# Estimation and Decomposition of Downside Risk for Portfolios with Non-Normal Returns

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## Abstract

We propose a new estimator for Expected Shortfall that uses asymptotic expansions to account for the asymmetry and heavy tails in financial returns. We provide all the necessary formulas for decomposing estimators of Value at Risk and Expected Shortfall based on asymptotic expansions and show that this new methodology is very useful for analyzing and predicting the risk properties of portfolios of alternative investments.

Keywords: alternative investments; component expected shortfall; Cornish-Fisher expansion; downside risk; expected shortfall; portfolio; risk contribution; value at risk.

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# 1  Introduction

Value at Risk (VaR) and Expected Shortfall (ES) have emerged as industry standards for measuring downside risk. Despite the variety of complex estimation methods based on Monte Carlo simulation, extreme value theory and quantile regression proposed in the literature (see Kuester et al., 2006, for a recent review), many practitioners either use the empirical or the Gaussian distribution function to predict portfolio downside risk. The potential advantage of using the empirical distribution function over the hypothetical Gaussian distribution function is that only the information in the return series is used to estimate downside risk, without any distributional assumptions. The disadvantage is that the resulting estimates of VaR and ES, called historical VaR and ES, typically have a larger variation from out of sample observations than those based on a correctly specified parametric class of distribution functions. J.P.Morgan/Reuters (1996) parametric VaR methodology assumes the Gaussian distribution function. Gaussian VaR and ES neglect the well established fact that many financial time series are skewed and fat tailed. It is intuitively clear that incorporating the asymmetry and the thickness of tails of the density function into the downside risk estimates should lead to more accurate risk forecasts. This statement has been empirically verified by Giot and Laurent (2003).

Ideally, estimators of portfolio downside risk provide not only accurate estimates of the downside risk of the whole portfolio, but also allow decomposition into the component risk contribution of individual portfolio assets. Estimates of how much risk each asset in the portfolio contributes to the total portfolio risk are extremely important for portfolio risk allocation and for portfolio risk monitoring. They constitute a central tool to help financial institutions enforce a risk budget stating the bounds within which the risky asset positions have to remain (Sharpe, 2002).

Garman (1997), Gouriéroux et al. (2000) and Hallerbach (2002) show that the derivative of VaR with respect to the portfolio weight of an asset multiplied by the portfolio weight of that asset qualifies as a good estimate for the risk contribution of that asset, which is called "Component VaR" by Garman (1997). Under this approach, risk contributions can be readily computed for Gaussian VaR, but not for historical VaR, since this risk measure cannot be expressed as an analytical function of the portfolio weights.

Modified VaR as proposed by Zangari (1996) is an estimator for VaR that corrects Gaussian VaR for skewness and excess kurtosis in the return series while preserving the ability to decompose portfolio risk into the components due to the different assets in the portfolio. It relies on adjusting the Gaussian quantile function for skewness and kurtosis, using the Cornish-Fisher expansion (Cornish and Fisher, 1937). Modified VaR has been used by Favre and Galeano

(2002), Gueyié and Amvella (2006) and by Amenc et al. (2003) to construct mean-VaR efficient portfolios and by Qian (2006) to do risk attribution for a balanced portfolio.

Motivated by the popularity of modified VaR in applied finance, we derive a definition for modified ES that, like modified VaR, uses asymptotic expansions to adjust the Gaussian distribution function for the non-normality in the observed return series. In both the theoretical case of a skewed student $t$ distribution and the empirical case of a portfolio of hedge fund style indices, we show the gain in estimation quality of using modified VaR and ES rather than Gaussian VaR and ES when estimating downside risk.

We also address the need of the financial industry to have decomposable risk measures that have a financial interpretation. We are the first to give all the formulas needed for decomposing modified VaR and ES into the risk contributions of the assets in the portfolio and to illustrate their practical use for a portfolio of alternative investments. An important finding for portfolio allocation is that our estimator for contribution to ES appears to be a good predictor of future contribution to ES. We believe that the practical application of risk decomposition to a portfolio containing multiple assets with non-normal distributions should allow for portfolio construction that more closely resembles investor preferences, and allow risk managers to better monitor and control risk in the portfolio.

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The remainder of this paper is organized as follows. In Section 2, we review important results from the literature on computation and attribution of portfolio risk. Section 3 first introduces the Edgeworth and Cornish-Fisher expansion and then uses them to define modified ES. Section 4 investigates how well modified VaR and ES approximate VaR and ES for the skewed Student $t$ distribution. For computing modified VaR and ES we need estimates of the multivariate moments of the return series. Since higher order moments such as skewness and kurtosis are extremely sensitive to outliers, we propose a robust estimation scheme in Section 5. The usefulness of this new methodology is illustrated in Section 6 where we analyze downside risk of the maximum Sharpe ratio portfolio of different hedge fund investment style indices. Section 7 summarizes our conclusions and outlines the implications for further research.

# 2 Computation and decomposition of portfolio risk

This section reviews useful results from the literature on computation and decomposition of portfolio risk. We focus on the parametric approach assuming the returns to be Gaussian distributed and the Cornish-Fisher approach under which the returns can be non-normal. We consider an investor who allocates his portfolio between $n$ assets, with weights $w = (w_1, ..., w_n)'$. We stack the $n$ asset returns into the random vector $r = (r_1, ..., r_n)'$, which we assume to be

strictly stationary with mean $\mu$ and covariance matrix $\Sigma$. It follows that the portfolio return $r_p$ has mean $w'\mu$ and variance $w'\Sigma w$. Under the additional assumption that the portfolio return distribution $F(\cdot)$ is continuous[1], the VaR and ES of $r_p$ as a function of $\alpha$, are defined as follows

$$\text{VaR}(\alpha) = -F^{-1}(\alpha)$$
$$\text{ES}(\alpha) = -E_F\left[r_p | r_p \leq F^{-1}(\alpha)\right],$$
(1)

with $F^{-1}(\cdot)$ the quantile function associated to $F(\cdot)$ and $E_F[\cdot|\cdot]$ the operator that takes the conditional expectation under $F(\cdot)$.

*Portfolio risk decomposition.* For the purpose of portfolio risk decomposition, we follow Martin et al. (2001) in requiring the risk measures to be 1-homogeneous, meaning that if the weight vector is multiplied by some scalar $b$, then also these risk measures are multiplied by $b$. From a mathematical perspective, risk decomposition is straightforward for such risk measures, thanks to Euler's homogeneous function theorem stating that for 1-homogenous $f(w)$, we have

$$f(w) = \sum_{i=1}^{n} w_i \partial_i f(w),$$

where $\partial_i f(w) = \partial f(w)/\partial w_i$. Under this decomposition, the Contribution of asset $i$ to the risk measure $f(w)$, $C_i f(w)$, and its percentage Contribution, $\%C_i f(w)$, which are also called the (percentage) Component of asset $i$ in the

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[1]Definitions of VaR and ES that apply also to discontinuous distribution functions can be found in Acerbi and Tasche (2002).

portfolio risk measure $f(w)$, equal

$$C_i f(w) = w_i \partial_i f(w) \quad \text{and} \quad \%C_i f(w) = C_i f(w)/f(w). \tag{2}$$

Sharpe (2002) warns that a mere mathematical decomposition of portfolio risk does not necessarily qualify as risk contribution. Gouriéroux et al. (2000) and Qian (2006), however, show that for VaR, this mathematical decomposition of portfolio risk has a financial meaning. It equals the negative value of the asset's expected contribution to the portfolio return when the portfolio return equals the negative portfolio VaR:

$$C_i \text{VaR}(\alpha) = w_i \partial_i \text{VaR}(\alpha) = -E\left[w_i r_i | r_p = -\text{VaR}(\alpha)\right]. \tag{3}$$

Scaillet (2002) proves that contribution to $\text{ES}(\alpha)$ can be interpreted as the expected contribution to portfolio return when the portfolio return is at least the negative value of $\text{VaR}(\alpha)$:

$$C_i \text{ES}(\alpha) = w_i \partial_i \text{ES}(\alpha) = -E\left[w_i r_i | r_p \leq -\text{VaR}(\alpha)\right]. \tag{4}$$

*Derivative of portfolio moments.* The definition of risk contributions in (2) will only be useful in practice for risk measures for which the derivative with respect to the portfolio weights can be easily computed. This paper considers downside risk measures that depend on the portfolio moments. It is computationally convenient to express the portfolio moments as a function of

the multivariate moments of the returns on the underlying assets, using the $N \times N^2$ co-skewness matrix

$$M_3 = E\left[(r - \mu)(r - \mu)' \otimes (r - \mu)'\right]$$

and $N \times N^3$ co-kurtosis matrix

$$M_4 = E\left[(r - \mu)(r - \mu)' \otimes (r - \mu)' \otimes (r - \mu)'\right],$$

where $\otimes$ stands for the Kronecker product (see e.g. Jondeau and Rockinger, 2006). Under this representation, the derivatives of the portfolio moments are easy to compute. Denote the $q$-th centered portfolio moment $m_q = E\left[(r_p - w'\mu)^q\right]$ and let $\partial_i m_q$ be its partial derivative with respect to $w_i$. We have that

$$
\begin{aligned}
m_2 &= w'\Sigma w & \partial_i m_2 &= 2(\Sigma w)_i \\
m_3 &= w'M_3(w \otimes w) & \partial_i m_3 &= 3(M_3(w \otimes w))_i \\
m_4 &= w'M_4(w \otimes w \otimes w) & \partial_i m_4 &= 4(M_4(w \otimes w \otimes w))_i.
\end{aligned}
\tag{5}
$$

