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A comment on “Intergenerational equity: sup, inf, lim sup, and lim inf”

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Abstract

We reexamine the analysis of Chambers (2009), that produces a characterization of a family of social welfare functions in the context of intergenerational equity: namely, those that coincide with either the sup, inf, lim sup, or lim inf rule. Reinforcement, ordinal covariance, and monotonicity jointly identify such class of rules. We show that the addition of a suitable axiom to this three properties permits to characterize each particular rule. A discussion of the respective distinctive properties is provided.

Key words: Social welfare function, Intergenerational equity, Lim sup, Lim inf

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1 Introduction

The resolution of distributional conflicts among an infinite and countable number of generations is subject to intense debate. When it comes to evaluating infinite utility streams different rules may be employed. Providing sets of properties that distinguish them is crucial as to the assessment of their normative appeal. After Koopman's (1960) axiomatic characterization of the discounting utilitarian rule (see also Lauwers, 1997) many authors have contributed to this aim. The Rawlsian infimum rule (also referred to as inf) is axiomatized in Lauwers (1997). Different versions of leximin and utilitarianism are characterized in Asheim and Tungodden (2004), and Basu and Mitra (2007) reobtain characterizations of the overtaking and catching up criteria that Asheim and Tungodden had axiomatized in terms of "preference continuity". Related references are d'Aspremont (2008), Asheim and Banerjee (2009), Bossert et al. (2007) among others.

Chambers (2009) has given a set of three conditions that identifies a family of criteria formed by the sup, inf, lim sup, and lim inf rules. Because those rules can not be differentiated according to Chambers' axiomatics, we build on Chambers (2009) in order to characterize the sup (resp. inf, lim sup, lim inf) rule in terms of suitable axioms.

We introduce our setting and properties in Section 2. Section 3 contains the characterizations as Corollaries to Chambers (2009), Theorem 1. Our conclusions are summarized in Section 4.

2 Notation and definitions

2.1 Chambers' characterization: the framework

Let $\mathbf{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$, that represents a domain of utility sequences or infinite-horizon utility streams. We restrict ourselves to study bounded real-valued sequences, that is, $\mathbf{X} = l_\infty$. The usual notation for utility streams applies: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. The constant sequence $(y, y, ...)$ is abbreviated as (y_{con}), and $(x, (y)_{con})$ holds for $(x, y, y, y, ...)$. We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

A *social welfare function* (SWF) is a function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$. Next we define particular SWFs with relevance in the literature.

Definition 1 *The Rawlsian infimum (or inf) rule is defined by:*

$$\text{for each } \boldsymbol{x} = (x_1, ..., x_n, ...) \in \boldsymbol{X},\ \boldsymbol{W}_R(\boldsymbol{x}) = inf\{x_1, ..., x_n, ...\}$$

Definition 2 *The supremum (or sup) rule is defined by:*

$$\text{for each } \boldsymbol{x} = (x_1, ..., x_n, ...) \in \boldsymbol{X},\ \boldsymbol{W}_s(\boldsymbol{x}) = sup\{x_1, ..., x_n, ...\}$$

Definition 3 *The lim inf rule is defined by:*

$$\text{for each } \boldsymbol{x} = (x_1, ..., x_n, ...) \in \boldsymbol{X},\ \boldsymbol{W}_{li}(\boldsymbol{x}) = lim\, inf\{x_1, ..., x_n, ...\}$$

Definition 4 *The lim sup rule is defined by:*

$$\text{for each } \boldsymbol{x} = (x_1, ..., x_n, ...) \in \boldsymbol{X},\ \boldsymbol{W}_{ls}(\boldsymbol{x}) = lim\, sup\{x_1, ..., x_n, ...\}$$

The following axiom captures efficiency displayed by each of these rules.

Monotonicity, also M. If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$.

Together with Monotonicity, Chambers (2009) uses the following two properties to axiomatize the family of rules formed by Definitions 1-4:

Reinforcement, also RI. For $K \in \mathbb{N}$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N} \times \{1, \ldots, K\}$, we write for each $i \in \mathbb{N}, \sigma(i) = (\sigma_1(i), \sigma_2(i))$, where $\sigma_1 : \mathbb{N} \to \mathbb{N}$ and $\sigma_2 : \mathbb{N} \to \{1, \ldots, K\}$.

Let $\{\mathbf{x}^j\}_{j=1}^K \subseteq \mathbf{X}$, where $K < \infty$. Suppose that $\mathbf{W}(\mathbf{x}^j) = \mathbf{W}(\mathbf{x}^k)$ for all $j, k \in \{1, \ldots, K\}$. Let $\sigma : \mathbb{N} \to \mathbb{N} \times \{1, \ldots, K\}$ be a bijection. Define $x^\sigma(i) \equiv x^{\sigma_2(i)}(\sigma_1(i))$. Then $\mathbf{W}(\mathbf{x}^\sigma) = \mathbf{W}(\mathbf{x}^1)$.

