Stable laws and the distribution of cash-flows

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Abstract

In a paper of 1994, Kozubowski and Rachev investigated the performance of stable laws and in particular geometric stable laws when modeling asset returns. One of their most important findings was that these models provide a good fit for financial data sets.

In the present contribution, we consider the present value of a series of cash flows under stochastic interest rates and we make use of stable laws to model these interest rates. Analogously to former papers, we will not try to calculate the exact analytical distribution for the cash-flow since this seems to be impossible. Instead we determine upper bounds which are easier to compute, and we derive results for the stop loss premium and distribution of these bounds.

1   Introduction

In some former contributions (see [1, 2, 5]) we investigated the present value of a series of non-negative payments at times 1 up to n

$$A = \sum_{j=1}^{n} c_j e^{-Y(t_j)}, \tag{1}$$

where $Y(t_j)$ represents the stochastic continuous compounded rate of return over the period $[0, t_j]$.

Up to now, the results based on comonotonic risks have been applied to the case of Wiener processes. In fact, we wrote the rate of return as the sum of increments over the previous periods,

$$Y(t_j) = \sum_{i=1}^{j} (Y(t_i) - Y(t_{i-1})) \tag{2}$$

where $0 = t_0 < t_1 < ... < t_n = t$, and where each component $Y(t_i) - Y(t_{i-1})$ denotes the rate of return for the period $[t_{i-1}, t_i]$. Remark that due to this characterization, the variables $Y(t_j)$ are mutually dependent.

Since we assumed the stochastic process $\{Y(s)\}_{s \geq 0}$ to be a Wiener process, the increments $Y(t_i) - Y(t_{i-1})$ are independent and normally distributed

$$Y(t_i) - Y(t_{i-1}) \sim N(\mu(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1})) \tag{3}$$

or

$$Y(t_i) - Y(t_{i-1}) \stackrel{d}{=} \mu(t_i - t_{i-1}) + (t_i - t_{i-1})^{1/2} \sigma Z_i \tag{4}$$

with $Z_i \sim N(0, 1)$ independent standard normal variates.

As a consequence of the properties of Wiener processes, for the compounded rates of return we have

$$Y(t_j) \sim N(\mu t_j, \sigma^2 t_j), \tag{5}$$

or

$$Y(t_j) \stackrel{d}{=} \mu t_j + t_j^{1/2} \sigma X_j \tag{6}$$

with $X_j \sim N(0, 1)$ again a standard normal variate. Note that the variables $X_1, ..., X_n$ are no longer independent.

In the present contribution, we will make two generalizations.

1. A normal distribution is a special case of a stable law, which in addition to the mean μ and deviation σ has two more parameters : $\alpha \in (0,2]$ being the index of stability, determining the tail of the distribution, and $\beta \in [-1,+1]$ being a skewness parameter. Using the (more general) stable law instead of a normal one, the distribution of the increments is a stable law, denoted as

$$Y(t_i) - Y(t_{i-1}) \sim S_\alpha(\sigma, \beta, \mu; t_i - t_{i-1}) \tag{7}$$

which means that equation (6) is changed into

$$Y(t_i) - Y(t_{i-1}) \overset{d}{=} \mu(t_i - t_{i-1}) + (t_i - t_{i-1})^{1/\alpha} \sigma Z_i \tag{8}$$

with $Z_i \sim S_\alpha(1, \beta, 0; 1)$ independent standard stable variates.

Since we work with stable processes, for the total rate of return we have

$$Y(t_j) \sim S_\alpha(\sigma, \beta, \mu; t_j), \tag{9}$$

or

$$Y(t_j) \overset{d}{=} \mu t_j + t_j^{1/\alpha} \sigma X_j \tag{10}$$

with $X_j \sim S_\alpha(1, \beta, 0; 1)$ again a standard stable variate.
Just as in the Wiener case, the variables $X_1, ..., X_n$ are dependent.

A justification of this result as well as more details with respect to this distribution and the meaning of the parameters are given in section 2.

For a choice of $\alpha = 2$, the normal model emerges.

2. Next, we introduce a risk parameter Θ. Conditionally on this risk parameter, the distribution of the increments is the one of a stable law. Thus for the compounded rate of return we have

$$Y(t_j)|_{\Theta = \theta} \sim S_\alpha(\sigma, \beta, \mu; t_j \theta) \ ; \tag{11}$$

equation (10) then yields

$$Y(t_j)|_{\Theta} \overset{d}{=} \mu t_j \Theta + (t_j \Theta)^{1/\alpha} \sigma X_j \tag{12}$$

with $X_j \sim S_\alpha(1, \beta, 0; 1)$, which is again a standard stable variate.