The portfolio skewness $s_p$ and excess kurtosis $k_p$ and their partial derivative are then given by

$$
\begin{aligned}
s_p &= m_3/m_2^{3/2} & \partial_i s_p &= \left(2m_2^{3/2}\partial_i m_3 - 3m_3 m_2^{1/2}\partial_i m_2\right)/2m_2^3 \\
k_p &= m_4/m_2^2 - 3 & \partial_i k_p &= (m_2\partial_i m_4 - 2m_4\partial_i m_2)/m_2^3.
\end{aligned}
\tag{6}
$$

*Gaussian VaR.* The portfolio standard deviation, VaR and ES are all 1-homogenous functions. For portfolio standard deviation, for example, the risk

contribution of asset $i$ is given by $0.5w_i\partial_i m_2/\sqrt{m_2}$. For many estimation methods, the computation of the derivative of the estimated VaR and ES is challenging because the estimator cannot be expressed as an explicit function of the portfolio weights. A notable exception is when VaR and ES are computed under the assumption of normality (Garman, 1997). Replacing $F(\cdot)$ by the Gaussian distribution function in (1), we obtain the following expressions for Gaussian VaR (GVaR) and ES (GES)

$$\text{GVaR}(\alpha) = -w'\mu - \sqrt{m_2}\Phi^{-1}(\alpha)$$
$$\text{GES}(\alpha) = -w'\mu + \sqrt{m_2}\frac{1}{\alpha}\phi[\Phi^{-1}(\alpha)],$$

(7)

where $\phi(\cdot)$, $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ are the standard Gaussian density, distribution and quantile functions, respectively. Note that Gaussian VaR and ES depend only on the portfolio mean and variance. Hence, risk contributions (2) can be readily computed, using the partial derivatives

$$\partial_i\text{GVaR}(\alpha) = -\mu_i - \frac{\partial_i m_2}{2\sqrt{m_2}}\Phi^{-1}(\alpha)$$
$$\partial_i\text{GES}(\alpha) = -\mu_i + \frac{\partial_i m_2}{2\sqrt{m_2}}\frac{1}{\alpha}\phi[\Phi^{-1}(\alpha)].$$

(8)

The partial derivative of the portfolio variance, $\partial_i m_2$, is given in (5).

*Modified VaR.* Gaussian VaR estimates VaR utilizing only the first two portfolio moments. This approach is no longer optimal for portfolios with non-normal returns. For this reason, Zangari (1996) proposed to generalize Gaussian VaR by correcting the Gaussian quantile for the portfolio skewness and excess kurtosis using series expansions around the normal distribution. For

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this, it is convenient to first express the portfolio return under its location-scale representation

$$r_p = w'\mu + \sqrt{m_2}u, \tag{9}$$

where $u$ is a zero mean, unit variance random variable with distribution function $G(\cdot)$. Under the location-scale representation (9), the VaR of the portfolio return equals

$$\text{VaR}(\alpha) = -w'\mu - \sqrt{m_2}\, G^{-1}(\alpha) \tag{10}$$

with $G^{-1}(\cdot)$ the quantile function associated to $G(\cdot)$. In an empirical setting, $G(\cdot)$ is generally assumed to be normal. The approximation can be improved by adjusting it for higher moments in the data. This can be done, using the $r^{th}$ order Edgeworth expansion of $G(\cdot)$ around the standard Gaussian distribution function $\Phi(\cdot)$:

$$G_r(z) = \Phi(z) - \phi(z)\sum_{i=1}^{r} P_i(z), \tag{11}$$

where $P_i(z)$ is a polynomial in $z$. The corresponding $r^{th}$ order Cornish-Fisher expansion of the quantile function $G^{-1}(\cdot)$ around the Gaussian quantile function $\Phi^{-1}(\cdot)$, equals

$$G_r^{-1}(\alpha) = z_\alpha + \sum_{i=1}^{r} P_i^*(z_\alpha), \tag{12}$$

with $z_\alpha = \Phi^{-1}(\alpha)$. Exact formulas for the first eight terms in the Edgeworth and Cornish-Fisher expansions can be found in Draper and Tierney (1973).

We only need the terms relating to the second order expansion:

$$P_1(z) = P_1^*(z) = \frac{1}{6}(z^2 - 1)s_p$$

$$P_2(z) = \frac{1}{24}(z^3 - 3z)k_p + \frac{1}{72}(z^5 - 10z^3 + 15z)s_p^2$$

$$P_2^*(z) = \frac{1}{24}\left(z^3 - 3z\right)k_p - \frac{1}{36}\left(2z^3 - 5z\right)s_p^2,$$

where $s_p$ and $k_p$ are the skewness and excess kurtosis of the portfolio return.

For a loss probability $\alpha$, Zangari (1996)'s new estimator for VaR, called modified VaR (mVaR), equals the negative value of the $\alpha$ second[2] order Cornish-Fisher quantile of the portfolio return distribution:

$$\text{mVaR}(\alpha) = -w'\mu - \sqrt{m_2}G_2^{-1}(\alpha) \tag{13}$$

$$= \text{GVaR}(\alpha) + \sqrt{m_2}\left[-\frac{1}{6}(z_\alpha^2 - 1)s_p - \frac{1}{24}(z_\alpha^3 - 3z_\alpha)k_p + \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)s_p^2\right].$$

Note that when skewness and excess kurtosis are zero, which is the case under normality, modified VaR equals Gaussian VaR. The component of asset $i$ in the portfolio's mVaR is fairly easy to compute, using the following partial derivative:

$$\partial_i\text{mVaR}(\alpha) = \partial_i\text{GVaR}(\alpha)$$

$$+ \frac{\partial_i m_2}{\sqrt{m_2}}\left[-\frac{1}{12}(z_\alpha^2 - 1)s_p - \frac{1}{48}(z_\alpha^3 - 3z_\alpha)k_p + \frac{1}{72}(2z_\alpha^3 - 5z_\alpha)s_p^2\right]$$

$$+ \sqrt{m_2}\left[-\frac{1}{6}(z_\alpha^2 - 1)\partial_i s_p - \frac{1}{24}(z_\alpha^3 - 3z_\alpha)\partial_i k_p + \frac{1}{18}(2z_\alpha^3 - 5z_\alpha)s_p\partial_i s_p\right] \tag{14}$$

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[2]Other authors have considered higher order Cornish-Fisher expansions but find that increasing the order $r$ in (12) does not necessarily improve the approximation, see e.g. Baillie and Bollerslev (1992) and Jaschke (2002).

It is thus possible both to compute mVaR numerically, and to decompose this into the risk contributions of the different assets in the portfolio using the equations presented above. This result should be very useful to a portfolio or risk manager wishing to understand the contribution to total risk from each element of the portfolio. This result may also be applied to a variety of portfolio construction and optimization approaches.

# 3   Modified expected shortfall

Downside risk measures describe the left tail of the return distribution. For a loss probability $\alpha$, it is interesting not only to estimate the location of this tail quantile, which is VaR, but also to have a central estimate of this tail, which is the Expected Shortfall (ES) as defined in Section 1. Given that ES is a useful downside risk measure, it is natural to consider an estimator of portfolio ES which, like modified VaR, uses asymptotic expansions to take into account the skewness and excess kurtosis in the asset returns.

Christoffersen and Gonçalves (2005) use the result that in the special case of the Gaussian density function, ES is given by $-w'\mu + \sqrt{m_2}\phi[\Phi^{-1}(\alpha)]/\alpha$. They propose to replace the Gaussian density and quantile functions in this formula by their asymptotic expansions. Their estimator for ES is defined as follows

$$\text{CGmES}(\alpha) = -w'\mu + \sqrt{m_2}\phi_{\text{gc}}[G_2^{-1}(\alpha)]/\alpha, \tag{15}$$

where $\phi_{\mathrm{gc}}(z) = \phi(z)[1 + s_p(z^2 - 1)/6 + k_p z(z^2 - 3)/24]$ is the Gram-Charlier approximation of the Gaussian density function and $G_2^{-1}(\alpha)$ is the $\alpha$ Cornish-Fisher quantile of order 2. Note that the Gram-Charlier expansion coincides with the Edgeworth expansion of order 2 in which the terms in $s_p^2$ have been removed. Such a definition of modified ES is not coherent with the general definition of ES in (1), since the expected value of the returns below the negative value of mVaR, calculated under the assumption that the standardized returns have density function $\phi_{\mathrm{gc}}(z)$, will, in general, not be equal to CGmES.

