CAPM EMPIRICAL PROBLEMS AND THE DISTRIBUTION OF RETURNS

FRANÇOIS DESMOULINS-LEBEAULT

Abstract. The CAPM is generally contested on an empirical basis. The tests conducted with data from financial markets do not generally imply the acceptance of the model as describing correctly the range of expected returns. When regressing returns of individual assets on the market portfolio, R^2 are generally not very good. The regression of the average returns against betas is even worse. Explaining these weak results by factors other than a temporary mispricing may allow us to understand better their origins. Specifically we will explore the relationship that may exist between the ineffectiveness of the CAPM and the non-normality of the returns distribution. More to the point, we find descriptive statistics of the returns distribution that partially predict the ineffectiveness of the beta for a given asset.

1. INTRODUCTION

1.1. **The CAPM challenged.** For many years, the CAPM has been one of the cornerstones of modern finance. Yet it is not empirically conclusive, and its hypotheses are very simplifying. Many researchers, who exposed the statistical difficulties inherent to the model, have tackled the empirical failure of the classical CAPM such as defined by Sharpe (1964) or Lintner (1965). They explain that the empirical results are not often conclusive because the market portfolio is unobservable, yet central to the specification of the various tests. Moreover, some researchers have shown in empirical tests that adding variables to the beta entails a better explanation of the variations of expected returns across the range of assets. Fama and French (1992, 1995, 1996), most notably have illustrated such facts in numerous articles. It seems that including variables such as the book to market equity ratio, the size of the stocks and some others, greatly enhance the explanatory power of the traditional CAPM regression.

The CAPM is therefore in a highly fragile position: if the tests including additional variables are justified, they show that the model is a failure. Yet, if, as some argue, the model cannot be tested effectively because of the unobservability of the market portfolio or biases in the data available to researchers, it cannot be of any use for practical applications and is therefore quite vain.

It seems that a more pragmatic approach leads one to understand that there are many biases involved when empirically testing the CAPM. There is, as Roll (1977) noted, no way we could conduct our tests with reference to the real market portfolio (to do this implies that the sample contains *all* investible assets in good proportions). Therefore, we know that all empirical results will not be entirely conclusive. Furthermore, the CAPM is based on anticipations and since the agents do not publish their beliefs about the future, tests can only be based on

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the assumption that the future will more or less reflect the past. These two problems imply that any test performed on the CAPM can, at best, only be partially conclusive.

However, if we identify the other aspects of the model that seem not to correspond tightly with the reality, we could test against the corresponding hypotheses and find out whether they really are a cause of the empirical failing of the CAPM. Indeed a model as such cannot describe exactly the facts and there always are trade-offs between the precision and the complexity. If we want to find a model explaining convincingly the assets expected returns, we need to determine which of the hypotheses are empirically more important.

The first steps in that direction were taken soon after the publication of the seminal CAPM paper. Lintner (1969) developed a version that accommodates heterogeneous beliefs. This was however very difficult to efficiently test since knowledge of the individual investors beliefs was required. Awhile later, Black (1972) introduced a CAPM with no unique lending-borrowing risk free rate. Kraus and Litzenberger (1976) developed a CAPM admitting returns distribution with a non-zero third central moment. A certain number of other researches have followed their path.

However pertinent it may seem to build modified CAPMs relaxing some of its hypotheses, it will not lead to a better understanding of the problems caused by these hypotheses. Even when testing a modified model against the original CAPM, one cannot be certain that the relative improvement is not caused by econometric artefacts or the specific structure of the data considered.

What we want to emphasise is that a better understanding of the causes of the empirical failing of the CAPM can come from an analysis of its errors. Testing the non-satisfying hypotheses against the actual errors of the tests will allow us to understand the impact of the hypotheses on the explanatory power of the model.

1.2. **Returns Distributions and the CAPM.**

One of the original hypotheses that are bound to pose important problems is that returns on all assets are joint normally distributed. This hypothesis can be levied to accommodate any sort of distribution if the agents express preferences only on the first two moments of the anticipated distribution of end of period wealth. We need to have at least one of these conditions met if the mean-variance efficient set mathematics (exposed by Merton (1972), Black (1972), Szegö (1975) or Roll (1977)) are to have any sense in portfolio selection.

Indeed, if the assets exhibit Gaussian returns, moments of higher order are completely determined by mean and variance, hence efficient portfolios are mean–variance efficient. On the other hand, if all the investors form their preferences only on the first two moments of the returns distribution, they will select mean–variance efficient portfolios, even if other sorts of efficient set are available.

However, it is hard to believe that agents do form their preferences on mean and variance only, especially in the light of the results of experimental finance (see Bossaerts, Plott, and Zame (2002)). This would be of no importance, should the returns be normally distributed. However, as we will see later, even if over longer periods we cannot reject the normal distribution hypothesis, this is absolutely not the case for shorter observation periods. One of the main characteristics of the Gaussian distribution being that it is stable under convolution, we therefore have to reject, for all maturities, the hypothesis of joint–normality for assets returns, unless prices follow a non–stationary process.

Therefore, we can suspect that the actual efficient set differs from the mean–variance efficient set, and hence that beta, measuring only the co-dependence of second order is not a sufficient measure of risk. These discrepancies between facts and theory are probably one of the sources of the empirical flaws of the CAPM. As we have underlined before, investigating the impact of this particular hypothesis on the real world performance of the CAPM is important.

In the following section, we will present the sample data with a special focus on their distributional properties, trying to gain a deeper insight on how the actual returns relate to their theoretical counterparts. For this purpose we use, among other statistics, semi-moments and introduce semi-cokurtosis. In the third section we will test for non-normal returns as an explanation of the weak R^2 of the betas determining regression and the residuals in the classical CAPM empirical test.

2. The Data and their Distributional Properties

2.1. **The Sample Data.** To study the distribution of assets' returns and their impact on the empirical effectiveness of the CAPM, we selected a two-fold sample. This sample is composed of stocks selected from the Standard & Poor's 500 (S&P 500) index, as of January 2nd, 2002. The second part of the sample is constituted of the same stocks, yet their returns are observed at a different time interval. The first part of the sample is therefore composed of daily returns, while the second part of the sample is composed of monthly returns.

We decided keep the same stocks in our sample, from start date to end, in order to have identical time series of returns for all stocks, so the estimates of their distributions have the same statistical significance. Therefore, all the stocks not existing for the entire test period were excluded of the sample. We perfectly understand how such a procedure may induce a survivor's bias and that no investor has the ability to determine if a stock will still be available on the market ten or twenty years from now. However, the purpose of this paper is not to perform classical tests of the CAPM but to evaluate if errors can be explained by the non-respect of one of the models' hypotheses.

It is nonetheless certain that the results will not be as good as they should. The reason is that the errors observed in our setting are not only due to model imperfections but also to statistical imperfections and probably significantly so. However, many articles have outlined the difficulties arising in specifying correctly the tests of the CAPM. Despite these difficulties, our purpose being mainly to show that the unrealistic distributional hypotheses of the CAPM have a measurable impact on the empirical efficiency of this model, we still can extract valuable information from our tests.

The samples were initially composed of the 500 stocks composing the S&P 500index on the 2nd of January 2002. From these stocks we kept all those for which we had prices from January 1st 1985 to January 2nd 2002. Then we again restricted the sample by eliminating every stock exhibiting a 0 return for more than 12% of the observations. Thus, we tried to restrict the effects of thin trading. Indeed, thin trading has a dissimulating effect on the distributional properties of the returns series. After these different restrictions on the sample, it was reduced to 239 individual stocks. As indicated before, we used these same stocks for both the daily returns' series and the monthly returns'.

