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by

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International Economics

Center for Economic Studies
Discussions Paper Series (DPS) 04.04
http://www.econ.kuleuven.be/ces/discussionpapers/default.htm

January 2004

DISCUSSION
PAPER

International Dynamic Asset Allocation and the Effect of the Exchange Rate

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Abstract

This paper analyzes a stylized theoretical framework to examine optimal portfolio selection in an international context with an explicit focus on the effect of the exchange rate. More specifically, we study how the elimination of the exchange rate induces shifts in the optimal international portfolio. We show that the effect of the elimination of the exchange rate on the optimal portfolio is twofold. First, the volatility of the international portfolio changes (volatility effect of the exchange rate), and second, the national market prices of risk converge to common international market prices of risk (price effect of the exchange rate). This induces important shifts in the optimal international portfolio.

JEL no. G11, G15, F31, F21.

Keywords: International Financial Markets, Portfolio Diversification, Foreign Exchange.

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1 Introduction

At the beginning of 1999 the euro was introduced creating the European Monetary Union (EMU). With the introduction, exchange rates, as separators of markets, disappeared. This was an important step in the ongoing process of economic and financial integration between the EMU member states. The current removal of financial, legal and information barriers implies that an EMU-wide economic and financial market is being created. Despite the large degree of integration, the goods markets are still segmented across countries. Business cycles and inflation rates in different countries are still highly dispersed, creating a wedge between the different economies (see Remsperger (2001)). Local economic factors might therefore still be an important driving force for an economy. This contrasts with the integration process in the financial markets. Money market rates across EMU members are equalized and bond market yields are converging. Furthermore, stock exchanges have set up initiatives to create a financial market with pan-European dimensions. The European financial market is thus, for a large extent, integrated (see Danthine et al. (2000)). This also implies that the different national financial markets share a common pricing structure.

The increased integration and the elimination of exchange rate variability surely has an impact on the investor's optimal investment strategies. This paper analyzes a stylized theoretical framework to examine optimal portfolio selection in an international context with an explicit focus on the exchange rate. More specifically, we study the effect of the elimination of the exchange rate on the optimal international portfolio. The international portfolio problem is studied in a stochastic environment[1]. We define separately a home and a foreign stochastic environment by introducing independent predictor variables in the home and foreign asset returns. In each market (country) the state variable is the driving force for the asset prices as well as for inflation. National markets are incomplete. This modelling assumption implies that the pricing structure in each market includes both the home and the foreign risk factors, but each asset market only spans the national risk factor. Using standard international no-arbitrage arguments, the home and foreign investment environments are linked. This creates the international stochastic environment. The international market is complete. Both the home and foreign risk factors can be fully hedged in the international financial market.

We analyze the international portfolio allocation of an investor, consuming only in his own country. The investor maximizes utility over real final wealth, defined as nominal wealth deflated by national inflation[2]. We show that the international portfolio allocation

[1] A number of recent papers address the portfolio choice problem of a multi-period investor. Among others, Brennan et al. (1997), Kim and Omberg (1996), Wachter (2002), Campbell and Viceira (2002), and Lynch (2001) analyze the consequences of predictability in asset returns for financial decisions.

[2] Brennan and Xia (2002) also account for inflation risk in a (national) dynamic portfolio problem.

is composed of three components: the nominal myopic demand, the inflation demand (real myopic demand), and the intertemporal hedging demand. The exchange rate dynamics affect all of the three portfolio components in terms of differences in market prices of risk assigned by the home and the foreign market. We distinguish two different effects of the exchange rate. The first effect is the volatility effect: exchange rate volatility is added to the portfolio. The second effect is the price effect: the national market prices of risk converge to common international market prices of risk. This framework is then used to examine explicitly the impact of the elimination of the exchange rate on the portfolio decision.

The contribution of this paper concerns the inclusion of an international investment dimension in the portfolio choice problem of a multi-period investor. Ang and Bekaert (1999) also solve the dynamic international portfolio choice, but their main focus is on the diversification benefits in a market characterized by time-varying correlations and volatilities. They do not look explicitly at the portfolio shifts induced by the elimination of the exchange rate.

The organization of the paper is as follows. In Section 2, we present the set-up of the international stochastic environment. Section 3 defines the intertemporal portfolio choice and solves the dynamic problem using the Bellman principle of optimality. An economic interpretation and the impact of the exchange rate is presented in Section 4 and in Section 5 we perform a calibration exercise. Finally, Section 6 concludes.

2 The International Stochastic Environment

We derive the optimal portfolio allocation for a two-country model, denoted as home and foreign.[3] The state of each market is described by an independent state variable. Assume that these state variables are perfectly captured by a two-dimensional vector of predictor variables $X(t) = \begin{bmatrix} X_1(t), & X_2(t) \end{bmatrix}'$. This vector of predictor variables (state variables) is driven by an Ornstein-Uhlenbeck process:

$$dX(t) = (k - KX(t))\,dt + \sigma_X dZ(t) \tag{1}$$

where k, K and σ_X are positive constants and K and σ_X are diagonal, implying independence of the processes X_1 and X_2; $Z(t) = \begin{bmatrix} Z_1(t), & Z_2(t) \end{bmatrix}'$ is a two-dimensional vector of independent Brownian motions. In this set-up, $Z_1(t)$ is the risk associated with the home state variable X_1 and Z_2 the risk associated with the foreign state variable X_2. For reasons that become clear later, the state variables are the dividend price ratios.

In each market, the investor can trade two assets, a risk-free asset and a risky asset. The

[3] Foreign variables are denoted with a superscript *.

return of the risk-free assets, $r(t)$ and $r^*(t)$, is an affine function of the state variables:

$$r(t)\,dt \equiv (\delta_{0,r} + \delta_{1,r}X_1(t))\,dt \tag{2}$$

$$r^*(t)\,dt \equiv \left(\delta^*_{0,r} + \delta^*_{1,r}X_2(t)\right)dt. \tag{3}$$

The home and foreign risk-free assets are driven by the home and foreign state variable, respectively. Define the home risky asset price as $P(t)$ and the foreign risky asset price as $P^*(t)$. The risky assets follow the diffusion process:

$$\frac{dP(t)}{P(t)} = (\delta_{0,P} + \delta_{1,P}X_1(t))\,dt - \sigma dZ(t) \tag{4}$$

$$\frac{dP^*(t)}{P^*(t)} = \left(\delta^*_{0,P} + \delta^*_{1,P}X_2(t)\right)dt - \sigma^* dZ(t) \tag{5}$$

where $\sigma = \begin{bmatrix} \sigma_1, & 0 \end{bmatrix}'$ and $\sigma^* = \begin{bmatrix} 0, & \sigma^*_2 \end{bmatrix}'$ are the positive volatilities of the risky assets. The home risky asset is determined by the home state variable and the foreign risky asset is determined by the foreign state variable. Given the independence of the two state variables, the home and foreign risky assets are then completely independent. Moreover, note that the home and foreign assets are perfectly negatively correlated with the home and foreign state variable, respectively. This perfect negative correlation is realistic. Empirical evidence suggests that asset returns exhibit mean reversion, and that asset returns are predictable by the dividend price ratio.[4] Perfect negative correlation between the shocks to the asset returns and the shocks to the expected excess returns then implies that stock returns are mean reverting. (see Wachter (2002), Campbell and Viceira (2002)).

The above set-up implies that in each market a single state variable, or single risk factor, is present in each risky asset. However, we assume that both state variables are priced in both the home and the foreign market. No-arbitrage assumptions now imply the following dynamics for the stochastic discount factors:

$$\frac{dM(t)}{M(t)} = -rdt - \Lambda(X(t))\,dZ \tag{6}$$

$$\frac{dM^*(t)}{M^*(t)} = -r^*dt - \Lambda^*(X(t))\,dZ \tag{7}$$

with $\Lambda(X(t))$ and $\Lambda^*(X(t))$ the market prices of risk. The market prices of risk in both

[4] Empirical evidence of mean reversion in asset returns is given by Poterba and Summers (1988). Campbell and Shiller (1988), and Fama and French (1989) show that the dividend yield predicts excess returns.

countries price both the home and the foreign risk factor:

$$\Lambda\left(X\left(t\right)\right) = \left[\begin{array}{cc}\Lambda_1\left(X\left(t\right)\right), & \Lambda_2\left(X\left(t\right)\right)\end{array}\right] \tag{8}$$

$$\Lambda^*\left(X\left(t\right)\right) = \left[\begin{array}{cc}\Lambda_1^*\left(X\left(t\right)\right), & \Lambda_2^*\left(X\left(t\right)\right)\end{array}\right] \tag{9}$$

This modelling of the stochastic discount factors implies complete international markets, while the national markets are incomplete.