At first sight, one could directly extend modified VaR to modified ES using the result that for continuous distribution functions

$$\mathrm{ES}(\alpha) = \frac{1}{\alpha} \int_{\beta=0}^{\alpha} \mathrm{VaR}(\beta) d\beta. \tag{16}$$

Modified ES would then be defined as the right hand side of the equation (16) in which $\mathrm{VaR}(\beta)$ is replaced with $\mathrm{mVaR}(\beta)$. In practice, this is not a good idea because as we will see in the next Section, $\mathrm{mVaR}(\beta)$ (and thus ES) is less reliable when $\beta$ is close to zero.

In this Section, we show that it is possible to combine the properties of the Cornish-Fisher and Edgeworth expansions, to derive a definition of modified ES that 1) for a loss probability $\alpha$, only depends on $\mathrm{mVaR}(\alpha)$ and not on $\mathrm{mVaR}(\beta)$ with $\beta < \alpha$ and 2) is coherent with the definition of ES in (1).

Under the location-scale representation (9), the ES of the portfolio return

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is given by

$$\mathrm{ES}(\alpha) = -w'\mu - \sqrt{m_2}E_G\left[z|z \leq G^{-1}(\alpha)\right].\tag{17}$$

For a loss probability $\alpha$, we define modified ES (mES) as the expected value of all returns below the $\alpha$ Cornish-Fisher quantile and where the expectation is computed under the second order Edgeworth expansion of the true distribution function $G(\cdot)$:

$$\mathrm{mES}(\alpha) \;=\; -w'\mu - \sqrt{m_2}\,E_{G_2}\left[z|z \leq g_\alpha\right],\tag{18}$$

with $g_\alpha = G_2^{-1}(\alpha)$. After tedious computations (see Appendix A for details), we obtain:

$$E_{G_2}\left[z|z \leq g_\alpha\right] \;=\; -\frac{1}{\alpha}\Bigg\{\phi(g_\alpha) + \frac{1}{24}\left[I^4 - 6I^2 + 3\phi(g_\alpha)\right]k_p + \frac{1}{6}\left[I^3 - 3I^1\right]s_p$$
$$+ \frac{1}{72}\left[I^6 - 15I^4 + 45I^2 - 15\phi(g_\alpha)\right]s_p^2\Bigg\}$$

where

$$I^q = \begin{cases} \sum_{i=1}^{q/2}\left(\frac{\prod_{j=1}^{q/2}2j}{\prod_{j=1}^{i}2j}\right)g_\alpha^{2i}\phi(g_\alpha) + \left(\prod_{j=1}^{q/2}2j\right)\phi(g_\alpha) & \text{for } q \text{ even} \\[2ex] \sum_{i=0}^{q^*}\left(\frac{\prod_{j=0}^{q^*}(2j+1)}{\prod_{j=0}^{i}(2j+1)}\right)g_\alpha^{2i+1}\phi(g_\alpha) - \left(\prod_{j=0}^{q^*}(2j+1)\right)\Phi(g_\alpha) & \text{for } q \text{ odd} \end{cases}$$

and $q^* = (q-1)/2$. In Appendix B, we provide a long but explicit formula for computing the derivative of mES. Although the resulting formulae are rather complex, they lend themselves to efficient translation into a simple algorithm that computes in less than a second mES and component mES, even for portfolios with a very large number of assets.[3]

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[3]The data can be downloaded from www.hedgeindex.com and the programs have been included in Carl and Peterson (2008)'s R package PerformanceAnalytics.

# 4 Approximation quality

In this Section we assess, in the absence of estimation error in the portfolio moments, how well modified VaR and ES approximate VaR and ES, when the portfolio return is distributed as a random variable, with zero mean and unit variance and various degrees of skewness and excess kurtosis. More specifically, we consider the skewed Student $t$ density function proposed by Fernández and Steel (1998) and standardized by Lambert and Laurent (2001) such that it has mean zero and unit variance. It has been used by Giot and Laurent (2003) among others, for forecasting one-day-ahead VaR for long and short trading positions on daily stocks and stock indices. Its density function has two parameters: $\xi$ and $\nu$. The parameter $\xi > 0$ is defined as the square root of the ratio of probability masses above and below the mode of the distribution such that the sign of $\log \xi$ indicates the sign of skewness. The third moment is positive (negative), and the density is skewed to the right (left), if $\log \xi > 0$ ($< 0$). The parameter $\nu > 0$ models the tail thickness. Holding $\xi$ fixed, we have that the smaller $\nu$ is, the thicker the tails are. When $\xi = 1$, the standardized skewed Student $t$ distribution coincides with the standardized Student $t$ distribution, and for $\nu \to \infty$, the standard Gaussian distribution is the limiting case.

In Appendix C we recall Lambert and Laurent (2001)'s expression for the

skewness, kurtosis and quantile function of this skewed Student $t$ distribution. Using these expressions, VaR and ES, and their estimators, modified VaR and ES can be computed at the theoretical values of the moments of the distribution. Let us now study how sensitive these statistics are to the parameters $\xi$ and $\nu$ of the skewed Student $t$ distribution function and the loss probability $\alpha$ for which they are computed.

In Table 1, we report the true, Gaussian and modified VaR and ES computed for $\alpha = 0.05$, for various values of $\xi$ and $\nu$. The benchmark values are $\xi = 1$ and $\nu = \infty$, for which the return distribution is the standard Gaussian one and for which, by definition, the Gaussian and modified estimators for VaR and ES coincide with the true VaR and ES. The more $\xi$ deviates from 1 and the smaller $\nu$ is, the higher skewness and excess kurtosis are in absolute values and the more distant the skewed Student $t$ will be from the Gaussian distribution. We find that for moderate values of skewness and kurtosis, modified VaR and ES are good approximations of the true VAR and ES, and they certainly do better than Gaussian VaR and ES which are independent of $\xi$ and $\nu$. The more extreme the skewness and excess kurtosis, the less reliable Gaussian and modified VaR and ES are. Modified ES is more sensitive to extreme deviations from normality than modified VaR, and therefore should only be used in the case of moderate deviation from normality. Note also that when the data is negatively skewed, modified VaR and ES tend to be pessimistic and

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Gaussian VaR and ES too optimistic. The opposite result is observed when the data is positively skewed. Note that in all cases mES is a better estimator for ES than CGmES in (15). This means that a more accurate estimate of ES is obtained by applying the Cornish-Fisher and Edgeworth/Gram Charlier approximations to the general definition of ES rather than to the formula of Gaussian ES as is done in Christoffersen and Gonçalves (2005).

Table 1: Sensitivity of skewness, excess kurtosis, VaR, ES and the estimation errors of Gaussian and modified VaR and ES (for $\alpha = 0.05$) to the parameters $\xi$ and $\nu$ of the skewed Student $t$ distribution function.

| $\xi$ | 0.5 | | | 1 | | | 1.5 | | |
|---|---|---|---|---|---|---|---|---|---|
| $\nu$ | 5 | 8 | $\infty$ | 5 | 8 | $\infty$ | 5 | 8 | $\infty$ |
| | | | | | | | | | |
| Skewness | -2.06 | -1.32 | -0.79 | 0 | 0 | 0 | 1.52 | 0.96 | 0.56 |
| Excess kurtosis | 14.54 | 3.53 | 0.51 | 6 | 1.5 | 0 | 10.42 | 2.53 | 0.24 |
| | | | | | | | | | |
| VaR | 1.82 | 1.87 | 1.88 | 1.56 | 1.61 | 1.64 | 1.27 | 1.34 | 1.43 |
| GVaR-VaR | -0.18 | -0.23 | -0.24 | 0.08 | 0.03 | 0 | 0.37 | 0.30 | 0.21 |
| mVaR-VaR | 0.04 | 0.05 | -0.03 | -0.04 | 0 | 0 | -0.31 | -0.04 | 0.05 |
| | | | | | | | | | |
| ES | 2.82 | 2.69 | 2.46 | 2.24 | 2.18 | 2.06 | 1.65 | 1.68 | 1.70 |
| GES-ES | -0.76 | -0.63 | -0.39 | -0.18 | -0.11 | 0 | 0.42 | 0.38 | 0.36 |
| mES-ES | 2.49 | 0.41 | -0.08 | 0.10 | 0.07 | 0 | -1.38 | -0.14 | 0.05 |
| CGmES-ES | -3.33 | -2.41 | -1.49 | 0.91 | 0.08 | 0 | 7.64 | 2.72 | 1.32 |

Figure 1 shows the sensitivity of the true and the approximated values of VaR and ES to the loss probability $\alpha$ for the skewed Student $t$ distribution with $\xi = 0.5$ and $\nu = 8$. Consistent with Jaschke (2002), we find that all approximations have the "wrong tail behavior" in the sense that the approximation

becomes less and less reliable for $\alpha \to 0$. For $\alpha \to 0$, mES drops to zero. This result occurs because the Edgeworth approximation to the density function tends to zero when evaluated at very large losses. As a method of avoiding the unwanted result that mES is smaller than mVaR, we will henceforth use the following *operational* definition of modified ES:

$$\text{mES}^*(\alpha) = -w'\mu - \sqrt{m_2} \ \min\{ \ E_{G_2}[z|z \leq g_\alpha] \ , \ g_\alpha \ \} \tag{19}$$

where $G_2$ and $g_\alpha$ are as defined in (18). In Figure 1 we see that by construction mES$^*$ coincides with mES as long as mES is greater than mVaR. For all values of $\alpha$, mES$^*$ is a better estimator of ES than GES.

