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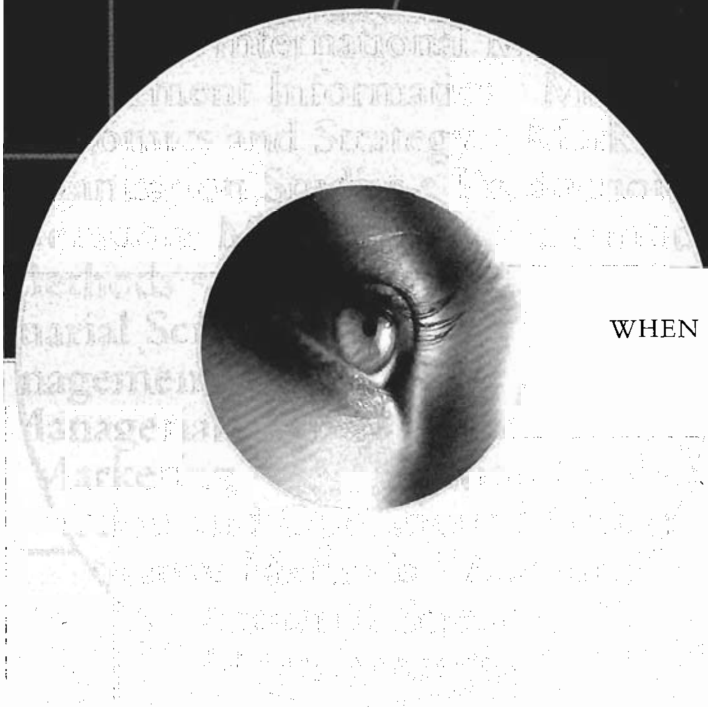

RESEARCH REPORT

AN INVESTIGATION ON THE USE OF COPULAS WHEN CALCULATING GENERAL CASH FLOW DISTRIBUTIONS

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An investigation on the use of copulas when calculating general cash flow distributions

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Abstract

In a paper of 2000, Kaas, Dhaene and Goovaerts investigate the present value of a rather general cash flow as a special case of sums of dependent risks. Making use of comonotonic risks, they derive upper and lower bounds for the distribution of the present value, in the sense of convex ordering. These bounds are very close to the real distribution in case all payments have the same sign; however, if there are both positive and negative payments, the upper bounds perform rather badly. In the present contribution we show what happens when solving this problem by means of copulas. The idea consists of splitting up the total present value in the difference of two present values with positive payments. Making use of a copula as an approximation for the joint distribution of the two sums, an approximation for the distribution of the original present value can be derived.

Keywords: cash flow, present value, convex order, copula, distribution.

1 Description of the problem

Consider a general series of deterministic payments α_1, α_2, ..., α_n due at times 1, 2, ..., n, which can be positive as well as negative. The present value for this cash flow can be written as

$$S = \sum_{i=1}^{n} \alpha_i \, e^{-Y(i)}, \tag{1}$$

where the stochastic variables $Y(i)$ are defined as $Y(i) = Y_1 + Y_2 + \cdots + Y_i$ and where the variables Y_i represent the stochastic continuous compounded rate of return over the period $[i-1, i]$.

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Following the classical assumption, the random variables Y_i are independent and normally distributed and hence the prices are lognormally distributed. For the parameters of the distributions, we use the notations

$$\mu_i = E[Y_i], \quad \sigma_i^2 = Var[Y_i] \tag{2}$$

and

$$\mu_{(i)} = E[Y(i)] = \sum_{j=1}^{i} \mu_j, \quad \sigma_{(i)}^2 = Var[Y(i)] = \sum_{j=1}^{i} \sigma_j^2. \tag{3}$$

In contrast to the risks Y_i, the variables $Y(i)$ and thus also the discounted payments $\alpha_i e^{-Y(i)}$ in the cash flow are mutually dependent. Therefore, it is nearly impossible to derive the exact distribution of the sum S.

In order to solve this problem, Kaas, Dhaene and Goovaerts [9] present bounds in convexity order that make use of the concept of comonotonic risks. This means that they replace the original sum S by a new sum, for which the components have the same marginal distributions as the components in the original sum, but with the most "dangerous" dependence structure (see Section 2 for details about these concepts). The advantage of working with a sum of comonotonic variables has to be found in the fact that the calculation of the distribution of such a sum is quite easy.

It is shown in [9] that the convex upper bound equals

$$S_u = \sum_{i=1}^{n} \alpha_i e^{-\mu_{(i)} + \operatorname{sign}(\alpha_i)\sigma_{(i)}\Phi^{-1}(U)}, \tag{4}$$

where U is a Uniform(0,1) random variable and where Φ is the standard normal cumulative distribution function.

If desirable and if more detailed information about the components in the sum is available, this upper bound can be improved by conditioning on a random variable Z that is defined as

$$Z = \sum_{i=1}^{n} \beta_i Y_i, \tag{5}$$

for which it is necessary to know the correlation with each risk $Y(i)$,

$$\rho_i = \operatorname{Corr}[Y(i), Z]. \tag{6}$$

Kaas, Dhaene and Goovaerts [9] show that by conditioning on this Z, the upper bound in (4) can be improved to the closer bound

$$S_{iu} = \sum_{i=1}^{n} \alpha_i e^{-\mu_{(i)} - \rho_i \sigma_{(i)} \Phi^{-1}(U) + \mathrm{sign}(\alpha_i) \sqrt{1-\rho_i^2}\, \sigma_{(i)} \Phi^{-1}(V)}, \tag{7}$$

where U and V are two mutually independent Uniform(0,1) random variables, independent of the variable Z.

Since successive variables $Y(i)$ represent sums that only differ in one term, they are rather strongly (and positively) dependent, explaining the good performance of both bounds. Indeed, the bounds make use of the "strongest possible" dependence between the discount factors. However, this strong affinity between exact and approximate distributions only holds in case all payments α_i have the same sign. When both positive and negative payments occur, the performance of the upper bound and of the improved upper bound is much worse. This is completely due to the negative dependence structure between terms with different signs.