Remark that in case Θ has all its mass in one, i.e. $Prob[\Theta = 1] = 1$, we recognize the first stable model.

The aim of this paper is the investigation of the present value of equation (1) when rates of return are modeled by means of the general model of equation (12). The paper is organized as follows. First we will give a summary of the concepts, properties and methods that are needed to reach our goal. In section 2 we briefly describe the stable laws, while section 3 is used to go through the methodology of bounds in convexity order. Afterwards in section 4, we will be able to present the results about the present value of equation (1). Finally section 5 is meant to numerically illustrate the results of section 4.

2   Stable laws

The most easy way to define a stable law starts from the standard stable law, which is best described through the characteristic function.

Definition 2.1 *A variable X is a standard stable variate, or*

$$X \sim S_\alpha(1, \beta, 0; 1) \tag{13}$$

if the characteristic function equals

$$\varphi(t) = E\left[e^{itX}\right] = \exp\left\{-|t|^\alpha\, \omega_{\alpha,\beta}(t)\right\} \tag{14}$$

where

$$\omega_{\alpha,\beta}(t) = \begin{cases} 1 - i\beta\, \mathrm{sign}(t)\tan(\pi\alpha/2) & \text{if } \alpha \neq 1 \\ 1 + i\beta\frac{2}{\pi}\, \mathrm{sign}(t)\log|t| & \text{if } \alpha = 1\,. \end{cases} \tag{15}$$

Definition 2.2 *A variable Y is a (general) stable variate, or*

$$Y \sim S_\alpha(\sigma, \beta, \mu; \theta) \tag{16}$$

if we have the equality in distribution

$$Y \overset{\mathrm{d}}{=} \begin{cases} \mu\theta + \theta^{1/\alpha}\sigma X & \text{if } \alpha \neq 1 \\ \mu\theta + \theta\sigma X + \theta\sigma\beta\frac{2}{\pi}\log|\theta\sigma| & \text{if } \alpha = 1 \end{cases} \tag{17}$$

with X a standard stable variate.

Remark that for $\alpha = 2$ the variable X is standard normally distributed, and the variable Y is normally distributed with mean $\mu\theta$ and variance $\sigma^2\theta$.

Without loss of generality, from now on we will assume that $\alpha \neq 1$ in order not to complicate the formulas. The case where $\alpha = 1$ can be written down in an analogous way.

The next lemma illustrates the "stability property" of random variables with stable distribution as defined in equation (12), and at the same time proves the result of equation (10).

Lemma 2.1 *Let the variables Y_1 and Y_2 be defined as*

$$Y_1|_{\Theta} = \mu\tau\Theta + (\tau\Theta)^{1/\alpha}\sigma X_1 \qquad (18)$$

$$Y_2|_{\Theta} = \mu(t-\tau)\Theta + ((t-\tau)\Theta)^{1/\alpha}\sigma X_2 \qquad (19)$$

with $0 \leq \tau \leq t$ and with X_1 and X_2 independent standard stable variates. Then conditionally on Θ, the sum $\tilde{Y} = Y_1 + Y_2$ in distribution equals

$$\tilde{Y}|_{\Theta} \stackrel{\mathrm{d}}{=} \mu t\Theta + (t\Theta)^{1/\alpha}\sigma\tilde{X} \qquad (20)$$

with $\tilde{X}$ a new standard stable variate.

Proof. Although this result is well known, we give a proof for the completeness.

Conditionally on Θ, the characteristic function can be written as

$$
\begin{aligned}
&E\left[e^{ik\tilde{Y}}|_{\Theta}\right] \\
&= E\left[e^{ik\left\{\mu\tau\Theta + (\tau\Theta)^{1/\alpha}\sigma X_1 + \mu(t-\tau)\Theta + ((t-\tau)\Theta)^{1/\alpha}\sigma X_2\right\}}\right] \qquad (21) \\
&= e^{ik\mu t\Theta} \cdot E\left[e^{ik(\tau\Theta)^{1/\alpha}\sigma X_1}\right] \cdot E\left[e^{ik((t-\tau)\Theta)^{1/\alpha}\sigma X_2}\right].
\end{aligned}
$$