If your data is fat-tailed and/or skewed enough to cause the breakdown in mVaR and mES, or if you want to estimate the very extreme downside risk ($\alpha = 0.01 - 0.001$ or smaller), a copula-based approach as proposed by Embrechts et al. (2001) may be a more appropriate estimator of VaR and ES. Under this approach there is, however, no straightforward procedure available for estimating Component VaR and Component ES. In cases where mVaR and mES are still providing a reliable estimation of downside risk, calculation of (Component) VaR and ES under the Cornish-Fisher and Edgeworth approach is certainly more computationally tractable and does not suffer the fitting subtleties of a copula approach.

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Figure 1: Sensitivity of true, Gaussian and modified VaR and ES to the loss probability $\alpha$ for the skewed Student $t$ distribution with $\nu = 8$ and $\xi = 0.5$.
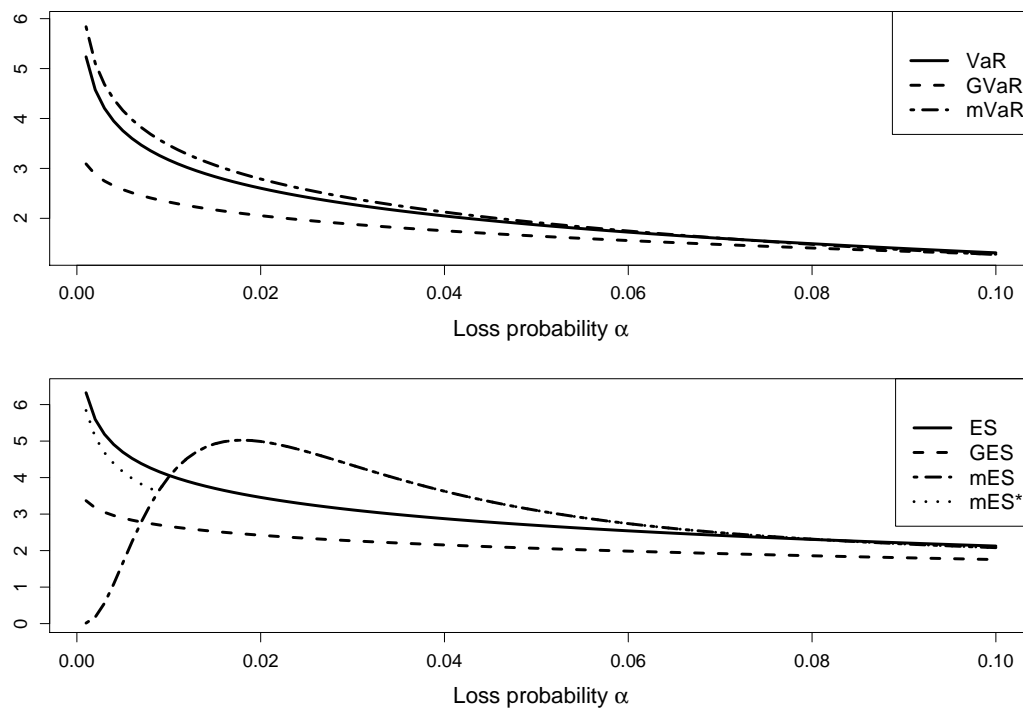
# 5 Robust estimation

Gaussian (modified) VaR and ES estimate VaR and ES using the first two (four) moments of the portfolio return distribution. Formulas (5)-(6) express the portfolio moments as a computationally convenient function of the multivariate moments of the return series of the assets in the portfolio. From the theoretical study in the previous section, it follows that modified VaR and ES are good approximations of VaR and ES, provided the true distribution function does not deviate too much from normality. For this reason, it seems appropriate to consider estimates of the multivariate moments that are robust to return observations that deviate extremely from the Gaussian distribution. There are two main approaches in defining robust alternatives to estimate the multivariate moments by their sample means (see e.g. Maronna et al., 2006). One approach is to consider a more robust estimator than the sample means. Another one is to first clean (in a robust way) the data and then take the sample means of the cleaned data.

Our cleaning method follows the second approach. It is designed in such a way that, if we want to estimate downside risk with loss probability $\alpha$, it will never clean observations that belong to the $1 - \alpha$ least extreme observations. Suppose we have an $n$-dimensional vector time series of length $T$: $r_1, ..., r_T$. We clean this time series in three steps.

1. *Ranking the observations in function of their extremeness.* Denote $\mu$ and $\Sigma$ the mean and covariance matrix of the bulk of the data and let $\lfloor \cdot \rfloor$ be the operator that takes the integer part of its argument. As a measure of the extremeness of the return observation $r_t$, we use its squared Mahalanobis distance $d_t^2 = (r_t - \mu)'\Sigma^{-1}(r_t - \mu)$. We follow Rousseeuw (1985) by estimating $\mu$ and $\Sigma$ as the mean vector and covariance matrix (corrected to ensure consistency) of the subset of size $\lfloor (1-\alpha)T \rfloor$ for which the determinant of the covariance matrix of the elements in that subset is the smallest. These estimates will be robust against the $\alpha$ most extreme returns. Let $d_{(1)}^2, ..., d_{(T)}^2$ be the ordered sequence of the estimated squared Mahalanobis distances such that $d_{(i)}^2 \leq d_{(i+1)}^2$.

2. *Outlier identification.* Return observations are qualified as outliers if their estimated squared Mahalanobis distance $d_t^2$ is greater than the empirical $1-\alpha$ quantile $d_{(\lfloor (1-\alpha)T \rfloor)}^2$ and exceeds a very extreme quantile of the Chi squared distribution function with $n$ degrees of freedom, which is the distribution function of $d_t^2$ when the returns are normally distributed. In the application we take the 99.9% quantile, denoted $\chi_{n,0.999}^2$.

3. *Data cleaning.* Similarly as in Khan et al. (2007) we only clean the returns that are identified as outliers in step 2 by replacing these returns

$r_t$ with

$$r_t\sqrt{\max(d^2_{(\lfloor(1-\alpha)T)\rfloor)}, \chi^2_{n,0.999})/d^2_t}.$$

The cleaned return vector has the same orientation as the original return vector, but its magnitude is smaller.

Note that the primary value of data cleaning lies in creating a more robust and stable estimation of the distribution generating the large majority of the return data. The increased robustness and stability of the estimated moments utilizing cleaned data should be used for portfolio construction. If a portfolio manager wishes to have a more conservative risk estimate, cleaning may not be indicated for risk monitoring. It is also important to note that the robust method proposed here does not remove data from the series, but only decreases the magnitude of the extreme events. It may also be appropriate in practice to use a cleaning threshold somewhat outside the VaR threshold that the manager wishes to consider. In actual practice, it is probably best to back-test the results of both cleaned and uncleaned series to see what works best with the particular combination of assets under consideration.

# 6    Application: component risk analysis of hedge fund portfolios

The correct measurement of financial risk of alternative investments is a concern for many portfolio managers. Because of the non-normality of these returns, they constitute an excellent case for illustrating the use of modified VaR and ES in a context of portfolio construction. Our data consists of monthly returns on 6 Credit Suisse/Tremont hedge fund investment style indices for the period January 1995 - August 2007. The advantages of the Credit Suisse/Tremont indices over their competitors and a description of the investment strategies can be found in Amenc and Martellini (2002). Before studying the estimates of modified VaR and ES for portfolios of these indices and the component risk allocation in the sample portfolios, we investigate the appropriateness of cleaning the data prior to estimating downside risk.