The returns' series for all samples were constructed from closing prices, corrected for dividends and capital operations. Though logarithmic returns present some interesting properties, which make them a natural choice in many studies, we used the somewhat rough division form: $r_{t,t+1} = (P_{t+1} - P_t)/P_t$. The reason for this choice is that the approximation provided by the logarithmic returns is quite good for values in the neighbourhood of zero, yet the errors become rapidly important for extreme deviations; moreover, the errors are not symmetric. As $\lim_{x\to 0} \ln(x) = -\infty$, we could have, for real returns close to -100%, logarithmic returns of -100000%, which artificially inflates the left tail of the distribution. This is particularly bad when studying the returns distributions and their effect on the effectiveness of the CAPM.

2.2. **Tests of Normality.**

To examine precisely the distributional properties of the data, we conducted several classes of tests. The first class, which is immediately relevant in a study of the impact of non-normality on the empirical aspect of the CAPM, is the tests of normality and other related tests of distribution. The tests of the second series are directly based on the estimation of the structural distributional properties of the series, like higher order moments and semi-moments, and the stability over time of these elements. In short, we first test for non-normality and then try to define in which way the series depart from normality.

2.2.1. *Kolmogorov–Smirnov Test.*

To test for the normality of the data, we used most of the classical tests. We started the study with the relatively imprecise Kolmogorov–Smirnov test. In this non-parametric test, which is designed to test for the fact that two samples have the same distribution, we compared each of the returns series to the Gaussian distributions with the same mean and variance.

For the daily sample of the S&P 500, the results were unequivocal; the hypothesis of normality was rejected for all the individual assets. Even for the ten portfolios built based on the capitalisations of the assets, the rejection was total. We should notice in table 1 that the Kolmogorov–Smirnov statistics are generally way above the critical value. The p-values are all really close to zero confirming the power of the rejection. Even for the asset closer to normality, the Johnson & Johnson stock, the p-value is of 3.8×10^{-5}.

Table 1. Kolmogorov–Smirnov Test (Daily Sample)

	Statistic	p-Value	Result	Critical Value
Providian Finl.	0.1233	2.58E-57	reject	0.020691
Xerox	0.11148	6.25E-47	reject	0.020691
Applera Appd.Bios.	0.10703	2.72E-43	reject	0.020691
Mean Result	0.0642166	1.93E-07	reject	0.020691
Pfizer	0.040063	1.97E-06	reject	0.020691
Merck	0.039093	3.82E-06	reject	0.020691
Johnson & Johnson	0.035519	3.80E-05	reject	0.020691

However, the results were not as strong in the case of the monthly returns series, as can be seen in table 2. Even if the ordering of stocks according to the p-value is quite similar in both cases, the Gaussian distribution hypothesis is rejected for only 8 stocks, representing 3.35% of the sample.

The smallest values of the Kolmogorov–Smirnov statistics, signaling the distributions that more closely resemble the Gaussian can be found for Johnson & Johnson, at the daily interval and for Johnson controls at the monthly interval. The stocks with the biggest p-value are generally from chemical or pharmaceutical companies, and the mean daily return of the 10 stock exhibiting the smallest statistic is 0.074%, as compared to the average mean return for the sample: 0.068%. On the other hand, the higher KS statistics can be found for the stock of companies in the high tech or financial sector. In section 2.3 we detail more of the origins and type of non-normality affecting these stocks returns. Notice however, that the mean daily return of the 10 stocks exhibiting the biggest Kolmogorov–Smirnov statistic is 0.061%.

Table 2. Kolmogorov–Smirnov Test (Monthly Sample)

	Statistic	p-Value	Result	Critical Value
Providian Finl.	0.12817	0.002038	reject	0.093776
Xerox	0.11903	0.0052578	reject	0.093776
Applera Appd.Bios.	0.10703	0.016395	reject	0.093776
Mean Result	0.058618	0.511565	3.347%[1]	0.093776
McDonalds	0.03339	0.97353	accept	0.093776
Wells Fargo & Co	0.032999	0.97639	accept	0.093776
Johnson Controls	0.03168	0.98446	accept	0.093776

The Kolmogorov–Smirnov test, as we said earlier, lacks discriminating power, so we can suspect that among the stocks returns series for which it could not reject the hypothesis, are a number of non-Gaussian distributed returns. More to the point, every single normality test available has specifics that put more value on certain kinds of departure from normality. To have as precise a picture as we could of the normality of the sample, we used two additional tests. First, we used the Shapiro–Francia test (more accurate when used on leptokurtic distribution than its Shapiro–Wilks sibling), then the Jarque–Bera test.

2.2.2. *Shapiro–Francia Test.* The Shapiro–Francia test is specifically designed for tests of the hypothesis of normal distribution in a sample. It uses order statistics of the sample and is quite akin to a regression of the data on the Gaussian line in a QQ-plot. One of the most powerful omnibus tests of normality, it offers a more severe vision of the samples.

The results for the daily sample (shown in table 3) are quite evidently similar to those obtained with the KS test, yet the ordering of p-values is slightly different. However, what is to notice principally is that the rejection of the Gaussian distribution of returns is extremely strong. The stocks for which the p-value is highest are still very far from the 5% we chose as confidence level.

The results are quite interesting for the monthly returns sample. When we could reject normality for only 3.35% of the sample using KS test, the Shapiro–Francia test reject the hypothesis for 76 stocks, representing 31.8% of the sample. The ordering of the level of rejection is again slightly different from what it was with the KS test, as can be observed in table 4, yet it does not fundamentally

Table 3. Shapiro–Francia Test (Daily Sample)

	Statistic	p-Value	Result
Masco	6.5152	3.63E-11	reject
Sealed Air	6.3586	1.02E-10	reject
Unilever NY	6.3149	1.35E-10	reject
Mean Result	4.736669038	1.59334E-05	reject
Micron Tech.	3.5368	0.00020254	reject
Kerr-McGee	3.4767	0.00025382	reject
KLA Tencor	3.188	0.00071625	reject

differ since there is no stock for which the hypothesis was rejected by the KS and accepted by the SF test.

Table 4. Shapiro–Francia Test (Monthly Sample)

	Statistic	p-Value	Result
Masco	3.6198	0.0001474	reject
Xerox	3.3926	0.00034622	reject
Providian Finl.	3.269	0.00053967	reject
Mean Result	1.301477	0.14074808	31.799%
Baxter Intl.	-0.04007	0.48402	accept
Comerica	0.023479	0.49063	accept
Walgreen	0.002701	0.49892	accept

2.2.3. *Jarque–Bera Test.* Again, to get a more precise picture of the normality in our sample we decided to conduct yet another test of normality, the Jarque–Bera test. This test is based upon the fact that for a Gaussian distribution, whatever its parameter, the skewness is 0 and the kurtosis 3. This test presents an advantage over the SF test in that it takes into account the fact that, as the number of observations in the sample grows, the difference in the estimates of the sample's moments and their theoretical value should tend towards zero. It is very suitable for large samples while the Shapiro–Francia tends to become lax for large sample sizes.

As our samples are quite large, especially in the case of daily returns, and still of a significant size in the case of the monthly returns, the Jarque–Bera test is expected to be more severe than the Shapiro–Francia test. The results we obtain tend to confirm that. The rejection of the Gaussian hypothesis is once again total for the daily sample (see table 5), with even more severity than for the two other tests, while the results for the monthly sample are quite significant.

