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PRODUCT MIXES AS OBJECTS OF CHOICE IN NONPARAMETRIC EFFICIENCY MEASUREMENT

by

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Abstract

Non-radial measures of technical efficiency essentially differ from their radial counterparts in that the product mix of the efficient reference is allowed to be different from the product mix of the evaluated observation. Whereas existing non-radial measures are still based on the product mix of the evaluated, i.e. possibly inefficient observation, we change the perspective and propose a measure based on the mix properties of the efficient reference. The resulting 'inverse' measure can be considered as complementary to the Färe-Lovell (or "Russell") efficiency measure.

Keywords: Data Envelopment Analysis, Non-radial efficiency measures, Product mix benchmarking

1 Introduction

The purpose of the original DEA model of Charnes, Cooper and Rhodes ([1], [2]) —as well as that of many of its successors— was to put into practice the notion of efficiency as expressed in the earlier work of Debreu [3] and Farrell [6]. Not surprisingly therefore, the associated "Debreu-Farrell" efficiency measure has become about as popular an analytical tool as DEA itself. As the inverse of Shephard's distance function the measure is closely linked to the microeconomic theory of production. Moreover, even when focusing on technical efficiency, Debreu-Farrell measures have a straightforward cost interpretation. Specifically, they can be written as the ratio of reference to actual costs (input orientation) or actual to reference revenues (output orientation), independently of the price vector that is used (see [10]).

This convenient property follows from the *radial* projection of inefficient observations on the reference frontier. The equiproportionate nature of comparisons has a drawback however, as it implies that Debreu-Farrell efficiency does not necessarily coincide with the more general Pareto-Koopmans efficiency concept (as introduced by Koopmans [8]). Radial measures face a slack problem: "efficient" projections can sometimes increase their technical efficiency by a further, non-radial change of some input or output dimensions.

Especially when slacks are large and occur frequently they could influence the efficiency scores considerably, which in turn could result in wrong management conclusions. In such cases it seems more appropriate to call for *non-radial* measures computed by comparing each observation to a reference observation that is not dominated in the Pareto-Koopmans sense. Non-radial measures essentially differ from their radial counterparts in that the input (output) mix of the input (output) efficient projection may differ from the mix of the inefficient observation. The Färe-Lovell [5] and the Zieschang [14] measures are two examples belonging to this class.[1] Färe-Lovell input efficiency scores, for example, are obtained starting from a "dimension-specific" correction of each component of the inefficient input mix (see below). So are the Zieschang scores as the latter are constructed as a simple product of a Debreu-Farrell and a Färe-Lovell component.

The central idea of this paper is that -once equiproportionality is no longer imposed- non-radial projection allows to measure inefficiency in two ways. The non-radial measures mentioned above still compute the distance between the inefficient observation and its efficient reference by starting from the mix of the inefficient observation. But one could equally well argue that the orientation may be reversed. Thus one can take the efficient reference itself rather than the inefficient observation as point of departure when computing the distance of an observation to the production frontier. In the next section we will propose an alternative non-radial measure which evaluates efficiency in this way. In a certain sense this opposed perspective "pushes the argument further" regarding the rationale of non-radial efficiency measurement. Any reference to properties of evaluated observations is further weakened and replaced by the search for a best practice benchmark on the basis of the latter's characteristics. The change of perspective is also clearly revealed by the formal representation of the associated efficiency measure which, in general, will lead to lower scores relative to its aforementioned counterparts. The rationale behind

[1] The former is sometimes also referred to as the Russell measure.

this result can also be shown graphically. Additional remarks are contained in the final section.

2 An 'inverse' Färe-Lovell measure

For ease of exposition we will concentrate on input efficiency in this section and postpone the discussion of the output-oriented version of the 'inverse' Färe-Lovell (FL) measure until the next section. First consider the original FL measure.[2] Let y represent a semi-positive output vector and x a strictly positive m-dimensional input vector and denote the input correspondence associated with each y by $L(y)$. The FL measure can now be defined as follows:

$$FL(x^o,y^o) = \min\left\{\sum_{i=1}^{m}\frac{\lambda_i}{m} \mid (\lambda_1 x_1^o,...,\lambda_i x_i^o,...,\lambda_m x_m^o)\in L(y^o),\ \ \lambda_i\in(0,1]\ \forall i\right\}$$

The FL measure minimises the arithmetic mean of the scalars λ_i, i.e. the proportional reduction in each input dimension. For an observation (x^o,y^o) the projection point (x^{o*},y^{o*}) is determined by scaling down each input by the corresponding element of the efficiency measure —i.e. $(x^{o*},y^{o*})=(\lambda_1^{fl}x_1^o,...,\lambda_m^{fl}x_m^o,y^o)$— and will always belong to the Pareto-Koopmans efficient subset of $L(y^o)$.