The home and the foreign stochastic environment taken together create the international stochastic environment. A home investor investing in an international environment is only interested in home currency payoffs. By means of the exchange rate he converts the foreign currency payoffs to home currency units. The assumption of international market integration determines the exchange rate[5]. Denote $S\left(t\right)$ the unit price of foreign currency in terms of domestic currency, international no-arbitrage implies:

$$E_t\left[\frac{M^*\left(t+\tau\right)}{M^*\left(t\right)}\frac{P^*\left(t+\tau\right)}{P^*\left(t\right)}\right] = E_t\left[\frac{M\left(t+\tau\right)}{M\left(t\right)}\frac{P^*\left(t+\tau\right)}{P^*\left(t\right)}\frac{S\left(t+\tau\right)}{S\left(t\right)}\right] \tag{10}$$

which is certainly satisfied for:

$$\frac{M^*\left(t+\tau\right)}{M^*\left(t\right)} = \frac{M\left(t+\tau\right)}{M\left(t\right)}\frac{S\left(t+\tau\right)}{S\left(t\right)} \tag{11}$$

The exchange rate thus ties the home and foreign stochastic discount factors. Applying Ito's lemma to $S\left(t\right) = \frac{M^*(t)}{M(t)}$ yields:

$$\frac{dS\left(t\right)}{S\left(t\right)} = \frac{dM^*\left(t\right)}{M^*\left(t\right)} - \frac{dM\left(t\right)}{M\left(t\right)} + \left(\frac{dM\left(t\right)}{M\left(t\right)} - \frac{dM^*\left(t\right)}{M^*\left(t\right)}\right)\left(\frac{dM\left(t\right)}{M\left(t\right)}\right) \tag{12}$$

Using (6) and (7) the dynamics of the exchange rate return can be rewritten as:

$$\frac{dS\left(t\right)}{S\left(t\right)} = \left(r\left(t\right) - r^*\left(t\right)\right)dt + \sigma_S\Lambda'\left(X\left(t\right)\right)dt - \sigma_S dZ \tag{13}$$

with

$$\begin{aligned}\sigma_S &= \left[\begin{array}{cc}\sigma_{S,1}, & \sigma_{S,2}\end{array}\right] \\ &= \left[\begin{array}{cc}\Lambda_1\left(X\left(t\right)\right) - \Lambda_1^*\left(X\left(t\right)\right), & \Lambda_2\left(X\left(t\right)\right) - \Lambda_2^*\left(X\left(t\right)\right)\end{array}\right]\end{aligned} \tag{14}$$

The exchange rate is fully determined by the home and foreign risk factors.

A home investor investing in the foreign risky asset passes through the exchange market. A cross-border investment is represented as $V\left(t\right) \equiv S\left(t\right)P^*\left(t\right)$. By Ito's lemma and using

[5] Among others, Brandt et al. (2001) and Backus et al. (2001) exploit this no-arbitrage equation to relate the exchange rate dynamics with the stochastic discount factor dynamics.

(5) and (13) yields:

$$\frac{dV(t)}{V(t)} = \frac{dP^*(t)}{P^*(t)} + \frac{dS(t)}{S(t)} + \frac{dS(t)}{S(t)}\frac{dP^*(t)}{P^*(t)} \tag{15}$$

$$\frac{dV(t)}{V(t)} = r(t)\,dt + (\sigma_S + \sigma^*)\,\Lambda'(X(t))\,dt - (\sigma_S + \sigma^*)\,dZ \tag{16}$$

$$\frac{dV(t)}{V(t)} = E_t\left(\frac{dV(t)}{V(t)}\right) - \sigma_V(X(t))\,dZ \tag{17}$$

The return of the foreign asset, converted to home currency units, equals the return on the foreign asset plus the return on the exchange rate plus the covariance between the exchange rate return and the foreign asset return (see (15)). Equation (16) shows that the expected return on a foreign asset, converted to home currency units, equals the home risk-free rate augmented with a risk-premium. The risk-premium is defined in a standard way as the amount of risk $(\sigma_S + \sigma^*)$ multiplied by the (home) price of risk $(\Lambda(X(t)))$. This shows the effect of the exchange rate as a converter of market prices of risk. That is, a home investor always prices based on his home market prices of risk, $\Lambda(X(t))$, independent of whether it concerns a home or a foreign investment opportunity. The above dynamics also show that investing in the foreign asset, an investor is able to hedge against the foreign risk. The crucial underlying assumption is that the foreign risk is priced in the home market. If the foreign risk factor is not priced in the home market, the home investor has no incentive to hold the foreign converted asset. He is not rewarded for taking the foreign risk, while he is faced with increased volatility.

The key property of an attractive international investment environment is the investor's ability to diversify away the home as well as the foreign risks. This implies non-zero market prices of risk. As is well understood, the market prices of risk are determined as the Sharpe ratio of the risky asset:

$$\Lambda_1(X(t)) = \frac{\delta_{0,P} + \delta_{1,P}X_1(t) - r(t)}{\sigma_1} \tag{18}$$

$$\Lambda_2^*(X(t)) = \frac{\delta_{0,P}^* + \delta_{1,P}^*X_2(t) - r^*(t)}{\sigma^*} \tag{19}$$

However, due to the modelling assumption that the national markets are incomplete, $\Lambda_2(X(t))$ and $\Lambda_1^*(X(t))$ cannot be derived from the observable security prices. International no-arbitrage conditions require that the exchange rate volatilities, $\sigma_{S,1}$ and $\sigma_{S,2}$, and the observable market prices of risk, $\Lambda_1(X(t))$ and $\Lambda_2^*(X(t))$, uniquely determine the market prices of risk that are unobservable from the national asset markets (see (14)):

$$\Lambda_1^*(X(t)) = \Lambda_1(X(t)) - \sigma_{S,1} \tag{20}$$

$$\Lambda_2(X(t)) = \Lambda_2^*(X(t)) + \sigma_{S,2} \tag{21}$$

The difference in market prices of risk is thus corrected by the volatility of the exchange rate. It immediately follows that zero exchange rate volatility equates market prices of risk across countries. This determines the international stochastic environment.

3 Intertemporal Portfolio Choice

In this section we determine the investor's dynamic portfolio problem. Following Merton (1990), we assume no transaction costs, no short-sale constraints and the investor is a price-taker. We take the viewpoint of a home investor who has the choice of investing in three assets: (1) the home risky asset, (2) the foreign risky asset (converted to home currency units), and (3) the home risk-free bond. He can invest a fraction $\alpha(t)$ in the home risky asset, a fraction $\alpha^*(t)$ in the converted foreign risky asset, and a fraction $(1-\alpha(t)-\alpha^*(t))$ in the home risk-free bond.