Making use of equations (14) and (15) for both X_1 and X_2, we find

$$
\begin{aligned}
&E\left[e^{ik\tilde{Y}}|_{\Theta}\right] \\
&= e^{ik\mu t\Theta} \cdot E\left[e^{-k^{\alpha}\tau\Theta\sigma^{\alpha}\left(1 - i\beta\,\mathrm{sign}(k(\tau\Theta)^{1/\alpha}\sigma)\tan(\pi\alpha/2)\right)}\right] \\
&\qquad \cdot E\left[e^{-k^{\alpha}(t-\tau)\Theta\sigma^{\alpha}\left(1 - i\beta\,\mathrm{sign}(k((t-\tau)\Theta)^{1/\alpha}\sigma)\tan(\pi\alpha/2)\right)}\right] \\
&= e^{ik\mu t\Theta} \cdot E\left[e^{-k^{\alpha}t\Theta\sigma^{\alpha}\left(1 - i\beta\,\mathrm{sign}(k(t\Theta)^{1/\alpha}\sigma)\tan(\pi\alpha/2)\right)}\right].
\end{aligned}
$$

$$(22)$$

From this intermediate result it is immediately clear that

$$E\left[e^{ik\tilde{Y}}|\Theta\right] = E\left[e^{ik\left\{\mu t\Theta + (t\Theta)^{1/\alpha}\sigma\tilde{X}\right\}}\right] \tag{23}$$

with $\tilde{X}$ a standard stable variate.

Q.E.D.

3 Convex upper bounds

In many financial and actuarial applications, the distribution of the (stochastic) quantity under investigation is too difficult to obtain. In the present case for example (see equation (1)), the stochastic variables $Y(t_j)$ are dependent, since they are constructed as successive sequences of several independent variables (see equation (2)).

In such cases, the method of convex upper bounds is extremely helpful. The idea consists of replacing the incalculable exact distribution by a simpler approximate distribution that is known to be associated with a quantity which is more dangerous than the original one.

The following theorem summarizes the most important result regarding this idea.

Proposition 3.1 *Consider a sum of functions of random variables*

$$V = \phi_1(X_1) + \phi_2(X_2) + ... + \phi_n(X_n), \tag{24}$$

where the functions $\phi_t : \Re \to \Re : x \mapsto \phi_t(x)$ are all increasing or all decreasing. The variable

$$W = \phi_1(F_{X_1}^{-1}(U)) + \phi_2(F_{X_2}^{-1}(U)) + ... + \phi_n(F_{X_n}^{-1}(U)) \tag{25}$$

with U an arbitrary random variable that is uniformly distributed on $[0,1]$ then defines an upper bound in convexity order, or

$$V \leq_{cx} W. \tag{26}$$

In the previous result, the notation $F_{X_j}(x)$ is used for the distribution function of X_j,

$$F_{X_j}(x) = Prob(X_j \leq x) \; ; \tag{27}$$

the inverse function is defined as $(p \in [0,1])$

$$F_{X_j}^{-1}(p) = \inf\{x \in \Re : F_{X_j}(x) \geq p\} \; . \tag{28}$$

For a proof of proposition 3.1, we refer to [1].

The meaning of the convex order becomes clear if we mention three different and equivalent characterizations of this concept.

We say that W is an upper bound for V in convexity order, $V \leq_{cx} W$, if

a) $E[u(V)] \leq E[u(W)]$ for each convex function u ;

since convex functions take on their largest values in the tails, the variable W is more likely to take on extreme values than the variable V and thus more dangerous.

b) $E[u(-V)] \geq E[u(-W)]$ for each concave function u ;

each risk averse decision maker prefers a loss V over a loss W, and thus the variable W is more dangerous.

c) $E[V] = E[W]$ and $E[(V - k)_+] \leq E[(W - k)_+]$ for each value of k ;

the financial loss of realizations exceeding a number k (the so-called stop-loss premium) is always larger for W than for V and thus the variable W is more dangerous.

As a consequence, replacing a variable V with unknown distribution by a variable W (satisfying one of the previous properties) with known distribution, can be seen as a prudent strategy.