Kerstens and Vanden Eeckaut [7] have shown that the FL projection has a cost interpretation when the 'implicit' cost prices of the evaluated observation are used. Indeed, the same projection point results from cost minimisation under the assumption that the relative factor prices are revealed by the inverse ratio of the input quantities used by observation (x^o,y^o). The entries of the price vector w can thus be represented as $w_i/w_m = x_m^o/x_i^o$, with x_m the numeraire input. This link can be made explicit by rewriting the FL measure as follows:

$$\begin{aligned} FL(x^o,y^o) &= \min_{x^{o*}}\left(\frac{\sum_{i=1}^{m} w_i x_i^{o*}}{\sum_{i=1}^{m} w_i x_i^{o}}\right) = \min_{x^{o*}}\left(\frac{\sum_{i=1}^{m} \frac{w_i}{w_m} x_i^{o*}}{\sum_{i=1}^{m} \frac{w_i}{w_m} x_i^{o}}\right) \\ &= \min_{x^{o*}}\left(\frac{\sum_{i=1}^{m} \frac{x_i^{o*}}{x_i^{o}}}{\sum_{i=1}^{m} 1}\right) = \min_{x^{o*}}\left(\frac{\sum_{i=1}^{m}\lambda_i}{m}\right) \end{aligned}$$

with $(\lambda_1^{fl}x_1^o,...,\lambda_i^{fl}x_i^o,...,\lambda_m^{fl}x_m^o)\in L(y)$ and $\lambda_i^{fl}\in(0,1]$ $(\forall i)$. Implicit cost prices follow directly from proportions between inputs, so that the above result again demonstrates that the FL measure is computed starting from the inefficient input mix.

As stated in the introduction, an alternative way to measure inefficiency is to start from the efficient input mix.[3] This is achieved by making the implicit factor prices corresponding to the efficient projection an object of choice when minimising the ratio of

[2] A thorough discussion of the FL measure is provided by Färe, Grosskopf and Lovell [4].

[3] To some extent, an analogy can be drawn between the problem studied in this paper and the concepts of equivalent and compensating variation in neoclassical consumer theory. Indeed, the latter are alter-

minimum to actual costs. The resulting 'inverse' *FL* measure will no longer minimise the arithmetic mean of the scalars λ_i as becomes clear from the following:

$$
\begin{aligned}
inverse\ FL(x^o, y^o) &= \min_{x^{o*}} \left(\frac{\sum_{i=1}^{m} w_i^* x_i^{o*}}{\sum_{i=1}^{m} w_i^* x_i^o} \right) = \min_{x^{o*}} \left(\frac{\sum_{i=1}^{m} \frac{w_i^*}{w_m^*} x_i^{o*}}{\sum_{i=1}^{m} \frac{w_i^*}{w_m^*} x_i^o} \right) \\
&= \min_{x^{o*}} \left(\frac{\sum_{i=1}^{m} 1}{\sum_{i=1}^{m} \frac{x_i^o}{x_i^{o*}}} \right) = \min_{x^{o*}} \left(\frac{m}{\sum_{i=1}^{m} \lambda_i^{-1}} \right)
\end{aligned}
$$

where $w_i^*/w_m^* = x_m^{o*}/x_i^{o*}$, with x_m^* being the numeraire input. Again it holds that $(\lambda_1^{ifl} x_1^o, ..., \lambda_i^{ifl} x_i^o, ..., \lambda_m^{ifl} x_m^o) \in L(y)$ and $\lambda_i^{ifl} \in (0,1]$ $(\forall i)$. The fact that in this case the harmonic mean of the λ_i's is obtained has a straightforward intuition. As inefficiency is measured starting from the efficient input combination, one seeks for feasible *expansions* in all input dimensions (i.e. the λ_i^{-1}'s) to get from the endogenous efficient reference to the inefficient observation. The resulting score for the inefficient observation thus equals the inverse of the (arithmetic) mean of these scalars. Two results directly follow from the general properties of both means. First, the inverse *FL* score will never exceed the *FL* score. This increased stringency is intuitive precisely since the inverse measure further weakens any reference to (input mix) properties of the inefficient observation. Second, the two measures will coincide if and only if their projections coincide with the radial projection (i.e. if $\lambda_i^{fl} = \lambda_i^{ifl} = \lambda, \forall i$).[4]