The conditioning on a random variable Z as in (5) can also be used to construct a lower bound for the original present value, corresponding to a sum that is less "dangerous" than the original sum. This lower bound equals

$$S_l = \sum_{i=1}^{n} \alpha_i e^{-\mu_{(i)} - \rho_i \sigma_{(i)} \Phi^{-1}(U) + \frac{1}{2}(1-\rho_i^2)\, \sigma_{(i)}^2}, \tag{8}$$

with U a Uniform(0,1) distributed random variable. In contrast with the upper bounds, this lower bound seems to perform much better for cash flows with payments with mixed signs. However, due to these mixed signs, the lower bound no longer consists of a sum of comonotonic risks. As a consequence, its distribution is more difficult to obtain.

In this paper we aim at deriving a new accurate and efficient approximation which can be used in case the payments do have such mixed signs. We will show how such an approximation can be constructed by the introduction of copulas within the framework of comonotonicity. The paper is organized as follows. First, the concepts of convex ordering and of copulas are explained in Section 2. Afterwards in Section 3, we introduce our methodology and we derive the approximation. Numerical illustrations are presented in Section 4.

2 More about the concepts

2.1 Convex order and comonotonic risks

Many financial and actuarial applications are faced with the difficulty or impossibility of finding an analytic expression for the distribution of a stochastic quantity. In many cases, this difficulty arises from the presence of dependent components in this quantity. Also in the current case, the stochastic variables $Y(i)$ in (1) and thus the discounted payments $\alpha_i e^{-Y(i)}$ are dependent, since they are constructed as successive series of the same sequence of independent variables.

The method of convex upper bounds is extremely helpful to deal with this kind of problems. As explained in the introduction, we replace the exact but incalculable distribution by an approximate and simpler distribution associated with a variable that is more dangerous than the original one. For details, see [2, 3, 7, 8, 15].

In order to illustrate the fact that convex order nicely suites the notion of dangerousness, we mention three equivalent characterizations of this concept where all the expectations are assumed to exist.

A variable W is said to be an upper bound for V in convexity order, notation $V \leq_{cx} W$, if

a) $E[u(V)] \leq E[u(W)]$ for each convex function $u : \mathbb{R} \rightarrow \mathbb{R}$;
since convex functions take on their largest values in the tails, the variable W is more likely to take on extreme values than the variable V and thus W is more dangerous.

b) $E[u(-V)] \geq E[u(-W)]$ for each concave function $u : \mathbb{R} \rightarrow \mathbb{R}$;
each risk averse decision maker prefers a loss V over a loss W and thus the variable W is more dangerous.

c) $E[V] = E[W]$ and $E[(V-k)_+] \leq E[(W-k)_+]$ for each value of k;
the financial loss of realizations exceeding a retention k, or stop-loss premium, is always larger for W than for V and thus the variable W is more dangerous.

The equivalence of conditions a) - c) is discussed in e.g. [2]. As a consequence, replacing a variable V with unknown distribution by a variable W with known distribution but larger in convex ordering can be seen as a prudent strategy.

If in addition to this upper bound W a lower bound can be found as well, this provides us with a measure for the reliability of the upper bound.

Returning to the cash flow problem, the following theorem summarizes an important result regarding this idea, in that it shows how to construct such convex larger sums. A proof can be found in [7].

Proposition 2.1. *Consider a sum of functions of random variables*

$$V = \phi_1(X_1) + \phi_2(X_2) + \cdots + \phi_n(X_n), \tag{9}$$

where the functions $\phi_t : \mathbb{R} \to \mathbb{R} : x \mapsto \phi_t(x)$ *are all increasing or all decreasing. The variable*

$$W = \phi_1(F_{X_1}^{-1}(U)) + \phi_2(F_{X_2}^{-1}(U)) + \cdots + \phi_n(F_{X_n}^{-1}(U)) \tag{10}$$

with U *an arbitrary random variable that is uniformly distributed on* $[0,1]$ *then defines an upper bound in convexity order, i.e.* $V \leq_{cx} W$.

The notation F_{X_j} is used for the distribution function of X_j, or

$$F_{X_j}(x) = Prob(X_j \leq x), \quad x \in \mathbb{R} \tag{11}$$

and the inverse function is defined as

$$F_{X_j}^{-1}(p) = \inf\{x \in \mathbb{R} : F_{X_j}(x) \geq p\}, p \in [0,1]. \tag{12}$$

One of the advantages of this method is the fact that, due to the construction of the variable W, the distribution of the bound can be determined rather easily by means of

$$F_W(s) = p_s \tag{13}$$

with p_s defined implicitly by

$$\sum_{i=1}^{n} \phi_i\left(F_{X_i}^{-1}(p_s)\right) = s. \tag{14}$$

2.2 Copulas

A copula C is a function that maps the marginals F_1 and F_2 of a bivariate distribution F to the joint distribution in a unique way:

$$C(u,v) : [0,1] \times [0,1] \to [0,1] : (u,v) \mapsto F(F_1^{-1}(u), F_2^{-1}(v)) \tag{15}$$

such that

$$C(F_1(x), F_2(y)) = F(x,y). \tag{16}$$

One of the most important examples of copulas is the Gaussian copula. In the bivariate case it can be parameterized by a single parameter ρ as follows:

$$C(u, v; \rho) = H_\rho\big(\Phi^{-1}(u), \Phi^{-1}(v)\big), \tag{17}$$

where $H_\rho(s_1, s_2)$ is a bivariate normal distribution function with mean 0 and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

This family arises naturally in the case of multivariate normal distributions. However they may also appear in many situations where the corresponding marginal distributions are not normal. Consider for example a multivariate normal vector $Y = (Y_1, Y_2, \ldots, Y_n)$. Then the vector

$$\exp(Y) = \big(\exp(Y_1), \exp(Y_2), \ldots, \exp(Y_n)\big) \tag{18}$$

will still have a Gaussian copula although the corresponding marginal distributions are now lognormal.

The family of Archimedean copulas is another very important class of copulas widely used in statistical applications. An Archimedean copula is given by the formula

$$C(u, v) = \psi^{-1}[\psi(u) + \psi(v)] \tag{19}$$

where the copula generator $\psi : [0, 1] \to [0, +\infty]$ is continuous, strictly decreasing and convex. Archimedean copulas are always symmetric (i.e. $C(u, v) = C(v, u)$), associative (i.e. $C(C(u, v), w) = C(u, C(v, w))$) and their diagonal section is always smaller than the identity functions (i.e. $C(u, u) \leq u$). It can be proved that the last two properties characterize the family of Archimedean copulas (see [11]).