4 Results for cash-flows

We now return to the present value under investigation, or

$$A = \sum_{j=1}^{n} c_j e^{-Y(t_j)}. \tag{29}$$

The variables $Y(t_j)$ $(t = 1, ..., n)$, representing the stochastic continuous compounded rate of return over the period $[0, t_j]$, are modeled as

$$Y(t_j) = \sum_{i=1}^{j}(Y(t_i) - Y(t_{i-1})) \qquad (0 = t_0 < t_1 < ... < t_n = t) \qquad (30)$$

with

$$Y(t_i) - Y(t_{i-1})|_{\Theta} \overset{d}{=} \mu(t_i - t_{i-1})\Theta + ((t_i - t_{i-1})\Theta)^{1/\alpha}\sigma Z_i ; \qquad (31)$$

the random variables Z_i are independent standard stable variates with distribution $S_\alpha(1, \beta, 0; 1)$, and the risk parameter Θ is independent of the variables Z_i.

As mentioned before, it follows from the model that

$$Y(t_j)|_{\Theta} \overset{d}{=} \mu t_j \Theta + (t_j \Theta)^{1/\alpha}\sigma X_j \qquad (32)$$

where now the variables X_j are dependent standard stable variates.

4.1   General results

Proposition 4.1 *Let U be a random variable which is uniformly distributed on $[0, 1]$. For the present value A in equation (29), the variable*

$$A_{upp} = \sum_{j=1}^{n} c_j e^{-\mu t_j \Theta - (t_j \Theta)^{1/\alpha}\sigma F^{-1}(U; \alpha, \beta)} \qquad (33)$$

where $F(x; \alpha, \beta) = Prob(X_j \leq x)$ denotes the distribution function of a standard stable variate, defines an upper bound in convexity order, or

$$A \leq_{cx} A_{upp} . \qquad (34)$$

Proof. This follows in a straightforward way from proposition 3.1.

Q.E.D.

Starting from this result for the boundary variable, we arrive at an expression for the stop-loss premiums.

Proposition 4.2 *The stop-loss premiums of the present value A in (29) are bounded from above by*

$$E\left[(A-k)_+\right]$$

$$\leq \int_0^{+\infty} dF_\Theta(\theta)\ \int_0^{x_\theta(d)} dF(x;\alpha,\beta)\ \left(\sum_{j=1}^n c_j e^{-\mu t_j\theta - (t_j\theta)^{1/\alpha}\sigma x} - k\right) \tag{35}$$

where for each value of k and θ the value $x_\theta(k)$ is defined implicitly through the equation

$$\sum_{j=1}^n c_j e^{-\mu t_j\theta - (t_j\theta)^{1/\alpha}\sigma x_\theta(k)} = k\ . \tag{36}$$

The function $F_\Theta(\theta)$ denotes the distribution function of the risk parameter Θ.

Proof. Because of the result of proposition 4.1, we know that

$$E\left[(A-k)_+\right] \leq E\left[(A_{upp}-k)_+\right] \tag{37}$$

with

$$E\left[(A_{upp}-k)_+\right]$$

$$= \int_0^{+\infty} dF_\Theta(\theta)\ \int_0^1 du\ \left(\sum_{j=1}^n c_j e^{-\mu t_j\theta - (t_j\theta)^{1/\alpha}\sigma F^{-1}(u;\alpha,\beta)} - k\right)_+\ . \tag{38}$$

Defining $x_\theta(k)$ as in equation (36), and making use of a substitution $u = F(x;\alpha,\beta)$ in the second integral, the desired result follows. Q.E.D.

Once the stop-loss premiums are found, the distribution function can be easily determined. Indeed, there is a well-known link between stop-loss premiums and distribution, stating that

$$\frac{d}{dk}E\left[(A-k)_+\right] = F_A(k) - 1\ , \tag{39}$$

where the notations are obvious.

Proposition 4.3 *The cumulative distribution for the quantity A_{upp} mentioned in proposition 4.1 can be calculated as*

$$F_{upp}(k) = Prob[A_{upp} \leq k] = 1 - \int_0^{+\infty} dF_\Theta(\theta)F(x_\theta(k);\alpha,\beta) \tag{40}$$

with $x_\theta(k)$ defined implicitly in equation (36).

Proof. This follows immediately when applying (39) to (35). Q.E.D.

4.2 Special cases

After presenting the general results, we want to specify the results for three special cases for the distribution of the variable Θ. We will use the same three cases for the numerical illustrations in the next section.