Suppose there are N observations and that y is an n-dimensional vector. The inverse *FL* measure for an observation o can then be computed by solving the following mathematical programming problem:[5]

$$
\begin{aligned}
inverse\ FL_{input}^o &= \min_{\lambda,\theta} \left(\frac{m}{\sum_{i=1}^{m} \lambda_i^{-1}} \right) \qquad (1) \\
\text{subject to} &: \\
\lambda_i x_i^o &= \sum_{k=1}^{N} \theta_k x_i^k \ , \ i = 1, ..., m \\
y_j^o &\leq \sum_{k=1}^{N} \theta_k y_j^k \ , \ j = 1, ..., n
\end{aligned}
$$

natives to measure distances between indifference curves that differ w.r.t. the prices used to define the relevant tangent lines of consumption bundles: "original" (equivalent variation/*FL* measure) or "final" prices (compensating variation/inverse *FL* measure).

[4] Färe and Lovell [5] arrived at their non-radial measure using an axiomatic approach. It can easily be demonstrated that the inverse FL measure satisfies the same properties as the original FL measure.

[5] Note that here we implemented the assumption of variable returns to scale by means of the convexity constraint $\sum_{k=1}^{N} \theta_k = 1$. Of course, other constraints concerning the θ_k can be implemented in a straightforwardly analogous way.

$$\sum_{k=1}^{N} \theta_k = 1$$
$$\theta_k \geq 0 \ , \ k = 1, ..., N$$
$$\lambda_i \leq 1 \ , \ i = 1, ..., m$$

While the above problem may seem somewhat complex, we show in the appendix that for each observation the inverse FL score can be computed following a simple three step procedure, comparable to the pairwise vector comparison algorithm used in the FDH procedure (as e.g. outlined in [13]). Specifically, it suffices to concentrate on the vertices of the part of the reference technology that Pareto-Koopmans dominates the evaluated observation. In a second step the harmonic mean of input reductions with respect to each of these vectors is computed. Finally, the minimum of these harmonic means equals the inverse FL score.

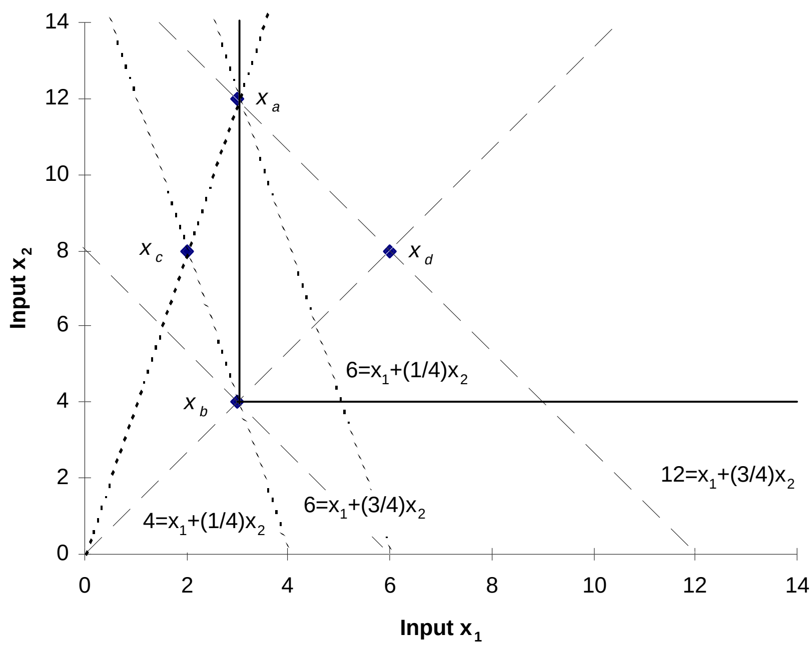

Figure 1: An illustrative example

The differences between both measures can also be represented graphically. In figure 1 the input vector x_a $(= (3, 12))$ is drawn. Suppose both the FL and the inverse FL measures have selected x_b $(= (3, 4))$ as its reference vector. The FL score will be $66,7\%$ whereas the inverse FL score only amounts to 50%. To interpret this result, we construct in a first step the implicit isocostlines associated with x_a and x_b. These lines take the form $\alpha_i = w_i^T x_i$ $(i = a, b)$ with w_a $(= (1, 3/12))$ and w_b $(= (1, 3/4))$ being the implicit cost price vectors.[6] We further construct an isocostline through x_a that is parallel to the implicit isocostline through x_b and, analogously, an isocostline through x_b that is parallel