Ordinal Covariance, also OC. Let $\mathbf{x} \in \mathbf{X}$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. Then $\mathbf{W}(\varphi(\mathbf{x})) = \varphi(\mathbf{W}(\mathbf{x}))$.

2.2 Additional properties

In order to complement Chambers' argument we make use of further axioms on a social welfare function $\mathbf{W}$.

Restricted Sensitivity, also RS. There are $y, x \in \mathbb{R}$ such that $y > x$ and $\mathbf{W}(y, (x)_{con}) > \mathbf{W}(x_{con})$.

Along Definitions 1-4, only the sup rule agrees with RS. In the presence of OC, RS is equivalent to the following stronger property:

Sensitivity, also S. Whenever $y > x$, $\mathbf{W}(y, (x)_{con}) > \mathbf{W}(x_{con})$.

S is weaker than Koopmans' sensitivity in the first coordinate, which requests $\mathbf{W}(y, x_1, x_2, ...) > \mathbf{W}(x, x_1, x_2, ...)$ whenever $y > x$ irrespective of $x_1, x_2, ...$

Restricted Lower Sensitivity, also RLS. There are $y, x \in \mathbb{R}$ such that $y > x$ and $\mathbf{W}(y_{con}) > \mathbf{W}(x, (y)_{con})$.

Along Definitions 1-4, only the inf rule agrees with RLS. In the presence of OC, RLS is equivalent to the following stronger property (that is called *Restricted Dominance* (RD) in Asheim et al., 2008).

Lower Sensitivity, also LS. Whenever $y > x$, $\mathbf{W}(y_{con}) > \mathbf{W}(x, (y)_{con})$.

Weak Sensitivity, also WS. There are $x, y \in \mathbb{R}$ and $\mathbf{z} = (z_1, ..., z_n, ...) \in \mathbf{X}$ such that $\mathbf{W}(x, z_1, ..., z_n...) > \mathbf{W}(y, z_1, ..., z_n...)$.

WS is weaker than RS, RLS, and LS. Both the inf and the sup rules agree with WS, but neither the lim inf nor the lim sup rule do.

All the axioms above in this Subsection are specifications of the Weak Dominance axiom, which reads as follows: if $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$, with $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$. None of the rules under inspection displays this kind of efficiency.

Independent Future, also IF. For each $\mathbf{x} = (x_1, ..., x_n, ...)$, $\mathbf{y} = (y_1, ..., y_n, ...)$, and $x \in \mathbb{R}$: $\mathbf{W}(x, x_1, ..., x_n, ...) \geqslant \mathbf{W}(x, y_1, ..., y_n, ...)$ if and only if $\mathbf{W}(x_1, ..., x_n, ...) \geqslant \mathbf{W}(y_1, ..., y_n, ...)$.

Weak Non-Substitution, also WNS. For each z, t, for each $y > x$: $\mathbf{W}(z, (y)_{con}) \geqslant \mathbf{W}(t, (x)_{con})$.

Both the lim inf and the lim sup rules agree with IF and WNS, but neither the inf nor the sup rule fulfil any of these two properties. WNS appears in Asheim et al. (2008) as a weaker version of Lauwers' (1998) Non-Substitution condition. It is weaker than Hammond Equity for the Future.

In order to distinguish the lim inf and the lim sup rules we introduce two reinforcements of Monotonicity that are weaker than the standard Pareto axiom. Although neither of the rules in Definitions 1 to 4 are Paretian –in the sense $\mathbf{x} > \mathbf{y}$ implies $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$– they do exhibit some Paretian efficiency that we intend to capture.

For any threshold $t \in \mathbb{R}$, with each $\mathbf{x} \in \mathbf{X}$ we associate the streams $\mathbf{x}^*(t)$ and $\mathbf{x}_*(t)$ whose i-th component are defined by:

$$\mathbf{x}^*(t)_i = \begin{cases} x_i + 1 & \text{if } x_i \geqslant t \\ x_i & \text{otherwise} \end{cases} \qquad \mathbf{x}_*(t)_i = \begin{cases} x_i - 1 & \text{if } x_i \leqslant t \\ x_i & \text{otherwise} \end{cases} \tag{1}$$

Above-t Monotonicity, also AtM. The SWF satisfies M and $\mathbf{W}(\mathbf{x}^*(t)) > \mathbf{W}(\mathbf{x})$ for each $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbf{X}$ such that $x_i \geqslant t$ for an infinite number of generations.

Below-t Monotonicity, also BtM. The SWF satisfies M and $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{x}_*(t))$ for each $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbf{X}$ such that $x_i \leqslant t$ for an infinite number of generations.

The SWFs defined by lim sup and supremum satisfy the AtM property, which is not satisfied by the lim inf and the infimum rules. Opposite, BtM is satisfied by the lim inf and the infimum rules, but not by the lim sup and the supremum.

Figure 1 shows the relationships among the efficiency axioms in use.