We solve the portfolio choice problem for an investor with power utility over real wealth at some finite horizon T. To derive the budget constraint, denote $W(t)$ the nominal wealth and $\overline{P}(t)$ the price level in the home economy. The dynamics of the nominal budget constraint and inflation are given by:

$$\frac{dW(t)}{W(t)} = \left\{A(t)'(\mu(X(t)) - r(t)\iota) + r(t)\right\} dt - A(t)'\Sigma dZ(t) \tag{22}$$

$$\frac{d\overline{P}(t)}{\overline{P}(t)} = \pi(X(t))\,dt + \sigma_{\overline{P}} dZ(t) \tag{23}$$

with $A(t) = \left[\begin{array}{cc} \alpha(t), & \alpha^*(t) \end{array}\right]'$ the risky portfolio weights, $\Sigma = \left[\begin{array}{cc} \sigma_1 & 0 \\ \sigma_{S,1} & \sigma_2^* + \sigma_{S,2} \end{array}\right]$ the portfolio's volatility, $\mu(X(t))\,dt = \left[\begin{array}{cc} E_t\left(\frac{dP(t)}{P(t)}\right), & E_t\left(\frac{dV(t)}{V(t)}\right) \end{array}\right]'$ the mean portfolio return and ι a two-dimensional unit vector. Expected inflation is denoted by $\pi(X(t))$ and we assume that it is an affine function of the state variables:

$$\pi(X(t)) = \delta_{0,\pi} + \delta_{1,\pi} X(t) \tag{24}$$

Moreover, the instantaneous unexpected inflation $\sigma_{\overline{P}} dZ(t)$ is perfectly correlated with the home risk factor and uncorrelated with the foreign risk factor. Formally, this implies $\sigma_{\overline{P}} = \left[\begin{array}{cc} \sigma_{\overline{P},1}, & 0 \end{array}\right]$. Using the nominal wealth dynamics (22) and the inflation dynamics (23), real wealth $Y(t) = \frac{W(t)}{\overline{P}(t)}$ can be derived by Ito's lemma:

$$dY(t) = Y(t)\left[\frac{dW(t)}{W(t)} - \frac{d\overline{P}(t)}{\overline{P}(t)} + \frac{d\overline{P}(t)}{\overline{P}(t)}\left(\frac{d\overline{P}(t)}{\overline{P}(t)} - \frac{dW(t)}{W(t)}\right)\right] \quad (25)$$

$$\begin{aligned} dY(t) &= Y(t)\left\{A(t)'(\mu(X(t)) - r(t)\iota) + r(t) - \pi(X(t)) + \sigma_{\overline{P}}\sigma'_{\overline{P}} + A(t)'\Sigma\sigma'_{\overline{P}}\right\}dt \\ &\quad - Y(t)\left(A(t)'\Sigma + \sigma_{\overline{P}}\right)dZ(t) \end{aligned} \quad (26)$$

We assume that the nationality of an investor is determined by the country he lives in, and thus where he consumes his wealth. This implies that the investor takes into account the national inflation rate when maximizing his wealth. The investor solves:

$$\begin{aligned} &\underset{A(t)}{max} E\left[U(Y(T))\right] \\ &s.t. \quad \text{equation (26)} \end{aligned}$$

where $U(Y(T)) = \frac{1}{1-\gamma}(Y(T))^{1-\gamma}$ is defined as power utility over real wealth, with γ the coefficient of relative risk aversion[6].

We use stochastic control to solve this problem[7]. Define $J(t, Y, X)$ as the maximized utility function (or value function). The Bellman principle of optimality implies the following Hamilton-Jacobi-Bellman equation (HJB herein)[8]:

$$0 = \underset{A(t)}{max}\left\{\mathcal{D}J(t, Y, X)\right\} \quad (27)$$

where $\mathcal{D}$ is the Dynkin operator, with $\mathcal{D}J(t, Y, X) = \frac{1}{dt}E_t\left[dJ(t, Y, X)\right]$. The value function is the utility obtained by the investor if he has followed the optimal portfolio policies. That is, the investor has allocated wealth optimally among the available assets, to achieve the highest real wealth at the end of the investment horizon T. Standard dynamic programming conditions result in the optimal portfolio allocation $A(t)^{opt}$:

$$\begin{aligned} A(t)^{opt} &= \frac{-J_Y}{Y(t)J_{YY}}(\Sigma\Sigma')^{-1}(\mu(X, t) - r(t)\iota) - \frac{J_Y}{Y(t)J_{YY}}(\Sigma\Sigma')^{-1}\Sigma\sigma'_{\overline{P}} \\ &\quad - (\Sigma\Sigma')^{-1}\Sigma\sigma'_{\overline{P}} + \frac{1}{Y(t)J_{YY}}(\Sigma\Sigma')^{-1}\Sigma\sigma'_X J_{XY}, \end{aligned} \quad (28)$$

[6] We focus on the case $\gamma > 1$. Kim and Omberg (1996) discuss the solutions for γ unrestricted.

[7] Kamien and Schwartz (1995) give a clear treatment of stochastic optimal control.

[8] For the complete derivation of the dynamic programming solution, see Appendix A.

and the partial differential equation (PDE):

$$\begin{aligned}
0 \;=\;& J_t + J_X' \left(k - KX\left(t\right)\right) + J_Y Y\left(t\right)\left(r\left(t\right) - \pi\left(X\left(t\right)\right)\right) \qquad (29)\\
& -\frac{1}{2} J_Y \left(\mu\left(X\left(t\right)\right) - r\left(t\right)\iota\right)' \left(\Sigma\Sigma'\right)^{-1} \left(\mu\left(X\left(t\right)\right) - r\left(t\right)\iota\right) \frac{J_Y}{J_{YY}} \\
& -J_Y \sigma_{\overline{P}} \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu\left(X,t\right) - r\left(t\right)\iota\right) \frac{J_Y}{J_{YY}} \\
& -J_Y Y\left(t\right) \sigma_{\overline{P}} \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu\left(X\left(t\right)\right) - r\left(t\right)\iota\right) \\
& +J_{XY}' \sigma_X \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu\left(X\left(t\right)\right) - r\left(t\right)\iota\right) \frac{J_Y}{J_{YY}} - \frac{1}{2} J_Y \sigma_{\overline{P}} \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}} \\
& -\frac{1}{2} J_{XY}' \sigma_X \sigma_X' \frac{J_{XY}}{J_{YY}} + \frac{1}{2} Tr\left(J_{XX'} \sigma_X \sigma_X'\right) + J_{XY}' \sigma_X \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}},
\end{aligned}$$

with terminal condition:

$$J\left(T,Y,X\right) = \frac{1}{1-\gamma}\left(Y\left(T\right)\right)^{1-\gamma}. \qquad (30)$$

To solve for the optimal investment allocation, the PDE (29) needs to be solved. Assume a trial solution of the form:

$$J\left(t,X,Y\right) = \frac{Y^{1-\gamma}}{1-\gamma}\Phi\left(X\left(t,T\right)\right) \qquad (31)$$

$$\Phi\left(X\left(t,T\right)\right) = exp\left(\mathcal{A}\left(t,T\right) + \mathcal{B}\left(t,T\right)X\left(t\right) + \frac{1}{2}X\left(t\right)'\mathcal{C}\left(t,T\right)X\left(t\right)\right) \qquad (32)$$

$$\mathcal{A}\left(T\right) = \mathcal{B}\left(T\right) = \mathcal{C}\left(T\right) = 0 \qquad (33)$$

Substitution of the trial solutions (31), (32) and (33) in the optimal investment allocation (28) yields:

$$\begin{aligned}
A\left(t\right)^{opt} \;=\;& \frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\left(\mu\left(X\left(t\right)\right) - r\left(t\right)\iota\right) + \left(\frac{1-\gamma}{\gamma}\right)\left(\Sigma\Sigma'\right)^{-1}\Sigma\sigma_{\overline{P}}' \qquad (34)\\
& -\frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\Sigma\sigma_X'\left(\mathcal{B}\left(t,T\right)' + \mathcal{C}\left(t,T\right)X\left(t\right)\right)
\end{aligned}$$

where $\mathcal{A}\left(t,T\right)$, $\mathcal{B}\left(t,T\right)$ and $\mathcal{C}\left(t,T\right)$ are the solutions to a system of ordinary differential equations (ODE's). This system of ODE's results from substituting the trial solution in the PDE (29). The system of ODE's can be solved numerically with the Runge Kutta simulation technique[9].

The optimal portfolio allocation consists of three components: (1) the nominal mean-variance component, (2) the inflation component (real mean-variance component), and (3) the intertemporal hedging component. The first two portfolio components are the so-called myopic demand. The first component is the standard nominal mean-variance portfolio weight. This is the portfolio that achieves the maximum Sharpe ratio. The higher the coefficient of relative risk aversion, the lower the asset demand induced by mean-variance trade-off. The second component is the inflation portfolio (real mean-variance component),

[9] An analytical solution can be found for the case where no cross-terms of the state variable $X\left(t\right)$ appear. The more general set-up has to be solved by simulation. See Judd (1998).

which adjusts for the fact that the mean-variance portfolio is in nominal rather than real terms. $(\Sigma\Sigma')^{-1}\Sigma\sigma_P'$ selects the portfolio that has maximum correlation with inflation. Given the negative correlation between inflation and asset returns, inflation decreases contemporaneous asset returns and wealth. This induces a negative inflation demand. The higher γ, the larger the negative inflation demand. To summarize, the two mean-variance components constitute the optimal allocation when the investor ignores changes about the future investment set.

