VALUATION OF DEFAULT SENSITIVE CLAIMS UNDER IMPERFECT INFORMATION

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Abstract

We propose an evaluation method for financial assets subject to default risk, when investors face imperfect information about the state variable triggering the default. The model we propose generalizes the one by Duffie and Lando (2001) in the following way: (i) it incorporates informational noise in continuous time, (ii) it respects the (H) hypothesis, (iii) it precludes arbitrage from insiders. The model is sufficiently general to encompass a large class of structural models. In this setting we show that the default time is totally inaccessible in the market's filtration and derive the martingale hazard process. Finally, we provide pricing formulas for default-sensitive claims and illustrate with particular examples the shapes of the credit spreads and the conditional default probabilities. An important feature of the conditional default probabilities is they are non Markovian. This might shed some light on observed phenomena such as the "rating momentum".

1 Introduction

Explaining the components of credit risk reflected in corporate bond yield spreads is certainly one of the most important questions in credit risk modeling. Since the seminal work of Merton (1974) that pioneered the "structural" representation in credit risk modeling, researchers have attempted to explain the size of the credit spread, without full success yet. The structural models relate the default event to a fundamental indicator of the financial health, usually the ratio between the total balance-sheet and the total debt outstanding. Then, option pricing theory is employed in order to price debt and derive spreads. In Merton's (1974) model, default may only occur at maturity of the debt claims. Black and Cox (1976) extended the model to allow default occurrence at a random time, in a first passage time model. But both models fail to produce spreads consistent with empirical observations: they predict lower spreads that decrease to zero for short maturities as documented by Jones, Manson and Rosenfeld (1984). This drawback explained by the fact that the default event is a predictable stopping time hence short-term default risk is not priced by the models; nevertheless in the real world investors price the risk of unexpected defaults. A second important drawback attached to this type of models is that the firm's value process is difficult to estimate precisely since investors face incomplete information regarding the firm's assets.

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The efforts deployed to address the deviations between observed and predicted credit spreads were essentially organized in two directions[1]. On the one hand, some models continued to improve structural modeling, keeping a first passage time definition of default but introducing new sources of risk in the analysis, in order to "boost" the spreads. Thus, interest rate risk was incorporated by Kim, Ramaswamy and Sundaresan (1993), Shimko, Tejima and Van Deventer(1993), Longstaff and Schwartz (1995); Zhou (1997) introduced jumps in assets value process and thus a positive probability of a downward drop even in the short term. The seminal paper by Leland (1994) introduces the notion of endogenous default, corresponding to a default barrier chosen to maximize equity value; hence higher spreads may be explained by the default not being an optimal choice for debtholders. Mella-Baral and Perraudin (1997) propose elaborate risk of recovery assuming that renegotiations in case of default may be expropriate debtholders of a part of their stake. All these models aimed at explaining default in a more elaborate way than Merton's model, but except for Zhou's model, the default event remained a predictable stopping time and short term spreads too low compared to the observed ones.

On the other hand, a new class of models appeared whose first goal was to fit the spreads; in this perspective, the primary focus was not the economic meaning of default. This approach, known as reduced-form modeling (or intensity approach) and was studied by Jarrow and Turnbull (1992, 1995), Lando (1998), Duffie and Singleton (1999), Eliott et al. (2000) among others. Since in the real world default often occurs as an unexpected event to the market, it is argued that the best way of modeling its arrival is to use the first jump of a Poisson process (or, more generally a Cox process), which is not necessarily adapted to the initial filtration of the non-defaultable assets. Thus, the default time is a totally inaccessible stopping time, quite appropriate to the situations where default comes as a "total surprise" to the market. Also, the parameter of the jump process, its intensity, may be directly calibrated to observed spreads, since it does not have to be constrained to fit any other economic fundamental. Reduced form modeling enjoyed an immediate success, among practitioners. This is due not only to the ability of the models to better fit observed short term spreads but also to the simplicity of the closed form formulas obtained, quite similar to those established for default-free bonds, leading in turn to the pricing of even complex instruments as credit derivatives.

However, despite their attractiveness, reduced-form models did not improve in any manner the economic understanding of the credit spread. In addition, many questions were left open: once the spreads calibrated, what is the predictive power of such models? Do the implied default probabilities correspond to historical ones? For instance, using a reduced-form model with standard credit risk premium adjustments, Jarrow, Lando and Yu (2001) find that bond-implied conditional default probabilities are in line with historical estimates for long maturities, but are too high at short maturities.

As of today, one may say that if the two approaches structural versus reduced-form may be considered as competing, empirical studies seem to indicate that in both cases a full understanding of the credit spread is far from being complete. In particular, is quite difficult to capture the behavior of the credit spread at short maturities.

Some light may arise from an emerging class of models which aims at bridging the gap between the two approaches by introducing incomplete information versions of standard structural models. For instance Duffie and Lando (2001) -hereafter DL(2001)-, Giesecke and Goldberg(2003), Cetin et al. (2004), Jeanblanc and Valchev (2005) or Guo et al. (2005) have proposed reduced-form models in which the intensity of default is determined endogenously as a function of the firm's

[1] Other reactions, not directly related to credit risk modeling, consisted in trying to identify other elements than default risk that are priced in the corporate credit spread, such as liquidity, tax or market factors. These approaches are very important because they clarify what exactly is the part of the spread that a credit model is expected to justify. See for instance Elton et al (2001), Delianedis and Geske (2001) or Huang and Huang (2003).

characteristics (like in structural models) and the level of information available in the credit risk market.

These models carry the strength of both structural and reduced form models while avoiding some important shortcomings. Thus, they provide an economic explanation of default events as structural models do, while recognizing the fact that market participants rely on imperfect information: the process driving the default event is unobservable, hence the distance to default is also uncertain. This impacts the short term spreads, because the imperfect information exacerbates investors' uncertainty as to when the default will be triggered: default becomes indeed a non -predictable event. As such, credit spreads will reflect the "imperfect information" risk premium in addition to the "structural" credit risk premium[2]. Hence, the models can account for short-term uncertainty inherent to the credit market and predict higher spreads than the original structural ones for short maturities. Furthermore, as in reduced-form models, a martingale hazard process may be characterized, hence tractable formulas from reduced form modeling may be used.

The first hybrid model based on incomplete information was proposed by DL(2001), who suppose that the market observes at discrete time intervals the firm value plus a noise. They use a classical structural model where investors do not have access to the "structural" filtration, i.e. the filtration where the default occurs as predicted by a structural model. Instead, they observe accounting reports and are trying to infer from these the probability of default. In this paper we propose an alternative model with noisy information, with the difference that the market can observe continuously the firm value plus a noise. The economic explanation behind is along the same lines: managers are not able or not willing to communicate the exact situation of the firm via the accounting reports. In our framework, as in Duffie and Lando, the default time becomes a totally inaccessible stopping time. Our results apply to a large class of continuous diffusions representing the fundamental process triggering default, so that many existing structural models may be embedded in our hybrid model. In this general framework, we are able to obtain explicit formulas for the hazard function of default and to price bonds and more general credit-related claims.

The reminder of the paper is organized as follows: Section 3.2 presents the modeling assumptions in a general case. Section 3.3 exhibits the characterization of the martingale hazard and the other analogies with a reduced form model. In Section 3.4 prices of defaultable bonds and other default-sensitive claims are provided. Finally, Section 3.5 is dedicated to the analysis of some particular examples of credit spreads.

2 The valuation framework

In this section we develop our model of default risk under the "real" probability measure instead of "risk-adjusted" probability measure, since firm's fundamentals such as accounting indicators are observed in the "real world", as well as events affecting the whole credit and equity markets. We won't introduce a change of measure before Section 3.4.

Our goal is to explain the credit spreads by two factors, or state variables: (i) a fundamental process measuring the credit quality (for instance the firm's assets or cash flows) and (ii) the quality of the information of market participants with regard to the fundamental process. Since they are not our first focus, default-free interest rates are supposed to be deterministic.

[2]Giesecke and Goldberg (2004) refer to these two different components of credit risk premium as "diffusive risk premium" and "default event risk premium", this decomposition being more general as it does not necessarily assume imperfect information.

2.1 The structural assumptions

The "structural" assumptions of the model correspond to a complete information case. Consider an economy where corporate default risk is measured by the distance of some fundamental process to a default threshold. Typically, such a fundamental process is the total value of assets, or alternatively, total cash flows[3]. We concentrate our analysis on a firm in this economy and suppose that its fundamental process, noted $X = (X_t)_{t \geq 0}$, is the solution of a stochastic differential equation of the form:

$$\begin{aligned} dX_t &= \mu(X_t, t)\, dt + \sigma(X_t, t)\, dB_t \\ X_0 &= x_0 \end{aligned} \tag{1}$$

with B a standard Brownian motion and diffusion coefficients suitably chosen for the equation to be well defined and with a unique strong solution (for instance continuous in t and uniformly Lipschitz in x). In addition, we state the following founding hypothesis:

(A) The solution of the stochastic differential equation (1) leads to a deterministic functional relation between X_t and B_t, namely $X_t = F(B_t, t)$, and F can be inverted. Without any loss of generality, we will consider that for any t, the function $x \to F(x, t)$ is increasing.

For simplicity, we choose to work under condition **(A)**. However, via a deterministic time-change, our results apply to the class of diffusions characterized in the following:

(A') There exists a martingale of the form: $m_t = \int_0^t h(s) dB_s$ with h being a Borel function, such that the solution of the stochastic differential equation (1) satisfies a deterministic functional relation between X_t and m_t, namely $X_t = F(m_t, t)$, and F can be inverted.

In Kloeden and Platen (1995) one may find several examples of stochastic differential equations satisfying our conditions **(A)** (or **(A')**).

Let $b(t)$, $t \geq 0$ be the default threshold representing a debt-covenant violation triggering default and depending on the liability structure of the firm. We suppose that this barrier is a continuous function of time, with $b(0) > x_0$. The default event is defined as in common structural models as the first passage time of the fundamental process value through the default barrier:

$$\tau = \inf\{t, X_t = b(t)\}.$$

But firm's value structural models with a stochastic barrier may sometimes be transformed to fit into this framework. Consider V_t being the asset's value and k_t being the stochastic barrier, then we may choose for the fundamental process, for instance, the log leverage process $X_t = \ln \frac{V_t}{k_t}$ or the distance to the barrier $X_t = V_t - k_t$. In these two cases, $b(t)$ is constant (and equal to 0). But the transformation chosen has to fulfill condition **(A)**. The notation (V_t, k_t) will be kept to stand for the assets value and assets' default point whenever necessary to be distinguished from $(X_t, b(t))$.

In the structural framework, the only state variable X explaining the default risk is generally unobservable. In practice, in order to implement a structural model, investors have to first estimate the value of the fundamental process X, hence they get exposed to a second source of risk, that of their estimation.

[3] Let us observe that even if very common, this assumption is simplifying. It is known that in the real world, default decision depends on more than a single factor. A recent work of Davydenko (2005) documents the fact that default may be caused either by low liquidity or low assets value, their relative importance depending on the firm's characteristics, especially the cost of outside financing.

2.2 The imperfect information model

We now account for the information imperfection: first we assume the filtration of the fundamental process X not to be publicly available and second, there is noise in the observation.

We suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is large enough to support two correlated Brownian motions, B and B'.

We define the process $(Y_t)_{t\geq 0}$ as a noisy signal of the fundamental value X and representing a publicly available information - thereafter Y will be called the *observation process*. Y could represent the accounting reports of the firm together with all releases of public information regarding the firm's assets or firm's financial health. Sometimes, in practice, when financial analysts dispose of poor financial information of the firm, the value of a similar but more transparent firm may be used to stand for the process Y. We suppose that the process Y follows a diffusion of the type:

$$\begin{aligned} dY_t &= \mu_1(Y_t, t)\, dt + \sigma(Y_t, t)\, dB_t + s(Y_t, t)\, dB'_t && (2) \\ &= \mu_1(Y_t, t)\, dt + \sigma_1(Y_t, t)\, d\beta_t && (3) \\ Y_0 &= y_0, \end{aligned}$$

where B and B' are correlated Brownian Motions, with $\langle B, B' \rangle_t = \rho t$, $|\rho| < 1$. The process β, defined as

$$\beta_t = \int_0^t \frac{\sigma(Y_u, u)\, dB_u + s(Y_u, u)\, dB'_u}{\sigma_1(Y_u, u)},$$

with $\sigma_1(Y_t, t) = \sqrt{\sigma(Y_t, t)^2 + s(Y_t, t)^2 + 2\rho\sigma(Y_t, t)\, s(Y_t, t)}$, is a Brownian motion in the filtration generated by the pairs (B, B'), since it is a martingale with bracket t.