*Out-of-sample evidence in favor of data cleaning.* The original and cleaned series are plotted in Figure 2. Note that cleaning the data leads to an important reduction in the magnitude of the extreme returns corresponding to the Russian financial crisis in August-October 1998. Because there is no similar succession of extreme returns in the time series, it is reasonable to impose that the effect of these returns on the multivariate moment estimates, and thus

Figure 2: Original and cleaned monthly returns on January 1995-August 2007
Credit Suisse/Tremont hedge fund investment style indices.
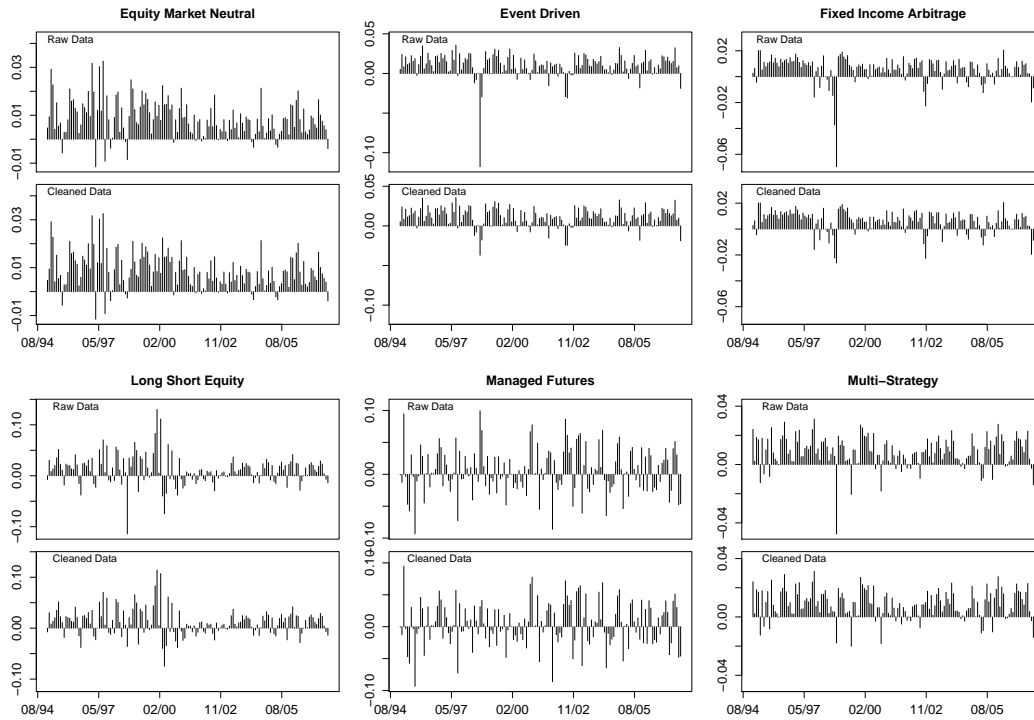


on the values of mVaR and mES, should be bounded. This reasoning is also
verified empirically. Table 2 compares the failure rate of the out-of-sample
Gaussian and modified VaR estimators ($\alpha = 0.05$) using the raw and cleaned
data sets for the period January 1999 - August 2007. Out-of-sample forecasts
of VaR for the month $t$ are obtained using all returns from January 1995 up
to the month $t - 1$. The failure rate is defined as the percentage of negative
returns smaller than the negative value of the out-of-sample one-step ahead

Table 2: Out-of-sample failure rate for one-step ahead VaR predictions ($\alpha = 0.05$) made by the Gaussian and modified VaR estimators for the period January 1999 - August 2007 for six Credit Suisse/Tremont hedge fund investment style indices.

| Hedge Fund Style Index | Raw data | | Cleaned data | |
|---|---|---|---|---|
| | GVaR | mVaR | GVaR | mVaR |
| Equity Market Neutral (EMN) | 0 | 0.029 | 0.010 | 0.029 |
| Event Driven (ED) | 0.039 | 0.019 | 0.049 | 0.049 |
| Fixed Income Arbitrage (FIA) | 0.029 | 0.019 | 0.068 | 0.039 |
| Long Short Equity (LSE) | 0.019 | 0.019 | 0.029 | 0.058 |
| Managed Futures (MF) | 0.039 | 0.049 | 0.049 | 0.049 |
| Multi-Strategy (MS) | 0.058 | 0.029 | 0.068 | 0.068 |

VaR forecast. If the VaR calculation method is accurate, then the failure rate should be close to $\alpha$. Because extreme returns blow up the sample standard deviation and kurtosis, we see in Table 2 that cleaning the data leads to a lower estimate of the Gaussian and modified VaR and thus a higher failure rate. Modified VaR is more affected than Gaussian VaR because it depends on the sample estimates of skewness and kurtosis which raise outliers to the third and fourth power. An important argument in favor of cleaning the data is that the failure rates obtained using the cleaned data to estimate the one-step ahead VaR forecast, are closer to $\alpha = 0.05$. For this reason we will use the cleaned data set in the remainder of the application. From Table 2 we cannot conclude whether the Gaussian or modified VaR estimator has a better out-of-sample performance.

Table 3: Sample mean, standard deviation, skewness, excess kurtosis, P-value of Jarque-Bera test for normality and 5% Gaussian and modified VaR and ES estimates for the January 1995-August 2007 cleaned monthly return series of Credit Suisse/Tremont hedge fund investment style indices.

| **Index** | Mean | Sd | Skew | Exc Kur | JB | GVaR | mVaR | GES | mES |
|---|---|---|---|---|---|---|---|---|---|
| EMN | 0.009 | 0.008 | 0.534 | 0.724 | 0.005 | 0.004 | 0.003 | 0.007 | 0.005 |
| ED | 0.011 | 0.012 | -0.807 | 1.477 | 0.000 | 0.009 | 0.011 | 0.014 | 0.021 |
| FIA | 0.006 | 0.008 | -1.294 | 2.237 | 0.000 | 0.008 | 0.010 | 0.011 | 0.017 |
| LSE | 0.012 | 0.026 | 0.597 | 2.233 | 0.000 | 0.032 | 0.026 | 0.043 | 0.033 |
| MF | 0.004 | 0.034 | -0.106 | -0.047 | 0.861 | 0.052 | 0.053 | 0.066 | 0.068 |
| MS | 0.009 | 0.010 | -0.320 | 0.150 | 0.255 | 0.007 | 0.008 | 0.012 | 0.013 |

*Downside risk analysis of individual hedge fund return series.* Table 3 reports a sample of descriptive statistics for the monthly return series of each of the six style indices. Note that the monthly returns of the equity market neutral, event driven, fixed income arbitrage and long short equity investment style indices are non-normal. Their skewness and excess kurtosis are high enough for the P-value of their Jarque-Bera test statistic for normality to be less than 1%. Their skewness and excess kurtosis are in the range of values for which we found in Section 4 that for $\alpha = 0.05$, modified VaR and ES are reliable estimators of VaR and ES and more accurate than Gaussian VaR and ES.

Regarding the sources of downside risk, we find that the mean and standard deviation of the return series are the main drivers of downside risk and

that skewness and excess kurtosis are important fine-tuning parameters for adjusting the Gaussian downside risk estimate for the non-normality in the return series. Indeed, the estimates of downside risk are the highest for the return series of the managed futures index, which has negligible skewness and excess kurtosis, but the lowest mean return and highest standard deviation. For $\alpha = 0.05$, its mVaR and mES[4] equal 5.3% and 6.8%, respectively. If the estimates are correct, this means that there is a 5% probability of observing monthly losses that exceed -5.3% and the expected value of these extreme losses equals -6.8%. When the return series exhibits significant skewness and excess kurtosis, modified VaR and ES can give very different estimates for VaR and ES than Gaussian VaR and ES. Since the returns on the equity market neutral index are positively skewed, modified VaR and ES are more optimistic about downside risk than Gaussian VaR and ES. The opposite result is found for the return series for the event driven and fixed income arbitrage indices. Since these returns are very much skewed to the left and heavy tailed, downside risk as estimated by modified VaR and ES is much greater than when the Gaussian estimators are used.

*Portfolio downside risk: in-sample analysis.* We will first discuss the portfolio construction for our application examples, and then discuss in more detail

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[4]In this Section, all values of mES as defined in (18) coincide with the values of mES$^*$ as defined in (19).

the risk decomposition of each portfolio. We will see that the contribution to risk of each portfolio component varies widely from the portfolio weights and standard deviation of the individual components, while following a pattern that can be intuited from the construction of the component risk model and the individual properties of the portfolio holdings.

For illustrating the computation and interpretation of portfolio modified VaR and ES, we consider the balanced Equal-Weighted (EW) portfolio, and the Maximum Sharpe Ratio (MSR) portfolios. We consider these portfolio constructions to be symbolically representative of actual practice at many hedge fund investors: investors often pursue either style diversification (thus our example of the equal-weighted balanced view), or they pursue the perceived best risk-reward ratio (represented by the MSR view). The Sharpe Ratio is the most popular risk-adjusted return measure reported by distributors of hedge funds (Amenc et al., 2003). Construction of portfolios that lie on the Markowitz efficient frontier is probably the most common portfolio optimization method employed in finance. For the purposes of this example, we have constrained the portfolio optimization to a long-only portfolio with minimum weights of 5% in each hedge fund style to avoid the creation of overly concentrated positions. Figure 3 compares the mean-variance characteristics of the hedge fund indices and the EW and MSR portfolios with the mean-variance efficient frontier of these portfolios. The long-only and 5% minimum constraint, as could

Figure 3: Mean and standard deviation of Credit Suisse/Tremont indices and of Equal-Weighted and Maximum Sharpe Ratio portfolios, together with the mean-variance efficient frontier of long-only portfolios with minimum weights of 5% in each hedge fund style (dashed).
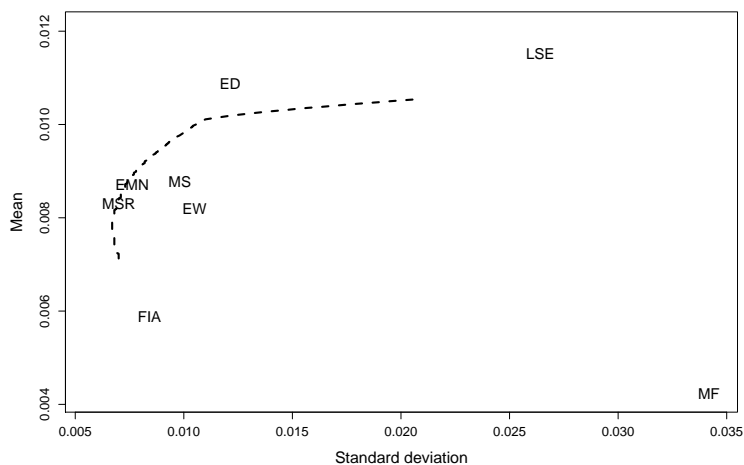


be expected, creates a smaller efficient frontier portfolio space than would be created in an unconstrained portfolio. As designed, these constraints elimi- nate "corner" portfolios where any single instrument can dominate the entire portfolio. Figure 4 plots the monthly returns of these portfolios.