Three special copulas are very illustrative:

- $C_1(u, v) = uv$

  represents the case of independent underlying variables;

- $C_2(u, v) = \min(u, v)$

  is an upper bound, representing the case of most related pair of variables with given marginals;

- $C_3(u,v) = \max(0, u+v-1)$
  is a lower bound, representing the case of most antithetic pair of variables.

If (X,Y) has the bivariate distribution function F, with marginals F_1 and F_2, and if C is a copula as in (16), then Spearman's rho is given by

$$\rho_s(X,Y) = 12 \iint_{(0,1)^2} C(u,v)\,du\,dv - 3. \tag{20}$$

The relation between Spearman's rho and Pearson correlation is given by

$$\rho_s(X,Y) = \rho_p(F_1(X), F_2(Y)), \tag{21}$$

see e.g. [4].

Note that also Kendall's τ can be expressed in terms of the copulas. The appropriate formula is given below:

$$\tau = 4 \iint_{[0,1]^2} C(u,v)dC(u,v) - 1. \tag{22}$$

There are several possibilities to generate copulas that correspond to couples of variables with given correlation ρ_s.

The first method is based on deriving the copula as a convex combination combination of the three copulas mentioned above,

$$C(u,v;\rho_s,\tau) = p_1 C_1(u,v) + p_2 C_2(u,v) + p_3 C_3(u,v). \tag{23}$$

One can try to solve the following system of equations derived explicitly from the definitions of the three copulas and formula (22):

$$\begin{cases} p_1, p_2, p_3 \geq 0 \\ p_1 + p_2 + p_3 = 1 \\ p_2 - p_3 = \rho_s \\ p_1^2 + 2p_2^2 + \frac{8}{3}p_1p_2 + \frac{4}{3}p_1p_3 + 2p_2p_3 - 1 = \tau. \end{cases} \tag{24}$$

Unfortunately the unique solution (in the case $\rho_s \neq 0$) of the last three equations:

$$\begin{cases} p_1 = \frac{3\rho_s - 3\tau}{\rho_s} \\ p_2 = \frac{3\tau - 2\rho_s + \rho_s^2}{2\rho_s} \\ p_3 = \frac{3\tau - 2\rho_s - \rho_s^2}{2\rho_s} \end{cases} \tag{25}$$

may not fall into the set $\mathcal{F} = \{(x,y,z)|x,y,z \geq 0\}$. In such a case, as well as in the case when $\rho_s = 0$, one may consider the following minimization problem. Define the set

$$\mathcal{S} = \{(x,y,z)|x,y,z \geq 0 \text{ and } x+y+z=1\}. \tag{26}$$

Then the weights (p_1, p_2, p_3) can be chosen as

$$\arg \min_{(x,y,z)\in\mathcal{S}} \left((y-z-\rho_s)^2 + (x^2+2y^2+\frac{8}{3}xy+\frac{4}{3}xz+2yz-1-\tau)^2\right). \tag{27}$$

A second natural approach is to fit an appropriate Gaussian copula (17). The most natural choice of Gaussian copula is based on fitting the Pearson's correlation coefficient ρ_p. However it is also possible to fit on the basis of Spearman's ρ_s or Kendall's τ because of the 1-1 correspondence with ρ given by the formulae:

$$\rho_s = \frac{6}{\pi} \arcsin\left(\frac{\rho_p}{2}\right) \tag{28}$$

and

$$\tau = \frac{2}{\pi} \arcsin(\rho_p) \tag{29}$$

(see e.g. [10]).

A third approach generates an Archimedean copula called the Gumbel copula

$$C(u,v;\rho_s) = \exp\left\{-\left((-\log u)^{1/\beta} + (-\log v)^{1/\beta}\right)^{\beta}\right\} \tag{30}$$

with $\beta \geq 0$ being the unique solution of the equation

$$\int_0^1 \int_0^1 e^{-\left((-\log u)^{1/\beta}+(-\log v)^{1/\beta}\right)^{\beta}} \, du \, dv = \frac{1}{12}(\rho_s+3). \tag{31}$$

Note that a positive correlation corresponds to a value of β smaller than one, a negative correlation to a value larger than one. More details about copulas can be found in [4, 5, 6, 13, 14].

3 Construction of the distribution function

3.1 Summary of the method

Since the approximation described in Section 1 performs excellent in case all payments are equally signed, it seems very reasonable to split up the total present value

$$S = \sum_{i=1}^{n} \alpha_i \, e^{-Y(i)} \tag{32}$$

into two separate parts, representing the positive and negative payments respectively. This means that we write the sum S as

$$S = S^+ - S^- \tag{33}$$

where the terms are defined as

$$S^+ = \sum_{i=1}^{n} (\alpha_i)_+ \, e^{-Y(i)} \tag{34}$$

and

$$S^- = \sum_{i=1}^{n} (-\alpha_i)_+ \, e^{-Y(i)}. \tag{35}$$

The expression $(x)_+$ is used as a short-hand notation for $\max(x, 0)$.

Since each of the sums S^+ and S^- refers to a situation with exclusively positive cash flow payments, the results of [9] can be used to find an adequate approximation for the distribution function of both sums.
Starting from these two approximate distribution functions, the idea then consists of constructing an adequate approximation for the joint distribution function of S^+ and S^-, for which we will use the notation H:

$$H(s^+, s^-) = \text{Prob}[S^+ \leq s^+, S^- \leq s^-]. \tag{36}$$

If this approximation is available, an integration leads to an (approximate) distribution for the difference of both sums.

