Hippolyte d'Albis • Emmanuelle Augeraud-Véron

Balanced cycles in an OLG model with a continuum of finitely-lived individuals

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Abstract This paper studies the intertemporal equilibrium of a barter economy populated with a continuum of finitely-lived overlapping generations. Assuming isoelastic preferences and zero endowments at the beginning and the end of the individuals' life-span, it proves the existence of an Hopf bifurcation and provides sufficient conditions on parameters for its occurrence.

Keywords: Overlapping-generations Models, Mixed-type Functional Differential Equations, Endogenous Fluctuations, Hopf Bifurcation.

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Hippolyte d'Albis GREMAQ, University of Toulouse I, 31000 Toulouse, FRANCE E-mail: hippolyte.d-albis@univ-tlse1.fr

Emmanuelle Augeraud-Véron LMA, University of La Rochelle, 17042 La Rochelle, FRANCE E-mail: eaugerau@univ-lr.fr

1 Introduction

This paper investigates the long-run fluctuations that may emerge in stationary OLG economies. To this end, we follow Demichelis and Polemarchakis [5] who study a continuous-time model with a continuum of finitely-lived individuals¹. The intertemporal equilibrium is shown to be the solution of a functional differential equation of mixed-type (MFDE). By extending this model to isoelastic preferences, the MFDE is nonlinear. The proof of the existence of long-run fluctuations then relies on Rustichini [10] and Benhabib and Rustichini [2]: it uses the linearized MFDE that characterizes the local motion of the economy and looks for solutions which have Hopf bifurcation values. Assuming zero discounting and a specific endowment distribution, we show that there exist some sets of parameters such that a barter economy exhibits a cycle on the neighborhood of a steady state. An elasticity of intertemporal substitution lower than one and zero endowments at beginning and end of the individuals' life-span are sufficient conditions to obtain this result. This paper consequently shows that an increase in the frequency of trade during the individual' life-span does not eliminate the possibility of endogenous fluctuations in OLG economies².

The sketch of the paper is as follows. Section 2 presents an overlappinggenerations model with continuous trading and finitely-lived individuals. Sec-

¹The pioneer OLG model of this kind is due to Cass and Yaari [4].

²Note that this issue is usually debated in discrete time environment: see notably Aiyagari [1] and Ghiglino and Tvede [6] in a framework with many generations and [8], [7] and [9] in models with two generations.

tion 3 characterizes the intertemporal equilibrium of a barter economy and analyses the existence and uniqueness properties of the steady state. Section 4 studies the spectral decomposition of the linearized dynamics in the neighborhood of a steady state and gives sufficient conditions for the existence of an Hopf bifurcation.

2 The model

Time is continuous and has a finite starting point; let $t \ge 0$ denote the time index. Individuals live for an interval of time of length 1. They only derive utility from consumption and have isoelastic preferences and no time discount. Let $c(s,t) \ge 0$ denotes the consumption of an individual who born at time s as of time t. Hence, the intertemporal utility of an individual who born at time $s \ge 0$, denoted as u(s), is:

$$u(s) = \int_{s}^{s+1} \frac{c(s,t)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} dt$$
 (1)

where $\sigma > 0$ stands for the elasticity of intertemporal substitution. During a lifetime, an age-dependent endowment is received; it is denoted w(t-s)and satisfies:

$$w(t-s) = \begin{cases} w(t) \text{ if } t \in [s+\alpha, s+\beta] \\ 0 \text{ otherwise} \end{cases}$$
(2)

with $0 \le \alpha < \beta \le 1$. Individuals have access to a competitive asset market that yields the interest rate r(t). Let a(s,t) denotes the real wealth of an individual who born at time s as of time t. The instantaneous budget

constraint is therefore:

$$\frac{\partial a\left(s,t\right)}{\partial t} = r\left(t\right)a\left(s,t\right) + w\left(t-s\right) - c\left(s,t\right) \tag{3}$$

Individuals are born with no financial assets and cannot die indebted. Therefore, initial and terminal conditions write:

$$a\left(s,s\right) = 0\tag{4}$$

$$a\left(s,s+1\right) \ge 0\tag{5}$$

It is assumed that a(s,t) and c(s,t) are $C^1(\mathbb{R}^2_+)$ and that r(t) and w(t-s) are continuous for all $t \in [s, s+1]$. The individual program is to maximize (1) subject to (3), (4) and (5).