While we could not reject the normality hypothesis for almost 70% of the stocks with the SF test, this time we reject the normality at a 5% confidence level for 75.73% of the individual assets' returns in our sample. Again, the ordering of the level of rejection, as measured by the p-value, is slightly different than the one observed with the other two tests, yet we can notice that it closely resembles the order given by the SF test. More over we can notice that while the ordering of the results tends to change quite significantly between the daily and the monthly

Table 5. Jarque–Bera Test (Daily Sample)

	Statistic	p-Value	Result	Critical Value
Masco	41801840.53	0	reject	5.9915
Sealed Air	15177711.56	0	reject	5.9915
Kroger	3770730.093	0	reject	5.9915
Mean Result	310415.7303	0	reject	5.9915
Wal Mart Stores	1214.9327	0	reject	5.9915
Abbott Labs.	1169.651	0	reject	5.9915
Micron Tech.	1063.159	0	reject	5.9915

samples for the stocks closer to Gaussian, the same stocks remain furthest apart from normality.

Table 6. Jarque–Bera Test (monthly Sample)

	Statistic	p-Value	Result	Critical Value
Masco	6162.9914	0	reject	5.9915
Sealed Air	1439.8673	0	reject	5.9915
Kroger	1390.5772	0	reject	5.9915
Mean Result	90.75721	0.09420	75.732%	5.9915
Pfizer	0.062743	0.96912	accept	5.9915
Heinz Hj	0.042134	0.97915	accept	5.9915
McDonalds	0.014308	0.99287	accept	5.9915

That confirms the fact that the distributions of returns do not seem to be Gaussian even if we cannot reject the hypothesis for many stocks on the monthly returns basis. However, that the time series of returns over a long period is not Gaussian does not really surprise us. More interesting is the determination of the sort of non-normality affecting the returns' series distribution.

2.3. **Characteristics of the Distributions of Returns.** Before studying the distributions of returns through their moments and half moments, it can be interesting to evaluate their normality on sub periods. Indeed, the non-normality that we have stressed in 2.2, can have different origins. It may be that the distributions are intrinsically non-Gaussian yet stable over time, or that the distributions are not so far from a Gaussian yet with parameters changing over time or even that the shapes of the marginals are time varying. As the period considered for this study is rather extended, it is highly probable that the economic conditions on the market have changed during the observation period. If that implies a change in the parameters of the distributions, it could mean that, over shorter periods, the returns are less non-normally distributed and present non-Gaussian features on the long range, as they become finite mixtures of other distributions.

2.3.1. *Subperiods Normality and the Stationarity of Returns.* To evaluate rapidly this hypothesis, we conducted our tests of normality on subperiods of time, first corresponding to the general trends in the market. In that setting, we decided to split our samples in three. The first part goes from January 1985 to December 1994, corresponding to a period of generally gently growing markets. The second period is from January 1995 to August 2000, corresponding to rapidly growing market conditions. The last subperiod is from September 2000 to January 2002 and is a period of rapidly falling prices.

TABLE 7. Percentages of Rejection of the Normality Hypothesis for Subperiods (equal and unequal)

	Daily Sample			Monthly Sample		
rejection percentage	KS test	JB test	SF test	KS test	JB test	SF test
entire period	100%	100%	100%	3.35%	75.73%	31.80%
01/1985 to 12/1995	100%	100%	100%	0.42%	57.74%	16.32%
01/1995 to 08/2000	98.33%	100%	100%	0.42%	40.59%	9.21%
09/2000 to 01/2002	28.03%	95.82%	74.06%	0%	2.93%	0.84%
01/1985 to 09/1990	100%	100%	100%	0.84%	45.61%	12.13%
10/1990 to 05/1996	99.16%	100%	91.21%	0.42%	17.57%	3.77%
06/1996 to 01/2002	95.82%	100%	100%	0%	33.89%	7.95%

The results from this first analysis are quite controversial, since the result show less non-normality for the last period, which is coincidentally the shortest. It even arrives to the point that the KS test cannot reject the hypothesis of normality for any of the stocks in the sample. Hence, the following question: are these results different for each period because the generating distribution of returns is different for each period or because of the different lengths of the periods.

To obtain a more precise view of this problem we decided to conduct the same tests on three subperiods of equal size. The results are shown in table 7 together with the results for unequal periods. When applied to the daily sample we recover a homogenous full rejection for the JB test, while the two other tests fail to reject the hypothesis for very few stocks if any. Yet, when considering the average p-values attached to the different tests, a more contrasted picture comes to view. While the rejection was extreme for the full period, it is significantly less assured for the subperiods, as can be seen in table 8(daily sample). The results for the monthly sample are quite similar.

TABLE 8. Average p-Value of Normality Tests for Subperiods (equal and unequal)

	Daily Sample		
average p-value	KS test	JB test	SF test
entire period	1.93E-07	0	1.59E-05
01/1985 to 12/1995	7.52E-06	0	0.00024313
01/1995 to 08/2000	0.0038948	3.01E-16	0.001724
09/2000 to 01/2002	0.22585	0.0088666	0.046643
01/1985 to 09/1990	0.0002021	0	0.00070533
10/1990 to 05/1996	0.0017975	8.94E-06	0.015945
06/1996 to 01/2002	0.0088732	1.32E-16	0.00142

These results hint at the possibility that the assets returns do have a switching regime generating process. That would yield to an ex-post distribution having the form of a finite mixture of distributions. This provides us with a good explanation of the sometimes almost multimodal empirical densities.

However interesting the results of theses tests, they are not precise proof of the non–stationarity of the returns series. To evaluate exactly to what extend the data are stationary or not we used a unit root test, the Augmented Dickey–Fuller test to determine whether the data series were integrated of order zero or no (weakly stationary or not stationary). A time series y_t is weakly stationary if:

(1) $E[y_t]$ is constant $\forall t$,
(2) $\text{var}(y_t) < \infty$ is constant $\forall t$,
(3) $\gamma_j = \text{cov}(y_t, y_{t-j})$ is constant $\forall t, j$

A time series having one (or more) unit roots is integrated of order one (or more). Therefore, to determine whether our data exhibit stationarity or not we tested for the existence of unit roots. In so doing, we used three confidence levels: 1%, 5% and 10%. The results are quite strong. For the monthly and the daily returns, we reject the hypothesis of weak stationarity at all three confidence intervals, even when using an additional number of lags (this corresponds to the hypothesis that the series generating process is autoregressive of order n if we use n lags). The results are summarised in table 9.

Table 9. Augmented Dickey-Fuller Test

	Daily Returns		Monthly Returns	
	no lag	10 lags	no lag	10 lags
Minimum ADF statistic	-71.2763	-29.9714	-17.9360	-7.5845
Mean ADF statistic	-64.3375	-25.7054	-14.8443	-5.6024
Maximum ADF statistic	-58.1958	23.0176	-11.6846	-3.8217
Critical value at 1%	-3.4583	-3.4583	-3.4926	-3.4926
Critical value at 5%	-2.871	-2.871	-2.876	-2.876
Critical value at 10%	-2.5937	-2.5937	-2.5688	-2.5688

We first notice that the rejection of stationarity is much stronger for the daily returns series. Even with 30 lags, the hypothesis is rejected for daily returns while with 10 lags there are a few stocks for which we cannot reject stationarity. The use of autoregressive hypothesis of various orders in our test is contrary to common financial theory since it is equivalent to predictable returns and thus to inefficient markets. However, the fact that using such models yields a less severe rejection of the stationarity hypothesis shows us that we need to relax at least one of the common hypothesis used in financial econometrics, either stationarity or unpredictability of the returns.