Figure 4: Bar plot of January 1995-August 2007 monthly return series for Equal-Weighted and Maximum Sharpe Ratio portfolios of Credit Suisse/Tremont indices and line graph of the negative value of out-of-sample one step ahead modified VaR forecast ($\alpha = 0.05$) for the monthly portfolio return in January 1999-August 2007.
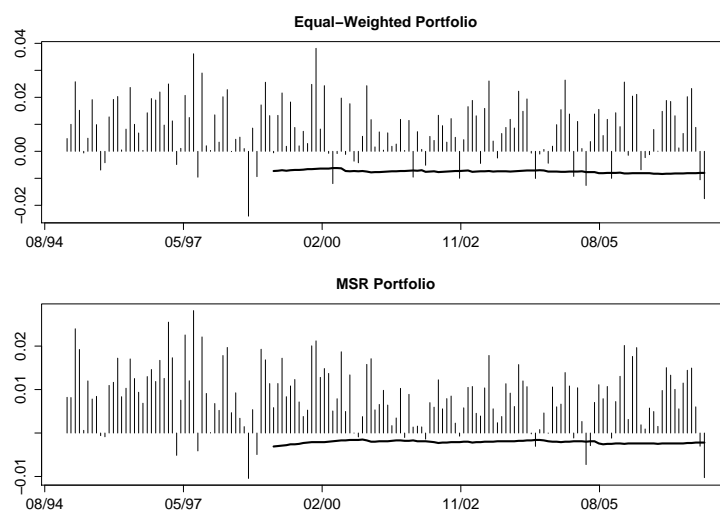
Table 4 reports the risk measures for the two portfolios and compares the portfolio holdings and percentage risk contributions calculated per (2). A direct comparison of all the portfolio risk measures (standard deviation, Gaussian VaR and ES, and modified VaR and ES) shows improvement for the MSR portfolio over the balanced portfolio, even though the mean return of the two portfolios is the same. We also find on examining the component decomposition of risk that for VaR and ES the capital allocation given by the portfolio weights can be very different from the risk exposures (Qian, 2006), and also that the risk allocation depends on the risk measure used.[5] This can be explained in two ways:

1. A mechanical explanation follows from the definition in (2) of asset $i$'s percentage risk contribution as the derivative of the risk measure with respect to the weight of that component multiplied by the component's weight in the portfolio and divided by the value of that risk measure. Since for Modified VaR and ES skewness and excess kurtosis can have an important impact on this derivative, we observe that for non-normal return the risk contributions predicted by Gaussian and Modified VaR and ES are different.

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[5]In Appendix D we show that for the unconstrained MSR portfolio the percentage contribution to portfolio standard deviation, Gaussian VaR and Gaussian ES coincide. This result may no longer hold under a long-only constraint.

2. A perhaps more intuitive understanding for the difference in the percentage risk contributions can be obtained using the financial interpretation given to VaR and ES in Section 2. Asset $i$'s contribution to VaR (ES) equals the negative value of the expected contribution to portfolio return when the portfolio return equals (is less than or equals) the negative value of VaR. For an equal-weighted portfolio, the expected contribution to the portfolio downside risk will be higher for assets with negatively skewed and thick-tailed returns than for assets with normal returns or positively skewed returns.

One of the first major observations on the sample portfolios is the impact the standard deviation has on both the risk decomposition and on the MSR portfolio construction. The managed futures style, with standard deviation (0.034) the largest of the styles, but minimal skewness (-0.106) and excess kurtosis (-0.047), accounts for the largest portion of both the Gaussian and the modified risk measures in the balanced portfolio. In the MSR portfolio, despite being penalized to the minimum 5% weight for its riskiness, the managed futures component of the portfolio still has the highest component risk contribution of any asset in the portfolio. The long short equity style, with the highest mean (0.012) and second highest standard deviation (0.026), and fat-tailed excess kurtosis (2.233), shows similar effects, being the second largest contributor on all component risk measures in both portfolios, and being sim-

Table 4: Portfolio totals, weights and percentage risk contributions for the equal-weighted and long-only 5% minimum weight constrained Maximum Sharpe Ratio portfolio of Credit Suisse/Tremont hedge fund investment style indices. Gaussian and modified VaR and ES are computed for $\alpha = 5\%$.

| Equal-Weighted Portfolio | | | | | | |
|---|---|---|---|---|---|---|
| | Mean | Sd | GVaR | GES | mVaR | mES |
| Total | 0.0083 | 0.0105 | 0.0091 | 0.0135 | 0.0089 | 0.0121 |
| | $w_i$ | % contribution: | | | | |
| EMN | 0.17 | 0.06 | -0.04 | -0.01 | -0.05 | -0.02 |
| ED | 0.17 | 0.13 | 0.05 | 0.08 | 0.06 | 0.08 |
| FIA | 0.17 | 0.06 | 0.00 | 0.02 | 0.01 | 0.02 |
| LSE | 0.17 | 0.31 | 0.38 | 0.36 | 0.30 | 0.28 |
| MF | 0.17 | 0.36 | 0.61 | 0.53 | 0.69 | 0.62 |
| MS | 0.17 | 0.08 | -0.01 | 0.02 | 0.00 | 0.03 |

| MSR Portfolio with 5% minimum weights | | | | | | |
|---|---|---|---|---|---|---|
| | Mean | Sd | GVaR | GES | mVaR | mES |
| Total | 0.0083 | 0.0070 | 0.0031 | 0.0060 | 0.0029 | 0.0053 |
| | $w_i$ | % contribution: | | | | |
| EMN | 0.49 | 0.44 | 0.25 | 0.34 | 0.04 | 0.22 |
| ED | 0.06 | 0.07 | 0.04 | 0.05 | 0.06 | 0.07 |
| FIA | 0.11 | 0.06 | 0.02 | 0.04 | 0.06 | 0.06 |
| LSE | 0.05 | 0.12 | 0.24 | 0.18 | 0.21 | 0.12 |
| MF | 0.05 | 0.11 | 0.32 | 0.21 | 0.37 | 0.29 |
| MS | 0.24 | 0.21 | 0.12 | 0.16 | 0.26 | 0.24 |

ilarly penalized in the MSR portfolio by a minimum weight.

The risk added by an asset to the portfolio depends on the asset's risk properties, relative to the risk properties of the other assets in the portfolio. This is very well illustrated by the fixed income arbitrage index. It has at the same time a lower portfolio weight and a higher risk contribution in the MSR portfolio than in the balanced portfolio. This is because assets such as the long

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short equity and managed futures indices with a relatively higher downside risk than the fixed income arbitrage index, also have a lower weight in the MSR portfolio. A similar effect may be noted with the multi-strategy style index, which has a negligible impact on all component VaR and ES risk measures in the balanced portfolio (-1% to 3%), but a much larger impact on all these measures in the MSR portfolio. The equity market neutral index shows the risk benefits of positive skew (0.534) and moderate excess kurtosis (0.724). In the balanced portfolio, it has a negative risk contribution to Gaussian and modified VaR. This means that it serves as a hedge to the VaR of the rest of the portfolio (Garman, 1997). The equity market neutral component risk measures increase in the MSR portfolio, but are still below the component weight of 49%.

*Portfolio downside risk: out-of-sample analysis.* We now compare the out-of-sample predictive performance of GVaR and mVaR for each month $t$ in the period January 1999-August 2007 using the cleaned returns from January 1995 up to month $t - 1$. Figure 4 compares the monthly returns on the equal-weighted and MSR portfolios with the negative value of their out-of-sample one step ahead mVaR forecast ($\alpha = 0.05$) for the period January 1999-August 2007. Since the number of out-of-sample observations equals 92, we expect that for a 95% VaR estimate there will be approximately 5 excessions. For our data set, the Gaussian and modified approach detect 9 excessions on the EW

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portfolio and 5 on the MSR portfolio. For each of these excessions we report in Table 5 the realized percentage contribution $(w_i r_i / r_p)$ and the one predicted by the mES measure ($\%C_i$mES). Any such examination of realized contribution is by nature imprecise, but this comparison should still be informative. We find that in almost all cases, the mES correctly predicts the largest contributors to potential losses.