3.2 Upper bounds for S^+ and S^-

As explained before, very close upper bounds for S^+ and S^- can be found by applying the method of [9]. This results in (see (4))

$$S_u^+ = \sum_{i=1}^{n} (\alpha_i)_+ \exp\left(-\mu_{(i)} + \sigma_{(i)} \Phi^{-1}(U_1)\right) \tag{37}$$

and

$$S_u^- = \sum_{i=1}^{n} (-\alpha_i)_+ \exp\left(-\mu_{(i)} + \sigma_{(i)} \Phi^{-1}(U_2)\right) \tag{38}$$

with U_1 and U_2 Uniform(0,1) random variables. Note that

$$S_u^+ - S_u^- \leq_{cx} S_u. \tag{39}$$

Due to the construction of these sums (see Section 2) their distribution can be derived rather easily. Indeed, we have

$$F_{S_u^+}(s) = \text{Prob}[S_u^+ \leq s] = \Phi(v_s) \tag{40}$$

and

$$F_{S_u^-}(s) = \text{Prob}[S_u^- \leq s] = \Phi(w_s), \tag{41}$$

with v_s and w_s defined implicitly by

$$\sum_{i=1}^{n} (\alpha_i)_+ \exp\left(-\mu_{(i)} + \sigma_{(i)} v_s\right) = s \tag{42}$$

and

$$\sum_{i=1}^{n} (-\alpha_i)_+ \exp\left(-\mu_{(i)} + \sigma_{(i)} w_s\right) = s. \tag{43}$$

3.3 Joint distribution of S^+ and S^-

The following step consists of mapping the two approximate distributions for the sums into an approximation for their joint distribution. This approach is very reasonable: the distributions of S_u^+ and S_u^- are both very close to the distributions of S^+ and S^- and in addition they are very well calculable in contrast with the exact distributions.
This mapping can be done by means of a copula if next to the approximate distributions the correlation of S^+ and S^-, or a good estimate, is known. We then get a bivariate distribution for which the marginals are equal to the approximate distributions of the two terms in the difference and for which the underlying variables have approximately the correct correlation.

In other words, we construct a copula $C(u, v; \hat{\rho}_S)$ with $\hat{\rho}_S$ the estimated correlation between S^+ and S^-, or

$$H(s^+, s^-) \approx C\left(F_{S_u^+}(s^+), F_{S_u^-}(s^-); \hat{\rho}_S\right). \tag{44}$$

The copula can be approximated by one of the methods described in Section 2.2, i.e. by fitting a Gaussian copula (17), by means of a suitable combination of the special copulas (23) and (24) or by means of a Gumbel copula (30). In the latter case, the value of β has to be determined as the unique solution of (31). In Section 3.4 we discuss how to choose the appropriate method.

The question still remaining is how to find an estimation $\hat{\rho}_S$ of the correlation ρ_s. We propose three alternatives.

1. Since S_u^+ and S_u^- are very close to the original sums S^+ and S^-, an obvious option is the use of the correlation $\rho_s(S_u^+, S_u^-)$, for which we know that

$$\rho_s\left(S_u^+, S_u^-\right) = \rho\left(F_{S_u^+}(S_u^+), F_{S_u^-}(S_u^-)\right). \tag{45}$$

Since both (approximate) distributions are known, a suitable value for this correlation can be found by simulation.

2. As this simulation can be rather time consuming, a better solution consists of an estimation by means of the correlation between the first order approximations of S^+ and S^-.

This can be done by defining the random variables

$$\begin{array}{l} \Lambda^+ = \sum_{j=1}^n \beta_j^+ Y_j \\ \Lambda^- = \sum_{j=1}^n \beta_j^- Y_j, \end{array} \tag{46}$$

with parameters

$$\begin{array}{l} \beta_j^+ = \sum_{k=j}^n (\alpha_k)_+ e^{-\mu_{(k)}} \\ \beta_j^- = \sum_{k=j}^n (\alpha_k)_- e^{-\mu_{(k)}}. \end{array} \tag{47}$$

In that case, Λ^+ and Λ^- are linear transformations of the first order approximations to S^+ and S^- respectively. Indeed, the sum S^+ e.g. can be written as

$$\begin{array}{rcl} S^+ & = & \sum_{i=1}^n (\alpha_i)_+ e^{-Y(i)} \\ & = & \sum_{i=1}^n (\alpha_i)_+ e^{-\mu_{(i)}} e^{-\sum_{j=1}^i (Y_j - \mu_j)}, \end{array} \tag{48}$$

and thus the first order approximation equals

$$\begin{array}{rcl} S^+ & \approx & \sum_{i=1}^n (\alpha_i)_+ e^{-\mu_{(i)}} \left(1 - \sum_{j=1}^i (Y_j - \mu_j)\right) \\ & \approx & C - \sum_{i=1}^n (\alpha_i)_+ e^{-\mu_{(i)}} \sum_{j=1}^i Y_j \\ & \approx & C - \sum_{j=1}^n Y_j \sum_{i=j}^n (\alpha_i)_+ e^{-\mu_{(i)}} \\ & \approx & C - \Lambda^+, \end{array} \tag{49}$$

with C chosen appropriately (see e.g. [16]). Note that this approximation is only accurate if the differences $Y_j - \mu_j$, or equivalently the volatilities σ_j, are sufficiently small.

The Pearson correlation of the couple (Λ^+, Λ^-) can be calculated easily as

$$\rho_p(\Lambda^+, \Lambda^-) = \frac{\sum_{i=1}^n \beta_i^+ \beta_i^-}{\sqrt{\sum_{i=1}^n (\beta_i^+)^2} \sqrt{\sum_{i=1}^n (\beta_i^-)^2}}. \tag{50}$$

Since (Λ^+, Λ^-) has a bivariate normal distribution, the Spearman correlation can then be found from formula (28).

As the numerical illustrations will indicate (see Section 4), this second estimate seems to perform excellent, in that it is just as accurate as the simulated value, but much more easy to calculate. This result should not surprise, because we expect that a sample of the variables distributed as (S^+, S^-) will have ranks very close to the corresponding sample of variables of the form (Λ^+, Λ^-). It is well known that empirical versions of ρ_s and τ depend only on the sample ranks.

3. Finally, one can simply use the Gaussian copula with the Pearson's correlation coefficient $\rho_p(\Lambda^+, \Lambda^-)$ (see (50)). This choice can be motivated as follows. As mentioned before the dependency structure of the bivariate distribution (S^+, S^-) is similar to the dependency structure of its first order approximation (Λ^+, Λ^-). Therefore, it looks reasonable to use the copula of (Λ^+, Λ^-) to approximate the joint distribution of discounted cash flows, which is precisely the Gaussian copula with parameter $\rho_p(\Lambda^+, \Lambda^-)$.