The third component is the intertemporal hedging portfolio. $(\Sigma\Sigma')^{-1}\Sigma\sigma_X'$ selects the portfolio that has maximum correlation with the state variables and $\left(\mathcal{B}(t)' + \mathcal{C}(t)X(t)\right)$ measures the sensitivity of wealth towards changes in the stochastic environment.[10] Based on the dynamics of the asset returns and the negative correlation between the shocks to the asset returns and the shocks to the state variables, an increase in the state variables decreases current asset returns, but increases future wealth, and thus a positive amount is allocated to the hedge portfolio. Put differently, when investment opportunities are poor, the risky assets pay off well. This provides a good hedge, and thus the investor increases his risky portfolio holdings relative to the myopic asset demand. This is a well-known result that investors want more wealth when marginal utility is higher. Finally, a more risk averse investor allocates less to the hedging component, reducing the total risky asset holdings.

The result in (34) is similar to the standard optimal dynamic portfolio weight. The exchange rate enters through the portfolio's mean and volatility and through the sensitivity of wealth towards changes in the stochastic environment. This effect is discussed in a next section, where we isolate the exchange rate effect by comparing the optimal portfolio in the presence of exchange rate to that in the absence of exchange rate variability.

4 Exchange Rate Effects on the Optimal Portfolio

Having defined the optimal portfolio, we analyze the impact of the exchange rate on the allocation. The exchange rate effect is isolated, comparing the optimal allocation with and without exchange rates. This sheds light on how the portfolio allocation changes by the removal of exchange rate variability. We start by deriving how the stochastic environment changes when exchange rate variability is eliminated. For ease of notation, denote the variables in the absence of exchange rate variability with a tilde. Elimination of exchange rate variability implies that the rate of return of the exchange rate becomes zero:

$$\frac{d\widetilde{S}(t)}{\widetilde{S}(t)} = 0 \tag{35}$$

[10] This term measures the sensitivity of the logarithm of marginal utility of wealth to a stochastic opportunity set.

For the exchange rate dynamics in (13), this implies that the drift and the diffusion of the exchange rate return should be zero:

$$0 = \left(\widetilde{r}(t) - \widetilde{r}^*(t)\right) + \sigma_S \widetilde{\Lambda}'(X(t)) \tag{36}$$

$$0 = \sigma_S \tag{37}$$

Using the no-arbitrage relation $\sigma_S = \widetilde{\Lambda}(X(t)) - \widetilde{\Lambda}^*(X(t))$, this can be further simplified to:

$$\widetilde{r}(t) = \widetilde{r}^*(t) \tag{38}$$

$$\widetilde{\Lambda}(X(t)) = \widetilde{\Lambda}^*(X(t)) \tag{39}$$

In the case of (irrevocably) fixed exchange rates, the risk-free rates of return and the market prices of risk of the home and the foreign country converge to each other.

Starting from the dynamics of the foreign asset return, converted to home currency units (16), and using the above restrictions on the risk-free rates of return (38) and on the market prices of risk (39), we derive the return of the foreign asset when exchange rate variability is eliminated:

$$\frac{d\widetilde{V}(t)}{\widetilde{V}(t)} = \left(\delta_0^* + \delta_1^* X_2(t)\right) dt - \sigma^* dZ(t) = \frac{dP^*(t)}{P^*(t)} \tag{40}$$

In the absence of exchange rates the price of the risky foreign asset should be the same in both countries. This clearly shows that elimination of exchange rate variability, together with standard no-arbitrage assumptions on an international level, imply that assets are priced as if priced in a single market.

Given the above discussion, we study now how the elimination of the exchange rate induces shifts in the optimal portfolio. We therefore analyze the change in the allocation in the absence of exchange rate variability, $\widetilde{\alpha}(t)$ and $\widetilde{\alpha}^*(t)$, with the allocation when exchange rate variability is present, $\alpha(t)$ and $\alpha^*(t)$:

$$\Delta\alpha(t) = \left(\frac{\sigma_{S,1}}{\sigma_1}\right)\alpha^*(t) + \frac{1}{\gamma}\left(\frac{\Delta\Lambda_1(X(t))}{\sigma_1}\right) - \frac{1}{\gamma}\left(\frac{\sigma_{X_1}}{\sigma_1}\right)\left(\Delta\mathcal{B}_1 + \Delta\mathcal{C}_1 X_1\right) \tag{41}$$

$$\Delta\alpha^*(t) = \left(\frac{\sigma_{S,2}}{\sigma_2^*}\right)\alpha^*(t) + \frac{1}{\gamma}\left(\frac{\Delta\Lambda_2(X(t))}{\sigma_2^*}\right) - \frac{1}{\gamma}\left(\frac{\sigma_{X_2}}{\sigma_2^*}\right)\left(\Delta\mathcal{B}_2 + \Delta\mathcal{C}_2 X_2\right) \tag{42}$$

with $\Delta x_i = \widetilde{x}_i - x_i$ the change in the variable x_i between a situation with no exchange rate variability and a situation with exchange rate variability. Equation (41) expresses the change in the optimal portfolio holding of the home risky asset as the sum of three components. The first component represents the shift induced by the change in the portfolios' exposure to X_1 risk, given equal pricing characteristics. This effect is called the volatility effect of the exchange rate. Elimination of exchange rate variability $\sigma_{S,1}$ implies a change in the portfolio's volatility due to X_1 risk. If, initially, the volatility of the exchange rate is positive, $\sigma_{S,1} > 0$, then elimination of exchange rate variability implies a decline in the portfolio's exposure to X_1 risk. However, given equal pricing structures, the portfolio's volatility to X_1 risk is too low. The investor makes up for this decline by increasing the demand of the home risky asset, whereby he increases the portfolio's exposure to X_1 risk. The opposite effect happens when the exchange rate volatility is negative, $\sigma_{S,1} < 0$. In this case, elimination of exchange rate variability increases the portfolio's exposure to X_1 risk. Accordingly, the investor decreases his home asset demand. Interestingly, in the case of a negative exchange rate variability, the exchange rate acts as an insurance. The exchange rate and the risky asset have an opposite exposure to the home risk factor. A shock to the asset is partly offset by the corresponding shock to the exchange rate and vice versa.

The second and the third components arise from a different cause, namely from a change in the market prices of risk. This effect is called the price effect of the exchange rate. Elimination of exchange rate variability implies a convergence in the market prices of risk between two countries, and thus new common market prices of risk. This change in the pricing characteristics alters the investment opportunities. The new risk-return relationship induces the investor to adjust his optimal home asset holdings. The third component represents the portfolio shift induced by myopic demand considerations. When the market price of risk in the presence of exchange rate variability is higher compared to the case without exchange rate variability, $\widetilde{\Lambda}_1 < \Lambda_1$, the investor decreases his myopic demand. This is reasonable, since the less the investor is rewarded for taking a certain amount of risk, the smaller the amount of risky assets the investor wants to hold. Alternatively, when the investor is rewarded more in the absence of exchange rate variability, $\widetilde{\Lambda}_1 > \Lambda_1$, the investor increases his myopic demand. Finally, the third component is the portfolio shift induced by hedging demand considerations. This change in the hedging demand originates from a change in the sensitivity of wealth of the investor towards changes in the investment environment. If we recall that the system of ODE's (see Appendix A) depends on the market prices of risk, the sensitivity of wealth of the investor towards changes in the investment environment alters. For example, when the sensitivity of the investors in the presence of exchange rates is larger compared to the case with no exchange rate variability, the investor wants to hedge less. Consequently, the hedging demand decreases. However, when the absence of exchange rate variability makes the investors' wealth more sensitive to changes in the investment set, he increases the hedging demand. To summarize, the effect of the exchange rate on the optimal home asset holdings is thus twofold. First, it alters the total exposure of the portfolio to

X_1 risk by $\sigma_{S,1}$ and second, it changes the investment opportunities by altering the market price of risk from Λ_1 to $\widetilde{\Lambda}_1$.