Since the percentage contribution to modified ES attempts to predict the *average* contribution to loss for the periods in which the loss exceeds the VaR estimate, it must be that the prediction bias is zero:

$$E[\%C_i\text{mES} - w_i r_i / r_p | r_p \leq -\text{mVaR}] = 0.$$

We test the null hypothesis of a zero prediction bias using the Student's $t$-statistic computed as the square root of the number of excessions times the average over the excessions of the difference between the predicted percentage contributions to modified ES and the realized ones, divided by the standard deviation of these differences. Under the null of no bias, this $t$-statistic is Student $t$ distributed with the number of excessions minus one degrees of freedom. We obtain that for almost all the assets, the $t$-statistic has a large enough p-value for the null hypothesis to be plausible. We also computed these $t$-statistics for the Gaussian approach and found that the p-values under the modified approach tend to be larger than under the Gaussian approach.

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This result is to be expected since the mES predicting process takes better the shape of past losses into account. Overall, Table 5 indicates that combining the capital allocation and the estimated risk contributions will help the risk manager in forming a better opinion of the sources and magnitude of future portfolio risk.

Another practical conclusion from examining the component risk contributions to the two sample portfolios would be that the long-term performance of the portfolios could be improved by adjusting the component weights to better match a deliberate risk profile that was complimentary to the investor's goals. In any portfolio holding a sufficient numbers of assets, there will be many possible portfolios with similar mean return and standard deviation, so additional information provided by the modified VaR and ES and portfolio risk decomposition techniques presented here adds significant information to the portfolio selection process. Further work should examine these techniques in relation to various risk budgeting and portfolio optimization methodologies.

Table 5: Realized percentage contribution to portfolio return $(w_i r_i / r_p)$ and predicted ones by mES $(\%C_i \text{mES})$ for the months in the period January 1999-August 2007, in which the portfolio loss exceeds the one-step ahead modified VaR forecast ($\alpha = 0.05$). The portfolio return $r_p$ is in parenthesis.

| | | EMN | ED | FIA | LSE | MF | MS |
|---|---|---|---|---|---|---|---|
| **EW Portfolio** | $w_i$ | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 |
| April 2000 | $w_i r_i / r_p$ | -0.20 | 0.10 | -0.13 | 1.04 | 0.32 | -0.13 |
| ($r_p = -0.012$) | $\%C_i \text{mES}$ | 0.03 | 0.10 | -0.03 | 0.36 | 0.64 | -0.11 |
| November 2001 | $w_i r_i / r_p$ | -0.14 | -0.15 | 0.05 | -0.19 | 1.50 | -0.07 |
| ($r_p = -0.010$) | $\%C_i \text{mES}$ | -0.02 | 0.11 | -0.05 | 0.52 | 0.50 | -0.07 |
| October 2002 | $w_i r_i / r_p$ | -0.07 | 0.03 | 0.38 | -0.01 | 0.84 | -0.17 |
| ($r_p = -0.010$) | $\%C_i \text{mES}$ | -0.02 | 0.09 | -0.04 | 0.44 | 0.59 | -0.05 |
| April 2004 | $w_i r_i / r_p$ | 0.06 | -0.08 | -0.22 | 0.23 | 1.07 | -0.05 |
| ($r_p = -0.010$) | $\%C_i \text{mES}$ | -0.04 | 0.06 | 0.01 | 0.36 | 0.67 | -0.06 |
| January 2005 | $w_i r_i / r_p$ | -0.06 | -0.04 | -0.02 | 0.15 | 0.97 | 0.00 |
| ($r_p = -0.009$) | $\%C_i \text{mES}$ | -0.03 | 0.06 | -0.01 | 0.34 | 0.69 | -0.05 |
| April 2005 | $w_i r_i / r_p$ | 0.03 | 0.08 | 0.08 | 0.20 | 0.45 | 0.14 |
| ($r_p = -0.013$) | $\%C_i \text{mES}$ | -0.03 | 0.05 | -0.01 | 0.33 | 0.70 | -0.04 |
| October 2005 | $w_i r_i / r_p$ | -0.14 | 0.30 | -0.04 | 0.38 | 0.33 | 0.17 |
| ($r_p = -0.010$) | $\%C_i \text{mES}$ | -0.02 | 0.05 | 0.00 | 0.31 | 0.68 | -0.02 |
| July 2007 | $w_i r_i / r_p$ | -0.06 | -0.15 | 0.31 | 0.11 | 0.76 | 0.04 |
| ($r_p = -0.010$) | $\%C_i \text{mES}$ | -0.03 | 0.06 | -0.01 | 0.33 | 0.66 | 0.00 |
| August 2007 | $w_i r_i / r_p$ | 0.04 | 0.18 | 0.08 | 0.13 | 0.44 | 0.13 |
| ($r_p = -0.018$) | $\%C_i \text{mES}$ | -0.03 | 0.05 | 0.01 | 0.32 | 0.65 | 0.00 |
| P-value test for no prediction bias | | 0.13 | 0.30 | 0.15 | 0.13 | 0.35 | 0.04 |

| | | EMN | ED | FIA | LSE | MF | MS |
|---|---|---|---|---|---|---|---|
| **MSR Portfolio** | $w_i$ | 0.49 | 0.06 | 0.11 | 0.05 | 0.05 | 0.24 |
| April 2004 | $w_i r_i / r_p$ | 0.55 | -0.09 | -0.49 | 0.23 | 1.05 | -0.24 |
| ($r_p = -0.003$) | $\%C_i \text{mES}$ | 0.40 | 0.07 | 0.03 | 0.18 | 0.26 | 0.06 |
| April 2005 | $w_i r_i / r_p$ | 0.15 | 0.05 | 0.10 | 0.11 | 0.24 | 0.36 |
| ($r_p = -0.007$) | $\%C_i \text{mES}$ | 0.36 | 0.06 | -0.01 | 0.18 | 0.32 | 0.09 |
| May 2005 | $w_i r_i / r_p$ | 0.58 | -0.11 | 0.48 | -0.09 | -0.62 | 0.76 |
| ($r_p = -0.003$) | $\%C_i \text{mES}$ | 0.32 | 0.06 | 0.02 | 0.15 | 0.30 | 0.14 |
| July 2007 | $w_i r_i / r_p$ | -0.65 | -0.18 | 0.72 | 0.11 | 0.79 | 0.20 |
| ($r_p = -0.003$) | $\%C_i \text{mES}$ | 0.27 | 0.06 | 0.02 | 0.16 | 0.29 | 0.20 |
| August 2007 | $w_i r_i / r_p$ | 0.19 | 0.10 | 0.09 | 0.07 | 0.22 | 0.32 |
| ($r_p = -0.010$) | $\%C_i \text{mES}$ | 0.24 | 0.05 | 0.05 | 0.16 | 0.30 | 0.20 |
| P-value test for no prediction bias | | 0.31 | 0.02 | 0.31 | 0.04 | 0.85 | 0.22 |

# 7 Concluding remarks

This paper contributes to the literature on downside risk measurement in multiple ways. First of all, we introduce a new estimator for Expected Shortfall (ES), called modified ES which is based on the Cornish-Fisher and Edgeworth approximations of the portfolio return quantile and distribution functions. The definition of this new estimator is consistent with Zangari (1996)'s definition of modified Value at Risk (VaR). Modified VaR and ES can be considered as parametric Gaussian VaR and ES corrected for skewness and excess kurtosis in the data. We investigate how well modified VaR and ES proxy the true VaR and ES for the skewed Student $t$ distribution function and find that for moderate values of skewness and excess kurtosis, modified VaR and ES are better estimators of VaR and ES than Gaussian VaR and ES, respectively. Some caution is necessary when modified VaR and ES are computed for very small loss probabilities $\alpha$ and for return distributions that deviate a lot from normality. We provide computationally convenient formulas for calculating these risk measures for portfolios and for decomposing them into the risk added to the portfolio by each of the assets in the portfolio.

We illustrate the usefulness of this new methodology for a set of hedge fund investment style indices. We investigate how the non-normality of the returns on these indices affects their downside risk as estimated by Gaussian

and modified VaR and ES. For the equal-weighted and maximum Sharpe ratio portfolios, we find that capital allocation can be very different from risk allocation and that the estimated risk allocation depends on the risk measure used. We conclude that estimating the risk contributions of the portfolio holdings will help the investor in adjusting the portfolio composition to better match the desired portfolio risk profile.