1. *The risk parameter Θ has all its mass in one, or $Prob[\Theta = 1] = 1$.*

   The model degenerates to the ordinary and unconditional stable model. The distribution function of the upper bound can be written as

   $$F_{upp}^{(1)}(k) = 1 - F(x(k); \alpha, \beta) \tag{41}$$

   with the values $x(k)$ defined implicitly through the equation

   $$\sum_{j=1}^{n} c_j e^{-\mu t_j - t_j^{1/\alpha} \sigma x(k)} = k . \tag{42}$$

   If α is chosen equal to 2, we recover the results as mentioned e.g. in [2].

2. *The risk parameter Θ is exponentially distributed, with unit mean.*

   The model is said to follow a geometric stable law. The variable $Y(t)$ can be seen as the sum of a stochastic number of independent standard stable variables, where the total number of terms follows a geometric distribution (see [3]).

   Now the distribution function of the upper bound can be written as

   $$F_{upp}^{(2)}(k) = 1 - \int_0^{+\infty} e^{-\theta} F(x_\theta(k); \alpha, \beta) \, d\theta \tag{43}$$

   with the values $x_\theta(k)$ defined in equation (36).

3. *The risk parameter Θ only appears in the volatility term.*

   In this case, the model slightly differs, and the rate of return $Y(t_j)$ is written as

   $$Y(t_j)|_\Theta \overset{d}{=} \mu t_j + (t_j \Theta)^{1/\alpha} \sigma X_j . \tag{44}$$

   The distribution function of the upper bound then equals

   $$F_{upp}^{(3)}(k) = 1 - \int_0^{+\infty} dF_\Theta(\theta) F(y_\theta(k); \alpha, \beta) \tag{45}$$

   with $y_\theta(k)$ defined implicitly through

   $$\sum_{j=1}^{n} c_j e^{-\mu t_j - (t_j \theta)^{1/\alpha} \sigma y_\theta(k)} = k . \tag{46}$$

5   Numerical illustration

In this section, we will present a few figures with graphs of the distribution functions of the upper bounds for the present value (29), as given in equations (41), (43) and (45).

The use of stable laws brings about a difficulty, which has to be found in the fact that we do not have a close formulation for the distribution function of standard stable variates, denoted as $F(x; \alpha, \beta)$. In order to solve this problem, we will make use of a numerical algorithm by Nolan (see [4]).

In Figure 1 we plot the distribution function of A_{upp}, in case of a cash-flow $c_t = 10$, $t = 1, \ldots, 10$, and with $Prob[\Theta = 1] = 1$. The parameters of the stable distribution are $\alpha = 1.8$ and $\beta = -0.05$, while μ and σ equal 0.07 and 0.10 respectively. The distribution function appears to be rather close to the distribution function of A, which was obtained by Monte Carlo simulation. In order to compare the accuracy in the tails, we construct a QQ-plot of the corresponding distributions. Figure 2 confirms the heavy-tailedness of the upper bound and indicates that the right quantiles are slightly overestimated. For instance, the relative error of the 99% quantile is approximately 8%.

Replacing the distribution of the risk parameter Θ by the Exp(1) distribution yields Figure 3 and in Figure 4 we turn to special case 3 with $\Theta \sim \chi_1^2$. Again, both upper bounds prove to be good approximations for the corresponding exact distributions.

In Figures 5 and 6, we use the same model as in Figure 1, but we change the cash-flow to $c_t = 1, \ldots, 10$ and $c_t = 10, \ldots, 1$ respectively. In case of an increasing cash-flow, the upper bound seems to approximate the exact distribution slightly better than in case of a decreasing cash-flow.