[6] Input 1 is taken as the numeraire.

to the original isocostline through x_a. This allows to identify x_c and x_d which lie on the intersection of these newly constructed isocostlines and the radials through x_a and x_b, respectively. In fact x_c (x_d) has the same cost level as x_b (x_a), evaluated against w_a (w_b). It is then easily seen that the *FL* score of x_a coincides with its Debreu-Farrell score if x_c would be the reference, while the inverse *FL* score of x_a equals the (lower) radial score of x_d (compared to x_b).

The input combination x_d can also be considered as a real observation, and then demonstrates what happens when both non-radial complements yield the same projection as the Debreu-Farrell procedure, viz. x_b. If this is the case, it does not matter whether the implicit price vector of the efficient combination or the implicit price vector of the inefficient vector is used. Indeed, both orientations yield the same value for the ratio expression, which illustrates the point we made before.

Evidently, the *FL* and the inverse *FL* projection point need not necessarily coincide. This is shown in the following example. Suppose an input vector x_a (=$(8,8)$) is to be evaluated and that there are two possible reference points, viz. x_b (=$(2,8)$) and x_c (=$(5,4)$).[7] Figure 2 presents the original *FL* case. Therefore figure 2 also presents the implicit isocostline through x_a and its two parallels through x_b and x_c. It is immediately clear that the *FL* measure will select x_c as the reference input vector for x_a. The associated *FL* score equals $56,3\%$.

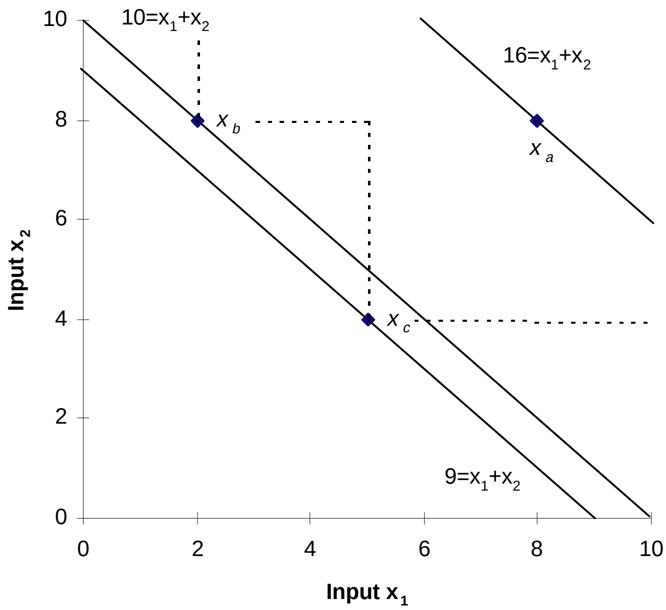

Figure 2: The *FL* case

The inverse *FL* measure will look for an efficient observation that minimises the same ratio of the reference to the actual cost level, but now evaluated using the reference's

[7] To facilitate discussion an FDH reference technology is assumed, since in this case only actually observed input combinations can serve as a reference point. FDH boils down to adding the integrality constraint $\theta_k \in \{0,1\}$ to the mathematical programming problem presented above (see Tulkens [13]).

implicit price vector. This case is presented in figure 3. The implicit isocostlines through x_b and x_c are drawn together with their parallels through x_a. The ratio value associated with x_b equals only 40% whereas the same value associated with x_c amounts to $55,5\%$. In fact the inverse FL measure penalises the inefficient observation x_a for using too much of the second input, which is accorded a relatively high implicit price by the efficient observation x_b. The inverse FL procedure will select x_b as the reference input vector, thus illustrating that the change of orientation may alter not only the efficiency scores but also the projection.