The fact that the rejection of stationarity is stronger for the data that are less Gaussian is quite significant. The non-normality seems therefore to be mainly coming from an ex-post mixture of distributions and more specifically a mixture of Gaussians. This confirms several analyses and results in the literature, like Ané and Labidi (2001).

2.3.2. *Moments and Semi–Moments of the Distribution.* Another point that can help characterise the distributions of returns is their moments and semi-moments. The general definition of the moments of order higher than two used in this study is the definition of standard central moments, that is:

Definition 2.1. Let ξ be a real valued random variable, then the expression:

$$\mu_n = \frac{E[(\xi - E[\xi])^n]}{\sigma^n}$$

where σ is the standard deviation of ξ gives the standard central n–th moment of ξ.

To complement these classical moments, we computed the semi–moments as well. The left (right) semi–moments are defined as the part of a moment corresponding to the observations located under (above) the mean. To be more precise, the definition, for standard signed semi–moments, with $x^- \equiv \min\{x, 0\}$ and $x^+ \equiv (-x)^-$, is:

Definition 2.2. Let ξ be a real valued random variable. Then:

$$m^n_-(\xi) = \frac{E\left[\{(\xi - E[\xi])^-\}^n\right]}{\sigma^n} \qquad \text{and} \qquad m^n_+(\xi) = \frac{E\left[\{(\xi - E[\xi])^+\}^n\right]}{\sigma^n}$$

are called the left and right signed standard semi–moment of n–th order of ξ, for $1 \leq n \leq \infty$, respectively.

Thus we obtain a decomposition of any standard central moment into components coming from the "bad" and "good" moves of the price. For symmetrically distributed returns, the left and right part should be equal. Yet, as we will see, this seldom happens in financial assets' returns. The measure of symmetry obtained with these semi–moments is more precise than the skewness alone. Indeed, this details the asymmetry of all components of the distribution.

Furthermore, it enables us to detect departure from the Gaussian with more precision. One may have a distribution with an apparent kurtosis of 3, and believe it to be similar to a Gaussian, when the left semi–kurtosis is of 2 instead of 3/2 and the right semi–kurtosis is of 1 instead of 3/2. This asset will be more risky than its Gaussian counterpart yet kurtosis alone would have failed to detect it.

The returns in our sample are not generally too asymmetric yet their moments and half moments confirm that the hypothesis of normality is not quite respected. As expected the daily sample exhibits really extreme values, especially for the kurtosis, and some individual assets have a strongly dissymmetric kurtosis. A summary of the results is presented in table 10 for the daily sample and in table 11 for the monthly sample.

Table 10. Moments and Semi–Moments of the Daily Sample

	skewness	abs. skew.	kurtosis	left semi kurt.	right semi kurt.
highest 1	12.69	12.69	486.03	291.38	483.93
highest 2	3.30	8.58	293.96	140.91	93.73
highest 3	1.29	4.67	147.98	72.53	53.78
Average	-0.16	0.47	18.89	11.25	7.64
lowest 3	-3.28	0.0034	5.55	2.24	2.33
lowest 2	-4.67	0.0011	5.47	1.92	2.24
lowest 1	-8.58	0.0009	5.41	1.74	2.21

The daily returns generally have a modest negative asymmetry, in the skewness sense, yet their kurtosis, which is always above the normal 3, are higher on the left part of the distribution, representing a big loss probability superior to the equivalent big gain (net of the average growth). Moreover, we notice that the mean of the absolute value of the skewness is noticeably higher that the mean of the skewness indicating that the returns do present asymmetries yet they are not systematically oriented in the same direction.

Table 11. Moments and Semi–Moments of the Monthly Sample

	skewness	abs. skew.	kurtosis	left semi kurt.	right semi kurt.
highest 1	3.12	3.12	29.35	14.34	28.25
highest 2	1.49	1.97	15.82	13.38	12.33
highest 3	1.44	1.67	15.50	10.66	10.09
Average	0.075	0.3003	4.97	2.31	2.66
lowest 3	-1.23	0.0049	2.88	0.87	1.00
lowest 2	-1.67	0.0030	2.84	0.82	0.99
lowest 1	-1.97	0.0022	2.83	0.77	0.93

The monthly returns present quite the same feature, with the difference that the asymmetries seem to be oriented in a positive direction. The skewness is positive on average and the asymmetry noticed through semi–kurtosis confirms it. Yet, as for the daily returns, the fact that the mean of the absolute values of skewness is much bigger than the absolute value of the mean skewness indicates that skewness is far from homogeneous in the sample.

This variability in the direction of asymmetries combined with our findings on the un-stationarity of the distribution leaded us to explore the stability over time of the different moments and semi–moments. Indeed, one of the key assumptions in the CAPM is that the risk measure (the beta) is an intertemporal constant. It is possible to accommodate non-Gaussian distribution in the model with a few limited modification as soon as they are elliptical, yet time varying distribution parameters are probably, if they exist, a cause of the CAPM failure.

2.3.3. *Intertemporal Stability of the Moments and Semi–Moments.* To assess the stability of the moments over time, we computed them on the sample divided into six approximately equal periods. The results confirm our intuition that the distributions' parameters change over time. However, the picture is a bit more contrasted than expected. If the skewness on the daily returns does not seem to have any stability at all, apart for its absolute value being not too important (it varies as well across the assets), the kurtosis seem more stable. However, these results (presented in table 12) are subject to sampling variations. We are not able to determine with a given confidence interval if our results are statistically different from one another. Tests exist for comparison between means or variance on normal samples, yet for higher order moments on non-Gaussian populations there are no such tests.

The first point of interest in these results is that the kurtosis is on average smaller on shorter periods than on the entire sample. This gives more weight to the notion of return being on observable periods a mixture of distributions. With the exception of the first period, the results for both the kurtosis and the semi-kurtosis are quite stable over time. On the other hand, the skewness is never really high and keeps changing signs. It does not seem quite stable.

Table 12. Moments and Semi–Moments for Subperiods (Daily Returns)

	Skewness	Kurtosis	Left Semi–K.	Right Semi–K.
Period 1	-0.73561	22.763	15.6734	7.0296
Period 2	-0.086016	9.3952	5.9489	3.4215
Period 3	0.16057	7.2481	3.5287	3.7003
Period 4	0.19531	7.657	3.2718	4.3649
Period 5	0.074462	7.6872	4.214	3.4529
Period 6	-0.013459	9.2868	5.3803	3.8694

The results are very similar when the same tests are conducted on the monthly returns (they are presented in table 13). The kurtosis is quite stable over time, with the exception, this time, of the second period. Both semi–kurtosis are not varying much over time. The skewness changes signs less often than in for daily returns, yet it still lacks the relative stability of the kurtosis. It is to be noted that these results are averages across all assets in the sample but they are quite representative of the behaviour of the moments for individual assets. However, we noticed that for individual stocks the relative stability of the kurtosis is not always present.

Table 13. Average Moments and Semi–Moments for Subperiods (Monthly Returns)

	Skewness	Kurtosis	Left Semi–K.	Right Semi–K.
Period 1	0.26184	3.4344	1.2	2.041
Period 2	-0.56474	4.4698	3.1095	1.1085
Period 3	0.14603	3.2691	1.3264	1.7532
Period 4	0.079741	2.9302	1.242	1.5183
Period 5	0.092703	3.5434	1.4982	1.8399
Period 6	0.26155	3.5734	1.2752	2.0912

The most striking finding in individual assets is that the right semi–kurtosis is generally the stable component of the kurtosis. It remains at a comparable level during all period while the left part may vary more extensively. This may be explained by the fact that really extreme events are more frequent for negative returns, yet they still do not happen really often and are present only on certain period, irregularly. There might be a high order asymmetry in returns, yet we do not dispose of enough data to detect them with statistical significance.