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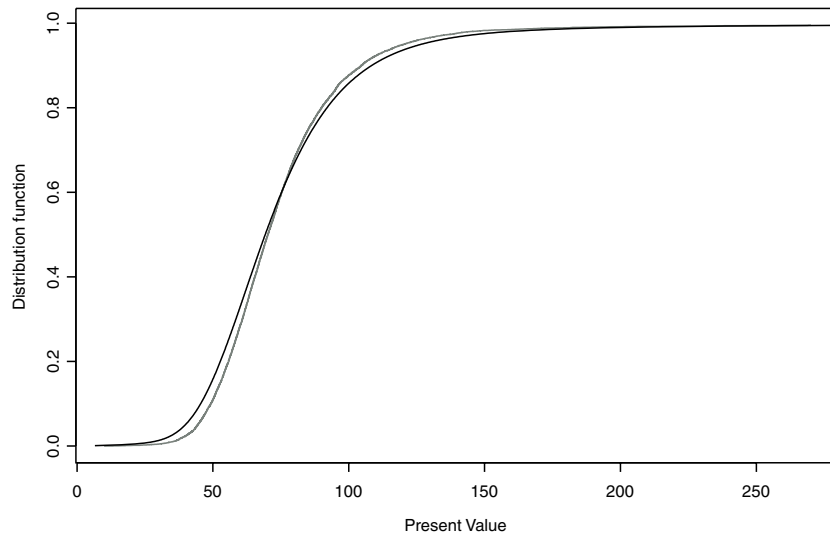
Figure 1: Distribution function of A_{upp} (black) for $c_t = 10$ ($t = 1, \ldots, 10$) and $Prob[\Theta = 1] = 1$, compared to a simulated distribution function of A (grey).
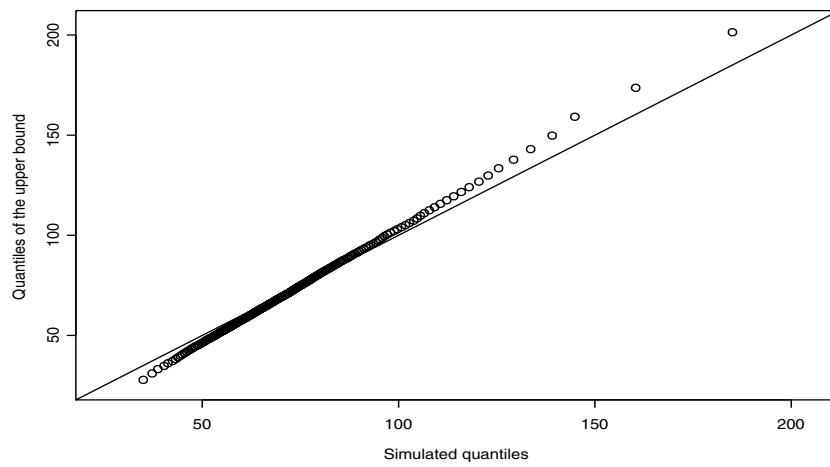
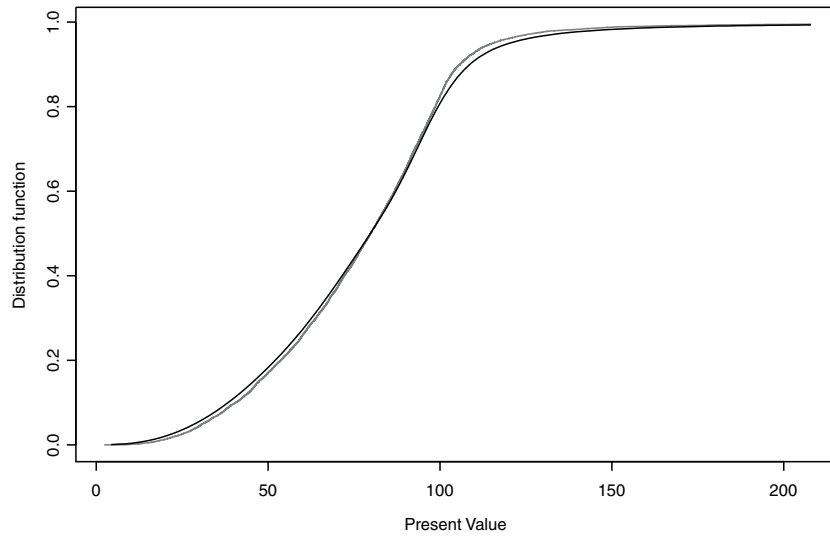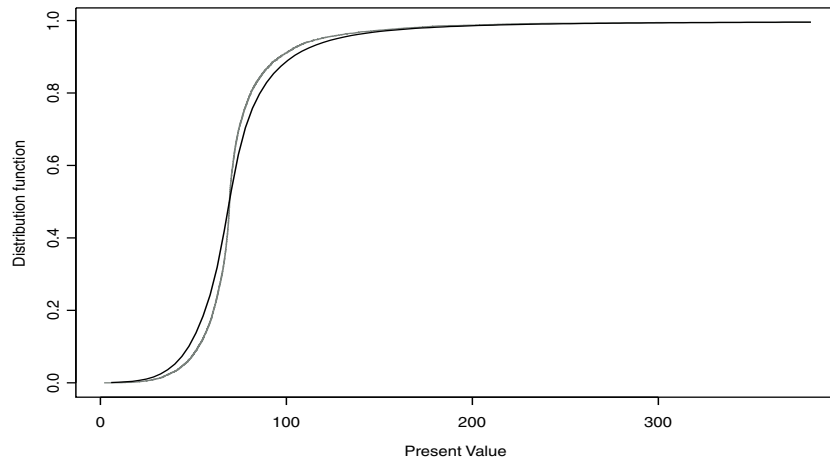Figure 2: QQ-plot of A_{upp} versus A, for $c_t = 10$ ($t = 1, \ldots, 10$) and $Prob[\Theta = 1] = 1$.