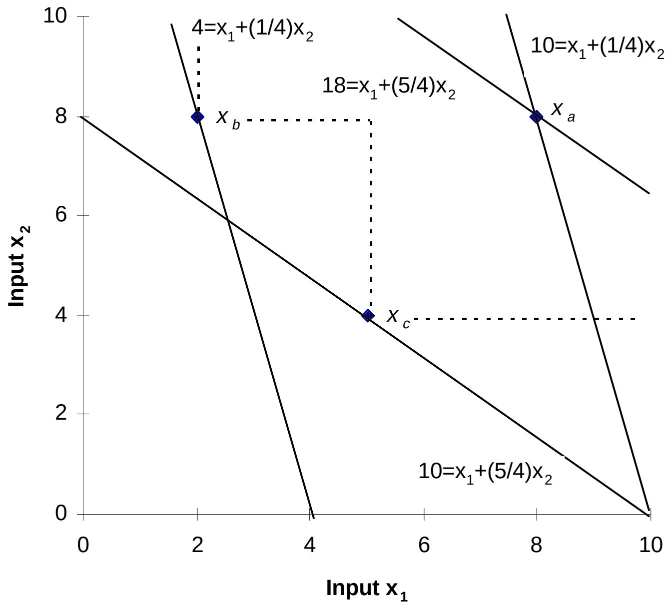

Figure 3: The inverse FL case

3 Additional remarks

In the previous section the input oriented version of the inverse FL measure was discussed. The output oriented version can be obtained in an analogous way. Suppose again that there are N observations for each of which a semi-positive m-dimensional input vector x and a strictly positive n-dimensional output vector y is observed. The inverse FL output score for an observation o is computed as the solution of the following mathematical programming problem:

$$\text{inverse } FL^o_{output} = \max_{\lambda,\theta} \left(\frac{n}{\sum_{j=1}^{n} \lambda_j^{-1}} \right) \tag{2}$$

subject to :

$$
\begin{aligned}
x_i^o &\geq \sum_{k=1}^{N} \theta_k x_i^k \ , \ i = 1, ..., m \\
\lambda_j y_j^o &= \sum_{k=1}^{N} \theta_k y_j^k \ , \ j = 1, ..., n \\
\sum_{k=1}^{N} \theta_k &= 1 \\
\theta_k &\geq 0 \ , \ k = 1, ..., N \\
\lambda_j &\geq 1 \ , \ j = 1, ..., n
\end{aligned}
$$

Again, we consider an equivalent formulation of this problem in the appendix. In contrast to the input oriented inverse FL measure, the output oriented version does *not* possess a straightforward implicit revenue price representation. The interpretation of the objective (2) is analogous to the one of (1), however. Inefficiency is again computed starting from the efficient observation and not by referring to the inefficient output mix.

By using the inverse input and output oriented FL measures to evaluate the efficiency of an observation, one considers the more demanding path to the frontier, which is clearly expressed by the objectives (1) and (2). As Lovell and Vanden Eeckaut [9] have shown for the original FL measure, one could also apply a less demanding procedure to analyse inefficiency if one is working in an FDH setting. A (non-convex) FDH reference technology is obtained by adding the integrality constraint $\theta_k \in \{0, 1\}$ to the respective mathematical models, and the set of possible references should be restricted to the undominated observations which at the same time dominate the o-th observation. Then, the direction of the objectives can be altered, which amounts to substituting (1) and (2) by respectively (3) and (4):[8]

$$
inverse\ FL^o_{input} = \max_{\lambda,\theta} \left(\frac{m}{\sum_{i=1}^{m} \lambda_i^{-1}} \right) \tag{3}
$$

$$
inverse\ FL^o_{output} = \min_{\lambda,\theta} \left(\frac{n}{\sum_{j=1}^{n} \lambda_j^{-1}} \right) \tag{4}
$$

A final note concerns the Zieschang [14] measure which is computed by first radially scaling down the input vector (scaling up the output vector) of an inefficient observation, so as to apply an FL input (output) projection in a second step. Zieschang scores are then computed as the product of the resulting Debreu-Farrell and FL scores. Clearly, an FL score different from unity is obtained only if the radial projection would result in a reference point lying on a Pareto-dominated part of the isoquant. It is evident then that

[8] In the example presented in figures 2 and 3 such a procedure would lead to a swap in the reference projections.

one could also construct an inverse Zieschang measure, computed as the same Debreu-Farrell score multiplied by an inverse FL score.[9]

Once the input-output mix of an inefficient observation has been allowed to vary when looking for better alternatives, at least to us it seems justifiable to minimise the "focal" character of this mix further. Thus, one can turn attention towards the mixes of observations that are known to perform better from the technical efficiency point of view. The concomitant change of perspective results in an 'inverse' FL measure that constitutes a valuable alternative for the original FL measure. It can be applied in exactly the same circumstances where the latter seems appropriate. Both alternatives provide the researcher with useful, complementary information about the non-radial efficiency of a Pareto-dominated observation.