2.4. **Synthesis of the findings.** The results of this analysis of the distributional properties of the returns confirm most of the general beliefs held by the finance community. Indeed, under no circumstances returns can be considered Gaussian on the daily returns scale. They present, at least for a few of them, some tendency towards normality when observed at a month interval. The more powerful tests on the sample were the Jarque–Bera tests, showing that the major difference between the empirical distribution of the returns and the Gaussian hypothesis lays in the tails of the distribution. Indeed, leptokurtosis seems to be a stable property of the data, while skewness might just be a local artefact due to mismeasurement and sample errors. This confirms the results of Peiró(1999, 2002) or Alles and

Kling (1994), who finds that there are only limited and unstable asymmetries in returns series.

Another point worth noticing is that the results hint generally at a non-Gaussian, non-stationary data generating process. There seems to be a regime switching process, the changes being frequent and not necessarily linked to general changes of conditions on the market. The most likely candidate for the ex–post distribution of returns, from the findings we have, is the finite components mixture of Gaussians. Indeed the results tend towards normality on a certain number of characteristics, they present a clear leptokurtosis and some unstable form of local asymmetry.

Thus the results confirm our hypothesis that the Normality Hypothesis in the CAPM does not hold in the real world. The question remain, however, whether this has an important impact on the empirical performances of the model. It may be that, as it is the case for numerous statistical procedures, the CAPM is robust to the modification of distributions. We need to evaluate the impact of the distributions of returns on its performances without specifying an alternative model, so as to have a precise understanding of the real consequences of the facts unwillingness to conform to the CAPM hypotheses.

3. Does non-Normality Explain the CAPM Weaknesses?

The purpose of this paper is to investigate the causes of the empirical limitations of the CAPM. We do not propose an alternative model designed to fit more precisely the data characteristics, but we try to evaluate if such a model is necessary. Most of the literature dealing with the CAPM is either empirical tests trying to determine whether the model describes correctly the facts, or extensions of the model trying to incorporate more realistic assumptions. The general conclusion one can draw from this literature is that the CAPM is generally not accepted as a good description of the facts and that most alternative models do perform slightly better.

Most alternative, if not all of them do propose a relationship between returns and risk which is an extension of the classical CAPM. Their general form is therefore the CAPM augmented by one or more additional term, generally the product between some complementary risk measure and the market price of this added risk. The fact that they are performing better than the CAPM is not really a surprise. We know that the variance-covariance is far from being the only dimension of risk agents bear when holding an asset. There are liquidity risks, operational risks, uncertainty risks coming from the absence of true model and the non homogenous nature of information. Since there is almost no strictly independent random variable in our economic world, it is almost certain that any additional variable will, at least locally, add to the explanatory power of the CAPM.

However, we believe that understanding the theoretical causes to the empirical weaknesses of a model is paramount in the design of a future, more precise model. To evaluate the impact of the hypothesis we study in this article, non-normality, we decided to take an alternate route. Instead of testing a model incorporating some additional parameters of the distributions, like higher order moments, we chose to see if these parameters have an explanatory power on the errors of the model. That is, in a perfect world the errors in the regressions used to test the

CAPM should be perfect white noise, orthogonal to any other random variable on the market. Thus the relation liking the errors and precision of the test to any extra parameter provide a proof of its insufficiency and allows to evaluate it.

3.1. **Weaknesses of the classical CAPM.** In order to obtain a basis test on which perform our analysis of the errors and precision, we conducted a rough test of the CAPM. We first established the linearity of the relation between individual returns and the market returns, thus determining the betas. This is the most widely test combined with the test on the intercepts of such regressions which should be zero when conducted on returns in excess of the risk free rate.

3.1.1. *The Linear Risk–Return Relation.* To obtain the historical betas, the returns of individual assets are regressed against the returns of the market portfolio. The general form of this regression is:

$$R_i - r_f = \alpha + \beta(R_M - r_f) + \varepsilon. \tag{3.1}$$

The basis of the test consists in estimating whether the α parameter is zero, which is predicted by the model, and whether the relationship between individual assets and the market portfolio is truly linear. Therefore, resting on the regression 3.1, we have two different ways to evaluate the performance of the CAPM, the first one being a correct estimation of the intercept (the hypothesis of equality to zero not being rejected) and that this linear description of the relationship between returns and the market portfolio is satisfactory (that the regression has a good R^2.) In a complete mean–variance efficiency world, with the good market portfolio, the results should be perfect as Roll (1977) has predicted.

On the assets of our sample, the results are generally not excellent... The average R^2 for these regressions with daily data is 0.215063, with a minimum of 0.004662 and a maximum of 0.574227. To improve these results, many articles use a reduced number of portfolios based on passed betas or capitalisation instead of individual assets, thus reducing the noise on the betas *and* the real errors coming from the model. For the monthly returns the results are slightly better, as the average R^2 of the regressions is 0.253011, with a minimum of 0.041162 and a maximum of 0.621712. Yet even these results are far from satisfactory and the weakest R^2 hint at a total absence of linear relation between market portfolio returns and the individual assets'.

For the daily returns, the average estimated intercept is of 0.0002020 and the attached t-statistic is of 0.69475, corresponding to a probability of 0.48156. That means that with a confidence interval of 48% we still could not reject the hypothesis that the intercept is zero. This seems to be a quite satisfactory result, yet one may notice that even in the maximum p-value is 0.9992, corresponding to an almost certain zero intercept, the minimum value is 0.0085 indicating quite probable non-zero intercept. The situation is quite similar for the intercept of the regressions on monthly returns. The average intercept is of 0.00433, with a mean t-statistic of 0.78774, a mean p-value of 0.44221. If these results appear to be slightly worse than those of the daily sample it is probably because of the limited sample size. Yet with a maximum p-value of 0.9973 and a minimum of 0.0055, the picture remains very similar.

A general comment is that the intercepts are quite satisfactorily close to zero, yet, in general, the relation between individual assets returns and the market portfolio returns is far from being only linear and the regressions have a quite weak R^2. This may be explained technically if the data generating process for the returns is not stationary. However, estimating the parameters in the linear equations with different techniques, supposedly more robust to non stationarity (Generalised Method of Moments) or even designed to fit it (ARCH-GARCH) does not yield considerable changes in the results. More precisely the estimated betas are very similar to the Ordinary Least Squares regressions estimates and the R^2 are generally similar or smaller.

Even these weak results, tainted as they may be by estimation errors and statistical imprecisions, provide us with a basis for conducting the explanation of the errors. We just need to remember that, due to the crude parameter estimation conducted, we try to explain a total amount of errors and imprecision well in excess of what really is the model error. Therefore, the quality of the explanation our non–Gaussian statistics will provide is probably underestimated in these tests.

Another classic test would then be to regress the mean returns on the betas. It allows us to estimate the predictive power of the betas. Indeed the CAPM predicts that returns depend on betas only and therefore the only possible explanation within this model for cross sectional variation of returns at equilibrium are the betas. Therefore, a cross sectional regression between the vector of expected returns and the vector of betas (in both cases approximated by their historical values) should have a strong R^2 (ideally 1). If the returns are net of the risk free rate (as is the case in this study) we should have an intercept not significantly different from zero.