Figure 3: Distribution function of A_{upp} (black) for $c_t = 10$ ($t = 1, \ldots, 10$) and $\Theta \sim$ Exp(1), compared to a simulated distribution function of A (grey).

Figure 4: Distribution function of A_{upp} (black) for $c_t = 10$ ($t = 1, \ldots, 10$) in special case 3 with $\Theta \sim \chi_1^2$, compared to a simulated distribution function of A (grey).
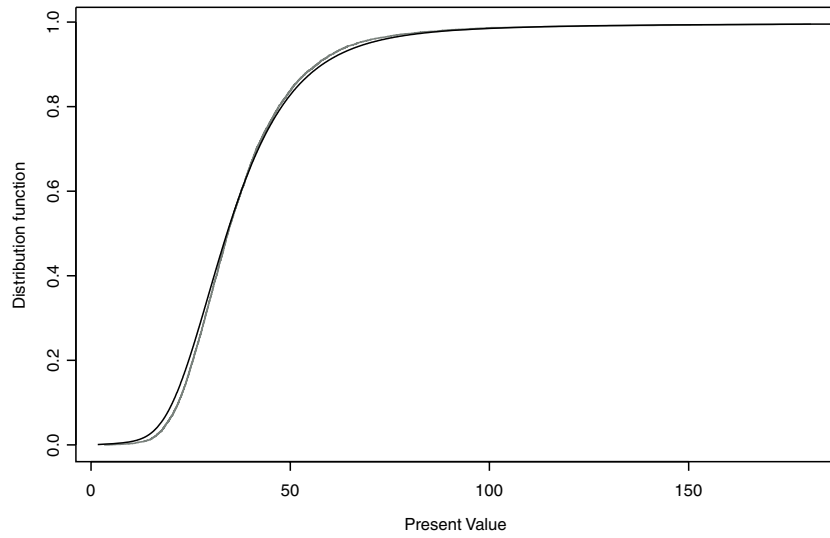
Figure 5: Distribution function of A_{upp} (black) for $c_t = 1, \ldots, 10$ and $Prob[\Theta = 1] = 1$, compared to a simulated distribution function of A (grey).
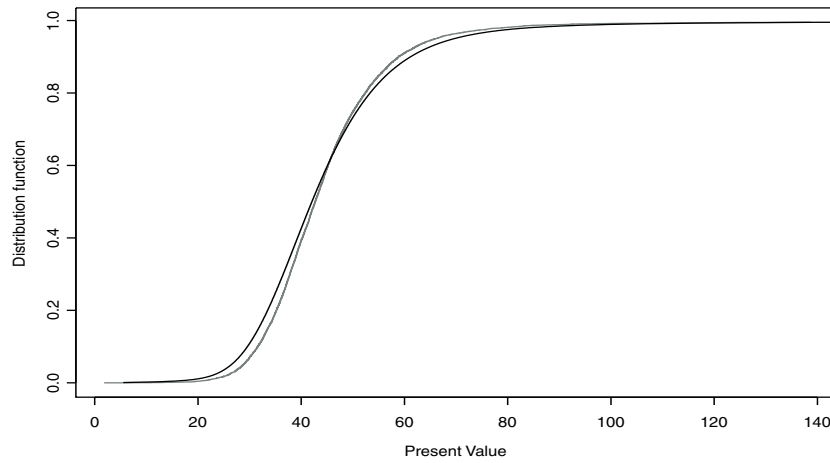

Figure 6: Distribution function of A_{upp} (black) for $c_t = 10, \ldots, 1$ and $Prob[\Theta = 1] = 1$, compared to a simulated distribution function of A (grey).