In equations (2) and (3), we require the functions $s(y, t)$ and $\sigma_1(y, t)$ to be strictly positive on $\mathcal{Y} \times [0, \infty)$, with $\mathcal{Y}$ being the domain of the process Y. Also, the equation (3) is supposed to have a strong solution, i.e. adapted to the filtration of β completed with respect to $\mathbb{P}$.

Remark that the drift function $\mu_1(x, t)$ is allowed to be different from $\mu(x, t)$ since it could possibly contain a premium for the supplementary risk attached to the observation process. But no particular relation is required to hold between $\mu_1(x, t)$ and $\sigma_1(x, t)$.

Let us now illustrate with an example:

Example 1 *Suppose that the fundamental process follows:*

$$dX_t = \frac{a^2}{2} X_t dt + a X_t dB_t, \quad X_0 = x$$

i.e., $X_t = x \exp\{aB_t\}$, *where* a *is a constant and that the observation process follows:*

$$\begin{aligned} dY_t &= \frac{a^2}{2} Y_t dt + a\left(Y_t dB_t + dB'_t\right) \\ &= \frac{a^2}{2} Y_t dt + a\sqrt{Y_t^2 + 1}\; d\beta_t, \quad Y_0 = 0, \end{aligned}$$

i.e., $Y_t = \sinh(a\beta_t)$, *where we considered* B *and* B' *to be independent, such that:*

$$\beta_t = \int_0^t \frac{Y_u dB_u + dB'_u}{\sqrt{Y_u^2 + 1}}.$$

Also, one may check that the following relation is holding:

$$Y_t = aX_t \int_0^t \frac{dB'_s}{X_s}. \tag{4}$$

We may notice that investors observing the process Y are also observing the process β, the function $\sinh(x)$ *being invertible. The equality (4) emphasizes the information about the process X which is contained in the observation process Y.*

2.3 Some remarks on the observation process Y

The modeling of the observation process Y is a very important point as it contains the form of the noise affecting the market perception of the firm's condition. Information quality in our model is measured by two parameters. First, the volatility parameter of the noise $s(Y_t, t)$: the higher the volatility the worse the quality of the information. Secondly, the correlation ρ between the two Brownian motions B' and B: a firm with highly intangible assets, could have $\rho > 0$, meaning that financial markets tend to over-react to releases of information, will this be good or bad news. This was the case of internet companies in the late '90s. On the opposite side, established blue chip firms could probably have $\rho < 0$, meaning that financial markets are confident in presence of bad news but do not expect the company to have an important growth in presence of good reports. This was the case of Enron.

It might be argued that such a model is difficult to implement: as the noise is by definition not observable, how could we capture its representation? In fact, for practical matters, information quality may be estimated in several ways. Yu (2003) has tested Duffie and Lando's model using the annual AIMR's Annual Reviews of Corporate Reporting Practices which provide corporate disclosure rankings as a proxy for the perceived precision of the reported firm value. An alternative and more complex ranking is provided by S&P Transparency and Disclosure. Khurana et al. (2003) use absolute values of analysts' earnings forecast error and firm's level R&D activity to capture the firm information precision. Also, for rated firms the size of reactions in bonds prices to rating change announcements may be an indicator of the noise magnitude.

2.4 The informational structure

We consider that date 0 is the last date when investors were completely informed: $X_0 = x_0$ being a constant. This date might be interpreted as the date of the firm's creation, when the market value of the firm equals the value of the funds raised. At date 0, the market-estimated probability of default can be computed as in a perfect information structural model.

We also suppose that investors know the functions $\mu(x,t)$ and $\sigma(x,t)$, but are unable to observe the true paths of the process X, that is they are only aware of the firm's profile of risk and return.

Let $\mathcal{N}$ denote the $\mathbb{P}$ null sets. We define:

$$(\mathcal{G}_t)_{t\geq 0} := \sigma(B_s, B'_s, s \leq t)_{t\geq 0} \vee \mathcal{N},$$

that is the information of an insider having both complete information of the fundamental process X and of the amount of noise affecting the market perception of this process. Note that τ is an $(\mathcal{G}_t)$-stopping time. An insider is able at any time to evaluate default probability as in a classical structural model.

Alternatively,

$$(\mathcal{F}_t)_{t\geq 0} := \sigma(\beta_s, s \leq t)_{t\geq 0} \vee \mathcal{N}$$

represents the filtration of the market-observed values with incomplete information, as the process Y is adapted to this filtration. Notice that τ is not an $(\mathcal{F}_t)$-stopping time. Also, the two filtrations $(\mathcal{G}_t)$ and $(\mathcal{F}_t)$ being generated by Brownian motions, all $(\mathcal{F}_t)$ and all $(\mathcal{G}_t)$-martingales are continuous.

We require that market investors are able to observe the process β, and the default state, so the market information filtration $(\mathcal{F}_t^\tau)_{t\geq 0}$ is such that, for every $t \geq 0$:

$$\mathcal{F}_t^\tau := \mathcal{F}_t \vee \sigma(s \wedge \tau, s \leq t).$$

$(\mathcal{F}_t^\tau)$ is the smallest filtration containing $(\mathcal{F}_t)$ and making τ a stopping time. Our definition of the market filtration implies that the values of all *traded* securities of the firm capital structure (bonds and equities) reflect the same average level of information, so that the different classes of firm's investors may be considered as uniformly informed; in particular, no informational asymmetry exists in average between bondholders and shareholders, if bonds and equities are traded. This condition is not in contradiction with the existence of some amount of insider trading, as long as this trading does not impact prices[4].

In short, we have constructed three different nested filtrations:

$$\mathcal{F}_t \subset \mathcal{F}_t^\tau \subset \mathcal{G}_t$$

for $t \geq 0$. In addition, the default time satisfies the following result:

Lemma 2 *The default time τ is a $(\mathcal{G}_t)$-predictable stopping time, for insiders; it is an $(\mathcal{F}_t^\tau)$-totally inaccessible stopping time, for ordinary market investors and is not an $(\mathcal{F}_t)$-stopping time.*

Proof. The proof uses some results not yet introduced, hence is postponed after the proof of the Proposition 5 below. ■

The filtrations $(\mathcal{F}_t^\tau)$ and $(\mathcal{F}_t)$ are of current use in reduced form modeling, and we dispose of mathematical tools which enable to make projections from one filtration to another. Moreover, relative to the first filtration $(\mathcal{G}_t)$ our model is a classical structural model with complete information, so here again, a number of results are available for pricing and computing the default probability, for some types of diffusion processes. One link is missing in the chain: projection formulas from filtration $(\mathcal{G}_t)$ to filtration $(\mathcal{F}_t)$ have to be established.

3 The "(H)-hypothesis" and the martingale hazard process

Because the default time τ is a totally inaccessible stopping time in the market filtration $(\mathcal{F}_t^\tau)$, the valuation issues could be addressed with similar tools as in reduced-form models of default. The object of this section is to adapt to our specific framework two important concepts from the reduced-form theory: the martingale hazard process and the so-called **(H)**-hypothesis.

Let us recall that the $(\mathcal{F}_t)$-martingale hazard process is defined as the continuous, increasing and predictable process Λ, such that $\Lambda_0 = 0$, and that the process $\mathbf{1}_{(\tau \leq t)} - \Lambda_{t\wedge\tau}$ is an $(\mathcal{F}_t^\tau)$-martingale.

The **(H)**-hypothesis states that all $(\mathcal{F}_t)$ square-integrable martingales remain square-integrable martingales in the enlarged filtration $(\mathcal{F}_t^\tau)$. This hypothesis is essential for pricing and has also important mathematical consequences, which were first studied by Brémaud and Yor (1978) and Mazziotto and Szpirglas (1979).

To understand these consequences, let us introduce the $(\mathcal{F}_t)$-conditional survival probability

$$Z_t^\tau := \mathbb{P}\left(\tau > t | \mathcal{F}_t\right). \tag{5}$$

[4] In fact, our model does not exclude possible insider trading of some small players and it will be shown later that prices generated by this model exclude arbitrage opportunities for this kind of insiders.

The process Z_t^τ is a supermartingale (called Azéma's supermartingale) which admits the Doob-Meyer decomposition:

$$Z_t^\tau = m_t^\tau - a_t^\tau \tag{6}$$

where a_t^τ is the $(\mathcal{F}_t)$-dual predictable projection of the process $\mathbf{1}_{(\tau \leq t)}$.

For the **(H)**-hypothesis to hold, a necessary -but not sufficient- condition is Z^τ to be a decreasing process. In addition, when $(\mathcal{F}_t)$ is the Brownian filtration, the process Z^τ is predictable hence the martingale part of the decomposition (6) is constant: $m_t^\tau \equiv 1$.

Also, the **(H)**-hypothesis has an equivalent formulation[5]:

$$\mathbb{P}\left(\tau \leq t | \mathcal{F}_t\right) = \mathbb{P}\left(\tau \leq t | \mathcal{F}_\infty\right)$$

which will reveal useful later on.

Finally, note that under the (H)-hypothesis the computation of the Azéma's supermartingale suffices for obtaining the value of the martingale hazard process. In the Brownian filtration, they are linked via the formula: $\Lambda_t = -\ln Z_t^\tau$.

Hence, it seems natural to begin by checking for the validity of the **(H)**-hypothesis in our framework and then find an estimate for the martingale hazard process.

The following proposition shows that not only is the **(H)**-hypothesis satisfied, but also the martingale property of prices is preserved in the larger filtration $(\mathcal{G}_t)$, meaning that conditional on an insider's knowledge, the discounted prices of the default-free claims remain martingales under a risk neutral measure. This ensures that the no-arbitrage condition holds even for insiders, who are observing at any time the true value of the fundamental process X.

Proposition 3 *The following martingale properties hold:*

*(i) All $(\mathcal{F}_t)$-square integrable martingales are $(\mathcal{F}_t^\tau)$-square integrable martingales, i.e., the **(H)**-hypothesis is satisfied.*

(ii) If M_t is an $(\mathcal{F}_t)$-local martingale, then M_t is also a $(\mathcal{G}_t)$-local martingale and $M_{t\wedge\tau}$is an $(\mathcal{F}_t^\tau)$-martingale.

Proof. We use the representation theorem of martingales in Brownian filtrations as integrals with respect to Brownian Motion: if M_t is a $(\mathcal{F}_t)$-local martingale, there exists an $(\mathcal{F}_t)$-predictable process h such that $M_t = \int_0^t h_u d\beta_u$. Since the process β is also a $(\mathcal{G}_t)$-Brownian motion, M is a $(\mathcal{G}_t)$-local martingale. It follows that the stopped process $(M_{t\wedge\tau})_{t\geq 0}$ is a $(\mathcal{G}_t)$-martingale, and satisfies for $T > t$:

$$M_{t\wedge\tau} = \mathbb{E}\left[M_{T\wedge\tau} | \mathcal{G}_t\right].$$

Taking expectation with respect to the filtration $(\mathcal{F}_t^\tau)$ we obtain:

$$M_{t\wedge\tau} = \mathbb{E}\left[M_{T\wedge\tau} | \mathcal{F}_t^\tau\right],$$

meaning that indeed $M_{t\wedge\tau}$ is an $(\mathcal{F}_t^\tau)$ martingale. Point (i) follows when applying the optional sampling theorem to the bounded martingale M ■

From the above proposition, we also know that the process Z^τ defined in (5) is decreasing since this is a necessary condition for the **(H)**-hypothesis to hold; the next step is to make it explicit.

[5]See for instance Dellacherie and Meyer (1978) for other equivalent formulations with proofs and for a summary of the results with financial interpretations, see Jeanblanc and Rutkowski (2000).

For now, let us introduce some other useful processes. First, the following $(\mathcal{G}_t)$-martingale:

$$D_t = \int_0^t \frac{\eta(Y_u, u)\, dB_u - s(Y_u, u)\, dB'_u}{\sigma_1(Y_u, u)} \tag{7}$$

with:

$$\eta(y,t) = s(y,t)\, \frac{\rho\sigma(y,t) + s(y,t)}{\sigma(y,t) + \rho s(y,t)}.$$

It can be checked that $d\langle \beta, D\rangle_t = 0$, meaning that D and β are orthogonal. As a consequence the $(\mathcal{G}_t)$-martingale D is not $(\mathcal{F}_t)$-adapted; we introduce $(\mathcal{D}_t)_{t\geq 0} = \sigma(D_u, u \leq t)_{t\geq 0}$.