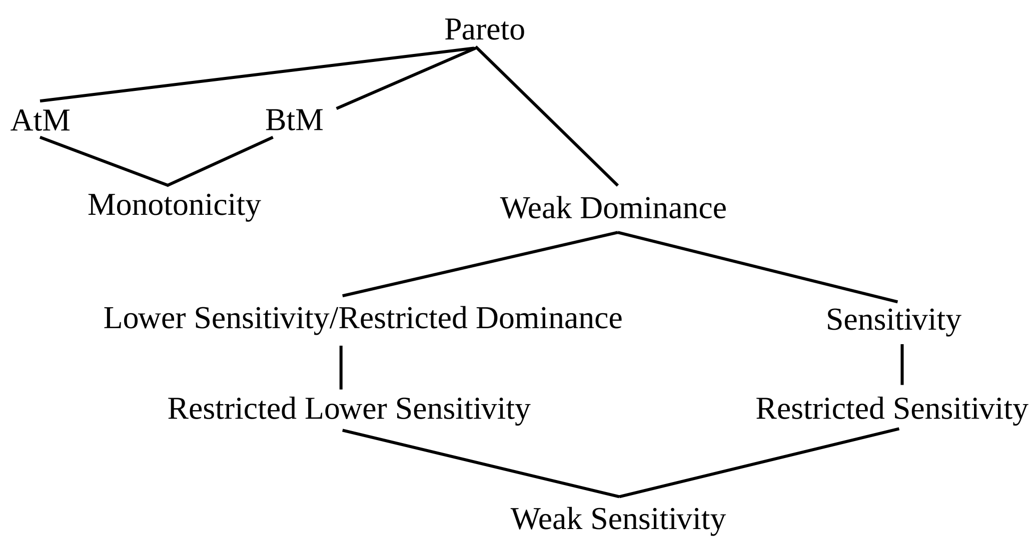

Fig. 1. The efficiency axioms under study.

3 Results

In this Section we provide Corollaries to the following result.

Theorem 1 (Chambers, 2009) *A SWF defined on $\boldsymbol{X} = l_\infty$ agrees with either the supremum, the infimum, the limit superior, or the limit inferior rule, if and only if it satisfies M, OC, and RI.*

From Theorem 1 the next Corollaries follow:

Corollary 1 *A SWF defined on $\boldsymbol{X} = l_\infty$ is the infimum rule if and only if it satisfies M, OC, RI, and RLS (resp., LS).*

Proof: The necessity part has been already stated. As for sufficiency, apply Theorem 1 and discard the supremum, lim inf, and lim sup because they do not satisfy RLS, which is equivalent to LS under M. $\triangleleft$

Corollary 1 adds to Lauwers' (1997) characterization of the infimum rule in terms of related axioms (v. Chambers, 2009, Section 4 for a discussion on this topic).

By using analogous direct arguments, further Corollaries identify the supremum / lim sup / lim inf rules.

Corollary 2 *A SWF defined on $\boldsymbol{X} = l_\infty$ is the supremum rule if and only if it satisfies M, OC, RI, and RS (resp., S).*

Corollary 3 *A SWF defined on $\boldsymbol{X} = l_\infty$ is the lim sup rule if and only if it satisfies AtM, OC, RI and WNS (or IF).*

Corollary 4 *A SWF defined on $\boldsymbol{X} = l_\infty$ is the lim inf rule if and only if it satisfies BtM, OC, RI and WNS (or IF).*

4 Summary of results and conclusions

Chambers (2009) has provided three axioms that, together with suitable comparisons of streams, permit to identify four rules for assessing infinite utility streams. For example, the infimum rule is characterized by M, OC, RI, and $\mathbf{W}_R(0,(1)_{con}) = \mathbf{W}_R((0)_{con})$. The supremum rule is characterized by M, OC, RI, and $\mathbf{W}_s(1,(0)_{con}) > \mathbf{W}_s((0)_{con})$. The lim inf rule is characterized by M, OC, RI, $\mathbf{W}_{li}(1,0,1,0,1,...) = \mathbf{W}_{li}((0)_{con})$, and $\mathbf{W}_{li}(0,(1)_{con}) > \mathbf{W}_{li}((0)_{con})$. Finally, the lim sup rule is characterized by M, OC, RI, $\mathbf{W}_{ls}(1,0,1,0,1,...) > \mathbf{W}_{ls}((0)_{con})$, and $\mathbf{W}_{ls}(1,(0)_{con}) = \mathbf{W}_{ls}((0)_{con})$. The distinction of cases in Chambers' proof already hints these consequences.

We have built on Chambers' theorem to produce characterizations in terms of normative axioms for the four rules under inspection. Table 1 collects the properties that have served us to state different Corollaries to that focal result.

	Axioms						
Criteria	*RLS/LS*	*RS/S*	*AtM*	*BtM*	*WS*	*IF*	*WNS*
inf	+	—	—	+	+	—	—
sup	—	+	+	—	+	—	—
lim inf	—	—	—	+	—	+	+
lim sup	—	—	+	—	—	+	+

Table 1. Properties of the criteria under inspection.

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