Lemma 1 The optimal consumption profile satisfies:

$$c(s,t) = \frac{\int_{s+\alpha}^{s+\beta} w(z) e^{-\int_{s}^{z} r(u)du} dz}{\int_{s}^{s+1} e^{-(1-\sigma)\int_{s}^{z} r(u)du} dz} e^{\sigma\int_{s}^{t} r(u)du}$$
(6)

<u>Proof</u>: The first order conditions are:

$$\frac{\partial c(s,t)}{\partial t} = \sigma r(t) c(s,t) \tag{7}$$

and a(s, s + 1) = 0. Integrating forward condition (3), and using the second optimal condition yields:

$$\int_{s}^{s+1} c(s,z) e^{-\int_{s}^{z} r(u)du} dz = \int_{s+\alpha}^{s+\beta} w(z) e^{-\int_{s}^{z} r(u)du} dz$$
(8)

Replacing (7) in (8) gives c(s, s) and consequently (6). \Box

The demographic structure is in overlapping generations. Each individual belongs to a cohort whose size is normalized to 1. There is no population growth and, at each point of time, a new cohort enters the economy since the oldest one leaves it. Hence, the aggregate counterpart, x(t), of any individual variable, x(s,t), is obtained by integrating over the birth date, such that:

$$x(t) = \int_{t-1}^{t} x(s,t) ds$$
(9)

Assume there exists a single non storable good and that the aggregate endowment equals the size of the population; then: $\int_{t-1}^{t} w(t-s) ds = 1$. Replacing the endowment distribution rule given by (2) yields: $w(t) = 1/(\beta - \alpha)$. Using (6), the aggregate consumption, denoted c(t), hence satisfies:

$$c(t) = \frac{1}{(\beta - \alpha)} \int_{t-1}^{t} \frac{\int_{s+\alpha}^{s+\beta} e^{-\int_{s}^{z} r(u)du} dz}{\int_{s}^{s+1} e^{-(1-\sigma)\int_{s}^{z} r(u)du} dz} e^{\sigma\int_{s}^{t} r(u)du} ds$$
(10)

There is no money in this economy and assets are constituted by consumption loans. The aggregate wealth is denoted a(t); using (3), (4) and the optimal condition: a(s, s + 1) = 0, its dynamics writes:

$$\frac{da(t)}{dt} = r(t) a(t) + 1 - c(t)$$
(11)

3 Equilibrium and steady states

Definition 1 An equilibrium with perfect foresight is a function $F(t) = (c(t), a(t), r(t)), F : \mathbb{R}_+ \to \mathbb{R}^2_+ \times \mathbb{R}, C^1(\mathbb{R}_+)$ such that (i) individuals maximize their utility subject to the budget constraint, (ii) the aggregate consumption equals the aggregate endowment (i.e. c(t) = 1), and (iii) the aggregate asset is zero (i.e. a(t) = 0).

Using (3), it can be inferred from definition 1, that an equilibrium reduces to a function r(t).

Definition 2 A steady state equilibrium is a triplet (c, a, r) that satisfies: (i) $c = \phi(r)$ and a = (c-1)/r, (ii) c = 1 and a = 0; where function $\phi : \mathbb{R} \to \mathbb{R}_{++}$ is such that:

$$\phi(r) = \frac{\int_0^1 e^{\sigma r z} dz}{(\beta - \alpha)} \frac{\int_\alpha^\beta e^{-r z} dz}{\int_0^1 e^{-(1 - \sigma) r z} dz}$$
(12)

Property 1 A steady state interest rate is a r that satisfies $\phi(r) = 1$ if $r \neq 0$ and $\phi'(r) = 0$ if r = 0.