Equation (42) expresses the optimal portfolio holding of the foreign risky asset. The shifts in the foreign portfolio holding are similar to the shifts in the home portfolio holdings. We see three different components. The first component is the volatility effect of the exchange rate. It represents the shift in the portfolio induced by the altered exposure to X_2 risk. When the total portfolio exposure to X_2 risk is increased (decreased) the investor decreases (increases) his holdings of the foreign risky asset. Finally, the second and the third components are the price effects of the exchange rate. They arise due to a change in the market price of X_2 risk, whereby the investment opportunities change. The second component changes the optimal myopic demand and the third component changes the optimal hedging demand. When the elimination of exchange rate variability increases the market price of risks (and makes wealth more sensitive to changes in the investment set) the investor will increase his demand of the risky foreign asset.

The removal of exchange rate variability thus induces shifts in the optimal allocation of both the home and the foreign risky asset. These shifts originate from two effects. First, the portfolios exposure to X_1 and X_2 risk changes, due to elimination of exchange rate variability. International no-arbitrage conditions imply that these exchange rate risk loadings are in fact determined as the differential in the home and foreign market prices of risk. Therefore, the impact of the volatility effect of the exchange rate crucially depends on the difference in market prices of risk before monetary unification. Secondly, the market prices of risk the investor faces have converged to new, common market prices of risk. This creates new investment opportunities. This makes clear that the importance of the portfolio shifts depends on the (differentials in) market prices of risk. When market prices of risk are (almost) equal in the home and foreign country in the presence of exchange rate variability, the effect of the elimination of the exchange rate on the optimal portfolio is negligible.

5 Calibration Exercise

To provide illustrative calculations of the exchange rate effect on the optimal international portfolio, we calibrate the above international portfolio model. Table 1 reports the parameter estimates of the stochastic environment, which we assume identical for the home and foreign market.

We estimate the optimal portfolio holdings for a home and foreign dividend yield of 0.4%. We take a coefficient of risk aversion of 2, 6 and 10 with a fixed investment horizon of 10 years. The stochastic environment for the home investor changes when exchange rate

Table 1: **Model Parameters**

Parameter	Estimate
constant state variable, k	7.3E-05
mean reversion parameter, K	0.0194
volatility of state variables, σ_X	0.0002
constant asset return, $\delta_{0,P}$	0.0031
time-varying asset return, $\delta_{1,P}$	1.2061
volatility of asset returns, σ	0.0581
constant risk-free rate, $\delta_{0,r}$	0.0027
time-varying risk-free rate, $\delta_{1,r}$	0.4648
constant inflation, $\delta_{0,\pi}$	-5.4E-05
time-varying inflation, $\delta_{1,\pi}$	0.0201
volatility of inflation, $\sigma_{\overline{P}}$	8.3E-05
Sharpe ratio, $\Lambda_1(X(t))$, $\Lambda_2^*(X(t))$	0.0597

All estimates are in monthly units. Parameters are based on estimations for Germany, using Morgan Stanley Capital International total market indices (net dividends included) and the one-month LIBOR rates. The state variable is the dividend yield. Price level data are used to calculate monthly inflation rates. The correlation between the state variable and the asset returns is -0.83. In the portfolio optimization, this is set to -1.

variability is eliminated. First, we assume a risk exposure of the exchange rate ranging from -0.02 up to 0.02, which disappears when exchange rate variability is eliminated[11]. Also, the market prices of risk the investor faces, $\Lambda_1(X(t))$ and $\Lambda_2(X(t))$, change. Furthermore, we assume for simplicity that the market prices of risk as determined by the Sharpe ratios, $\Lambda_1(X(t))$ and $\Lambda_2^*(X(t))$ do not change following the elimination of exchange rate variability. With this assumption, a negative exchange rate risk sensitivity, $\sigma_{S,2} < 0$, implies an increase in the home market price of risk of the X_2 risk. On the contrary, when the exchange rate volatility is positive, $\sigma_{S,2} > 0$, the home market price of risk of the X_2 risk decreases. The portfolio shifts, thus, originate from a change in the home market price of risk of the foreign risk factor, $\Lambda_2(X(t))$, and from the elimination of exchange rate variability, $\sigma_{S,1}$ and $\sigma_{S,2}$. With these simplifying assumptions, the exchange rate effect in the home risky asset (41) reduces to the volatility effect $ER_{\sigma_{S,1}} = \frac{\sigma_{S,1}}{\sigma_1}\alpha^*(t)$. For the foreign risky asset, (42) shows that the total exchange rate effect is given by the volatility effect $ER^*_{\sigma_{S,2}} = \left(\frac{\sigma_{S,2}}{\sigma_2^*}\right)\alpha^*(t)$, plus the myopic price effect $ER^*_{\Delta\Lambda_2,M} = \frac{1}{\gamma}\left(\frac{\Delta\Lambda_2(X(t))}{\sigma_2^*}\right)$ and the intertemporal price effect $ER^*_{\Delta\Lambda_2,H} = -\frac{1}{\gamma}\left(\frac{\sigma_{X_2}}{\sigma_2^*}\right)(\Delta\mathcal{B}_2 + \Delta\mathcal{C}_2 X_2)$. Schematically, we summarize the exchange rate effects in table 2.

[11] This sensitivity of the exchange rate is reasonable; De Santis and Gerard (1998), and De Santis, Gerard and Hillion (1999) estimate exchange rate volatilities up to 0.03.

Table 2: **Signs exchange rate effects**

	volatility effect	price effect	
$\sigma_{S,i} < 0$	-	+/0	$\Delta\Lambda_i \geq 0$
$\sigma_{S,i} > 0$	+	-/0	$\Delta\Lambda_i \leq 0$

Table 3 summarizes the optimal investment strategies for different levels of exchange rate volatilities and coefficients of relative risk aversion. The investment horizon is fixed at $T = 10$ years. The portfolio weights add up to 100%. The middle column, under the exchange rate loading of zero, reports the optimal portfolio weights when exchange rate variability is eliminated. Given the symmetrical structure for the home and foreign stochastic environment, the portfolio weight of the risky home asset and the risky foreign asset are almost identical. The small difference originates from the inflation demand, inducing the investor to hold less of the home risky asset.

The exchange rate component in the optimal international portfolio is significant, as high as 22%. Elimination of exchange rate variability, thus, induces large shifts in the optimal international portfolio. The exchange rate effect is decreasing over the coefficient of relative risk aversion. For initial exchange rate sensitivities $\sigma_{S,1}$ and $\sigma_{S,2}$ smaller (larger) than zero, the optimal portfolio demand decreases (increases) following the elimination of exchange rate variability. For the home risky asset, this result is clear. Given the equal market price of risk $\Lambda_1 (X(t))$ with and without exchange rate variability, there is only the volatility effect of the exchange rate. When the total portfolio exposure increases (decreases) due to elimination of exchange rate variability, the investor hedges this additional volatility away by reducing (increasing) the optimal allocation to the home risky asset. For the foreign risky asset, the exchange rate does not only have this volatility effect, but also a price effect. When the exchange rate volatility is negative (positive), the market price of risk in the presence of exchange rate variability is lower (higher) compared to the market prices of risk in the absence of exchange rate variability. Faced with a higher (lower) market price of risk following the elimination of exchange rate variability, the investor increases (decreases) his myopic and hedging demand. For the calibration reported in Table 3, we see that the volatility exchange rate effect dominates in the foreign allocation. That is, when an initial negative exposure of the exchange rate is eliminated, the investor decreases the total allocation to the foreign risky asset, while he increases the total allocation to the foreign risky asset for an initial positive exchange rate volatility. Table 4 reinforces these results. This table reports the magnitude of the different exchange rate components.