3.4 A choice of the fitting method - a simulation study

In this section we perform a simulation study of a simple cash flow of future payments of the form:

$$S = \exp(-Y_1) - \exp(-Y_1 - Y_2) + \exp(-Y_1 - Y_2 - Y_3) - \exp(-Y_1 - Y_2 - Y_3 - Y_4), \quad (51)$$

where Y_i are independent, normally distributed, with the mean $\mu = 0.07$ and the standard deviation $\sigma = 0.1$. It is a special case of our situation of interest, however it is representative in this sense that some features of the copula (like the tail behavior) observed in this simplified situation will be expected to hold true in general.

We consider the joint distribution (S^+, S^-), where

$$S^+ = \exp(-Y_1) + \exp(-Y_1 - Y_2 - Y_3), \quad (52)$$

$$S^- = \exp(-Y_1 - Y_2) + \exp(-Y_1 - Y_2 - Y_3 - Y_4). \quad (53)$$

We have simulated 10.000.000 random pairs (S^+, S^-). In our simulation study we perform the analysis on the basis of so-called dependence measures χ and $\bar{\chi}$ (see [1]) for the empirical copula and theoretical copulas considered in the paper.

3.4.1 The dependence measure χ

A definition of the measure χ arises naturally from the concept of tail independence. Consider any bivariate random vector (X, Y). One says that (X, Y) is tail independent if

$$\chi = \lim_{u \to 1} \Pr(V > u | U > u) = 0, \tag{54}$$

where $U = F_X(X)$ and $V = F_Y(Y)$. One can easily verify that $\Pr(V > u|U > u) \sim 2 - \frac{\log(C(u,u))}{\log(u)}$, where $C(\cdot, \cdot)$ denotes an appropriate copula. In practice it is often convenient to study a function:

$$\chi(u) = 2 - \frac{\log(C(u,u))}{\log(u)} \tag{55}$$

(obviously $\lim_{u \to 1} \chi(u) = \chi$). Note that the values of function $\chi(u)$ belong to the interval $(-\infty, 1)$.
It is straightforward to show that for the Gumbel (extreme value) copula $\chi(u)$ is constant. Thus studying empirical estimates of the function $\chi(u)$ provides an excellent diagnostic check about the adequacy of fitting Gumbel copula.

3.4.2 The dependence measure $\bar{\chi}$

The function $\chi(u)$ is not a sufficient tool to study the tail dependence of an underlying bivariate distribution. The most difficult problem to overcome is the estimation of very high quantiles (our simulation allows for reliable estimates up to the 99.999% quantile and it does not seem to suffice). To get some complementary information it is useful to study a dual function $\bar{\chi}(u)$ defined as follows:

$$\bar{\chi}(u) = \frac{2\log(1-u)}{\log(\bar{C}(u,u))} - 1, \tag{56}$$

where $\bar{C}(u, v) = 1 - u - v + C(u, v)$. Some properties of the function $\bar{\chi}$ are given below:

- $-1 \leq \bar{\chi} \leq 1$
- For tail dependent r.v.'s $\lim_{u \to 1} \bar{\chi}(u) = 1$
- For tail independent r.v.'s $\limsup_{u \to 1} \bar{\chi}(u) < 1$.

Thus the tail independence can be characterized by the following equivalent conditions:

$$(X, Y) \text{ is tail independent } \Longleftrightarrow \lim_{u \to 1} \chi(u) = 0 \Longleftrightarrow \limsup_{u \to 1} \bar{\chi}(u) < 1. \tag{57}$$

3.4.3 An analysis of measures $\chi(u)$ and $\bar{\chi}(u)$ for (S^+, S^-)

On Figure 1 the graphs of χ and $\bar{\chi}$ of the empirical copula of the pair (S^+, S^-) are compared to the copulas fitted on the basis of Spearman's ($\rho_s = 0.88998$) and/or Kendall's ($\tau = 0.71312$) coefficients, as described in Section 2.2. The graphs were obtained on the basis of arguments up to 0.99999 which has significant influence on the obtained picture - it is well-known that χ for the Gaussian copula converges to 0 (the Gaussian copula is tail independent) while $\bar{\chi}$ for the Gumbel copula and the mix of copulas has to converge to 1 - thus the final values visible on the graphs are still quite far from the limits.
The picture is very suggestive: the fits provided by a mix of copulas and Gumbel copula is in this case very bad. In particular the hypothesis of χ being constant for the pair (S^+, S^-) (what is true for Gumbel copula) has to be rejected. On the other hand the fit provided by a Gaussian copula is excellent.
However one thing requires a further investigation. In general it is not clear whether the pair (S^+, S^-) is tail independent or not. For higher quantiles one of two things given below must happen:

- if (S^+, S^-) is tail independent then χ converges to 0 while $\bar{\chi}$ converges to $b < 1$;
- if (S^+, S^-) is tail dependent then χ converges to $a > 0$ while $\bar{\chi}$ converges to 1.

From the picture there are no clear premises which of the two possibilities holds true. It is well known that the Gaussian copula itself is tail independent, see e.g. [1].

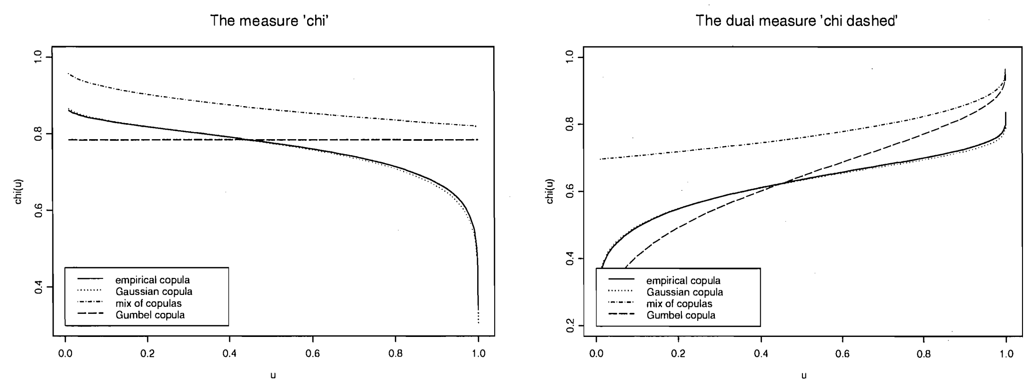

Figure 1: Functions χ and $\bar{\chi}$ for the empirical copula of (S^+, S^-)

In Table 1 some empirical results for higher quantiles are provided. One can observe that the results are not so smooth anymore. While the sample size seems to be large enough to estimate with a reasonable precision univariate higher quantiles, an estimation of the bivariate cumulative distribution function becomes very inadequate.