There are various remaining research questions that we leave open for future research. First of all, we assume throughout the paper that the portfolio return distribution is continuous and the conditional portfolio moments are constant. The relaxation of these two assumptions deserves to be studied further. Second, as already discussed in Section 6, modified ES and Component VaR and ES can be used as portfolio optimization criterion. A first successful attempt in this direction was made in Boudt et al. (2008). Further work should also test implementation of risk monitoring and portfolio construction systems that use the formulae given in this paper to assure that the actual risk positions remain within the bounds stated in the risk budget.

# A    Modified expected shortfall

Here we show how

$$
\begin{aligned}
E_{G_2}[z|z \le g_\alpha] &= \frac{1}{\alpha} \int_{-\infty}^{g_\alpha} z dG_2(z) = \frac{1}{\alpha} \int_{-\infty}^{g_\alpha} zd \left[ \Phi(z) - \phi(z) \sum_{i=1}^{2} P_i(z) \right] \\
&= -\frac{1}{\alpha} \int_{-\infty}^{g_\alpha} \phi'(z) dz + zd \left[ \phi(z) \sum_{i=1}^{2} P_i(z) \right]
\end{aligned}
$$

can be expressed as a polynomial in $g_\alpha$ with coefficients that depend on the portfolio skewness $s_p$ and excess kurtosis $k_p$ and on the standard Gaussian density function $\phi(\cdot)$. Using the property $\phi'(z) = -z\phi(z)$, we first rewrite the differentials $zd\phi(z)P_i(z)$ $(i = 1, 2)$ as a function of $z^q \phi'(z)dz$, with $q$ a positive integer:

$$
\begin{aligned}
zd\phi(z)P_1(z) &= \frac{s_p}{6} \left( z^3 - 3z \right) \phi'(z)dz \\
zd\phi(z)P_2(z) &= \frac{k_p}{24} \left( z^4 - 6z^2 + 3 \right) \phi'(z)dz \\
&\quad + \frac{s_p^2}{72} \left( z^6 - 15z^4 + 45z^2 - 15 \right) \phi'(z)dz.
\end{aligned}
$$

Through integration by parts, we find that for $q = 1$,

$$
I^1 = \int_{-\infty}^{g_\alpha} z\phi'(z)dz = g_\alpha \phi(g_\alpha) - \Phi(g_\alpha)
$$

and for $q > 1$:

$$
I^q = \int_{-\infty}^{g_\alpha} z^q \phi'(z)dz = g_\alpha^q \phi(g_\alpha) + q \int_{-\infty}^{g_\alpha} z^{q-2} \phi'(z)dz.
$$

From the development of this recursive formula and using the result that by l'Hopital's theorem, $z^q \phi(z)$ is zero for $z = -\infty$, expression (18) follows straight-forwardly.

# B    Derivative of modified expected shortfall

As with modified VaR, the derivative of modified ES can be computed analytically. Using the property $\phi'(z) = -z\phi(z)$, we obtain

$$\partial_i \mathrm{mES}(\alpha) = -\mu_i - \frac{\partial_i m_2}{2\sqrt{m_2}} E_{G_2}\left[z | z \le g_\alpha\right] + \sqrt{m_2}\,\frac{1}{\alpha}\Bigg\{\frac{1}{24}\left[I^4 - 6I^2 + 3\phi(g_\alpha)\right]\partial_i k_p$$

$$+\frac{1}{6}\left[I^3 - 3I^1\right]\partial_i s_p + \frac{1}{36}\left[I^6 - 15I^4 + 45I^2 - 15\phi(g_\alpha)\right]s_p\partial_i s_p$$

$$+\partial_i g_\alpha\Bigg[-g_\alpha\phi(g_\alpha) + \frac{1}{24}\left[\partial_i I^4 - 6\partial_i I^2 - 3g_\alpha\phi(g_\alpha)\right]k_p + \frac{1}{6}\left[\partial_i I^3 - 3\partial_i I^1\right]s_p$$

$$+\frac{1}{72}\left[\partial_i I^6 - 15\partial_i I^4 + 45\partial_i I^2 + 15g_\alpha\phi(g_\alpha)\right]s_p^2\Bigg]\Bigg\}.$$

Let $z_\alpha = \Phi^{-1}(\alpha)$, then

$$\partial_i g_\alpha = \frac{1}{6}(z_\alpha^2 - 1)\partial_i s_p + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)\partial_i k_p - \frac{1}{18}(2z_\alpha^3 - 5z)s_p\partial_i s_p.$$

For $q$ even, we have

$$\partial_i I^q = \sum_{i=1}^{q/2}\left(\frac{\prod_{j=1}^{q/2}2j}{\prod_{j=1}^{i}2j}\right)g_\alpha^{2i-1}(2i - g_\alpha^2)\phi(g_\alpha) - \left(\prod_{j=1}^{q/2}2j\right)g_\alpha\phi(g_\alpha)$$

and for $q$ odd

$$\partial_i I^q = \sum_{i=0}^{q^*}\left(\frac{\prod_{j=0}^{q^*}(2j+1)}{\prod_{j=0}^{i}(2j+1)}\right)g_\alpha^{2i}(2i + 1 - g_\alpha^2)\phi(g_\alpha) - \left(\prod_{j=0}^{q^*}(2j+1)\right)\phi(g_\alpha),$$

with $q^* = (q-1)/2$. At first sight, this expression may seem daunting, but it is fairly easy to implement using the computationally convenient formulas for portfolio skewness and excess kurtosis and their derivative in (6).

## C Skewed Student $t$

Let $t_\nu(\cdot)$ and $T_\nu^{-1}(\cdot)$ be the density and quantile functions of the classical, non standardized Student $t$ density functions with $\nu$ degrees of freedom, mean zero and standard deviation $\sigma = \sqrt{\nu/(\nu-2)}$. The random variable $z$ is said to be (standardized) skewed Student $t$ distributed with $\nu$ degrees of freedom and skewness parameter $\xi$, if its density function equals

$$
t_{\xi,\nu}(z) = \begin{cases} \frac{2s\sigma}{\xi+\xi^{-1}} t_\nu \left[\sigma\xi(sz+m)\right] & \text{if} \quad z < -m/s \\[2ex] \frac{2s\sigma}{\xi+\xi^{-1}} t_\nu \left[\sigma(sz+m)/\xi\right] & \text{if} \quad z \geq -m/s, \end{cases}
$$

where $m$ and $s$ are the mean and standard deviation of the non-standardized skewed Student, respectively:

$$
\begin{aligned}
m &= \left[\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1} \Gamma\left(\frac{\nu-1}{2}\right) \sqrt{\nu-2}\left(\xi - \frac{1}{\xi}\right) \\
s &= \sqrt{\xi^2 + \xi^{-2} - 1 - m^2}.
\end{aligned}
$$

The skewness and kurtosis of $z$ equal its third and fourth uncentered moment

$$
E(z^q) = \int_{-\infty}^{\infty} z^q t_{\nu,\xi}(z)dz,
$$

which we compute by numerical integration. Lambert and Laurent (2001) show that the quantile function of the standardized skewed Student $t$ equals

$$
T_{\nu,\xi}^{-1}(\alpha) = \begin{cases} \left[(\sigma\xi)^{-1}T_\nu^{-1}\left(\frac{\alpha}{2}(1+\xi^2)\right) - m\right]/s & \text{if} \quad \alpha < 1/(1+\xi^2) \\[2mm] \left[-(\xi/\sigma)T_\nu^{-1}\left(\frac{1-\alpha}{2}(1+\xi^{-2})\right) - m\right]/s & \text{if} \quad \alpha \geq 1/(1+\xi^2). \end{cases}
$$

# D  Percentage contribution MSR portfolio

The Maximum Sharpe Ratio (MSR) portfolio maximizes the ratio between the portfolio mean $\mu_p$ and the portfolio standard deviation $\sigma_p$. This implies the first order condition

$$
\partial_i(\mu_p/\sigma_p) = 0 \Rightarrow \partial_i\mu_p/\partial_i\sigma_p = \mu_p/\sigma_p.
$$

It follows that

$$
\begin{aligned}
\frac{\partial_i\text{GVaR}}{\text{GVaR}} &= \frac{-\partial_i\mu_p - \Phi^{-1}(\alpha)\partial_i\sigma_p}{-\mu_p - \Phi^{-1}(\alpha)\sigma_p} = \frac{\partial_i\sigma_p}{\sigma_p}\left(\frac{-\partial_i\mu_p/\partial_i\sigma_p - \Phi^{-1}(\alpha)}{-\mu_p/\sigma_p - \Phi^{-1}(\alpha)}\right) = \frac{\partial_i\sigma_p}{\sigma_p} \\
&= \frac{\partial_i\sigma_p}{\sigma_p}\left(\frac{-\partial_i\mu_p/\partial_i\sigma_p + (1/\alpha)\phi[\Phi^{-1}(\alpha)]}{-\mu_p/\sigma_p + (1/\alpha)\phi[\Phi^{-1}(\alpha)]}\right) = \frac{\partial_i\text{GES}}{\text{GES}}.
\end{aligned}
$$

Hence, the percentage risk contributions of portfolio standard deviation, GVaR and GES coincide for the MSR portfolio.

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