However, even if the CAPM was the model describing perfectly the data, the results would be tainted and certainly not perfect. Indeed, the theory predicts the following relation :

$$E[R_i] - r_f = \beta \cdot (E[R_M] - r_f).$$

Therefore, it should hold only for expectations, the beta being an intertemporal constant. Yet, we have seen that indeed the distributions of returns do not present a character of stationarity and not even weak stationarity. That means that the variance–covariance structure is time dependent, and therefore that betas cannot possibly be constant over time. And more, the expectation are possibly very different from the historical results. Yet, this test is important in the sense that it allows us to test for the general usefulness of the CAPM. If the relation between the returns net of the risk free rate and the betas is not present in the data, the model is definitely useless, at least in terms of practical uses.

Knowing that the result will not be very convincing, because of the elements mentioned above, we still need this regression performed, in order to evaluate afterwards if different non–Gaussian statistics have any explanatory power, on its errors. The results we obtained were, as expected, not excellent.

A relation exists between returns and betas, yet betas are far from explaining well the dispersion of the returns across assets. Arguably a part of the error comes from the fact that the betas were estimated using a proxy for the market portfolio which is possibly not mean–variance efficient and almost certainly not the real market portfolio. An other part of the residuals probably comes from the error made in estimating expected returns by the historical values. Yet if the process generating the returns is a martingale under the physical probability, it should not be a problem.

The results of this regression on the daily sample are given in table 14. The good point is that it is probable that the intercept is zero, as predicted. Yet, the betas have a weak explanatory power on the returns and their coefficient does not correspond to the mean net return of the market portfolio.

TABLE 14. Daily Returns Cross–Sectional Regression on Betas

Ordinary Least-Squares Estimates			
R-squared	= 0.2328		
Rbar-squared	= 0.2295		
Durbin-Watson	= 2.0641		
Variable	Coefficient	t-statistic	t-probability
constant	-0.000081	-1.231381	0.219401
betas	0.000609	8.479966	0.000000

For the monthly returns the result is quite similar, yet a bit less convincing, probably because the estimates of the betas were less precise due to the inferior number of observations. The results of this regression for monthly returns are shown below:

Ordinary Least-Squares Estimates			
R-squared	= 0.1847		
Rbar-squared	= 0.1813		
Durbin-Watson	= 1.9773		
Variable	Coefficient	t-statistic	t-probability
constant	0.001440	1.257972	0.209640
betas	0.007792	7.328337	0.000000

The intercept are close to zero which is good, yet the explanatory power of these regressions is quite weak. We could wonder if these rather limited R^2 are not caused by the non-normality of the distributions of returns. Indeed the CAPM relies on mean-variance efficient sets where co-dependencies of order higher than two do not matter. In the context of non-Gaussian distributions and agents expressing their preferences on more than two moments, it is probable that distributional parameters other than the first two moments have an impact on the relations on the market.

3.2. **Non Gaussian Explanations.**

3.2.1. *The individual assets regression.* The first step towards determining if the non-Gaussian characteristics of the returns have an influence on the empirical performances we have obtained is to test for a relation between the R^2 of the initial regressions descriptive statistics of the distributions that escape the mean–variance framework. One of the main characteristics of asset returns is leptokurtosis. It could be tempting to test for the influence of kurtosis on the regressions. Yet, even in a completely general setting, it is logical that the statistics of importance is not the kurtosis or the skewness themselves but rather the cokurtosis or coskewness.

A problem arises when trying to determine the coskewness and cokurtosis as their are not unique as the co–variance. Indeed, the skewness (respectively the kurtosis) of a sum of random variables involves 2 by 2 and 3 by 3 dependencies (respectively 2 by 2 and 3 by 3 and 4 by 4). However, there exists an expression of the coskewness or cokurtosis of a random variable with respect to an other one.

We formulate a coskewness between returns on asset i and on portfolio p as:

$$c\xi_{i,p} = \frac{E[(R_i - E(R_i)) \times (R_p - E(R_p))^2]}{\sigma_i \sigma_p^2},$$

It mesures the co-dependencies of order three between two of the individual assets. Similarly a cokurtosis between returns on asset i and on the portfolio, p, can be defined as:

$$c\kappa_{i,p} = \frac{E[(R_i - E(R_i))^2 \times (R_p - E(R_p))^2]}{\sigma_i^2 \sigma_p^2}.$$

It is a measure of the co-dependencies of order four between the returns of two assets. Racine (1998) presents coskewness and cokurtosis as the ability of the asset i to hedge shocks on the portfolio p returns' variance and skewness respectively. In that sense, it is important in terms of asset pricing when returns are not Gaussian, if the agents express preferences on higher order moments.

Just as variance in itself has no influence on expected returns, skewness and kurtosis should not have neither. The multimoment-CAPM literature explains that abundantly. Hence the choice of coskewness and cokurtosis. Indeed, they should not be concerned by the moments of individual assets, if there exist a co–dependency, but the higher order moments of their portfolio are of importance.

Then, we regress the R^2's of the betas regressions on these coskewness and cokurtosis. The results are again quite questionable in terms of precision of the regressions. Indeed, in the case of non–Gaussian returns and preferences for higher order moment, the multimoment CAPM literature show (e.g. Kraus and Litzenberger (1976), Jurczenko and Maillet (2002)) that there should be more than one market portfolio. In that sense, the use of a unique (and possibly not correctly identified) market portfolio in the regression weakens the results. However, it would not make sense to test the classical CAPM against multiple market portfolios. The estimation error that may reduce the explanatory power of the coskewness and cokurtosis is therefore unavoidable. Generally the results we obtain are weaker than what they could be in a perfect setting, yet the reason of this study is the absence of this perfect setting.

The results of the regression for the daily returns are shown in table 15. We can notice that the R^2 of this regression is quite correct, given the specifications of the regression and the regressors:

Quite predictably the intercept is non zero, as a part of the explanatory power of the regression comes from the covariance measured by the beta. The coskewness has almost no importance and its coefficient is non significant, probably because the skewness is not really a form of risk. It may as well be that the skewness and coskewness are not stable and in many cases might just be measurement errors. However, as the average skewness is negative and thus not desired, its coefficient is positive. The cokurtosis, on the other hand, has a more important effect and a positive coefficient, as it is a dimension of risk.

Table 15. Explanation of the R^2 of the Individual Regressions .

Ordinary Least-Squares Estimates (Daily returns)			
R-squared =	0.2155		
Rbar-squared =	0.2089		
Durbin-Watson =	1.9306		
Variable	Coefficient	t-statistic	t-probability
constant	0.161573	9.167208	0.000000
coskewness	0.009160	0.251354	0.801759
cokurtosis	0.005210	3.173505	0.001706

The results of the regression for the monthly returns are shown in table 16. They are better than with the daily sample, probably because at monthly horizons the noises tend to be reduced. The returns series are closer to stationarity at the month horizon and probably the statistical methods used for the estimation of the regressions are less flawed on this sample. This time the coefficient of the coskewness is negative as the average skewness is positive and, as such, desired by the investors. The effect of cokurtosis on the fitting of the model is really apparent, and its coefficient fully significant.

Table 16. Explanation of the R^2 of the Individual Regressions .