Let us now construct the following two orthogonal $(\mathcal{G}_t)$-martingales: for $t \geq 0$

$$M_t = \int_0^t \frac{\sigma_1(Y_u, u)}{\sigma(Y_u, u) + \eta(Y_u, u)} d\beta_u \tag{8}$$

$$N_t = \int_0^t \frac{\sigma_1(Y_u, u)}{\sigma(Y_u, u) + \eta(Y_u, u)} dD_u. \tag{9}$$

Notice that M is also an $(\mathcal{F}_t)$-martingale. We remark that

$$B_t = M_t + N_t$$

and that:

$$B'_t = \int_0^t \frac{\eta(Y_u, u)}{s(Y_u, u)} dM_u - \int_0^t \frac{\sigma(Y_u, u)}{s(Y_u, u)} dN_u.$$

We deduce that:

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t,\ t \geq 0.$$

Our aim is to give an approximation scheme for $\mathbb{P}(\tau \leq t | \mathcal{F}_t) = 1 - Z_t^\tau$. In order to establish this result, we need a preliminary technical and important lemma:

Lemma 4 *Conditionally on $\mathcal{F}_\infty$, the process $(N_t)_{t\geq 0}$ is a Gaussian martingale.*

Proof. Let $\langle D\rangle_t = \int_0^t \delta(Y_u, u)\, du$ be the quadratic variation of the process D defined in (7), with $\delta(Y_u, u) = \frac{\eta(Y_u,u)^2 + s(Y_u,u)^2 - 2\rho\eta(Y_u,u)s(Y_u,u)}{\sigma_1(Y_u,u)^2} > 0$. It follows that the process:

$$W_t^D = \int_0^t \frac{dD_u}{\sqrt{\delta(Y_u, u)}}$$

is a $(\mathcal{G}_t)$-Brownian motion, according to Lévy's characterization theorem. Moreover, since D and β are orthogonal, we have:

$$\left\langle W^D, \beta \right\rangle_t = 0,$$

for any t, meaning that the two $(\mathcal{G}_t)$-Brownian motions W^D and β are independent. As a consequence, we also have the property:

$$\text{The processes } W^D \text{ and } Y \text{ are independent,}$$

since by hypothesis the filtration generated by the process Y is contained in the filtration generated by the process β. We now emphasize the fact that the process N can be written as:

$$N_t = \int_0^t f(Y_u, u) dW_u^D.$$

with $f(Y_u, u) = \frac{\sigma_1(Y_u,u)\sqrt{\delta(Y_u,u)}}{\sigma(Y_u,u)+\eta(Y_u,u)}$. Using the independence of the processes W^D and Y, we find that conditionally to $\mathcal{F}_\infty$, N_t is a Wiener integral.

Note that the result may also be obtained using the theorem of Knight on the representation of two continuous orthogonal martingales as independent Brownian motions time changed with their respective increasing processes. ∎

We are now able to enounce one of our main results, recalling that we are working under assumption **(A)**, which was defined in section 2.1.

Proposition 5 *Under the assumption **(A)**, the $(\mathcal{F}_t)$-conditional default probability at time t is given by the following formula:*

$$\mathbb{P}(\tau \leq t | \mathcal{F}_t) = \lim_{\Delta t \to 0} \sum_{i=1}^{k} q_i \tag{10}$$

where $k = t/\Delta t$, and for $i = 2, ..., k$ and for $j = 1, ..., i-1$, we have :

$$\begin{aligned} q_1 &= \Phi(a_1) \\ q_i &= \Phi(a_i) - \sum_{j=1}^{i-1} \Phi(b_{i,j}) q_j \\ a_i &= \frac{c_{i\Delta t}}{\sqrt{\langle N \rangle_{i\Delta t}}} \\ b_{i,j} &= \frac{c_{i\Delta t} - c_{j\Delta t}}{\sqrt{\langle N \rangle_{i\Delta t} - \langle N \rangle_{j\Delta t}}} \end{aligned}$$

with:

$$c_t = F^{-1}(b(t), t) - M_t.$$

Φ stands for the cumulative function of the standard normal law.

Proof. First remark that the right hand side in formula (10) is indeed $(\mathcal{F}_t)$-measurable as the processes M and the quadratic variation $\langle N \rangle$ are $(\mathcal{F}_t)$-adapted. Indeed, $\langle N \rangle_t = \int_0^t \frac{\eta(Y_u,u)^2 + s(Y_u,u)^2 - 2\rho\eta(Y_u,u)s(Y_u,u)}{(\sigma(Y_u,u)+\eta(Y_u,u))^2} du$ is $(\mathcal{F}_t)$-adapted, even if the process N is obviously not.

Now, we write:

$$\begin{aligned} \mathbb{P}(\tau \leq t | \mathcal{F}_t) &= \mathbb{P}(\exists u \in [0,t], F(B_u, u) < b_u | \mathcal{F}_t) \\ &= \mathbb{P}\left(\exists u \in [0,t], B_u < F^{-1}(b_u, u) | \mathcal{F}_t\right) \\ &= \mathbb{P}\left(\exists u \in [0,t], N_u < F^{-1}(b_u, u) - M_u | \mathcal{F}_t\right) \end{aligned}$$

In addition, the **(H)** hypothesis implies that:

$$p(t) := \mathbb{P}\left(\tau \leq t | \mathcal{F}_t\right) = \mathbb{P}\left(\tau \leq t | \mathcal{F}_\infty\right)$$

or equivalently:

$$\mathbb{P}\left(\exists u \in [0,t]\,, N_u < c_u | \mathcal{F}_t\right) = \mathbb{P}\left(\exists u \in [0,t]\,, N_u < c_u | \mathcal{F}_\infty\right),$$

with

$$c_u := F^{-1}(b\left(u\right), u) - M_u.$$

We remark that $p(u)$ is the distribution function of the first-passage time of a Gaussian martingale (the process N conditional to $\mathcal{F}_\infty$) through a deterministic barrier, representing an observed path of the process $(c_t)_{t\geq 0}$.

Because conditionally to $\mathcal{F}_\infty$ the process N may be seen as a frozen time-change of a Brownian motion, we will use first passage time formulas of a Brownian motion through a deterministic barrier, that we recall now[6].

Let $(W_t)_{t\geq 0}$ be a Brownian motion and $h(t)$ a continuous function with $h\left(0\right) < 0$. We introduce the following hitting time:

$$T_h = \inf\left\{t : W_t \leq h(t)\right\}$$

and define the following functions:

$$\begin{aligned} \pi^h(t) &:= \mathbb{P}(T_h \leq t) \\ f\left(t,x\right) &:= \mathbb{P}\left(W_t \leq x\right) = \Phi\left(x/\sqrt{t}\right) \qquad t \geq 0 \\ g\left(t,x,u\right) &:= \mathbb{P}\left(W_t \leq x | T_h = u\right) \qquad u,t \geq 0 \end{aligned}$$

where Φ stands for the cumulative function of the standard normal law. Due to the strong Markov property of the Brownian motion, we have for $u < t$:

$$g\left(t,x,u\right) = f\left(t-u, x-h(u)\right) = \Phi\left(\frac{x-h(u)}{\sqrt{t-u}}\right). \tag{11}$$

Also, for $x < h(t)$ the distribution of the hitting time T_h satisfies the following integral equation (due to Fortet, 1943):

$$\begin{aligned} f\left(t,x\right) &= \int_0^t g\left(t,x,u\right) d\pi^h\left(u\right) \\ &= \int_0^t f\left(t-u, x-h(u)\right) d\pi^h\left(u\right), \end{aligned} \tag{12}$$

which is a Volterra equation of the first type. Now, we define the increasing process $\left(\varphi\left(t\right)\right)_{t\geq 0}$ by:

$$\varphi\left(t\right) = \inf\left\{u \geq 0, \langle N\rangle_u > t\right\}$$

and remark that $\varphi\left(t\right)$ is an $\left(\mathcal{F}_{\varphi(t)}\right)$-stopping time. Also, we set

$$h(t) = c_{\varphi(t)},$$

or, equivalently,

$$h(\langle N\rangle_t) = c_t.$$

[6] For a more developed treatment of the subject, see Fortet (1943), Buonocore et al. (1987), and, more recently, Peskir (2002).

From the Lemma 4 and Knight theorem, we know that there exist a $(\mathcal{G}_t)$-Brownian motion W, independent from β and such that $N_t = W_{\langle N \rangle_t}$, $t \geq 0$. Let $\langle N \rangle_t = l$, which is $\mathcal{F}_\infty$ measurable. We obtain:

$$\begin{aligned}\tau &= \inf\{t : N_t \leq c_t\} = \inf\{\varphi(l) : W_l \leq c_{\varphi(l)}\} \\ &= \inf\{\varphi(l) : W_l \leq h(l)\} = \varphi(T_h)\end{aligned}$$

or, equivalently:

$$\langle N \rangle_\tau = T_h.$$

Consequently:

$$\begin{aligned}\mathbb{P}(N_t \leq c_t | \mathcal{F}_\infty) &= f(\langle N \rangle_t, c_t) = \Phi\left(c_t \Big/ \sqrt{\langle N \rangle_t}\right) \\ \mathbb{P}(N_t \leq c_t | \mathcal{F}_\infty \vee \sigma(\tau)) &= g(\langle N \rangle_t, c_t, \langle N \rangle_\tau) \\ \mathbb{P}(N_t \leq c_t | \mathcal{F}_\infty \vee \sigma(\tau))|_{\tau=u} &= g(\langle N \rangle_t, c_t, \langle N \rangle_u).\end{aligned}$$

Applying the equality 11, we obtain:

$$\begin{aligned}g(\langle N \rangle_t, c_t, \langle N \rangle_u) &= f(\langle N \rangle_t - \langle N \rangle_u, c_t - c_u) \\ &= \Phi\left(\frac{c_t - c_u}{\sqrt{\langle N \rangle_t - \langle N \rangle_u}}\right).\end{aligned}$$

Also, $p(u) = \pi^h(\langle N \rangle_u)$, and using (12), we conclude that $p(u)$, $u \in [0,t]$ satisfies the integral equation:

$$\mathbb{P}(N_t \leq c_t | \mathcal{F}_\infty) = \int_0^t \mathbb{P}(N_t - N_u \leq c_t - c_u | \mathcal{F}_\infty)\, dp(u) \tag{13}$$

or, equivalently:

$$\Phi\left(\frac{c_t}{\sqrt{\langle N \rangle_t}}\right) = \int_0^t \Phi\left(\frac{c_t - c_u}{\sqrt{\langle N \rangle_t - \langle N \rangle_u}}\right) dp(u). \tag{14}$$

An approximate formula for the $(\mathcal{F}_t)$-conditional default distribution can be obtained if we discretize[7] time interval $[0,t]$ into n equal intervals Δt. We define $a_i = \frac{c_{i\Delta t}}{\sqrt{\langle N \rangle_{i\Delta t}}}$ and $b_{i,j} = \frac{c_{i\Delta t} - c_{j\Delta t}}{\sqrt{\langle N \rangle_{i\Delta t} - \langle N \rangle_{j\Delta t}}}$ for $j < i$ and $b_{i,i} = \infty$. We approximate the equation (14) considering default may arrive at ends of intervals in the following way:

$$\Phi(a_n) = \sum_{i=1}^{n} \Phi(b_{n,i})\, \mathbb{P}(\tau \in ((i-1)\Delta t, i\Delta t] | \mathcal{F}_\infty). \tag{15}$$

From the above equation, a recursive system of n equations with n unknowns $q_i = \mathbb{P}(\tau \in ((i-1)\Delta t, i\Delta t] | \mathcal{F}_i \Delta t)$, $i = 1, ..., n$, is obtained. Thus, for the first time interval we have:

$$\Phi(a_1) = \mathbb{P}(\tau \in (0, \Delta t] | \mathcal{F}_\infty) = q_1.$$

[7] The idea of discretizing the differential equation in order to obtain first passage time density was already employed in the default models of Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). See also the applications of this method in insurance (Bernard et al., 2005).

For two intervals, we have:

$$\begin{aligned}\Phi(a_2) &= \Phi(b_{2,1})\,\mathbb{P}(\tau\in(0,\Delta t]\,|\mathcal{F}_\infty) + \mathbb{P}(\tau\in(\Delta t,2\Delta t]\,|\mathcal{F}_\infty)\\ &= \Phi(b_{2,1})\,q_1 + q_2.\end{aligned}$$

Continuing in this manner and solving, we obtain the proposed solution. ■

Now, we give the proof of the Lemma 2.

Proof of Lemma 2. It is obvious that $\{\tau \le t\} \notin \mathcal{F}_t$, hence τ is not an $(\mathcal{F}_t)$-stopping time. Since the filtration $(\mathcal{G}_t)$ is Brownian, τ is a $(\mathcal{G}_t)$-predictable stopping time. Also, by definition of the filtration $(\mathcal{F}_t^\tau)$, τ is an $(\mathcal{F}_t^\tau)$-stopping time and in the remainder we prove that it is totally inaccessible. Like all stopping times, τ has a unique decomposition in an accessible stopping time, say τ_A, and a totally inaccessible stopping time, say τ_B, such that:

$$\tau = \tau_A \wedge \tau_B.$$

(see Dellacherie, 1972). Let us remark that from the construction of $(\mathcal{F}_t^\tau)$ it follows that τ_A is also an $(\mathcal{F}_t)$-stopping time[8], hence an $\mathcal{F}_\infty$-measurable random variable.

