

$$\mathrm{CTE}_p[X] = \mu + \sigma \frac{c}{2(1-p)} \int_{Q_p^2\left[\frac{X-\mu}{\sigma}\right]}^{\infty} g(x)\ dx, \qquad p \in (0,1), \tag{18}$$

with c being the appropriate normalizing constant as defined in (11).

In the following example, we consider the conditional tail expectation $\mathrm{CTE}_p[X]$ of a normally distributed random variable.

Example 2.2. *In case $X \sim N(\mu, \sigma^2)$, we have that*

$$\int_a^{\infty} g(x)\ dx = 2e^{-\frac{a}{2}}.$$

From this expression and (18), we find that the conditional tail expectation $CTE_p[X]$ is given by

$$CTE_p[X] = \mu + \sigma \frac{\Phi'\left(\Phi^{-1}(p)\right)}{1-p}, \qquad p \in (0,1). \tag{19}$$

where Φ denotes the standard normal cumulative distribution function (cdf) and Φ' the standard normal pdf.

2.3 Denault's allocation formula for multivariate elliptical risks

Recently, several authors propose Conditional Tail Expectation as a suitable risk measure for setting aggregate capital requirements of a financial institution, see for instance Wang (2002). For a discussion on the suitability of this risk measure, see Dhaene, Laeven, Vanduffel, Darkiewicz & Goovaerts (2005).

By the additivity property of the expectation operator, the conditional tail expectation allows for a natural allocation of the total capital among its various constituents. Indeed, based on the observation that

$$\mathrm{CTE}_p[X] = \sum_{k=1}^{n} \mathrm{E}[X_k | S > Q_p[S]], \tag{20}$$

where $S = X_1 + \cdots + X_n$, Denault (2001) proposes the allocation rule where the amount $E[X_k | S > Q_p[S]]$ is attribued to the k-th risk. Panjer (2002) derives closed-form expressions for the allocation $E[X_k|S > Q_p[S]]$ in the case of the risk vector $(X_1, X_2, \ldots, X_n)$ being multivariate normally distributed. Landsman & Valdez (2004) generalize Panjer's result to the class of multivariate elliptical distributions. In the following theorem we restate their result. We give an elegant and short proof for their result.

Theorem 2.1. *Let* $\mathbf{X} = (X_1, X_2, ..., X_n)^T \sim E_n(\mu, \mathbf{\Sigma}, g)$ *and let* $S = X_1 + \ldots + X_n$*. Then the contribution* $E[X_k|S > Q_p[S]]$ *of the* k*-th risk,* $1 \leq k \leq n$*, to the Conditional Tail Expectation* $CTE_p[X]$ *is given by*

$$E[X_k|S > Q_p[S]] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S^2}(CTE_p[S] - \mu_S), \qquad p \in (0,1). \tag{21}$$

Proof. From the Law of Total Probability, we find that

$$E[X_k|S > Q_p[S]] = \int_{Q_p[S]}^{\infty} E[X_k|S = s] \, dF_S(s \mid S > Q_p[S]). \tag{22}$$

Substituting the expression (8) for $\mathrm{E}[X_k|S = s]$ in (22) leads to (21). □

Substituting the expression (18) for $\mathrm{CTE}_p[S]$ in (21) leads to the following explicit expression for the contribution of the k-th risk:

$$E[X_k|S > Q_p[S]] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S} \times \frac{c}{2(1-p)} \int_{Q_p^2\left[\frac{X-\mu}{\sigma}\right]}^{\infty} g(x) \, dx, \qquad p \in (0,1). \tag{23}$$

As a special case of the previous theorem, we consider the multivariate normal distribution in the following example.

Example 2.3. *In case* $\mathbf{X} = (X_1, X_2, ..., X_n)^T \sim \mathbf{N}_n(\mu, \mathbf{\Sigma})$*, we find from (19) and (21) that* $E[X_k|S > Q_p[S]]$ *is given by*

$$E[X_k|S > Q_p[S]] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S} \times \frac{\Phi'(\Phi^{-1}(p))}{1-p}, \qquad p \in (0,1). \tag{24}$$

This expression for Denault's capital allocation rule in the multivariate normal case can be found in Panjer (2002).

3 Log-elliptical distributions

3.1 Definitions and some properties

The multivariate log-elliptical distribution is a natural generalization of the multivariate lognormal distribution.

For any n-dimensional vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ with positive components x_i, we define

$$\ln \mathbf{x} = (\ln x_1, \ln x_2, ..., \ln x_n)^T.$$

The random vector $\mathbf{X}$ is said to have a multivariate log-elliptical distribution if $\ln \mathbf{X}$ has an elliptical distribution.