	Empirical		Gaussian	
u	$\chi(u)$	$\bar{\chi}(u)$	$\chi(u)$	$\bar{\chi}(u)$
0.9999900	0.3599948	0.8369867	0.3012353	0.8112380
0.9999925	0.3733295	0.8458796	0.2946738	0.8123443
0.9999950	0.3799975	0.8531032	0.2857220	0.8138429
0.9999965	0.2857121	0.8186440	0.2781215	0.8151063
0.9999975	0.2799985	0.8203572	0.2711759	0.8162541
0.9999980	0.1499984	0.7473788	0.2666854	0.8169930
0.9999985	0.0666653	0.6639739	0.2610278	0.8179206
0.9999990	0.0999991	0.7142857	0.2532967	0.8191825
0.9999993	0.1428566	0.7585434	0.2467212	0.8202513
0.9999995	0.1999996	0.8002943	0.2407034	0.8212261

Table 1: Estimates of χ and $\bar{\chi}$ for high quantiles

The results contained in Table 1 suggest that for the pair (S^+, S^-) it is more likely that χ converges to 0 than $\bar{\chi}$ converges to 1. The values of the function $\bar{\chi}$ also vary significantly, however they do not have any tendency to increase any more while the values of χ apparently decrease. The last increase seems to result from estimation inadequacies. One has to note that a pace of convergence of χ for the Gaussian copula is very slow. We can expect that the copula of (S^+, S^-) will behave similarly.

There are other heuristic arguments supporting the Gaussian copula.

a) The copula of the random vector

$$\big(\exp(-Y_1), \exp(-Y_1 - Y_2), \exp(-Y_1 - Y_2 - Y_3), \exp(-Y_1 - Y_2 - Y_3 - Y_4)\big) \quad (58)$$

is Gaussian itself (compare to (18)). So the couple (S^+, S^-) is created as a linear transformation of a random vector with the dependence structure given by a Gaussian copula. One can expect that the copula of (S^+, S^-) should inherit some properties of the Gaussian copula, especially the tail behavior.

b) Recall that the estimated values of Spearman's and Kendall's correlations for (S^+, S^-) are given by $\rho_s = 0.88998$ and $\tau = 0.71312$. Suppose that we treat

ρ_s as fixed. We compute τ_{Gauss} from the formulae (28) and (29) as follows:

$$\tau_{\text{Gauss}} = \frac{2}{\pi} \arcsin\big(2 \sin\big(\frac{\pi \rho_s}{6}\big)\big) = 0.71085, \tag{59}$$

which is surprisingly close to the estimated value of τ.

In this analysis we have used the values of ρ_s and τ estimated from the simulated sample. However very close results can be obtained by approximations derived from a first order approximation described in Section 3.3. Indeed, using the formula (50) one obtains $\rho_p^\Lambda = 0.90067$, which gives consequently $\rho_s^\Lambda = 0.89218$ and $\tau^\Lambda = 0.71385$, which are very close to the simulated values. Thanks to this approximation the whole methodology does not depend on simulation.

3.5 Distribution of the present value

Suppose that the exact bivariate distribution of S^+ and S^- is expressed by means of a copula as

$$\begin{aligned} H(s^+, s^-) &= \text{Prob}[S^+ \le s^+, S^- \le s^-] \\ &= C\left(F_{S^+}(s^+), F_{S^-}(s^-); \rho_s\right). \end{aligned} \tag{60}$$

Then, starting from this joint distribution, the cumulative distribution function of the difference $S = S^+ - S^-$ can be written as

$$\begin{aligned} F_S(s) &= \text{Prob}[S^+ - S^- \le s] \\ &= \int_0^\infty \int_0^\infty \frac{\partial^2}{\partial s^+ \partial s^-} H(s^+, s^-)\, \mathcal{U}(s - s^+ + s^-)\, ds^+ ds^- \\ &= \int_0^1 \int_0^1 \frac{\partial^2}{\partial u \partial v} C(u, v; \rho_s)\, \mathcal{U}(s - F_{S^+}^{-1}(u) + F_{S^-}^{-1}(v))\, du\, dv \\ &= \int_0^1 du \int_{F_{S^-}\left(F_{S^+}^{-1}(u) - s\right)}^1 \frac{\partial^2}{\partial u \partial v} C(u, v; \rho_s)\, dv, \end{aligned} \tag{61}$$

where $\mathcal{U}(x)$ denotes the Heaviside function, or

$$\mathcal{U}(x) = \begin{cases} 1 & \text{if} \quad x \ge 0 \\ 0 & \text{if} \quad x < 0. \end{cases} \tag{62}$$

Splitting up the integration over u, and carrying through the integration over v, this results in

$$\begin{aligned}
F_S(s) &= F_{S^+}(s) + \int_{F_{S^+}(s)}^{1} du \int_{F_{S^-}\left(F_{S^+}^{-1}(u)-s\right)}^{1} \frac{\partial^2}{\partial u \partial v} C(u,v;\rho_s)\, dv \\
&= F_{S^+}(s) \\
&\quad + \int_{F_{S^+}(s)}^{1} du \left[\frac{\partial}{\partial u} C(u,1;\rho_s) - \frac{\partial}{\partial u} C\left(u, F_{S^-}\left(F_{S^+}^{-1}(u) - s\right); \rho_s\right)\right] \\
&= 1 - \int_s^\infty dF_{S^+}(k) \frac{\partial}{\partial u} C\left(F_{S^+}(k), F_{S^-}(k-s); \rho_s\right). \qquad (63)
\end{aligned}$$

Substituting S_u^+ and S_u^- for S^+ and S^- in (63) respectively provides us with a appropriate approximation $F_{S_{cop}}$ for the distribution of the present value (1):

$$F_{S_{cop}}(s) = 1 - \int_s^\infty dF_{S_u^+}(k) \frac{\partial}{\partial u} C\left(F_{S_u^+}(k), F_{S_u^-}(k-s); \hat{\rho}_S\right). \qquad (64)$$