From the preceding proof, $p(t) = \mathbb{P}(\tau \le t|\mathcal{F}_t)$ was shown to be the distribution of the first passage time of a Gaussian martingale (the process N conditional to $\mathcal{F}_\infty$) through a deterministic barrier, representing an observed path of the process $(c_t)_{t\ge 0}$. Hence, we have shown that:

$$p(t) = \pi^h\left(\langle N\rangle_t\right),$$

where $T_h = \inf\{t : W_t < h(t)\}$ with $h(t) = c_{\varphi(t)}$. Remark that T_h is not a bounded stopping time, i.e., $\pi^h(t) < 1$, $\forall t \ge 0$. However, on $\{\tau_A < \infty\}$, we have $p(t) = 1$ for $t \ge \tau_A$, implying that conditionally on $\mathcal{F}_\infty$, T_h is bounded by $\langle N\rangle_{\tau_A}$ (which is fixed conditionally to $\mathcal{F}_\infty$). This being impossible, $\mathbb{P}(\tau_A < \infty) = 0$, and $\tau = \tau_B$ *a.s.* which proves the result. ■

For pricing purposes, one also needs an estimate for the $(\mathcal{F}_t)$-conditional probability that the default arrives before a fixed maturity time, $T > t$ (i.e., the maturity of a claim):

Proposition 6 *Under assumption **(A)**, $(\mathcal{F}_t)$-conditional default probability on the interval $[t,T]$ is given by the following formula:*

$$\mathbb{P}(t < \tau \le T|\mathcal{F}_t) = \lim_{\Delta t\to 0} \sum_{j=1}^{n} p_j \tag{16}$$

where $n = (T-t)/\Delta t$, and:

$$p_1 = \int_{-\infty}^{\infty}\left[\Phi(A_1) - \sum_{i=1}^{k}\Phi(C_{i,1})\,q_i\right]\varphi(x)dx$$

$$p_j = \int_{-\infty}^{\infty}\left[\Phi(A_j) - \sum_{i=1}^{k}\Phi(C_{i,j})\,q_i\right]\varphi(x)dx - \sum_{i=1}^{j-1}p_i\Phi(B_{i,j}),\quad j = 2,...,n$$

[8] To understand this, note first that before τ all predictable $(\mathcal{F}_t^\tau)$-stopping times are also $(\mathcal{F}_t)$-stopping times (since before τ, the filtration of all $(\mathcal{F}_t^\tau)$ predictable processes is contained in $(\mathcal{F}_t)$, see Jeulin and Yor (1979)) and second that an accessible time will be equal to some predictable time on a partition of Ω, except negligible sets, implying that it will also be an $(\mathcal{F}_t)$-stopping time.

with:

$$
\begin{aligned}
A_i &= \frac{F^{-1}\left(b(t+i\Delta t)\right) - M_t - x\sqrt{i\Delta t}}{\sqrt{\langle N\rangle_t}}, \quad 1 \le i \le n \\
B_{i,j} &= \frac{F^{-1}\left(b(t+j\Delta t)\right) - F^{-1}\left(b(t+i\Delta t)\right)}{\sqrt{(j-i)\,\Delta t}}, \quad 0 \le i < j \le n \\
C_{i,j} &= \frac{F^{-1}\left(b(t+j\Delta t)\right) - M_t - c_{i\Delta t} - x\sqrt{j\Delta t}}{\sqrt{\langle N\rangle_t - \langle N\rangle_{i\Delta t}}}, \quad 1 \le i \le k-1, \quad 1 \le j \le n \\
C_{k,j} &= B_{0,j}.
\end{aligned}
$$

Φ stands for the cumulative function of the standard normal law and φ for its derivative. The variables q_i, $i = 1, ..., k$, are computed as defined in Proposition 5.

Proof. We shall use the results and notations from the proof of Proposition 5. We begin by proving two lemmas:

Lemma 7 *For $u < T$, denote:*

$$\Theta\left(T,u\right) = \mathbb{P}\left(N_T - N_u \le c_T - c_u | \mathcal{F}_\infty\right).$$

The following holds:

$$\mathbb{E}\left[\Theta\left(T,u\right)|\mathcal{F}_u\right] = \mathbb{E}\left[\Theta\left(T,u\right)\right].$$

Proof. We denote $F^{-1}\left(b(t),t\right) = g(t)$. In the case $u < T$:

$$
\begin{aligned}
\mathbb{E}\left[\Theta\left(T,u\right)|\mathcal{F}_u\right] &= \mathbb{P}\left(N_T - N_u \le c_T - c_u | \mathcal{F}_u\right) \\
&= \mathbb{P}\left(B_T - B_u \le g(T) - g(u) | \mathcal{F}_u\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(B_T - B_u \le g(T) - g(u) | \mathcal{G}_u\right)|\mathcal{F}_u\right] \\
&= \mathbb{E}\left(\Phi\left(\frac{g(T)-g(u)}{\sqrt{T-u}}\right)|\mathcal{F}_u\right) = \Phi\left(\frac{g(T)-g(u)}{\sqrt{T-u}}\right).
\end{aligned}
$$

■

Lemma 8 *For $T > t$:*

$$
\begin{aligned}
\mathbb{P}\left(X_T \le b(T)|\mathcal{F}_t\right) &= \int_0^T \mathbb{E}\left[\Theta\left(T,u\right)|\mathcal{F}_t\right] d\mathbb{P}\left(\tau \le u|\mathcal{F}_t\right) \qquad (17) \\
&= \int_0^t \mathbb{E}\left[\Theta\left(T,u\right)|\mathcal{F}_t\right] dp(u) + \int_t^T \mathbb{E}\left[\Theta\left(T,u\right)\right] d\mathbb{P}\left(\tau \le u|\mathcal{F}_t\right).
\end{aligned}
$$

Proof. Using equation (13) for the interval $[0,T]$ and conditioning with respect to $\mathcal{F}_t$, we obtain:

$$
\begin{aligned}
\mathbb{P}\left(N_T \le c_T)|\mathcal{F}_t\right) &= \mathbb{E}\left\{\int_0^T \mathbb{P}\left(N_t - N_u \le c_t - c_u|\mathcal{F}_\infty\right) dp(u)|\mathcal{F}_t\right\} = \mathbb{E}\left\{\int_0^T \Theta\left(T,u\right) dp(u)|\mathcal{F}_t\right\} \\
&= \int_0^t \mathbb{E}\left(\Theta\left(T,u\right)|\mathcal{F}_t\right) dp(u) + \mathbb{E}\left\{\int_t^T \Theta\left(T,u\right) dp(u)|\mathcal{F}_t\right\}.
\end{aligned}
$$

For computing the last term we discretize the interval $[t,T]$ with $n = (T-t)/\delta$, and denote $\mathbb{P}(\tau \in (t+(j-1)\delta, t+j\delta] \,|\mathcal{F}_\infty) = \tilde{q}(t+j\delta)$. We find that:

$$
\begin{aligned}
\mathbb{E}\left\{\int_t^T \Theta(T,u)\,dp(u)|\mathcal{F}_t\right\} &= \mathbb{E}\left\{\lim_{\delta\to 0}\sum_{j=1}^{n}\Theta(T,t+j\delta)\,\tilde{q}(t+j\delta)\,|\mathcal{F}_t\right\} \\
&= \lim_{\delta\to 0}\mathbb{E}\left\{\sum_{j=1}^{n}\mathbb{E}\left[\Theta(T,t+j\delta)\,\tilde{q}(t+j\delta)\,|\mathcal{F}_{t+j\delta}\right]|\mathcal{F}_t\right\} \\
&= \lim_{\delta\to 0}\mathbb{E}\left\{\sum_{j=1}^{n}\mathbb{E}\left[\Theta(T,t+j\delta)\,|\mathcal{F}_{t+j\delta}\right]\tilde{q}(t+j\delta)\,|\mathcal{F}_t\right\} \\
&= \lim_{\delta\to 0}\mathbb{E}\left\{\sum_{j=1}^{n}\mathbb{E}\left[\Theta(T,t+j\delta)\right]\tilde{q}(t+j\delta)\,|\mathcal{F}_t\right\} \qquad (18) \\
&= \lim_{\delta\to 0}\sum_{j=1}^{n}\mathbb{E}\left[\Theta(T,t+j\delta)\right]\mathbb{E}\left[\tilde{q}(t+j\delta)\,|\mathcal{F}_t\right]. \qquad (19)
\end{aligned}
$$

The second equality is obtained using Lebesgue bounded convergence. The equality (18) is justified by the Lemma 7. The result follows when writing the limit of the sum in (19) as an integral. ■

We now find an expression for the expectations appearing in the integral equation (17). We denote $F^{-1}(b(t),t) = g(t)$. The left hand side of equation (17) equals:

$$
\begin{aligned}
\mathbb{P}(X_T \le b(T)|\mathcal{F}_t) &= \mathbb{E}\left(\mathbb{P}(X_T \le b(T)|\mathcal{G}_t)\,|\mathcal{F}_t\right) \\
&= \mathbb{E}\left(\mathbb{P}(B_T - B_t \le g(T) - B_t|\mathcal{G}_t)\,|\mathcal{F}_t\right) \\
&= \mathbb{E}\left(\Phi\left(\frac{g(T)-B_t}{\sqrt{T-t}}\right)|\mathcal{F}_t\right) \\
&= \int_{-\infty}^{\infty}\mathbb{P}\left(x \le \frac{g(T)-B_t}{\sqrt{T-t}}|\mathcal{F}_t\right)\varphi(x)dx \\
&= \int_{-\infty}^{\infty}\mathbb{P}\left(N_t \le g(T) - M_t - x\sqrt{T-t}|\mathcal{F}_t\right)\varphi(x)dx \\
&= \int_{-\infty}^{\infty}\Phi\left(\frac{g(T)-M_t-x\sqrt{T-t}}{\sqrt{\langle N\rangle_t}}\right)\varphi(x)dx.
\end{aligned}
$$

Let us now turn to the right hand side of the equation (17). It has a different expression depending on the position of u with respect to t, and we already saw in Lemma 7 that in the case $u \ge t$

$$
\mathbb{E}\left[\Theta(T,u)\,|\mathcal{F}_t\right] = \mathbb{E}\left[\Theta(T,u)\right] = \Phi\left(\frac{g(T)-g(u)}{\sqrt{T-u}}\right)
$$

On the other hand, in the case $u < t$, we have:

$$
\begin{aligned}
\mathbb{E}\left[\Theta\left(T,u\right)|\mathcal{F}_t\right] &= \mathbb{P}\left(N_T - N_u \leq c_T - c_u|\mathcal{F}_t\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(B_T - B_u \leq g\left(T\right) - g\left(u\right)|\mathcal{G}_t\right)|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\Phi\left(\frac{g\left(T\right) - g\left(u\right) + B_u - B_t}{\sqrt{T-t}}\right)|\mathcal{F}_t\right] \\
&= \int_{-\infty}^{\infty}\mathbb{P}\left(x \leq \frac{g\left(T\right) - g\left(u\right) + B_u - B_t}{\sqrt{T-t}}|\mathcal{F}_t\right)\varphi(x)dx \\
&= \int_{-\infty}^{\infty}\mathbb{P}\left(B_t - B_u \leq g\left(T\right) - g\left(u\right) - x\sqrt{T-t}|\mathcal{F}_t\right)\varphi(x)dx \\
&= \int_{-\infty}^{\infty}\mathbb{P}\left(N_t - N_u \leq g\left(T\right) - M_t + M_u - g\left(u\right) - x\sqrt{T-t}|\mathcal{F}_t\right)\varphi(x)dx \\
&= \int_{-\infty}^{\infty}\Phi\left(\frac{g(T) - M_t - c_u - x\sqrt{T-t}}{\sqrt{\langle N\rangle_t - \langle N\rangle_u}}\right)\varphi(x)dx.
\end{aligned}
$$

Plugging all these results in equation (17) leads to:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\Phi\left(\frac{g(T) - M_t - x\sqrt{T-t}}{\sqrt{\langle N\rangle_t}}\right)\varphi(x)dx &= \int_0^t\int_{-\infty}^{\infty}\Phi\left(\frac{g(T) - M_t - c_u - x\sqrt{T-t}}{\sqrt{\langle N\rangle_t - \langle N\rangle_u}}\right)\varphi(x)dxdp(u) \\
&+ \int_t^T\Phi\left(\frac{g(T) - g(u)}{\sqrt{T-u}}\right)d\mathbb{P}\left(\tau \leq u|\mathcal{F}_t\right)
\end{aligned}
$$

and rearranging terms:

$$
\begin{aligned}
\int_t^T\Phi\left(\frac{g(T) - g(u)}{\sqrt{T-u}}\right)d\mathbb{P}\left(\tau \leq u|\mathcal{F}_t\right) = &\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{g(T) - M_t - x\sqrt{T-t}}{\sqrt{\langle N\rangle_t}}\right) - \right. \\
&\left. -\int_0^t\Phi\left(\frac{g(T) - M_t - c_u - x\sqrt{T-t}}{\sqrt{\langle N\rangle_t - \langle N\rangle_u}}\right)dp(u)\right\}\varphi(x)dx
\end{aligned}
$$