Given the assumption that the market price of risk $\Lambda_1 (X(t))$ does not change by the elimination of exchange rate variability, the allocation to the home risky asset only has

Table 3: **Optimal Portfolio Strategy (%)**

γ		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
		-0.02	-0.002	-0.002	0	0.0002	0.002	0.02
			$\Delta\Lambda_2 > 0$				$\Delta\Lambda_2 < 0$	
2	$\alpha(t)$	83.55	63.24	61.34	61.13	60.92	59.03	40.72
	$\alpha^*(t)$	65.13	61.47	61.23	61.21	61.18	60.69	59.29
	$100-\alpha(t)-\alpha^*(t)$	-48.68	-24.71	-22.57	-22.34	-22.10	-19.72	-0.01
6	$\alpha(t)$	38.94	28.70	27.82	27.73	27.63	26.78	18.93
	$\alpha^*(t)$	32.58	28.18	27.89	27.86	27.83	27.56	25.56
	$100-\alpha(t)-\alpha^*(t)$	28.48	43.12	44.29	44.41	44.54	45.66	55.51
10	$\alpha(t)$	29.13	21.12	20.48	20.41	20.34	19.71	14.13
	$\alpha^*(t)$	25.33	20.88	20.58	20.55	20.52	20.25	18.22
	$100-\alpha(t)-\alpha^*(t)$	45.54	58.00	58.94	59.04	59.14	60.04	67.65

This table reports the optimal strategies for an investor with different values of exchange rate volatility and risk aversion, for a fixed investment horizon of 10 years; $\alpha(t)$ and $\alpha^*(t)$ are the proportional allocation to the home and foreign risky assets, respectively; $100-\alpha(t)-\alpha^*(t)$ is the proportion allocated to the riskless asset. The middle column reports the optimal allocation when exchange rate variability is eliminated.

a single exchange rate component. An initial negative exchange rate volatility leads to a lower weight of the home risky asset. An initial positive exchange rate volatility leads to a higher home portfolio weight. The exchange rate component of the foreign asset allocation is more interesting. We clearly see two opposing effects of the exchange rate: for an initial negative exchange rate sensitivity, the volatility effect decreases the demand of the foreign risky asset, while the price effect leads to an increase in the foreign asset demand. In this example, the volatility effect dominates the price effect. An investor faced with an initial negative exchange rate loading, decreases the holdings of the foreign risky asset. A positive exchange rate effect induces the investor to increase the amount of foreign risky assets in his optimal portfolio. Finally, note that the price effect of the exchange rate induced by hedging considerations is small relative to the volatility effect and the price effect of the myopic demand. The volatility effect is thus mainly offset by the myopic demand price effect.

The above calibration is repeated for the optimal investment policies for a fixed coefficient of relative risk aversion ($\gamma = 6$), and for different investment horizons. The results of this calibration are similar as the previous results presented above. They can be found in Appendix B.

To see the importance of the exchange rate on the optimal allocation in terms of wealth, we compare the certainty equivalent of wealth for the optimal portfolio in the presence of

Table 4: **Exchange Rate Components (as % of total portfolio weights)**

γ		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
		-0.02	-0.002	-0.0002	0	0.0002	0.002	0.02
			$\Delta\Lambda_2 > 0$				$\Delta\Lambda_2 < 0$	
2	$ER_{\sigma_{S,1}}$	-22.42	-2.12	-0.21	0	0.21	2.10	20.41
	$ER^*_{\sigma_{S,2}}$	-32.13	-2.18	-0.21	0	0.21	2.04	15.67
	$ER^*_{\Delta\Lambda_2,M}$	26.25	1.78	0.17	0	-0.17	-1.66	-12.80
	$ER^*_{\Delta\Lambda_2,H}$	1.96	0.13	0.01	0	-0.01	-0.12	-0.96
6	$ER_{\sigma_{S,1}}$	-11.21	-0.97	-0.10	0	0.10	0.95	8.80
	$ER^*_{\sigma_{S,2}}$	-14.62	-0.99	-0.10	0	0.10	0.93	7.13
	$ER^*_{\Delta\Lambda_2,M}$	8.75	0.59	0.06	0	-0.06	-0.55	-4.27
	$ER^*_{\Delta\Lambda_2,H}$	1.16	0.08	0.01	0	-0.01	-0.07	-0.57
10	$ER_{\sigma_{S,1}}$	-8.72	-0.72	-0.07	0	0.07	0.70	6.27
	$ER^*_{\sigma_{S,2}}$	-10.79	-0.73	-0.07	0	0.07	0.68	5.26
	$ER^*_{\Delta\Lambda_2,M}$	5.25	0.36	0.03	0	-0.03	-0.33	-2.56
	$ER^*_{\Delta\Lambda_2,H}$	0.76	0.05	0.01	0	-0.00	-0.05	-0.37

exchange rates to that of the optimal portfolio when exchange rate variability is eliminated[12]:

$$ERG = \frac{CE}{\widetilde{CE}}$$

This ratio of certainty equivalents reflects the effect of exchange rate variability on the investor's welfare. If the ratio is larger than one, the investor will be made worse off by the elimination of exchange rate variability. Adversely, if the ratio is smaller than one, the investor will be made better off by the elimination of exchange rate variability. Table 5 reports these results. The variable ERG is the ratio of the certainty equivalent of the optimal portfolio in the presence of exchange rate variability to the certainty equivalent of the optimal portfolio when exchange rate variability is eliminated.

Table 5: **Ratio of Certainty Equivalents of Wealth**

γ	ERG	Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
		-0.02	-0.002	-0.002	0	0.0002	0.002	0.02
			$\Delta\Lambda_2 > 0$				$\Delta\Lambda_2 < 0$	
2		0.94	0.99	1.00	1.00	1.00	1.01	1.10
6		0.97	1.00	1.00	1.00	1.00	1.00	1.04
10		0.98	1.00	1.00	1.00	1.00	1.00	1.03

The ratio of the certainty equivalents, when initial exchange rate risk sensitivities are negative, are smaller than one, while they are larger than one when initial exchange rate volatilities are positive. From the previous discussion, we know that an initial negative

[12]The certainty equivalent is the amount of wealth such that the investor is indifferent between receiving it for sure at the horizon, and having his current wealth today invested optimally up to the horizon.

exchange rate sensitivity implies that the price of risk the investor faces when exchange rate variability is eliminated has increased. For a given amount of risk, the investor is rewarded more. The investor is then better off when exchange rate variability is eliminated. On the contrary, an initial positive exchange rate volatility implies that the price of risk the investor faces has decreased. The investors is then worse off when exchange rate variability is eliminated.

The gain or loss over the elimination of exchange rate variability is substantial when the exchange rate volatility is large: when the exchange rate volatility is -0.02, the gain ranges from 2 percent to 6 percent, and the loss ranges from 3 percent to 10 percent when the exchange rate volatility is 0.02.

6 Conclusion

This paper solves the dynamic portfolio allocation for an international market where asset prices are characterized by mean reversion. The model derives an analytical solution for the optimal portfolio in a two-country, two-state variable model, in which an investor can allocate his wealth between a home and a foreign asset. Moreover, inflation dynamics are included to determine the nationality of an investor. Our model is derived under the assumption that the home and foreign assets are influenced by independent sources of risk, but that both markets do price both risks. We then have incomplete national markets, but a complete international market. Once converted to home currency units, the foreign asset depends on both the home and the foreign risk factors. This setting allows us to study the effect of the exchange rate in isolation.

Using international no-arbitrage conditions, we link the home and foreign stochastic environment, and we show that the optimal international portfolio consists of three components: a nominal myopic demand, an inflation demand (real myopic demand) and a hedging demand. We show that with the elimination of exchange rate variability, market prices of risk and riskfree interest rates converge to each other and the portfolios' exposure to the risk factors change. These changes in the investment set have a double effect. First there is a volatility effect, as the elimination of exchange rate volatility induces the investor to adjust his portfolio to obtain a new optimal exposure for the given market prices of risk. Second, the investor faces new investment opportunities through the change in the market prices of risk. Accordingly, he adjusts his myopic and hedging demand optimally. This is the price effect of the exchange rate.

To determine the importance of the elimination of the exchange rate in an international portfolio, we calibrate our model for an international portfolio using the dividend yield as the state variable. We compare the optimal allocation in the presence of exchange rate variability, with the optimal allocation when exchange rates are artificially eliminated. We find that the exchange rate effect on the portfolio allocation is significant. In response to the volatility and price effect, the investor optimally reallocates his international portfolio. We show that the volatility effect of the exchange rate and the myopic demand price effect constitute the main driving forces of these portfolio shifts.