We will denote $\ln \mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, g)$ as $\mathbf{X} \sim LE_n(\mu, \mathbf{\Sigma}, g)$. In the particular case that $\mathbf{X}$ is multivariate lognormally distributed, we write $\mathbf{X} \sim LN_n(\mu, \mathbf{\Sigma})$. One can prove that the density of $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c}{\sqrt{|\mathbf{\Sigma}|}} \left(\prod_{k=1}^{n} x_k^{-1}\right) g\left[(\ln \mathbf{x} - \mu)^T \mathbf{\Sigma}^{-1} (\ln \mathbf{x} - \mu)\right], \tag{25}$$

see e.g. Fang, Kotz & Ng (1987).

In the case of $\ln \mathbf{X}$ being characterized by its characteristic generator ϕ, we will denote $\ln \mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$ as $\mathbf{X} \sim LE_n(\mu, \mathbf{\Sigma}, \phi)$. Notice that $\mathbf{X} \sim LE_n(\mu, \mathbf{\Sigma}, \phi)$ implies that for any k, one has that $X_k \sim LE_1(\mu_k, \sigma_k^2, \phi)$.

Let $X \sim LE_1(\mu, \sigma^2, \phi)$. In Valdez & Dhaene (2004), the following expression is derived for the Conditional Tail Expectations of X:

$$\mathrm{CTE}_p[X] = \frac{e^{\mu}}{1-p} \phi\left(-\sigma^2\right) \Pr\left[Z^* > Q_p[Z]\right], \qquad 0 < p < 1, \tag{26}$$

where $Z = \frac{\ln X - \mu}{\sigma}$ and Z^* is a random variable with density given by

$$f_{Z^*}(x) = \frac{f_Z(x) e^{\sigma x}}{\phi(-\sigma^2)}.$$

The lognormal case is considered in the following example.

Example 3.1. *When $\ln X \sim N(\mu, \sigma^2)$, we write $X \sim LN(\mu, \sigma^2)$. In this case, we have that $\phi(-\sigma^2) = e^{\frac{\sigma^2}{2}}$ and $Z^* \sim N(\sigma, 1)$. From (26), we find that the Conditional Tail Expectations are given by*

$$CTE_p[X] = \frac{e^{\mu + \frac{\sigma^2}{2}}}{1-p} \Phi\left(\sigma - \Phi^{-1}(p)\right), \qquad p \in (0,1). \tag{27}$$

where Φ and Φ^{-1} denote the the standard normal cdf and its quantile function.

3.2 Denault's allocation formula for multivariate logelliptical risks

In this section, we consider Denault's allocation formula for a lognormal random vector $\mathbf{X} \sim LN_n(\mu, \mathbf{\Sigma})$. As before let S denote the aggregate risk $\sum_{k=1}^{n} X_k$. As it is not possible to derive an analytical expression for $\mathrm{E}[X_k \mid S > Q_p[S]]$ in this case, we will consider approximations for the contribution of the k-th risk to $\mathrm{CTE}_p[S]$.

We propose to approximate the random variable S by the random variable

$$S^l = \mathrm{E}[S \mid \Lambda] = \sum_{k=1}^{n} E[X_k \mid \Lambda], \tag{28}$$

where the conditioning random variable Λ is determined by

$$\Lambda = \sum_{k=1}^{n} \beta_k \ln(X_k) \tag{29}$$

for an appropriate choice of the coefficients β_k.

Since it obviously holds that $S \equiv$E$[S|S]$, one may expect that the random variable E$[S|\Lambda]$ will be 'close' to the random variable S, provided Λ is 'close' to S. We propose to approximate E$[X_k \mid S > Q_p[S]]$ by E$[X_k \mid S^l > Q_p[S^l]]$. Intuitively, it is clear that this approximation will perform well, provided the conditioning random variable Λ is a good approximation for S. We refer to Dhaene et al (2002b), Vanduffel et al (2005a) or Vanduffel et al (2005b) for discussions on how to choose Λ.

In the remainder of the paper we will assume that Λ is such that the correlation coefficients r_k defined by

$$r_k = \text{corr}[X_k, \Lambda], \qquad k = 1, 2, ..., n, \tag{30}$$

are non-negative. From Kaas et al. (2000) it follows that S^l is a sum of comonotonic risks in this case. This implies that any distortion risk measure $\rho[S^l]$ related to the sum S^l can be determined as the sum of the risk measures of the components in this sum:

$$\rho[S^l] = \sum_{k=1}^{n} \rho[E[X_k \mid \Lambda]], \tag{31}$$

see e.g. Dhaene et al (2002a,b). The accuracy of the approximation S^l of S is investigated in Dhaene et al. (2005) and Vanduffel et al. (2005).

The approximation E$[X_k \mid S^l > Q_p[S^l]]$ for the k-th contribution to CTE$_p[S]$ gives rise to a closed-form expression as shown in the next theorem.