We discretize the above expression, considering that default may arrive only at ends of intervals, and that $T = (k+n)\,\Delta t$ and $t = k\Delta t$ Let:

$$
p_i = \mathbb{P}(\tau \in \left(t + (i-1)\,\Delta t, t + i\Delta t\right]|\mathcal{F}_t), i = 1, ..., n
$$

and, as in Proposition 5,

$$
q_i = \mathbb{P}\left(\tau \in (i-1)\,\Delta t, i\Delta t|\mathcal{F}_t\right), i = 1, ..., k
$$

We obtain:

$$
\begin{aligned}
\sum_{i=1}^{n}\Phi\left(\frac{g(t+n\Delta t) - g\left(t+i\Delta t\right)}{\sqrt{(n-i)\,\Delta t}}\right)p_i = &\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{g(t+n\Delta t) - M_t - x\sqrt{n\Delta t}}{\sqrt{\langle N\rangle_t}}\right)\right. \\
&\left. -\sum_{i=1}^{k}\Phi\left(\frac{g\left(t+n\Delta t\right) - M_t - c_{i\Delta t} - x\sqrt{n\Delta t}}{\sqrt{\langle N\rangle_t - \langle N\rangle_{i\Delta t}}}\right)q_i\right\}\varphi(x)dx.
\end{aligned}
$$

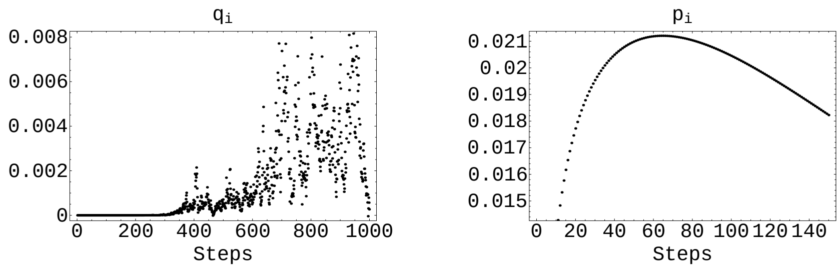

Figure 1: Left: Conditional default probabilities q_i for 1000 time steps. Right: Conditional probabilities p_i for one year time to maturity, using 150 time steps.

Using the notation from the statement of the proposition, the equation becomes:

$$\sum_{i=1}^{n}\Phi\left(B_{i,n}\right)p_i = \int_{-\infty}^{\infty}\left[\Phi\left(A_n\right) - \sum_{i=1}^{k}\Phi\left(C_{i,n}\right)q_i\right]\varphi(x)dx.$$

We are able to obtain a recursive system of n equations with $p_i, i = 1, ..., n$ being the unknowns, and $q_i, i = 1, ..., k$, previously computed using Proposition 5. Thus, we obtain:

$$p_1 = \int_{-\infty}^{\infty}\left[\Phi\left(A_1\right) - \sum_{i=1}^{k}\Phi\left(C_{i,1}\right)q_i\right]\varphi(x)dx$$

$$p_2 = \int_{-\infty}^{\infty}\left[\Phi\left(A_2\right) - \sum_{i=1}^{k}\Phi\left(C_{i,2}\right)q_i\right]\varphi(x)dx - p_1\Phi\left(B_{1,2}\right)$$

$$p_n = \int_{-\infty}^{\infty}\left[\Phi\left(A_n\right) - \sum_{i=1}^{k}\Phi\left(C_{i,n}\right)q_i\right]\varphi(x)dx - \sum_{i=1}^{n-1}p_i\Phi\left(B_{i,n}\right).$$

■

Remark 9 (Condition (A')) *In fact, as we announced in the previous section, similar formulas are obtained for the more general class of diffusions, defined as the* ***(A')*** *condition:* $X_t = F\left(m_t, t\right)$ *where* m_t *is a* $(\mathcal{G}_t)$*-martingale of the form:* $m_t = \int_0^t h(s)dB_s$ *with* h *being a Borel function. We remark that we can recover the situation from condition* ***(A)*** *via a deterministic time change, since* m_t *is Gaussian. The corresponding default probabilities are simply obtained by replacing in the above formulas* N_t *by* $N_t' = \int_0^t h(s)dM_s$ *and* M_t *by* $M_t' = \int_0^t h(s)dM_s$*. In addition, in Proposition 6 we need to replace* $\sqrt{i\Delta t}$ *with* $\sqrt{\langle m\rangle_{i\Delta t}}$ *and similarly* $\sqrt{(j-i)\,\Delta t}$ *with* $\sqrt{\langle m\rangle_{(j-i)\Delta t}}$*. An illustration is provided in Section 5, for the Ornstein-Uhlenbeck process.*

Our closed form formulas for the default probabilities imply recursive formulas which are straightforward to implement for numerical purposes. Even if the results are defined as limits, the number of steps needed is generally not high: 2000 steps for $\mathbb{P}\left(\tau \leq t|\mathcal{F}_t\right)$ and 200 steps for

$\mathbb{P}(t < \tau \leq T|\mathcal{F}_t)$ permit generally to obtain a high accuracy, as illustrated in Figure 2, where we have chosen t=1 and T=1. The rest of parameters are those from sub-section 3.5.1. Figure 1 shows the corresponding values of q_i in basis points, using 1000 steps and the values of p_i in basis points, using 150 steps.

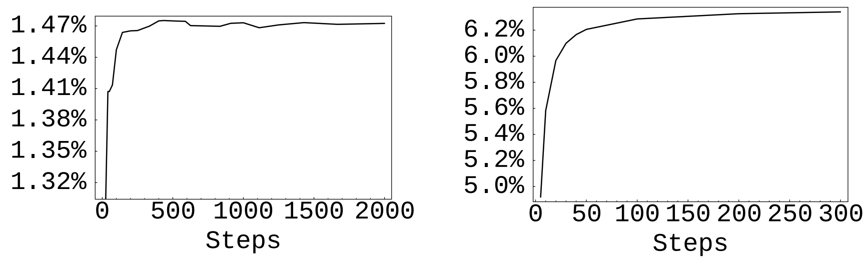

Figure 2: Impact of the number of steps (n) on the value of the sum $\sum_1^n p_i$ (on the left) and on$\sum_1^n q_i$ (on the right).

4 Valuation of default-sensitive claims

In this section we consider an arbitrage-free financial market, composed of three types of securities: default-free and default-sensitive and defaultable, defined to be compatible with our imperfect information model.

We show how to price even complex financial products, making use of the above results. The main idea is to consider the loss given default process as a default-free security. Then, we show that a default-sensitive security, may be considered as a default-free security with the same characteristics, but which pays a flow of "negative dividends". This "dividends" are proportions of the loss given default process and are payed whenever the $(\mathcal{F}_t)$ conditional default probability is increasing, i.e., on the set $\{t : d\mathbb{P}(\tau \leq t|\mathcal{F}_t) > 0\}$. In the case of intensity-based models, the flow is continuous.

4.1 Assumptions on the default-free market

We suppose that the current date is t and we fix a finite horizon date T, such that $0 < t < T$. We consider that there exist a default-free market, composed of the savings account, defined as

$$B(t) = \exp \int_0^t r_u du,$$

with a deterministic risk-free interest rate r_t. Let $B(t,T)$ be the price at date t of a non defaultable bond with a nominal value of one dollar and maturity date T. In the case of deterministic interest rates, its value is simply: $B(t,T) = \frac{B(t)}{B(T)}$. Also, we suppose there exist some default-free securities, defined in the following

Definition 10 *For a fixed maturity T, a default-free $\mathcal{F}_T$-contingent claim is any nonnegative, square integrable, $\mathcal{F}_T$-measurable random variable ξ_T. A default-free security $(\xi_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-adapted process describing the price of a default-free $\mathcal{F}_T$-contingent claim ξ_T.*

Hence, the default-free securities are measurable with respect to the filtration generated by the Brownian motion β, since this represents the only available information for market investors, which is pertinent for pricing default-free claims (notice that the information relative to the default time does not impact default-free securities under **(H))**.

Now, we assume that the default-free market is complete[9] and arbitrage-free and let $\mathbb{P}^*$ be equivalent to $\mathbb{P}$ on $\mathcal{F}$ such that its restriction to $\mathcal{F}_\infty$ is the unique probability measure under which the discounted default-free securities are $(\mathcal{F}_t)$-martingales for $0 \le t \le T$:

$$\xi_t = B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[\xi_T|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[\xi_T|\mathcal{F}_t^\tau\right].$$

The last equality is due to the validity of the **(H)**-hypothesis in our model.

If the hypothesis **(A)** or, more generally **(A')** holds for the dynamics of the process X under the equivalent measure $\mathbb{P}^*$, then the results of the preceding section still apply under this probability $\mathbb{P}^*$. It is useful to add some assumption regarding the dynamics of the process X under the martingale measure. This is possible, via an intermediate assumption regarding the process V representing the value of the assets of the firm, commonly used in structural models:

(E) The discounted process $\left(\frac{V_t}{B(t)}\right)_{t\ge 0}$ is a martingale under the equivalent martingale measure, $\mathbb{P}^*$.

Whenever the fundamental process represents the market value of assets, assumption **(E)** directly applies to X. Even if the process X is neither directly traded nor observable, this assumption is reasonable[10]. For the situations where the process X does not represent the value of assets, one has to define how X is related to the value of assets, using a function which relates the two processes. Then, given **(E)**, it suffices to apply Itô's lemma to find the dynamics of X under $\mathbb{P}^*$.

In the next section, we will consider that: (i) condition **(A')** holds for the dynamics of the process X under $\mathbb{P}^*$ (ii) condition **(E)** holds and (iii) for ease of exposition we put $X \equiv V$.

4.2 The market of defaultable securities

Before default occurs, market participants try to infer the true value of the fundamental process X from their available information, namely the process Y and the fact that the process X has not yet crossed the default barrier.

Definition 11 *We introduce the risk-neutral estimate of the variable X_T as the $\mathcal{F}_T$ measurable random variable $\hat{X}_T$ defined by the equality:*

$$\hat{X}_T = \frac{1}{Z_T^\tau}\mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{(\tau>T)}X_T|\mathcal{F}_T\right], \tag{20}$$

where we consider $Z_T^\tau = \mathbb{P}^*\left(\tau > T|\mathcal{F}_T\right)$.

The name of the variable $\hat{X}_T$ is due to the following property:

---

[9]This assumption implies that at least one default-free security except the savings account is traded in the market. Examples of default-free securities are delayed to the next sub-section.

[10]To understand this point, we can observe that at date 0, the investment in the firm's assets X_0 was accepted by completely informed investors; thus the no-arbitrage condition must have prevailed at that date. As the risk of the business measured by the volatility function $\sigma(x,t)$, is considered as known by investors for all values of x and t, the drift coefficient of the process X should reflect the market price for the firm's risk process B_t.

Lemma 12 *The risk-neutral estimate of X_T satisfies the following relation:*

$$\mathbf{1}_{(\tau>T)}\mathbb{E}^{\mathbb{P}^*}\left[X_T|\mathcal{F}_T^\tau\right] = \mathbf{1}_{(\tau>T)}\hat{X}_T. \tag{21}$$

Proof. This is a consequence of the following projection formula (see Jeanblanc and Rutkowski (2000) for the proof):

Proposition 13 (Projection formula) *Let A be a bounded, $\mathcal{F}$-measurable random variable and τ a random time, such that $\mathbb{P}(\tau > t|\mathcal{F}_t) \neq 0$. Then, for every $t \leq T$:*

$$\mathbb{E}[\mathbf{1}_{(\tau>T)}A|\mathcal{F}_t^\tau] = \mathbf{1}_{(\tau>t)}\frac{\mathbb{E}\left(\mathbf{1}_{(\tau>T)}A|\mathcal{F}_t\right)}{\mathbb{P}\left(\tau > t|\mathcal{F}_t\right)}.$$

■

The variables $\hat{X}_t$, $t > 0$ enjoy several interesting properties. First, for any fixed maturity T, the variable $\hat{X}_T$ is a default-free claim, because it is a non-negative, $\mathcal{F}_T$-measurable random variable. Second, before default, the process $\hat{X} = \left(\hat{X}_t\right)_{t\geq 0}$ represents the best estimate of the unobservable process X, since from the equation (21), on $\{\tau > t\}$ every random variable $\hat{X}_t$ is the estimate of the variable X_t, given the market information available at that time. Due to this property, the process $\hat{X}$ should play an important role for pricing, as it will be shown in the following definition of defaultable claims.