Note that in case we use a Gaussian parameterized by $\rho_p = \rho$, the partial derivative of (17) can be written in a very simple form

$$\frac{\partial}{\partial u} C(u,v;\rho) = \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{(1-\rho^2)}}\right). \qquad (65)$$

The derivatives of the other copulas considered can also be expressed easily. For the mix of special copulas, we have

$$\frac{\partial}{\partial u} C(u,v;p_1,p_2,p_3) = p_1 u + p_2 \mathcal{U}(v-u) + p_3 \mathcal{U}(u+v-1) \qquad (66)$$

and for the Gumbel copula we find

$$\begin{aligned}
\frac{\partial}{\partial u} C(u,v;\rho_s) &= \frac{1}{u}(-\log u)^{1/\beta - 1}\left((-\log u)^{1/\beta} + (-\log v)^{1/\beta}\right)^{\beta - 1} \\
&\quad \exp\left\{-\left((-\log u)^{1/\beta} + (-\log v)^{1/\beta}\right)^{\beta}\right\}. \qquad (67)
\end{aligned}$$

4 Numerical illustration

In this last section, we examine the accuracy and efficiency of our approximation, compared to the exact distribution of the present value obtained by Monte-Carlo simulation and also compared to the comonotonic bounds in [9]. We show results for the cumulative distribution function of the present value in two examples with different cash flow structures:

- $\alpha_i = \begin{cases} -1 & i = 1, \ldots, 5 \\ +1 & i = 6, \ldots, 20, \end{cases}$
- $\alpha_i = \begin{cases} -1 & i = 1, 3, 5, \ldots, 19 \\ +1 & i = 2, 4, 6, \ldots, 20. \end{cases}$

The parameters of the lognormal distributions are chosen as in [9], i.e. $\mu_i = \mu = 0.07$ and $\sigma_i = \sigma = 0.1$. For the construction of the copula, we decided to use a Gaussian copula (see Subsection 3.4). Following the methods of Subsection 3.3, for the first cash flow, the estimated values for the correlation and for the corresponding parameters ρ computed from (28) are

- $\rho_s(S_u^+, S_u^-) = 0.6391160$ and $\rho_p(S_u^+, S_u^-) = 0.656859$;
- $\rho_s(\Lambda^+, \Lambda^-) = 0.6344521$ and $\rho_p(\Lambda^+, \Lambda^-) = 0.6522439$.

Due to the specific structure of the second cash flow, the correlation is much higher:

- $\rho_s(S_u^+, S_u^-) = 0.9929377$ and $\rho_p(S_u^+, S_u^-) = 0.9935884$;
- $\rho_s(\Lambda^+, \Lambda^-) = 0.9928121$ and $\rho_p(\Lambda^+, \Lambda^-) = 0.9934742$.

For both examples, we first computed the quantiles of the variables S_u^+ and S_u^-. Afterwards, a simulation provided us with the first estimator $\rho_s(S_u^+, S_u^-)$, while the second estimator $\rho_s(\Lambda^+, \Lambda^-)$ obviously followed from (28) and (50). Note that the estimates $\rho_s(S_u^+, S_u^-)$ and $\rho_s(\Lambda^+, \Lambda^-)$ are very similar.

Figure 2 shows the quantiles for the present value of the cash flow with negative payments in the beginning and positive payments afterwards. One can see that the convex upper bound S_u performs very badly. On the other hand, our copula approximations S_{cop} (we have calculated the approximation only for $\rho_s(\Lambda^+, \Lambda^-)$) seem to be very accurate in approximating the exact distribution. The convex lower bound S_l performs best.
The same and more pronounced observations can be made for the cash flow with payments with alternating signs for which graphs are shown in Figure 3.
In Table 2 and 3 the numerical values for some upper quantiles are provided. In general they confirm the observations made on the basis of graphical illustrations.

q	LowB	MCSim	CopAppr	ComUpB
0.750	3.5159	3.5136	3.5843	4.2861
0.900	4.9045	4.8963	5.1536	6.4487
0.950	5.8851	5.8847	6.2964	7.9282
0.975	6.8406	6.8500	7.3559	9.3450
0.990	8.0885	8.0885	8.8204	11.1716
0.995	9.0300	9.0902	9.9422	12.5400
0.999	11.2519	11.3996	12.6233	15.7310

Table 2: Upper quantiles of the approximations derived for the first cashflow

q	LowB	MCSim	CopAppr	ComUpB
0.750	−0.2585	−0.2610	−0.2494	1.5399
0.900	−0.1640	−0.1638	−0.1350	3.3359
0.950	−0.1100	−0.0983	−0.0534	4.4781
0.975	−0.0523	−0.0365	0.0278	5.5249
0.990	0.0108	0.0442	0.1365	6.8233
0.995	0.0551	0.1036	0.2207	7.7667
0.999	0.1498	0.2441	0.4241	9.8955

Table 3: Upper quantiles of the approximations derived for the second cashflow

5 Some comments on the convex order and conclusions

In Kaas, Dhaene and Goovaerts [9] both convex upper and lower bounds have been derived for a present value of future cash flow with stochastic interest rates, i.e. one has

$$S_l \leq_{cx} S \leq_{cx} S_u. \tag{68}$$

There are several benefits of their approach. As mentioned before, the random distribution of S is not mathematically easy tractable. One possibility to solve this problem is to substitute the distribution of S by a handy distribution of its lower bound S_l, which provides an excellent approximation. In actuarial applications however the upper bound S_u should draw even more attention, because it is more consistent with the principle of "actuarial prudence". In the case of annuities with stochastic interest rates the analysis relies for example on the assumption that the interest rates follow a Black-Scholes model, i.e. the volatility is constant and that the logreturns are normally distributed. These assumptions have been questioned in the financial literature. Also the estimates of μ and σ are burdened with an error. For these reasons it is recommended to use the upper bound S_u as an approximation of the real distribution, because it allows to take possible additional negative discrepancies into account.