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A Finding optima for the inverse FL problems

In this appendix we will discuss the solutions for problems (1) and (2). In section A.1 a procedure is presented to obtain the input oriented inverse FL score. We show that it is sufficient to compare evaluated observations only with the vertices of the Pareto-Koopmans dominant reference frontier. This procedure is very similar to the one which is used in an FDH framework and thus attractive given its ease of implementation. In appendix A.2 we introduce a problem which is equivalent to problem (2). It is shown that for this problem the necessary and sufficient Kuhn-Tucker optimality conditions are fulfilled. We also introduce the solution procedure under the assumption of a constant, a non-increasing and a non-decreasing returns to scale technology.

A.1 Input orientation

Clearly, minimising objective (1) is equivalent to maximising its inverse, which in turn can be expressed as:

$$\frac{1}{m}\sum_{i=1}^{m}\frac{x_i^o}{\left(\sum_{k=1}^{N}\theta_k x_i^k\right)}$$

Given the last (equivalent) representation, we now show that in an optimum for problem (1) the reference input vector will always be a vertex of the part of the efficient reference frontier that dominates the DMU to be evaluated. In other terms, the inverse

[9] Again, the inverse Zieschang measure satisfies the same axiomatic properties as its original counterpart.

FL projection will always be either an actually observed Pareto-Koopmans dominant point or a 'dimension specific efficient projection'. An example is provided in figure 4, where points 2 and 3 are the dimension specific efficient projections for observation 1. Figure 4 further illustrates that any potential reference for observation 1 (i.e. any element of the dominant part of the input production frontier) can be constructed as a convex combination of the vertices 2, 3 and 4, a result which also applies in general. The above assertion thus implies that to obtain the inverse FL score for observation 1, one only has to take the minimum of the inverse FL scores associated with references 2, 3 and 4.

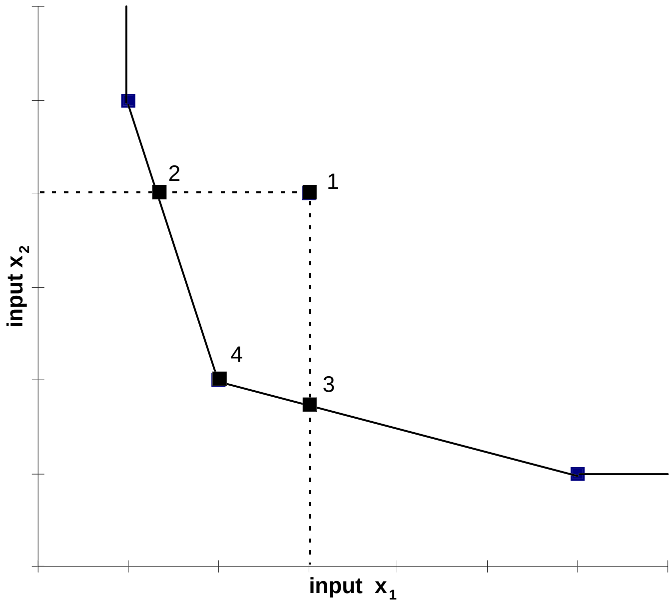

Figure 4

To see this, first consider a situation where an input vector x^o has to be compared only to convex combinations of two different dominant vectors x^1, x^2. In an optimum the reference will maximise the following function with respect to $\theta \in [0, 1]$:

$$g(\theta) = \sum_{i=1}^{m} \frac{x_i^o}{(\theta x_i^1 + (1-\theta)x_i^2)}$$

Clearly $g(1)$ and $g(0)$ give the values of the above function when x^o is evaluated against x^1 and x^2, respectively. Taking the second order derivative with respect to θ, one gets:

$$\frac{\partial^2 g(\theta)}{\partial \theta^2} = 2\sum_{i=1}^{m} x_i^o \frac{(x_i^1 - x_i^2)^2}{(\theta x_i^1 + (1-\theta)x_i^2)^3} > 0$$