<u>Proof</u>: For $r \neq 0$, the property is an immediate implication of definition 2; for r = 0, just observe that $\phi(0) = 1$ and use l'Hôpital's Rule. \Box

Lemma 2 r = 0 is a steady state interest rate if and only if $\alpha + \beta = 1$.

<u>Proof</u>: Replace r = 0 in $\phi'(r)$ and use property 1 to conclude.

Corollary 1 The individual consumption is age-independent if and only if $\alpha + \beta = 1$.

Lemma 3 There exist (α, β, σ) such that there is no steady-state equilibrium.

<u>Proof</u>: It can be easily shown that there is no solution $r \neq 0$ such that $\phi(r) = 1$ if $\alpha = 0$ and $\beta \in (0, 1 - \sigma]$ or if $\beta = 1$ and $\alpha \in [\sigma, 1)$. \Box

Lemma 4 There exist (α, β, σ) such that there are multiple steady-state equilibria. <u>Proof</u>: Suppose $\beta = 1 - \alpha$ and recall with lemma 2 that r = 0 is a steady state; it can be easily shown that there exist at least two other solutions $r \neq 0$ such that $\phi(r) = 1$ if $\sigma \in (\alpha; 2\alpha (1 - \alpha))$. \Box

4 Endogenous cycles

The equilibrium satisfies the following non linear dynamics:

$$\frac{1}{(\beta - \alpha)} \int_{t-1}^{t} \frac{\int_{s+\alpha}^{s+\beta} e^{-\int_{s}^{z} r(u)du} dz}{\int_{s}^{s+1} e^{-(1-\sigma)\int_{s}^{z} r(u)du} dz} e^{\sigma \int_{s}^{t} r(u)du} ds = 1$$
(13)

In this section, it is the local dynamics around steady-state r^* which is studied; it is the one of x(t) defined such that $r(t) = r^* + \epsilon x(t)$.

Property 2 The characteristic function of x(t) is denoted $Q(\lambda)$ and satisfies:

$$Q(\lambda) = -\frac{\int_{\alpha}^{\beta} e^{(\lambda - r^*)s} dz}{\int_{\alpha}^{\beta} e^{-r^*z} dz} + (1 - \sigma) \frac{\int_{0}^{1} e^{(\lambda - (1 - \sigma)r^*)z} dz}{\int_{0}^{1} e^{-(1 - \sigma)r^*z} dz} + \sigma \frac{\int_{0}^{1} e^{\sigma r^*z} dz}{\int_{0}^{1} e^{(\sigma r^* - \lambda)z} dz}$$
(14)

<u>Proof</u>: Replace $r(t) = r^* + \epsilon x(t)$ in (13). Then do a Taylor expansion in the neighborhood of $\epsilon = 0$ and rearrange using $\phi(r) = 1$. Define $X(t) = \int_0^t x(u) \, du$. It yields:

$$X(t) = \frac{\int_{t-1}^{t} e^{\sigma r^{*}(t-s)} \left(\frac{\int_{s+\alpha}^{s+\beta} e^{-r^{*}(z-s)} X(z) dz}{(\beta-\alpha)} - \frac{(1-\sigma) \int_{s}^{s+1} e^{-(1-\sigma)r^{*}(z-s)} X(z) dz}{\int_{0}^{1} e^{\sigma r^{*}z} dz} \right) ds}{\sigma \int_{0}^{1} e^{-(1-\sigma)r^{*}z} dz}$$
(15)

Finally, $Q(\lambda)$ is obtained by the following change of variable: $x(t) = e^{\lambda t}$ and rearranging using $\phi(r) = 1$. \Box