This model allows us to assess the portfolio shifts following the introduction of the Euro. Given the small difference in market prices of risk across the different EMU members, even before the monetary unification (see Dewachter et al. (2003)), it can be expected that the portfolio shifts are limited. Furthermore, in this paper we derived the effect of the elimination of exchange rate variability on the optimal international portfolio assuming that international markets are complete, and that there are no exchange rate specific diversification benefits. The next challenge is to extend the international portfolio model to take account of an additional, priced exchange rate specific stochastic component. To fully hedge all the risks away, the optimal portfolio then consists of the home risky asset (hedging the home risk factor), the foreign converted risky asset (hedging the foreign risk factor) and the currency (hedging the exchange rate risk factor). In this case, the removal of exchange rate variability has an additional effect of removing the specific exchange rate risk factor.

A Stochastic Control Problem

We use stochastic control to solve the investors' portfolio problem. Define $J(t,Y,X)$ as the maximized utility function (or value function). The Bellman principle of optimality implies the following Hamilton-Jacobi-Bellman (HJB herein):

$$0 = \max_{A(t)} \{\mathcal{D}J(t,Y,X)\} \tag{43}$$

where $\mathcal{D}$ is the Dynkin operator, with $\mathcal{D}J(t,Y,X) = \frac{1}{dt}E_t[dJ(t,Y,X)]$. Using Ito's lemma to compute $dJ(t,Y,X)$, the control problem becomes:

$$0 = \max_{A(t)} \frac{1}{dt} E_t \left[\begin{array}{c} J_t dt + J_Y dY + J_X' dX + \frac{1}{2} J_{YY} dY dY' \\ +\frac{1}{2} Tr\left(J_{XX'} dX dX'\right) + J_{XY}' dX dY \end{array} \right] \tag{44}$$

with the terminal condition:

$$J(T,Y,X) = \frac{1}{1-\gamma}(Y(T))^{1-\gamma} \tag{45}$$

where J_d is the partial derivative of J with respect to d. Using the real wealth dynamics (26), the state variable dynamics (1) and taking expectations yields:

$$0 = \max_{A(t)} \left[\begin{array}{c} J_t + J_X'(k - KX(t)) + J_Y Y(t)(r(t) - \pi(X(t))) \\ +J_Y Y(t) A(t)' (\mu(X(t)) - r(t)\iota) + J_Y Y(t) \sigma_{\overline{P}} \sigma_{\overline{P}}' \\ +J_Y Y(t) A(t)' \Sigma \sigma_{\overline{P}}' + \frac{1}{2} J_{YY} Y(t) A(t)' \Sigma\Sigma' A(t) Y(t)' \\ +J_{YY} Y(t) A(t)' \Sigma \sigma_{\overline{P}}' Y(t)' + \frac{1}{2} J_{YY} Y(t) \sigma_{\overline{P}} \sigma_{\overline{P}}' Y(t)' \\ +\frac{1}{2} Tr\left(J_{XX'} \sigma_X \sigma_X'\right) - J_{YX}' \sigma_X \Sigma' A(t) Y(t)' - J_{XY}' \sigma_X \sigma_{\overline{P}}' Y(t)' \end{array} \right] \tag{46}$$

From the HJB equation, we can now compute the first-order optimality condition (FOC) for the portfolio allocation:

$$\begin{aligned} A(t)^{opt} \quad = \quad & \frac{-J_Y}{Y(t) J_{YY}} (\Sigma\Sigma')^{-1} (\mu(X,t) - r(t)\iota) - \frac{J_Y}{Y(t) J_{YY}} (\Sigma\Sigma')^{-1} \Sigma \sigma_{\overline{P}}' \\ & - (\Sigma\Sigma')^{-1} \Sigma \sigma_{\overline{P}}' + \frac{1}{Y(t) J_{YY}} (\Sigma\Sigma')^{-1} \Sigma \sigma_X' J_{XY} \end{aligned} \tag{47}$$

Substitution of the first order condition (47) back into the HJB (46) equation yields the following partial differential equation:

$$\begin{aligned} 0 = {} & J_t + J_X' (k - KX(t)) + J_Y Y(t)(r(t) - \pi(X(t))) \\ & -\frac{1}{2} J_Y (\mu(X(t)) - r(t)\iota)' (\Sigma\Sigma')^{-1} (\mu(X(t)) - r(t)\iota) \frac{J_Y}{J_{YY}} \\ & - J_Y \sigma_{\overline{P}} \Sigma' (\Sigma\Sigma')^{-1} (\mu(X,t) - r(t)\iota) \frac{J_Y}{J_{YY}} \\ & - J_Y Y(t) \sigma_{\overline{P}} \Sigma' (\Sigma\Sigma')^{-1} (\mu(X(t)) - r(t)\iota) + J_{XY}' \sigma_X \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}} \\ & + J_{XY}' \sigma_X \Sigma' (\Sigma\Sigma')^{-1} (\mu(X(t)) - r(t)\iota) \frac{J_Y}{J_{YY}} \\ & - \frac{1}{2} J_Y \sigma_{\overline{P}} \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}} - \frac{1}{2} J_{XY}' \sigma_X \sigma_X' \frac{J_{XY}}{J_{YY}} + \frac{1}{2} Tr(J_{XX'} \sigma_X \sigma_X') \end{aligned} \tag{48}$$

The next step in solving this PDE is to guess a trial solution for the value function $J(t, X, Y)$. We conjecture that:

$$J(t, X, Y) = \frac{Y^{1-\gamma}}{1-\gamma} \Phi(X(t,T)) \tag{49}$$

where $\Phi(X(t,T)) = \Phi(X(T-t))$ represents the contribution to the investor's expected utility of the remaining investment opportunities up to the horizon. From (18)-(21) in the main text, we know that the market prices of risk are affine functions of the state variables. Therefore, define:

$$\Lambda(X(t)) = \lambda_0' + X(t)' \lambda_1' \tag{50}$$

$$\Lambda^*(X(t)) = \lambda_0^{*\prime} + X(t)' \lambda_1^{*\prime} \tag{51}$$

Furthermore, using the fact that $(\mu(X(t)) - r(t)\iota) = \Sigma\Lambda' = \Sigma(\lambda_0' + X'(t)\lambda_1')$ and that $\pi(X(t)) = \delta_{0,\pi} + \delta_{1,\pi} X(t)$, and substituting the trial solution (49) in the PDE (48) yields after simplification:

$$\begin{aligned} 0 = {} & -\Phi_t + \Phi_X' (k - KX(t)) + \left(\frac{1-\gamma}{2\gamma}\right) \Phi(X(t,T)) \lambda_0' \lambda_0 \\ & + \left(\frac{1-\gamma}{\gamma}\right) \Phi(X(t,T)) X(t)' \lambda_1' \lambda_0 + \left(\frac{1-\gamma}{2\gamma}\right) \Phi(X(t,T)) X(t)' \lambda_1' \lambda_1 X(t) \\ & + (1-\gamma) \Phi(X(t,T)) (\delta_{0,r} + \delta_{1,r} X(t) - \delta_{0,\pi} - \delta_{1,\pi} X(t)) \\ & + \frac{(1-\gamma)^2}{\gamma} \Phi(X(t,T)) \sigma_{\overline{P}} \lambda_0 + \frac{(1-\gamma)^2}{\gamma} \Phi(X(t,T)) \sigma_{\overline{P}} \lambda_1 X(t) \\ & + \left(\frac{1-\gamma}{2\gamma}\right) \Phi(X(t,T)) \sigma_{\overline{P}} \sigma_{\overline{P}}' + \left(\frac{1-\gamma}{2\gamma}\right) \Phi_X' \sigma_X \sigma_X{}' \left(\frac{\Phi_X}{\Phi(X(t,T))}\right) \\ & + \frac{1}{2} Tr(\Phi_{XX'} \sigma_X \sigma_X') - \left(\frac{1-\gamma}{\gamma}\right) \Phi_X' \sigma_X \lambda_0 \\ & - \left(\frac{1-\gamma}{\gamma}\right) \Phi_X' \sigma_X \lambda_1 X(t) - \left(\frac{1-\gamma}{\gamma}\right) \Phi_X' \sigma_X \sigma_{\overline{P}}' \end{aligned} \tag{52}$$