That is, the function $g(\theta)$ is strictly convex. We thus have for $0 < \theta < 1$:

$$g(\theta) < \theta g(1) + (1-\theta)g(0) \tag{5}$$

So, if $g(0) \geq g(1)$ $[g(1) \geq g(0)]$, then $g(0)$ $[g(1)]$ maximises $g(\theta)$ for all possible $\theta \in [0,1]$, which implies that one only has to consider $g(0)$ and $g(1)$ when looking for a maximum : x^o has to be compared with x^1 and x^2 only. Equivalently, one only has to take the minimum over the harmonic means of input reductions of x^0 with reference to x^1 and x^2.

The extension to an arbitrary number N_o of vertices of the dominant part of the reference frontier is straightforward. Let X be an $N_o \times m$ matrix with rows corresponding to the vertex input vectors x^i $(i = 1, ..., N_o)$, and $\theta^T = (\theta_1, \theta_2, ..., \theta_{N_o})$ a row vector of intensity parameters with $\sum_{k=1}^{N_o} \theta_k = 1$. The objective to be maximised looks as follows:

$$g(\theta) = \sum_{i=1}^{m} \frac{x_i^o}{\left(\sum_{k=1}^{N_o} \theta_k x_i^k\right)}$$

The reference input vector$X^T\theta$ can be written as a linear combination of the vector x^1 and an input vector $(1/1-\theta_1)\, X_{-1}^T \theta_{-1}$ with $\theta_{-1}^T = (\theta_2, ..., \theta_{N_o})$ and X_{-1} an $(N_o - 1) \times m$ matrix with rows corresponding to $x^2, ..., x^{N_o}$:

$$X^T\theta = \theta_1 x^1 + (1-\theta_1)\left[\frac{1}{1-\theta_1} X_{-1}^T \theta_{-1}\right]$$

But then, applying the above reasoning either $\theta_1 = 1$ or $1 - \theta_1 = 1$. In the former case x^1 is the reference for x^o with respect to which the minimum inverse FL score is obtained. On the other hand, if $\theta_1 = 0$ one can restrict attention to the vector $X_{-1}^T \theta_{-1}$. We can then repeat the exercise to check whether $\theta_2 = 1$, and so on. In conclusion, there will be one input vector x^l $(l \in \{1, ..., N_o\})$ with $\theta_l = 1$ while for all other vectors the corresponding intensity parameter is zero. Therefore, it suffices to take the minimum of the harmonic means of input reductions with reference to the different rows of X. Convex combinations should not be considered as these would never generate a lower inverse FL score.

The three step procedure just described (identifying efficient vertices of the feasible region of problem (1) using e.g. the algorithm outlined in [11] (which is employed in the ADBASE code [12]), calculating the harmonic mean of input reductions with respect to each of these vectors, selecting the minimum of these harmonic means) is very similar in terms of computational requirements to the FDH procedure outlined in [13]. In the latter case, an equally simple vector comparison algorithm (identifying Pareto dominant DMU's, calculating the maximum dimension specific input shrinkage factor with respect to each of these vectors, selecting the minimum of these maximum factors) suffices to obtain efficiency measures.

So far we have restricted attention to a technology characterised by variable returns to scale, with $\sum_{k=1}^{N} \theta_k = 1$. However, the above algorithm applies equally well for technologies with non-increasing returns to scale ($\sum_{k=1}^{N} \theta_k \leq 1$), non-decreasing returns to scale ($\sum_{k=1}^{N} \theta_k \geq 1$) or constant returns to scale ($\sum_{k=1}^{N} \theta_k$ unbounded). Obviously, different scores may be obtained when the set of efficient vertices of the Pareto-Koopmans dominant region alters.