Lemma 5 The characteristic function $Q(\lambda)$ has an infinity of complex roots with negative real parts and an infinity of complex roots with positive real parts. <u>Proof</u>: Roots of $Q(\lambda)$ are asymptotic to those of the following equations³:

$$P_{1}(\lambda) = -\lambda \left(\sigma \lambda^{2} \int_{0}^{1} e^{\sigma r^{*} z} dz + \frac{(1-\sigma) e^{\lambda - (1-\sigma)r^{*}}}{\int_{0}^{1} e^{-(1-\sigma)r^{*} z} dz} \right)$$
(16)

$$P_2(\lambda) = -\lambda \left(\sigma \lambda^2 \int_0^1 e^{\sigma r^* z} dz + \frac{(1-\sigma) e^{-\lambda + \sigma r^*}}{\int_0^1 e^{-(1-\sigma)r^* z} dz} \right)$$
(17)

Asymptotically, roots of $P_1(\lambda)$ have a positive real part while those of $P_2(\lambda)$ have a negative real part. \Box

Corollary 2 The dynamics is generically characterized by oscillations that decrease in magnitude and eventually disappear.

Consider now the case $\beta = 1 - \alpha$ and focus on the neighborhood of the steady state $r^* = 0$. The characteristic function (14) rewrites:

$$Q(\lambda) = -\frac{\int_{\alpha}^{1-\alpha} e^{\lambda z} dz}{(1-2\alpha)} + (1-\sigma) \left(\int_{0}^{1} e^{\lambda z} dz\right) + \sigma \frac{1}{\left(\int_{0}^{1} e^{-\lambda z} dz\right)}$$
(18)

Lemma 6 There exist (α, β, σ) such that there are pure imaginary roots which are Hopf bifurcation values.

<u>Proof</u>: Let $\lambda = p + iq$. The proof proceeds in two steps: (i) it supposes $\lambda = iq$ and proves that for $\sigma < 2\alpha (1 - \alpha)$, there exists a q > 0 such that Q(iq) = 0; it hence defines $(\alpha_0, \sigma_0(\alpha_0))$ the pair of parameters for which this root does exist; (ii) it uses σ as a bifurcation parameter and shows that there exists a neighborhood of σ_0 such that $d \operatorname{Re}(Q(\lambda))/d\sigma$ is not equal to zero.

(i) Replace $\lambda = iq$ in (18) and define $\tilde{Q}(q)$ such that:

$$\tilde{Q}(q) = \frac{\cos(\alpha q) - \cos((1 - \alpha)q)}{1 - 2\alpha} + (1 - \sigma)[\cos(q) - 1] - \frac{\sigma q^2}{2}$$
(19)

 $^{^3 \}mathrm{See}$ Bellman and Cooke (1963) p. 410.

A Taylor expansion in the neighborhood of q = 0 yields:

$$\tilde{Q}(q) = \frac{q^4}{12} \left(\alpha \left(1 - \alpha \right) - \frac{\sigma}{2} \right) + O\left(q^6 \right) > 0$$
(20)

Moreover, $\lim_{q\to+\infty} \tilde{Q}(q) = -\infty$. Therefore, there exists a q > 0 such that $\tilde{Q}(q) = 0$.

(ii) From (18), it yields:

$$\frac{d\operatorname{Re}\left(Q\left(\lambda\right)\right)}{d\sigma}\bigg|_{\sigma_{0}} = \frac{\sin\left(q\right)}{q}\left(\frac{q^{2}}{2\left(1-\cos\left(q\right)\right)}-1\right)$$
(21)

The roots of this latter function are those of $\sin(q)$ and $\cos(q) = 1 - q^2/2$. They are the $\{2\pi + k\pi, k \in \mathbb{Z}\} \cup \{q_1, q_2, q_3, q_4\}$, where q_1, q_2, q_3 and q_4 are the 4 roots of the following polynomial:

$$\left(1 - \frac{q^2}{2}\right)^2 + \left(1 - \frac{(q - \frac{\pi}{2})^2}{2}\right)^2 = 1$$
(22)

Since elements of $\{2\pi + k\pi, k \in \mathbb{Z}\} \cup \{q_1, q_2, q_3, q_4\}$ are independent with respect to a_0 , there exists a_0 , such that $a_0 \notin \{2\pi + k\pi, k \in \mathbb{Z}\} \cup \{q_1, q_2, q_3, q_4\}$. \Box

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