Theorem 3.1. *Let* $\mathbf{X} = (X_1, X_2, ..., X_n)^T \sim LN_n(\mu, \mathbf{\Sigma})$ *and* $S = X_1 + \cdots + X_n$. *Furthermore, let* S^l *be defined by (28) with the conditioning variable* Λ *given by (29).*
Assuming that the correlation coefficients r_k *defined in (30) are non-negative, we have that*

$$E[X_k \mid S^l > Q_p[S^l]] = \frac{e^{\mu_k + \frac{\sigma_k^2}{2}}}{1-p} \Phi[r_k\, \sigma_k - \Phi^{-1}(p)], \qquad k = 1, 2, ..., n. \tag{32}$$

Proof. It is straightforward to prove that E$[X_k \mid \Lambda = \lambda]$ and S^l are given by

$$E[X_k \mid \Lambda = \lambda] = e^{\mu_k + \left(1 - r_k^2\right)\frac{\sigma_k^2}{2} + \rho_k \frac{\sigma_k}{\sigma_\Lambda}(\lambda - E[\Lambda])}$$

and

$$S^l = \sum_{k=1}^{n} e^{\mu_k + \left(1 - r_k^2\right)\frac{\sigma_k^2}{2} + r_k\, \frac{\sigma_k}{\sigma_\Lambda}(\Lambda - E[\Lambda])},$$

respectively. The assumption of non-negativity of the correlation coefficients r_k also implies that the quantiles of S^l are given by

$$Q_p[S^l] = \sum_{k=1}^{n} e^{\mu_k + \left(1 - r_k^2\right)\frac{\sigma_k^2}{2} + r_k\, \frac{\sigma_k}{\sigma_\Lambda}(Q_p[\Lambda] - E[\Lambda])},$$

for more details, see e.g. Dhaene et al. (2002b). Hence, it follows that the two following events are equivalent:

$$S^l > Q_p[S^l] \Leftrightarrow \Lambda > Q_p[\Lambda].$$

From these observations, we find

$$\begin{aligned}
\mathrm{E}[X_k \mid S^l > Q_p\left[S^l\right]] &= \mathrm{E}[X_k \mid \Lambda > Q_p\left[\Lambda\right]] \\
&= \int_{Q_p[\Lambda]}^{\infty} \mathrm{E}[X_k \mid \Lambda = \lambda]\ dF_\Lambda\left(\lambda \mid \Lambda > Q_p\left[\Lambda\right]\right) \\
&= e^{\mu_k + \left(1-r_k^2\right)\frac{\sigma_k^2}{2}} \int_{Q_p[\Lambda]}^{\infty} e^{r_k\ \sigma_k \frac{\lambda - \mathrm{E}[\Lambda]}{\sigma_\Lambda}}\ dF_\Lambda\left(\lambda \mid \Lambda > Q_p\left[\Lambda\right]\right) \\
&= e^{\mu_k + \left(1-r_k^2\right)\frac{\sigma_k^2}{2}} \mathrm{E}\left[e^{r_k\ \sigma_k \frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}} \mid \Lambda > Q_p\left[\Lambda\right]\right].
\end{aligned}$$

Notice that $Y = e^{r_k\ \sigma_k \frac{\Lambda - \mathrm{E}[\Lambda]}{\sigma_\Lambda}} \sim LN(0, r_k^2\ \sigma_k^2)$. The expression above can be rewritten as

$$\mathrm{E}[X_k \mid S^l > Q_p\left[S^l\right]] = e^{\mu_k + \left(1-r_k^2\right)\frac{\sigma_k^2}{2}} \mathrm{E}\left[Y \mid Y > Q_p\left[Y\right]\right]$$

Expression (32) follows then from expression (27) for the Conditional Tail Expectation of the lognormal distribution. □

The results of the theorem above can be used to construct approximations for Conditional Tail Expectations of sums of non-independent lognormal random variables. Using the same notations as in the theorem, one could approximate $\mathrm{CTE}_p\left[S\right]$ by $\sum_{k=1}^{n} \mathrm{E}[X_k \mid S^l > Q_p\left[S^l\right]$ with the components in this sum given by (32). Alternatively, following Dhaene et al. (2006), one could directly approximate $\mathrm{CTE}_p\left[S\right]$ by $\mathrm{CTE}_p\left[S^l\right]$. Making use of the additivity property of CTE for comonotonic risks, the latter approach leads to the following analytical expression:

$$CTE_p\left[S^l\right] = \sum_{k=1}^{n} \frac{e^{\mu_k + \frac{\sigma_k^2}{2}}}{1-p} \Phi[r_k\ \sigma_k - \Phi^{-1}(p)]. \tag{33}$$

Obviously, both approximation techniques are identical.

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