Ordinary Least-Squares Estimates (Monthly returns)			
R-squared =	0.3484		
Rbar-squared =	0.3428		
Durbin-Watson =	2.1704		
Variable	Coefficient	t-statistic	t-probability
constant	0.057934	3.132022	0.001955
coskewness	-0.016653	-0.345450	0.730064
cokurtosis	0.061729	6.882303	0.000000

A possible extension of these results comes from the use of semi–moments. As the coskewness adds little explanatory power, we shall concentrate on the cokurtosis. The semi-moments have never been applied to co–dependencies so their definition is still not quite precise. However, they may allow us to gain a better understanding of the situation as they may give hints about the fifth order moment. To be more precise, the definition we used for standard signed semi–co-kurtosis is:

Definition 3.1. Let ξ and ν be two real valued random variable. Then:

$$c\kappa_-(\xi,\nu) = \frac{E\left[\{(\xi - E[\xi])^-\}^2 \cdot \{(\nu - E[\nu])^-\}^2\right]}{\sigma_\xi^2\sigma_\nu^2} \tag{3.2}$$

and

$$c\kappa_+(\xi,\nu) = \frac{E\left[\{(\xi - E[\xi])^+\}^2 \cdot \{(\nu - E[\nu])^+\}^2\right]}{\sigma_\xi^2\sigma_\nu^2} \tag{3.3}$$

are called the left and right signed standard semi cokurtosis of ξ and ν, respectively.

This allows us to separate the fourth order dependency between a down and a up components. The left semi cokurtosis is a representation of tail dependency between two assets when returns are inferior to their average value. This is probably very important for investors. On the other hand the right semi cokurtosis indicates the same thing when returns are above their mean value. Again an important indicator for any investor: will a positive shock on an asset benefit my portfolio?

As the semi cokurtosis have different meaning than the cokurtosis itself, we decided to conduct our regression using these two statistics as additional regressors. We hopped that the the fact that the cokurtosis is mainly composed of left cokurtosis, as the tail dependency seems to be strongest when returns are inferior to their mean. This conforts the idea that in crisis the correlations increase. The results, presented in table 17 for the daily sample and in table 18 for the monthly sample, were very good yet quite surprising in terms of coefficient signs...

TABLE 17. Explanation of the R^2 of the Individual Regressions (Daily returns, with semi cokurtosis).

Ordinary Least-Squares Estimates			
R-squared =	0.5134		
Rbar-squared =	0.5050		
Durbin-Watson =	1.9404		
Variable	Coefficient	t-statistic	t-probability
constant	0.151300	6.265312	0.000000
coskewness	-0.074578	-1.864950	0.063440
cokurtosis	-0.192896	-11.02743	0.000000
left cokurtosis	0.191111	10.956716	0.000000
right cokurtosis	0.229440	11.967020	0.000000

The explanatory power increases drastically with the inclusion of the semi cokurtosis, reaching an interesting 51.3% for the daily returns and a very strong 71.2% in the case of monthly returns. Again the difference is probably due to the fact that the monthly returns are slightly less noisy and closer to stationarity. However interesting these results are, we should notice that the signs of the coefficients are quite surprising. The coskewness and cokurtosis are negative is both cases, while the average sign of skewness and coskewness changes between the two samples. On the other hand both semi cokurtosis have positive coefficients. In terms of preferences analysis they should have opposite signs. Indeed, the agents want to be hedged against negative extreme variations and not against positive extreme variations.

Another very interesting point on these results is that the coefficients are almost all statistically significant. The constant looses its significativeness for the monthly sample, while the coskewness on the other hand regains its significativeness. However, one point may help understand the results. The coefficients of the cokurtosis and the left semi cokurtosis are extremely close in absolute value, on all regressions performed (with a different set of regressors, on different sub-samples etc.)...

TABLE 18. Explanation of the R^2 of the Individual Regressions (Monthly returns, with semi cokurtosis).

Ordinary Least-Squares Estimates			
R-squared	= 0.7120		
Rbar-squared	= 0.7071		
Durbin-Watson	= 2.0185		
Variable	Coefficient	t-statistic	t-probability
constant	0.048424	2.057621	0.040735
coskewness	-0.247665	-4.692194	0.000005
cokurtosis	-0.468004	-11.039966	0.000000
left cokurtosis	0.474712	10.607139	0.000000
right cokurtosis	0.647340	16.407777	0.000000

3.2.2. *The general regression.* The single regression of the vector of expected returns against the estimated betas is much more problematic than the individual assets regression already tested. It is supposed to hold with anticipations and not passed returns. In this regression betas are considered as intertemporal constants. Yet, the simple fact that returns series are not weakly stationary implies the betas are probably time dependant. These factors, added to the fact that the regression performed here is cross sectional and has not as many inputs as desired, will probably give weak results.

To try to test this regression for a relation between its bad performances and the non-Gaussian statistics of the assets we must forget about the beta and concentrate on the size of the residuals. Therefore, we tested for a relation between the size of the residual attached to each asset and the coskewness and cokurtosis of this asset. As for the individual assets regression, the results obtained with simple skewness or kurtosis are far less conclusive and, as such, omitted here. Moreover, the use of skewness and kurtosis is theoretically unjustified.

The results of the regression for the daily returns are shown in table 19. Surprisingly enough the coskewness seems to be more significant than the cokurtosis. However, the level of the R^2 is quite low and the t-statistics not extremely good.

TABLE 19. Explanation of the residuals of the General Regression (Daily Returns).

Ordinary Least-Squares Estimates			
R-squared	= 0.1048		
Rbar-squared	= 0.0972		
Durbin-Watson	= 2.1422		
Variable	Coefficient	t-statistic	t-probability
constant	0.000317	4.848879	0.000002
coskewness	0.000516	3.817194	0.000173
cokurtosis	0.000014	2.233690	0.026441

Even if at a five percent confidence level we can reject the hypothesis that the coefficients are zero, the t-statistics are not extremely convincing. Indeed, the coefficient for the cokurtosis is very weak. Probably the limited amount of inputs in the initial regression makes it rather noisy and limits the explanatory power of the non-Gaussian statistics. Moreover, local mispricings may induce a part of these residuals. The most surprising part of the results is that the only coefficient almost certainly different from zero is the intercept, while residuals are supposed to be distributed asymptotically as a Gaussian with mean zero and variance one. Probably this results comes from the fact that the initial regression is quite weak and that obviously the model tested here is incomplete.

The results are even less convincing with the monthly returns sample (presented in table 20, which is not surprising since the original regression was already very noisy. This time the coefficient for the cokurtosis is negative, which is surprising, yet this result is not really significant as their is a high probability that this coefficient be zero. Again, the only coefficient very significantly distinct from zero is the intercept.

TABLE 20. Explanation of the Residuals of the General Regression (Monthly Returns).

Ordinary Least-Squares Estimates			
R-squared =	0.0760		
Rbar-squared =	0.0682		
Durbin-Watson =	2.0318		
Variable	Coefficient	t-statistic	t-probability
constant	0.005139	4.150571	0.000046
coskewness	0.006838	2.119020	0.035134
cokurtosis	-0.000496	-0.825881	0.409706

Even if the results for the monthly sample are less significant than the results for the daily sample, they exhibit the same sort of relation, with a cokurtosis of small significance and a coskewness surprisingly more certainly different from zero. The results of both the monthly and daily samples suggest that the model tested by our regressions is incomplete. Considering the impact the introduction of semi cokurtosis had on the individual regressions explanation, we use these same parameters again.

TABLE 21. Explanation of the Residuals of the General Regression (Daily Returns with Semi–Cokurtosis).

Ordinary Least-Squares Estimates			
R-squared =	0.1325		
Rbar-squared =	0.1176		
Durbin-Watson =	2.1578		
Variable	Coefficient	t-statistic	t-probability
constant	0.000155	1.387180	0.166707
coskewness	0.000350	1.887703	0.060303
cokurtosis	0.000215	2.646739	0.008679
left cokurtosis	-0.000209	-2.580218	0.010485
right cokurtosis	-0.000177	-1.988114	0.047964

The use of left and right semi-cokurtosis has on the general regression a less impressive impact than on the beta determining regression. Yet, the results improve slightly in terms of explanatory power. These results are exposed in table 21. They present some more convincing t-statistics, especially in terms of the intercept which loses its significance. Co-skewness, exactly like in the individual regressions, is no longer significantly different from zero. On the other hand co-kurtosis regains its significance, and its coefficient is positive as expected for a factor of mispricing.