Definition 14 *For a fixed maturity T, a default sensitive contingent claim is an $\mathcal{F}_T^\tau$-measurable random variable defined by:*

$$c_T := \mathbf{1}_{(\tau>T)}\xi_T + \mathbf{1}_{(\tau\leq T)}\xi_T',$$

where ξ_T and ξ_T' are two default-free, $(\mathcal{F}_T)$-contingent claims. A defaultable contingent claim is an integrable, $\mathcal{F}_T^\tau$-measurable random variable of the form:

$$d_T := \mathbf{1}_{(\tau>T)}f\left(\hat{X}_T\right) + \mathbf{1}_{(\tau\leq T)}g\left(X_\tau\right)B(\tau,T), \tag{22}$$

where f and g are two Borelian functions.

Note that in our definition of defaultable claims, we assume that in the case of default, the recovery payment $g(X_\tau)$ is immediately invested up to time T either in default-free zero-coupon bonds or in the money market account (this are equivalent in the case of deterministic interest rates).

We emphasize we consider defaultable claims as a different class from the default sensitive claims. Thus, corporate bonds, equity, and their derivatives issued by the firm are -as expected-defaultable claims, as their values are functions of the market estimate process, and in the case of default, the recovery is established as a function of the defaulted firm value and the priority rules. But a portfolio containing a corporate bond secured by a credit default swap is only a default sensitive claim: the holder receives a complete compensation in case of default, thus the portfolio is independent of the value of the defaulted firm. Nevertheless, this portfolio is not default-free, as its composition changes once the default event has occurred. Finally, a corporate bond secured with a total return swap behaves like a default-free bond: the return on the portfolio does not depend on the occurrence of the default event.

The main difference with the complete information models is that in presence of imperfect information, we suppose the defaultable claims to be evaluated using the estimation $\hat{X}_T$ whenever the firm is not in the default state. Let us provide some simple examples in order to gain intuition of the method.

Consider an unlevered firm, with only equity in its balance sheet. This firm never defaults because it has no contractual payments to make, hence $\tau = \infty$ *a.s.* and $Z^\tau \equiv 1$. Also, the filtrations $(\mathcal{F}_t)$ and $(\mathcal{F}_t^\tau)$ coincide. Structural models of default with perfect information state that, for this unlevered firm:

$$E_T^u = X_T,$$

where E_T^u stands for the price of equity at any fixed time T and the superscript remembers that we are analyzing the unlevered firm. Instead, using our definition, we obtain:

$$\hat{E}_T^u = \mathbf{1}_{(\tau>T)}\hat{X}_T = \hat{X}_T^u = \mathbb{E}^{\mathbb{P}^*}\left[X_T|\mathcal{F}_T\right] = \mathbb{E}\left[E_T^u|\mathcal{F}_T\right].$$

Here also the superscript of $\hat{X}^u$ remembers that $\tau = \infty$ *a.s.* We see that whenever the assets of the firm stand for the fundamental process X driving the default, the discounted process $\left(\hat{X}_t^u/B(t)\right)$ is an $(\mathcal{F}_t)$-martingale, since, for $0 \le t \le T$[11]:

$$\hat{X}_t^u/B(t) = \mathbb{E}^{\mathbb{P}^*}\left[X_t/B(t)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}^*}\left[X_T/B(T)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}^*}\left[\hat{X}_T^u/B(T)|\mathcal{F}_t\right]$$

and $\hat{X}^u$ represents the market estimated value of the unlevered firm.

At this point, we are able to identify several default-free securities, and explain how to extract the market risk premium.

First, we the process $\hat{X}^u$ representing the market value of the unlevered firm behaves like default-free asset, since its discounted value is an $(\mathcal{F}_t)$-martingale. Considering $\hat{X}^u$ to be traded presents however the same shortcomings as those of structural models with complete information considering X is traded: this is rarely the case. Given its economic interpretation as the best pre-default estimate of the firm's value assets and equation (22), the process $\hat{X}$ -even if not directly traded- can be inferred from the price of a defaultable security, for instance stocks. This approach is similar that of all structural models. Secondly, a portfolio containing a defaultable claim secured with a total rate of return swap is a default-free security, but perhaps not enough liquid. Also, whenever the process Y is traded (for instance, when it represents the value of a similar but more transparent firm) it is a default-free security and may be used to complete the market.

Now, let us consider a levered firm with very simple capital structure composed of equity and of zero bonds with maturity T and nominal value $b(T)$. Suppose also that in case of default the liquidation costs are a fixed proportion l of the value of the firm at date T. Structural models with complete information will value the payoffs at time T, respectively for equity and debt as follows:

$$E_T = \mathbf{1}_{(\tau>T)}\left(X_T - b(T)\right)$$
$$D(T,T) = \mathbf{1}_{(\tau>T)}b(T) + \mathbf{1}_{(\tau\le T)}\left(1-l\right)X_T,$$

hence,

$$E_T + D(T,T) = X_T\left(1 - \mathbf{1}_{(\tau\le T)}l\right).$$

Instead, our formulation incorporates the fact that the true value of assets is revealed only in case of default, hence we propose the representation:

$$\hat{E}_T = \mathbf{1}_{(\tau>T)}\left(\hat{X}_T - b(T)\right) = \mathbb{E}\left[E_T|\mathcal{F}_T^\tau\right]$$
$$D(T,T) = \mathbf{1}_{(\tau>T)}b(T) + \mathbf{1}_{(\tau\le T)}\left(1-l\right)X_T$$

[11]Remember we are considering the assumption (E) to hold true with $X \equiv V$.

and hence, the value of the outstanding firm's debt and equity is given by:

$$\hat{E}_T + D(T,T) = \mathbf{1}_{(\tau>T)}\hat{X}_T + \mathbf{1}_{(\tau\leq T)}\,(1-l)\,X_T.$$

Of course, besides the different representation of the payoffs of defaultable claims, our approach fundamentally differs from the perfect information case in the way this payoffs are priced, since we shall use a different filtration.

Proposition 15 (Prices of default-sensitive claims) *Considering that at time $t<T$ the default has not yet occurred, the price of a default-sensitive claim is given by:*

$$c_t = \xi_t - \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T \frac{B(t,u)}{Z_t^\tau}\left(\xi_u - \xi_u'\right) d\mathbb{P}\left(\tau \leq u|\mathcal{F}_u\right)|\mathcal{F}_t\right]$$

where $(\xi_t - \xi_t')_{t\geq 0}$ is the loss given default process.

Proof. $(\mathcal{F}_t^\tau)$-informed agents will evaluate the payoff of default-sensitive claims under the risk-neutral expectation conditional to the filtration $(\mathcal{F}_t^\tau)$. On $\{\tau > t\}$, we obtain:

$$\begin{aligned}
c_t &= B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{(\tau>T)}\xi_T + \mathbf{1}_{(\tau\leq T)}\xi_T'|\mathcal{F}_t^\tau\right] \\
&= B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[\xi_T|\mathcal{F}_t^\tau\right] - B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{(t<\tau\leq T)}\left(\xi_T - \xi_T'\right)|\mathcal{F}_t^\tau\right] \\
&= \xi_t - \frac{B(t,T)}{Z_t^\tau}\mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{(t<\tau\leq T)}\left(\xi_T - \xi_T'\right)|\mathcal{F}_t\right] \\
&= \xi_t - \frac{B(t,T)}{Z_t^\tau}\mathbb{E}^{\mathbb{P}^*}\left[\mathbb{P}^*\left(t<\tau\leq T|\mathcal{F}_T\right)\left(\xi_T - \xi_T'\right)|\mathcal{F}_t\right]
\end{aligned}$$

We integrate by parts the product which appear in the expectation, observing that the process $(\mathbb{P}^*\left(t<\tau\leq u|\mathcal{F}_u\right))_{u\in(t,\infty)}$ is increasing. We obtain the relation:

$$\begin{aligned}
\mathbb{P}^*\left(t<\tau\leq T|\mathcal{F}_T\right)\left(\frac{\xi_T - \xi_T'}{B(T)}\right) &= \int_t^T \mathbb{P}^*\left(t<\tau\leq u|\mathcal{F}_u\right) d\left(\frac{\xi_u - \xi_u'}{B(u)}\right) \\
&\quad + \int_t^T \left(\frac{\xi_u - \xi_u'}{B(u)}\right) d\mathbb{P}^*\left(t<\tau\leq u|\mathcal{F}_u\right).
\end{aligned}$$

The martingale property of the discounted process $\frac{\xi_t - \xi_t'}{B(t)}$ implies that the first term in the above expression is an $(\mathcal{F}_t)$-local martingale. It is in fact a true martingale, because $\mathbb{P}^*\left(t<\tau\leq u|\mathcal{F}_u\right)$ is bounded, hence has a null $\mathcal{F}_t$-conditional expectation. Taking the $\mathcal{F}_t$-conditional expectation in the formula above, we obtain:

$$\mathbb{E}^{\mathbb{P}^*}\left[\mathbb{P}^*\left(t<\tau\leq T|\mathcal{F}_T\right)\left(\xi_T - \xi_T'\right)|\mathcal{F}_t\right] =$$

$$\begin{aligned}
&= \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T B(T)\left(\frac{\xi_u - \xi_u'}{B(u)}\right) d\mathbb{P}^*\left(t<\tau\leq u|\mathcal{F}_u\right)|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T \frac{\xi_u - \xi_u'}{B(u,T)} d\mathbb{P}^*\left(t<\tau\leq u|\mathcal{F}_u\right)|\mathcal{F}_t\right]
\end{aligned}$$

which completes the proof. ■

Defaultable claims valuation

As in the case of default-sensitive claims, we may obtain the price of a defaultable claim as an expectation:

$$\begin{aligned} d_t &:= B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{(\tau>T)} f\left(\hat{X}_T\right) + \mathbf{1}_{(t<\tau\le T)}\frac{g\left(X_\tau\right)}{B(\tau,T)}|\mathcal{F}_t^\tau\right] \\ &= B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[f(\hat{X}_T)|\mathcal{F}_t\right] \\ &- \frac{B(t,T)}{Z_t^\tau}\mathbb{E}^{\mathbb{P}^*}\left[\int_t^T \left(f\left(\hat{X}_T\right) - \frac{g\left(b(u)\right)}{B(u,T)}\right) d\mathbb{P}\left(\tau\le u|\mathcal{F}_u\right)|\mathcal{F}_t\right] \end{aligned}$$

The first term corresponds to the price at date t of a default-free security which pays $f(\hat{X}_T)$ at date T. We denote $v_f(t)$ this price and we obtain:

$$d_t = v_f(t) - \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T \frac{B(t,u)}{Z_t^\tau}\left[v_f\left(u\right) - g\left(b(u)\right)\right] d\mathbb{P}\left(\tau\le u|\mathcal{F}_u\right)|\mathcal{F}_t\right].$$

Zero-coupon bond valuation

Let $D(t,T)$ be the price of a defaultable zero-coupon bond with nominal one dollar value and maturity date T. We assume that in case of default a recovery payment $\delta_\tau\in[0,1]$ is instantaneously made, with $\delta_\tau = \frac{\alpha b(\tau)}{N}$, N representing the nominal value of all outstanding debt and $\alpha\in[0,1]$ being a parameter depending on the seniority of the bond[12]. We note $\pi_t = 1-\delta_t$ the "loss given default" process.

Then, the payoff at maturity of the defaultable bond is:

$$D(T,T) = \mathbf{1}_{(\tau>T)} + \frac{\delta_\tau}{B(\tau,T)}\mathbf{1}_{(\tau\le T)} \tag{23}$$

which is a special case of formula (22), where $f(x)=1$ and $g(x)=\frac{\alpha x}{N}$. The time t price of the zero-coupon bond, conditional on the default having not occurred is:

$$D(t,T) = B(t,T) - \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T \frac{B(t,u)}{Z_t^\tau}\pi_u d\mathbb{P}\left(\tau\le u|\mathcal{F}_u\right)|\mathcal{F}_t\right], \quad \tau>t.$$

5 Some particular examples

Using the general results obtained in the previous sections, we now provide some numerical illustrations for the shape of $(\mathcal{F}_t)$-conditional default probabilities and of credit spreads, for the cases where the process X is a geometrical Brownian motion or an Ornstein-Uhlenbeck process.

But this section is also aimed to illustrate the complexity of the situations in which our framework apply. Not only existing structural models may be embedded in our model, but also new models can easily be created as long as the condition **(A')** is fulfilled for the process X. There is no need to know the distribution of some hitting time for the process X. For instance, we propose a

[12] In general, the value of the default barrier is chosen so that $X_\tau < N$, so that the recovery rate will be inferior to 100% for all values of $\alpha\in[0,1]$.