A.2 Output orientation

The problem as stated in (2) is equivalent to the programming problem below:

$$\left(1 \ / \ inverse \ FL^o_{output}\right) = \min_{\theta} \frac{1}{n} \sum_{j=1}^{n} \frac{y_j^o}{\left(\sum\limits_{k=1}^{N} \theta_k y_j^k\right)} \tag{6}$$

subject to :

$$x_i^o \geq \sum_{k=1}^{N} \theta_k x_i^k \ , \ i = 1, ..., m \tag{7}$$

$$y_j^o \leq \sum_{k=1}^{N} \theta_k y_j^k \ , \ j = 1, ..., n \tag{8}$$

$$\sum_{k=1}^{N} \theta_k = 1 \tag{9}$$

$$\theta_k \geq 0 \ , \ k = 1, ..., N$$

Clearly the constraints in this problem are all linear, so that strict convexity of the objective function is the only requirement for the Kuhn-Tucker conditions to be necessary and sufficient for a unique global minimum. Above we already showed that this function is convex if θ is two-dimensional. This result can be used to establish strict convexity with respect to a vector θ of dimension N. Indeed, strict convexity holds if for any two vectors θ_1 and θ_2 $(\theta_i' = (\theta_{i1}, ..., \theta_{iN}), \sum_k \theta_{ik} = 1 \ (i = 1, 2), \theta_1 \neq \theta_2)$ and a value $0 < \kappa < 1$ we have:

$$\kappa \sum_{j=1}^{n} \frac{y_j^o}{\left(\sum\limits_{k=1}^{N} \theta_{1k} y_j^k\right)} + (1-\kappa) \sum_{j=1}^{n} \frac{y_j^o}{\left(\sum\limits_{k=1}^{N} \theta_{2k} y_j^k\right)} > \sum_{j=1}^{n} \frac{y_j^o}{\left(\kappa \sum\limits_{k=1}^{N} \theta_{1k} y_j^k + (1-\kappa) \sum\limits_{k=1}^{N} \theta_{2k} y_j^k\right)}$$

The latter immediately follows from (5) as the $\sum_{k=1}^{N} \theta_{ik} y_j^k \ (i = 1, 2)$ are two elements of the production set. We conclude that for problem (6) the Kuhn-Tucker conditions are necessary and sufficient to characterise a unique global optimum. Denote the Lagrange multipliers corresponding to restrictions (7) and (8) by $\tau_i \ (i = 1..., m)$ and $\sigma_j \ (j = 1, ...n)$, respectively. With the convexity constraint (9) we associate the dual variable u. An optimal θ for problem (6), which also maximises objective (2), will solve the following system:

$$-\frac{1}{n} \sum_{j=1}^{n} \frac{y_j^l y_j^o}{\left(\sum\limits_{k=1}^{N} \theta_k y_j^k\right)^2} \geq \sum_{j=1}^{n} \sigma_j y_j^l - \sum_{i=1}^{m} \tau_i x_i^l + u \ , \ l = 1, ..., N$$

$$\theta_l \left[-\frac{1}{n} \sum_{j=1}^{n} \frac{y_j^l y_j^o}{\left(\sum\limits_{k=1}^{N} \theta_k y_j^k\right)^2}\right] = \theta_l \left[\sum_{j=1}^{n} \sigma_j y_j^l - \sum_{i=1}^{m} \tau_i x_i^l + u\right] \ , \ l = 1, ..., N$$

$$
\begin{aligned}
x_i^o &\geq \sum_{k=1}^{N} \theta_k x_i^k \ , \ i = 1, ..., m \\
\tau_i x_i^o &= \tau_i \sum_{k=1}^{N} \theta_k x_i^k \ , \ i = 1, ..., m \\
y_j^o &\leq \sum_{k=1}^{N} \theta_k y_j^k \ , \ j = 1, ..., n \\
\sigma_j y_j^o &= \sigma_j \sum_{k=1}^{N} \theta_k y_j^k \ , \ j = 1, ..., n \\
\sum_{k=1}^{N} \theta_k &= 1 \\
\theta_l &\geq 0 \ , \ l = 1, ..., N \\
\tau_i &\geq 0 \ , \ i = 1, ..., m \\
\sigma_j &\geq 0 \ , \ j = 1, ..., n \\
u &= free
\end{aligned}
$$

Analogously with the input oriented case, the above system also characterises the optimal solution under the assumptions of constant, non-increasing and non-decreasing returns to scale, after appending respectively $u = 0$, $u \geq 0$ and $u \leq 0$ and replacing $\sum_{k=1}^{N} \theta_k = 1$ by $\sum_{k=1}^{N} \theta_k = free$, $\sum\limits_{k=1}^{N} \theta_k \leq 1$ and $\sum_{k=1}^{N} \theta_k \geq 1$, in that order.

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