More surprising is the fact that semi co-kurtosis have both a negative coefficient. We have detected earlier that they were a sign of ineffectiveness of the beta estimation and therefore should coincide with a larger pricing error. Yet they seem to reduce the size of the residual attached to an asset. Quite interestingly the right co-kurtosis, corresponding to growing markets, is not highly significant, in any case less significant than the left co-kurtosis. This correspond to the intuition that risk is considered by agents mainly on the downside.

Table 22. Explanation of the Residuals of the General Regression (Monthly Returns with Semi–Cokurtosis).

Ordinary Least-Squares Estimates			
R-squared =	0.0973		
Rbar-squared =	0.0818		
Durbin-Watson =	2.0359		
Variable	Coefficient	t-statistic	t-probability
constant	0.001452	0.610690	0.541998
coskewness	0.001187	0.222564	0.824069
cokurtosis	0.009289	2.169350	0.031063
left cokurtosis	-0.010609	-2.346785	0.019771
right cokurtosis	-0.007666	-1.923642	0.055613

The results for the monthly sample, presented in table 22, are quite similar to those obtained on the daily sample (in both cases the R^2 increases by approximately 30%). Nevertheless, one point is to be noticed: this time the intercept and the co-skewness coefficients are close to be almost certainly zero. All other t-statistics are affected as well, probably because the original regression is of lesser quality. The semi cokurtosis still have these curious negative coefficients, and this time the right cokurtosis can be considered zero at a 5% significance level, reinforcing our remark on the attitude of agents towards risk being different on the downside from what it is on the upside.

4. Concluding Remarks

We have confirmed that the returns' distributions are far from Gaussian, more so at shorter time horizons. As the returns time series seems to be more non-stationary at the daily horizon than at the monthly horizon, it is probable that the non-normality stems from a non-stationary process. The main characteristic of non-normality for the marginals is their leptokurtic nature. There is a higher order co-dependency between the assets. This proves that, even if the marginals may sometime appear as close to Gaussians, the joint distribution is not a multivariate normal.

We have shown that the non-normality of returns has a direct influence on the tests of the classical CAPM. The results are more significant for the tests of market linearity. The significance improving drastically with the inclusion of the semi cokurtosis imply that this parameter, which is yet to be included in any asset pricing or portfolio selection theory, has an important role to play.

All of the results obtained in this work indicate that at least one of the non respected assumptions of the CAPM has an important empirical impact. Yet, not all of the results are intuitive and some may appear contradictory with the finding of theoretical works on multimoment CAPMs. The recent reemergence of this area of finance will probably soon provide us with the answers to the question of the relative importance of different co–moments and semi co–moments in asset pricing and of why the impact is more important in individual relations than on the general relations. Moreover, we have highlighted the possible relation between the non–stationarity of returns series and their non–normality, hence it seems important that some future work enquires more in that direction.

References

1. A.L. Alles and J.L. Kling, *Regularities in the variation of skewness in asset returns*, Journal of Financial Reaserch **17** (1994), 427–438.
2. Y. Amihud, B.J. Christensen, and H. Mendelson, *Further evidence on the risk-return relationship*, Working Paper, New York University, 1992.
3. T. Ane and C. Labidi, *Revisitng the finite mixture of gaussian distributions with application to futures market*, Journal of Future Markets **21** (2001), no. 4, 347–376.
4. F. Black, *Capital market equilibrium with restricted borrowing*, Journal of Business **44** (1972), 444–455.
5. P. Bossaerts, C. Plott, and W. R. Zame, *Prices and portfolio choices in financial markets: Theory and experimental evidence*, presented at NBER Conference Seminar on General Equilibrium Theory, May 10-12, 2002.
6. J. Campbell, A. Lo, and C. MacKinlay, *The Econometrics of Financial Markets*, Princeton University Press, Princeton, 1997.
7. L. Chan, Y. Hamao, and J. Lakonishok, *Fundamentals and stock returns in japan*, Journal of Finance **46** (1991), 1739–1764.
8. H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, 1946.
9. E. Elton and M. Gruber, *Modern portfolio theory and investment analysis*, 5th ed., Princeton University Press, Princeton, 1995.
10. E. Fama and K. French, *The cross-section of expected stock returns*, Journal of Finance **47** (1992), 427–465.
11. ———, *Size and book to market factors in earnings and returns*, Journal of Finance **50** (1995), 131–155.
12. ———, *Multifactor explanations of asset pricing anomalies*, Journal of Finance **51** (1996), 55–84.
13. I. Friend and R. Westerfield, *Co-skewness and capital asset pricing*, Journal of Finance **35** (1980), 897–913.
14. C.R. Harvey and A. Siddique, *Conditional skewness in asset pricing tests*, Journal of Finance **55** (2000), 1263–1295.
15. E. Jurczenko and B. Maillet, *The four moment capital asset pricing model: Some basic results*, Procedings of the Multimoment CAPM and Related topics Conference, Finance-sur-Seine, Paris, 2002.
16. M. Kendall and A. Stuart, *The Advanced Theory of Statistics, Vol. 1: Distribution Theory*, 4th ed., MacMillan, New York, 1977.
17. A. Kraus and R. Litzenberger, *Skewness preference and the valuation of risk assets*, Journal of Finance **31** (1976), 1085–1100.

18. J. Lintner, *The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets*, Review of Economics and Statistics **47** (1965), 13–37.
19. H. Markowitz, *Portfolio selection*, Journal of Finance **7** (1952), 77–91.
20. R.C. Merton, *An analytic derivation of the efficient portolio frontier*, Journal of Financial and Quantitative Analysis **7** (1973), 1851–1872.
21. A. Peiró, *Skewness in financial returns*, Journal of Banking and Finance **23** (1999), 847–862.
22. ______, *Skewness in individual stocks at different investment horizons*, Procedings of the Multimoment CAPM and Related topics Conference, Finance-sur-Seine, Paris, 2002.
23. J. Pratt, *Risk aversion in the small and in the large*, Econometrica **32** (1964), 122–136.
24. M. Racine, *Asset valuation and coskewness in canada*, Working Paper Series 9802, University of Wilfrid Laurier, 1992.
25. R. Roll, *A critique of the asset pricing theory's tests*, Journal of Financial Economics **4** (1977), 129–176.
26. ______, *Ambiguity when performance is measured by the securities market line*, Journal of Finance **33** (1978), 1051–1069.
27. S. Ross, *The arbitrage theory of capital asset pricing*, Journal of Economic Theory **13** (1976), 341–360.
28. ______, *The capital asset pricing model, short sales restriction and related issues*, Journal of Finance **32** (1977), 177–183.
29. R. Scott and P. Horvath, *On the direction of preferences for moments of higher order than the variance*, Journal of Finance **35** (1980), 915–919.
30. W. Sharpe, *Capital asset prices: a theory of market equilibrium under conditions of risk*, Journal of Finance **19** (1964), 425–442.
31. G.P. Szegö, *Nuovi risultati analitici nella teoria della selezione del portafoglio*, (1975), (C.N.R., Comitato Scienze Economiche, Sociologiche e Statistiche).
32. L. Tibiletti, *Higher order moments and beyond*, Procedings of the Multimoment CAPM and Related topics Conference, Finance-sur-Seine, Paris, 2002.

PhD. Student, CEREG, Université de Paris Dauphine
E-mail address: `fd-l@caramail.com`