In this contribution we have considered cashflows with payments of mixed signs for which the comonotonic upper bound S_u performs rather poorly. We propose to substitute this upper bound by a new approximation $S_{cop} \leq_{cx} S_u$. This approach allows for significant improvement of the fit to the original distribution. Although one cannot prove that $S \leq_{cx} S_{cop}$, it is intuitively clear that S_{cop} is more risky than the original distribution and that it can play a role of an upper bound. Indeed, if the pairs (X, Y) and (X_u, Y_u), with $X \leq_{cx} X_u$ and $Y \leq_{cx} Y_u$, have identical conditional increasing copulas then $X + Y \leq_{cx} X_u + Y_u$, see [12]. In this setting X and Y play the role of S^+ and $-S^-$ respectively. The Gaussian copula is conditional increasing if $\rho \geq 0$. For both cashflows given, the convex ordering can be easily verified from the graphs as the distribution function of S_{cop} crosses the distribution of S only once. According to [2] this condition is sufficient to have convex ordering. An overview of stop-loss premiums for both cases provided in Tables 4 and 5 also suggests that the idea of substituting S by S_{cop} is reasonable.

d	$E(S-d)_+$	$E(S_{cop}-d)_+$
0	2.5666	2.5693
2	0.9626	1.0219
4	0.2610	0.3217
6	0.0617	0.0922
8	0.0133	0.0254

Table 4: Stop-loss premiums for the first cashflow and their approximations

d	$E(S-d)_+$	$E(S_{cop}-d)_+$
-0.50	0.1546	0.1578
-0.25	0.0216	0.0281
-0.10	0.0042	0.0080
0.00	0.0014	0.0034
0.05	0.0008	0.0022

Table 5: Stop-loss premiums for the second cashflow and their approximations

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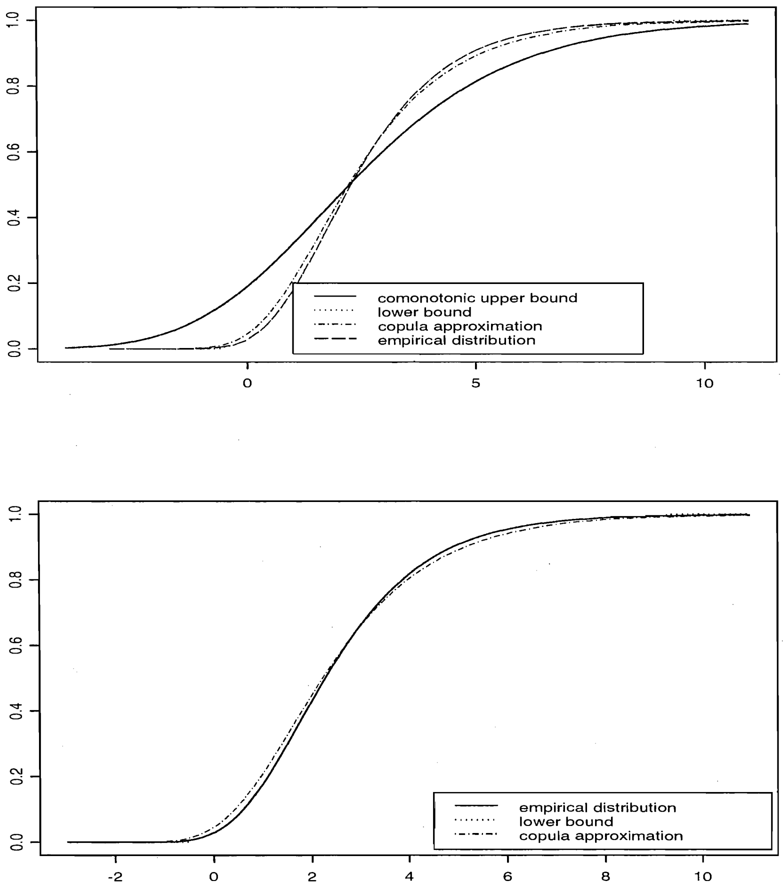

Figure 2: *Cumulative distribution for the present value of the first cash flow, with and without the convex upper bound*

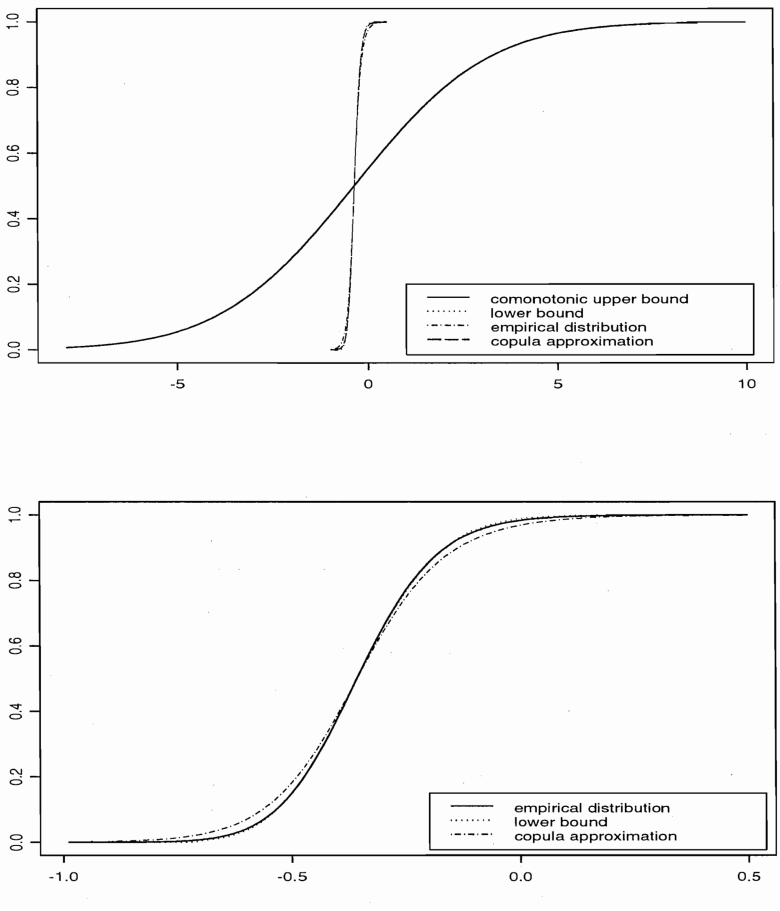

Figure 3: *Cumulative distribution for the present value of the second cash flow, with and without the convex upper bound*