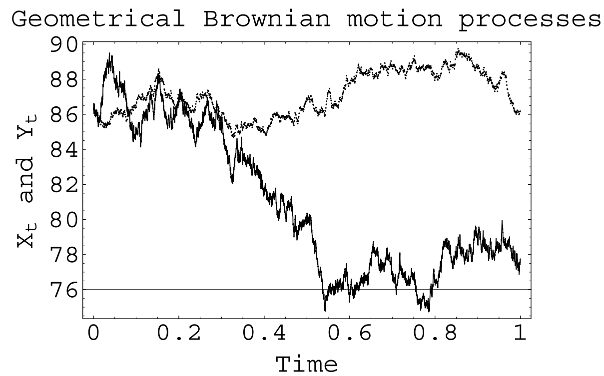

Figure 3: A realization of the process Y (solid path) and X (dashed path) up to the current time $t = 1$.

model with business cycles. Also, we present a generalization of the model of Giesecke and Goldberg (2001) for a stochastic unobservable barrier.

To simplify the analysis, we will next consider zero-recovery rates for bonds, leading to defaultable bond formula:

$$D(t,T) = B(t,T)\mathbb{P}^*\left(\tau > T|\mathcal{F}_t^{\tau}\right) = B(t,T)\frac{\mathbb{P}^*\left(\tau > T|\mathcal{F}_t\right)}{\mathbb{P}^*\left(\tau > t|\mathcal{F}_t\right)}.$$

so that the spread for the maturity T is given by:

$$S(t,T) = -\ln\left(\frac{\mathbb{P}^*\left(\tau > T|\mathcal{F}_t\right)}{\mathbb{P}^*\left(\tau > t|\mathcal{F}_t\right)}\right)/(T-t).$$

5.1 The value of the assets is a geometrical Brownian motion

Here, we analyze the model of DL(2001) in our framework. Suppose that the fundamental process X is the value of the assets of the firm, and follows a geometrical Brownian motion, under the risk neutral measure:

$$\begin{aligned} dX_t &= rX_t dt + \sigma X_t dB_t \\ X_0 &= x_0, \end{aligned}$$

where r is a constant instantaneous interest rate. This equation satisfies condition **(A)** with $F^{-1}(x,t) = \frac{1}{\sigma}\left(x - \log v_0 - mt\right)$.

We also consider a constant default barrier: $b = x_b \in (0, x_0)$ and chose the following form for the observation process:

$$dY_t = rY_t dt + \sigma_1 Y_t d\beta_t$$

where: $\sigma_1 = \sqrt{\sigma^2 + s^2 + 2\rho\sigma s}$ and $\beta_t = \frac{\sigma B_t + sB_t'}{\sigma_1}$. This means that the noise affects the returns of the assets of the firm.

Our base case parameters are in the same lines with those of DL(2001):

$$t = 1; \quad \sigma = 0.05; \quad s = 0.1; \quad r = 0.03; \quad \rho = 0; \quad x_0 = 86.3; \quad b(t) = 76.$$

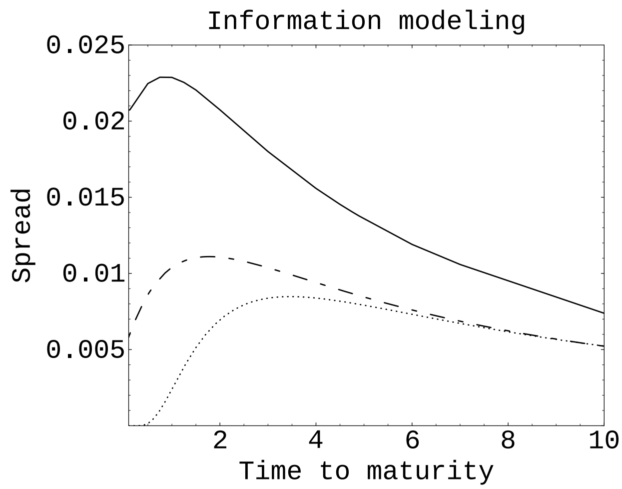

Figure 4: Credit spreads generated by our model (solid curve), compared to spreads obtained with complete information (dashed curve) and to spreads generated by DL model (dot dashed curve). Base case parameters.

Let us suppose that at time $t = 1$ we observe the realization of the process Y showed in Figure 3 (solid curve). An unobservable for the investors realization of the true value of assets, X (dashed path) and the level of the default barrier are also displayed in Figure 3.

Figure 4 displays the credit spreads generated by our model (solid path), by the DL(2001) model (dot dashed curve) and by the corresponding complete information model (dashed curve). In our model the entire path of the process Y on $[0, 1]$ is the available information used for computing the default probabilities; for DL model, the available information contains only the points Y_0 and Y_1 and for the complete information case, we have assumed that the process X is observable.

We remark that the spreads generated by our model are above the DL(2001) spreads, which are above the complete information spreads. This configuration is of course not always true, depending on the realizations of the observation process Y. What is always true is that the short term spreads are higher for our model and DL(2001) compared to the complete information short term spreads. However, it can happen that the firm is really close to the default barrier (X close to b) and in this case, not knowing the truth (that is observing only Y) may lead to thinner spreads for medium term that those computed with insider information. In this way, we capture the situations when default comes a surprise and produces large losses.

Also, comparing our model and DL(2001) we can say that our model keeps the memory of all the observed path of the process Y when computing spreads and default probabilities. Thus if between Y_0 and Y_1 the process had "bad" excursions the spreads will be larger than DL(2001); the contrary will happen if over the period of observation the process Y was "being well" (also note that this effect will be reduced when the correlation of the noise is negative).

This path dependence feature of the default probabilities in our model is very important. Note that it is implicit in reduced-form models, since their implementation needs calibration of the hazard process to historical data on bonds. The advantage of our method is to model explicitly the role of the past information in the current prices combined with the use of a fundamental economic process for predicting the default event. Structural models usually don't present the

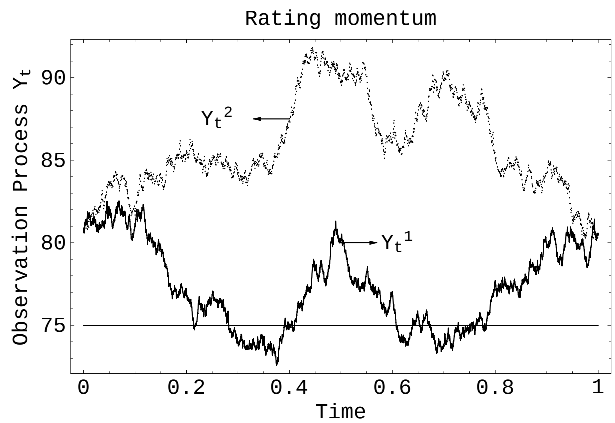

Figure 5: The conditional default probabilities depend on the entire path of the observation process.

path dependent feature since the fundamental process is considered to be Markov, according to the theory of the markets efficiency. Thus, with complete information, the current state of the diffusion is the only information impacting the default probability. Our result seem to indicate that imperfect information may explain the non Markovian patterns of the default probabilities documented in the literature, such as the "rating momentum", which is the dependence of the new rating from the previous ones, not only the last evaluated[13].

Figure 5 displays the "rating momentum": supposing two alternative paths Y^1 and Y^2 for the observation process, but having the same current state, i.e. $Y_t^1 = Y_t^2$, we obtain different default probabilities. Thus, with current time $t = 1$, maturity $T = 2$, and setting the default barrier at $b = 75$ we obtain for the observation Y^1: $\mathbb{P}(\tau \leq t|\mathcal{F}_t) = 15.71\%$ and $\mathbb{P}(t < \tau \leq T|\mathcal{F}_t) = 6.91\%$, while for the observation Y^2, we obtain: $\mathbb{P}(\tau \leq t|\mathcal{F}_t) = 1.18\%$ and $\mathbb{P}(t < \tau \leq T|\mathcal{F}_t) = 4.57\%$.

Figure 6 compares the base case term structure (solid curve) with the term structure that would apply with correlated noise. All spreads are obtained using the realized observation Y from Figure 3. Independent noise produces the highest spreads, since any correlation in the noise permits to better estimate the unobservable process X and reduces the risk of estimation for market investors. In addition, positive correlation seems to imply higher spreads than negative correlation. The explanation is as follows: since the path of the process Y used here is rather a bad outcome, negative correlation permits to hope a better outcome for the unobservable process X. But, more exactly, what really counts is the sign of $\langle X, Y \rangle_t$ (rather than the sign of ρ) and which is given in this example by the sign of $\sigma + \rho s$. Indeed, this is negative for $\rho = -0.7$ and positive for $\rho = 0.5$.

Figure 7 shows the impact of the two variables t (time since the last complete information) and s (the volatility of noise) on the default probability. Both variables have a negative impact on the accuracy of the observation process and $t = 0$ or $\sigma = 0$ correspond to the complete information case.

Figures 8 and 9 show that the impact of the firm's fundamentals μ and σ on the default probability is similar to structural models: μ is negatively related and σ is positively related to $\mathbb{P}(\tau \leq t|\mathcal{F}_t)$.

Finally, let us remark, that our model permits to obtain more variate credit spreads than the DL(2001) model. First, the DL(2001) model does not permit correlated noise. Secondly, we use the whole trajectory of the observation process Y for computing the default probabilities.

[13]See for instance Bahar and Nagpal (1999), Kavvathas (2000) or Lando and Skodeberg (2001).

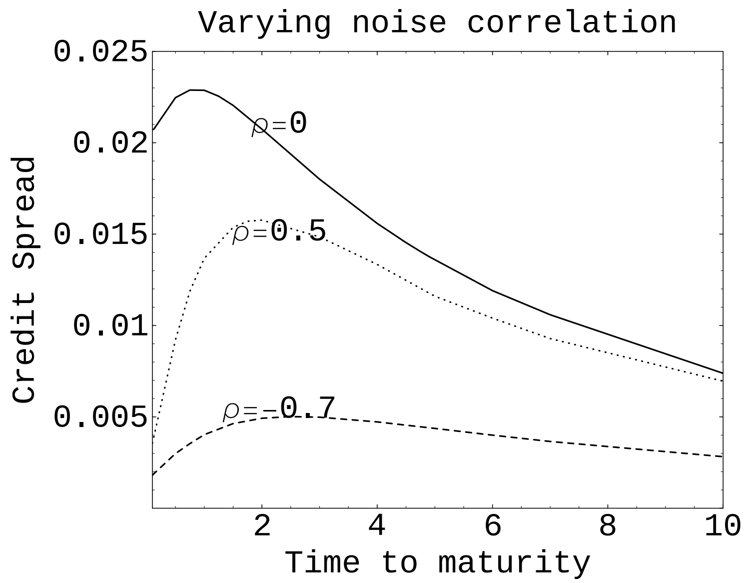

Figure 6: Credit spreads when only ρ varies.

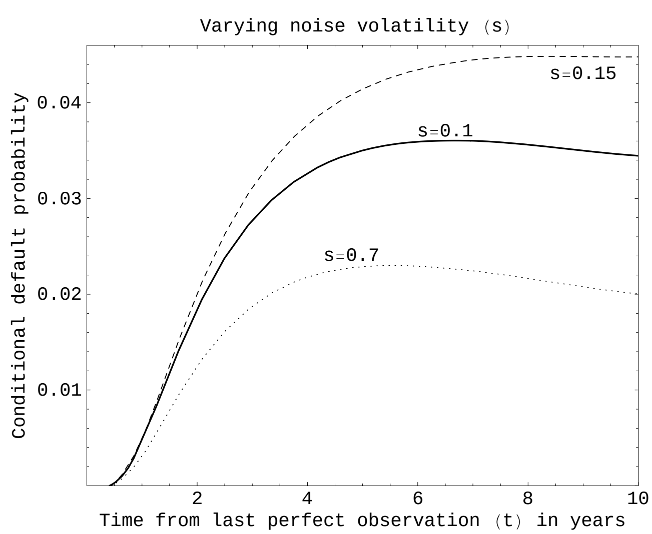

Figure 7: Impact of the variables t and s on the process $\mathbb{P}(\tau \leq t|\mathcal{F}_t)$, with $t \in [0, 10]$ years.