This equation is a linear second-order PDE, which is analytically hard to solve. Therefore, we follow the approach of Liu (2001), and rewrite the above PDE into a system of ODE's. This is done by guessing a functional form for $\Phi(X(t,T))$. Assume the trial solution:

$$\Phi(X(t,T)) = exp\left(\mathcal{A}(t,T) + \mathcal{B}(t,T)X(t) + \frac{1}{2}X(t)'\mathcal{C}(t,T)X(t)\right) \tag{53}$$

and we impose that $\Phi(X(T)) = 1$, which implies boundary conditions:

$$\mathcal{A}(T) = \mathcal{B}(T) = \mathcal{C}(T) = 0 \tag{54}$$

Substitution of this trial solution in the HJB (52) results in a quadratic equation in X. Its three coefficients need to be zero. Therefore, collecting terms in the constant, X and $X'X$ results in the following system of first-order nonlinear equations:

$$\begin{aligned} 0 &= -\mathcal{A}_t + \mathcal{B}(t,T)k + \left(\frac{1-\gamma}{2\gamma}\right)\lambda_0'\lambda_0 + \frac{(1-\gamma)^2}{\gamma}\sigma_{\overline{P}}\lambda_0 - \left(\frac{1-\gamma}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\lambda_0 \\ &\quad + (1-\gamma)(\delta_{0,r}-\delta_{0,\pi}) + \left(\frac{1-\gamma}{2\gamma}\right)\sigma_{\overline{P}}\sigma_{\overline{P}}' + \left(\frac{1}{2\gamma}\right)\mathcal{B}(t,T)\sigma_X\sigma_X'\mathcal{B}(t,T)' \\ &\quad + \frac{1}{2}Tr\left(\mathcal{C}(t,T)\sigma_X\sigma_X'\right) - \left(\frac{1-\gamma}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\sigma_{\overline{P}}' \end{aligned} \tag{55}$$

$$\begin{aligned} 0 &= -\mathcal{B}_t - \mathcal{B}(t,T)K + k'\mathcal{C}(t,T) + \left(\frac{1-\gamma}{\gamma}\right)\lambda_0'\lambda_1 + \frac{(1-\gamma)^2}{\gamma}\sigma_{\overline{P}}\lambda_1 \\ &\quad - \left(\frac{1-\gamma}{\gamma}\right)\lambda_0'\sigma_X\mathcal{C}(t,T) - \left(\frac{1-\gamma}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\lambda_1 + (1-\gamma)(\delta_{1,r}-\delta_{1,\pi}) \\ &\quad - \left(\frac{1}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\sigma_X'\mathcal{C}(t,T) - \left(\frac{1-\gamma}{\gamma}\right)\sigma_{\overline{P}}\sigma_X'\mathcal{C}(t,T) \end{aligned} \tag{56}$$

$$\begin{aligned} 0 &= -\mathcal{C}_t - 2\mathcal{C}(t,T)'K + \left(\frac{1-\gamma}{\gamma}\right)\lambda_1'\lambda_1 + \left(\frac{1}{\gamma}\right)\mathcal{C}(t,T)'\sigma_X\sigma_X'\mathcal{C}(t,T) \\ &\quad -2\left(\frac{1-\gamma}{\gamma}\right)\mathcal{C}(t,T)'\sigma_X\lambda_1 \end{aligned} \tag{57}$$

To solve for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ we start from the ODE (57), then solve for (56), and finally for (55). We use the fourth order Runge Kutta simulation technique. Finally, substitution of the trial solutions (49) and (53) into the portfolio first-order condition (47) determines the optimal portfolio allocation:

$$\begin{aligned} A(t)^{opt} &= \frac{1}{\gamma}(\Sigma\Sigma')^{-1}(\mu(X(t)) - r(t)\iota) + \left(\frac{1-\gamma}{\gamma}\right)(\Sigma\Sigma')^{-1}\Sigma\sigma_{\overline{P}}' \\ &\quad -\frac{1}{\gamma}(\Sigma\Sigma')^{-1}\Sigma\sigma_X'\left(\mathcal{B}(t,T)' + \mathcal{C}(t,T)X(t)\right) \end{aligned} \tag{58}$$

where $\mathcal{B}$ and $\mathcal{C}$ are the solutions to the system of ODE's.

B Calibration

Tables 6, 7 and 8 summarize the results of the optimal portfolio policies for an investment horizon of 1, 15 and 20 years and a fixed coefficient of relative risk aversion of 6.

Table 6: **Optimal Portfolio Strategy (%)**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
T		-0.02	-0.002	0.0002	0	0.0002	0.002	0.02
			$\Delta\Lambda_2 > 0$				$\Delta\Lambda_2 < 0$	
1	$\alpha\left(t\right)$	26.57	20.09	19.48	19.41	19.34	18.74	12.90
	$\alpha^*\left(t\right)$	20.80	19.62	19.54	19.53	19.52	19.45	18.91
	$100-\alpha\left(t\right)-\alpha^*\left(t\right)$	52.63	60.29	60.98	61.06	61.14	61.81	68.19
15	$\alpha\left(t\right)$	40.34	29.68	28.77	28.67	28.57	27.69	19.61
	$\alpha^*\left(t\right)$	33.90	29.15	28.84	28.81	28.77	28.48	26.32
	$100-\alpha\left(t\right)-\alpha^*\left(t\right)$	25.76	41.17	42.39	42.52	42.56	43.83	54.07
20	$\alpha\left(t\right)$	40.84	30.03	29.11	29.01	28.91	28.02	19.86
	$\alpha^*\left(t\right)$	34.38	29.50	29.18	29.15	29.11	28.82	26.60
	$100-\alpha\left(t\right)-\alpha^*\left(t\right)$	24.78	40.47	41.71	41.84	41.98	43.16	53.54

This table reports the optimal strategies for an investor with different values of exchange rate volatility and investment horizons, for a fixed coefficient of relative risk aversion of 6; $\alpha\left(t\right)$ and $\alpha^*\left(t\right)$ are the proportional allocation to the home and foreign risky assets, respectively. The middle column reports the optimal allocation when exchange rate variability is eliminated.

Table 7: **Exchange Rate Components (as % of total portfolio weights)**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
T		-0.02	-0.002	-0.0002	0	0.0002	0.002	0.02
1			$\Delta\Lambda_2 > 0$				$\Delta\Lambda_2 < 0$	
	$ER_{\sigma_{S,1}}$	-7.16	-0.68	-0.07	0	0.07	0.67	6.51
	$ER^*_{\sigma_{S,2}}$	-10.25	-0.70	-0.07	0	0.07	0.65	5.00
	$ER^*_{\Delta\Lambda_2,M}$	8.25	0.59	0.06	0	-0.06	-0.55	-4.27
	$ER^*_{\Delta\Lambda_2,H}$	0.23	0.02	0.00	0	-0.00	-0.01	-0.11
15								
	$ER_{\sigma_{S,1}}$	-11.67	-1.00	-0.10	0	0.10	0.98	9.06
	$ER^*_{\sigma_{S,2}}$	-15.12	-1.03	-0.10	0	0.10	0.96	7.38
	$ER^*_{\Delta\Lambda_2,M}$	8.75	0.59	0.06	0	-0.06	-0.55	-4.27
	$ER^*_{\Delta\Lambda_2,H}$	1.28	0.09	0.01	0	-0.01	-0.08	-0.62
20								
	$ER_{\sigma_{S,1}}$	-11.83	-1.02	-0.10	0	0.10	0.99	9.16
	$ER^*_{\sigma_{S,2}}$	-15.30	-1.04	-0.10	0	0.10	0.97	7.46
	$ER^*_{\Delta\Lambda_2,M}$	8.75	0.59	0.06	0	-0.06	-0.55	-4.27
	$ER^*_{\Delta\Lambda_2,H}$	1.32	0.09	0.01	0	-0.01	-0.08	-0.64

Table 8: **Ratio of Certainty Equivalents of Wealth**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
T	ERG	-0.02	-0.002	-0.002	0	0.0002	0.002	0.02
			$\Delta\Lambda_2 > 0$				$\Delta\Lambda_2 < 0$	
1		1.00	1.00	1.00	1.00	1.00	1.00	1.00
15		0.96	0.99	1.00	1.00	1.00	1.01	1.06
20		0.94	0.99	1.00	1.00	1.00	1.01	1.09

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