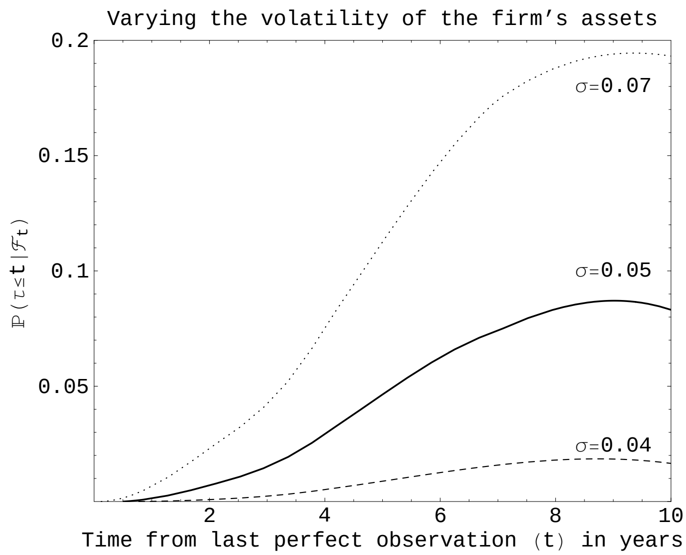

Figure 8: Impact of the volatility of assets σ on the process $\mathbb{P}(\tau \leq t|\mathcal{F}_t)$, with $t \in [0, 10]$ years.

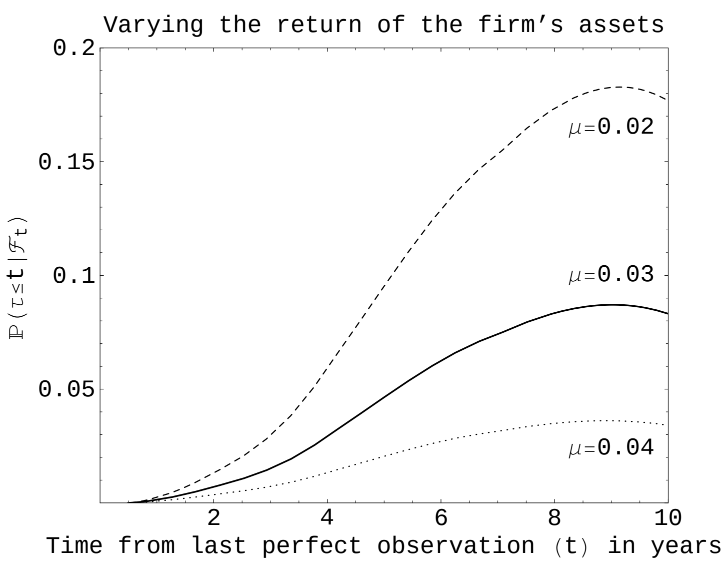

Figure 9: Impact of the expected return on assets μ on the process $\mathbb{P}(\tau \leq t|\mathcal{F}_t)$ with $t \in [0, 10]$ years.

5.2 Ornstein-Uhlenbeck process

Let us now suppose that the fundamental process follows an Ornstein-Uhlenbeck (OU) process, under the risk-neutral measure:

$$dX_t = \lambda(\theta - X_t)dt + \sigma dB_t \tag{24}$$

with θ, λ and σ are constants. We define $\tau = \inf(t : X_t = 0)$, so that the default barrier is $b(t) \equiv 0$.

The solution of (24) is given by:

$$X_t = \theta + \left(x_0 - \theta + \sigma \int_0^t e^{\lambda s} dB_s\right) e^{-\lambda t}. \tag{25}$$

A structural model of this type is presented in Collin-Dufresne and Goldstein (2001), where the fundamental process is the log leverage process ($\log \frac{k_t}{V_t}$), negative up to the default barrier (we recall that V is the assets value and k is the assets' default point). In this model it is considered that the firm continuously adjust its leverage ratio towards a target level. We will keep the same framework but we rather consider $X_t = \log \frac{V_t}{k_t}$ so that fundamental process is positive up to the default time. It follows that the parameter θ has here a nice financial interpretation, since $-\theta$ is the expected long term log-leverage ratio of the firm.

Let us also remark that the condition **(A')** is satisfied with: $X_t = F(m_t, t) = \theta + (x_0 - \theta + m_t) e^{-\lambda t}$ where $m_t = \sigma \int_0^t e^{\lambda s} dB_s$ is a $(\mathcal{G}_t)$-martingale. In consequence, $F^{-1}(x, t) = (x - \theta) e^{\lambda t} - (x_0 - \theta)$.

We choose to define the observation process as:

$$\begin{aligned} dY_t &= \lambda(\theta - Y_t)dt + \sigma dB_t + s dB'_t \\ &= \lambda(\theta - Y_t)dt + \sigma_1 d\beta_t. \end{aligned} \tag{26}$$

where: $\sigma_1 = \sqrt{\sigma^2 + s^2 + 2\rho\sigma s}$ and $\beta_t = \frac{\sigma B_t + s B'_t}{\sigma_1}$.

The processes defined in the Remark 9 take here the particular forms:

$$\begin{aligned} M'_t &= \frac{\sigma\sigma_1}{\sigma + \eta} \int_0^t e^{\lambda u} d\beta_u \\ N'_t &= \frac{\sigma\eta}{\sigma + \eta} \int_0^t e^{\lambda u} dB_u + \frac{\sigma s}{\sigma + \eta} \int_0^t e^{\lambda u} dB'_u \end{aligned}$$

with $\eta = s \frac{\rho\sigma + s}{\sigma + \rho s}$ and leading to

$$\langle N' \rangle_t = \frac{\sigma^2 \left(\eta^2 + s^2 - 2\rho\eta s\right)}{2\lambda (\eta + \sigma)^2} \left(e^{2\lambda t} - 1\right).$$

To illustrate our model we consider the following base case parameters:

$$t = 1; \quad x_0 = 0.35; \quad \lambda = 0.18; \quad \theta = 0.35; \quad \sigma = 0.12; \quad s = 0.16; \quad \rho = 0$$

so that the observable process Y has diffusion parameters in similar lines as those estimated in Collin-Dufresne and Goldstein [7] under the risk-neutral measure.

Figure 10 displays a path of the observable process Y and of the unobservable process X generated using the base case parameters.

The corresponding credit spreads are displayed in Figure 11, where we also display the spreads computed using an insider information (i.e. the process X is observable) and spreads generated by

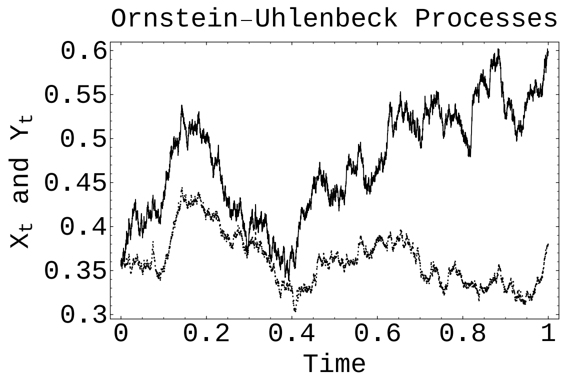

Figure 10: A realization of the process Y (solid path) and X (dashed path) up to the current time $t = 1$.

the structural model (i.e. only the process Y is observable, but we do not take into account the noise when pricing). For the last two curves, we use the first passage time distribution for an OU process over a constant barrier[14]. But, if for the insider, the starting value of the process is X_t, for the structural model, the initial value is Y_t. Because in our example $Y_t > X_t$ (see Figure 10), the insider spreads are above the structural spreads.

5.3 Modeling business cycles

Some firms are subject to known in advance business cycles, like tourism, agriculture, constructions, some transport firms, among other businesses. We propose a model aimed to handle such situations. Let X represent the value of the firm, described by the following SDE under the historical probability measure, $\mathbb{P}$:

$$\begin{aligned} \frac{dX_t}{X_t} &= [\mu + \lambda \sin(\pi n t)]\, dt + \sigma dB_t \\ X_0 &= x_0 \end{aligned}$$

where $n > 0$ represents the number of cycles within a year, λ is a parameter accounting for the severity of the cycles, μ is the expected growth rate in absence of cycles and σ the volatility parameter. Note that the equation above satisfies condition **(A)** with $F(x,t) = \exp\left[\left(\mu - \frac{\sigma^2}{2} + \frac{1-\cos \pi n t}{\pi n t}\right) t + \sigma x\right]$ which is indeed invertible in x. Figure 12 presents a path of the process X with cyclical effects (solid curve) compared to an Geometrical Brownian motion path (dashed curve).

5.4 A model with stochastic and unobservable barrier

Lack of transparency may also obscure the company's level of debt. Giesecke and Goldberg (2004) - hereafter GG(2004) - model this situation considering that the market can observe the true value of the firm's assets, but not the default threshold, which is a random variable with known distribution.

Our framework permits to generalize GG(2004) model such that the unobservable barrier follows a stochastic process with known diffusion parameters. A a firm who does not reveal its leverage is

[14] We use the same methodology as [7]. See also the approximation formulas proposed in [1].

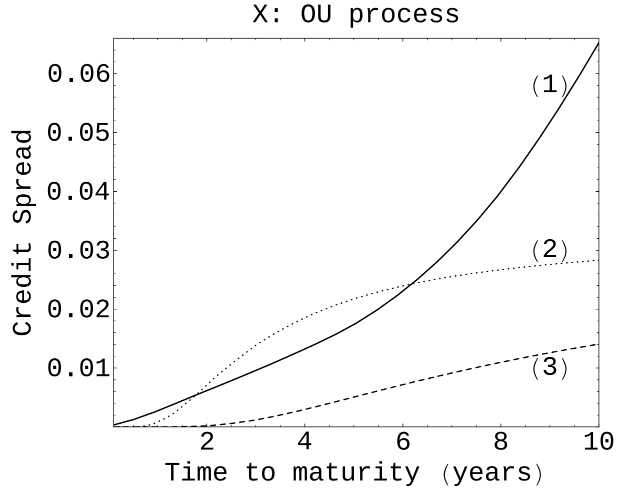

Figure 11: Credit spreads generated by our model: (1) market spreads with imperfect observation; (2) spreads computed with insider information versus (3) spreads with structural modeling.

likely to change as often as possible. Thus, we also avoid having zero short term spreads whenever firm's assets are above their historical minimum level, which was the case of GG(2004) model.

Several processes could stand for the firm's value or for the default threshold, but our purpose here is to illustrate with one example how to fit such a model in our framework.

Consider that under the historical probability measure, $\mathbb{P}$, the firm's value is an observable process following a Brownian motion with drift:

$$dV_t = \mu_1 dt + \sigma_1 d\beta_t$$

while the default threshold, k is unobservable for market participants and follow a mean-reverting process:

$$dk_t = \lambda(V_t - k_t - \theta)dt + sdB'_t.$$

The interpretation of the default threshold process is as follows: as long as k_t is less then the target level $V_t - \theta$, the firm tend to increase k_t and vice-versa. This reflect the fact that firms with an already high leverage are reluctant to issue new debt, while firms with low leverage have more chances to increase it. Due to the flexibility of our framework, we may set the fundamental process to:

$$X_t = V_t - k_t$$

and the barrier to $b(t) \equiv 0$. Remark that the process X is indeed unobservable, since k is unobservable. Finally, we set:

$$Y_t = V_t$$

which is by hypothesis observable. From Itô's lemma, the process X_t follows

$$dX_t = \mu\left(X_t\right) dt + \sigma dB_t$$

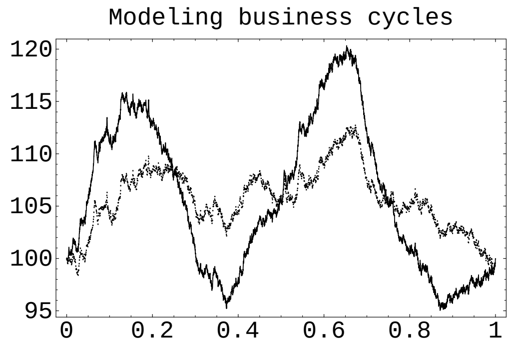

Figure 12: Accounting for cyclical effects. The figure displays at one year horizon a geometrical Brownian motion path (dashed curve) and the corresponding cyclical model path (solid curve), for 2 cycles within a year ($n = 2$) and $\lambda = 0.07$. The other parameters: $\sigma = 10\%$, $\mu = 3\%$, $x_0 = 100$.

where

$$\mu(x) = (\mu_1 + \lambda\phi) - \lambda x$$
$$\sigma = \sqrt{\sigma_1^2 + s^2 + 2\rho\sigma_1 s}$$
$$B_t = \frac{\sigma_1\beta_t + sB'_t}{\sigma}$$

and respects the assumption **(A')**.

6 Conclusions

We provided a framework for modeling informational noise on the market regarding the firm's fundamentals. Also, the corresponding pricing formulas were derived.

Incorporating such information imperfections modify the investors' perception about the real risks they face, hence it is not surprising the model predicts credit spreads much different from the perfect information models. An important feature is investors will always price a short-term default-risk premium, since the default time becomes totally inaccessible. A second important feature is the fact that the default probabilities are non-Markovian, a feature observed in practice, but for which no model-based explanation was given.

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