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The PQ-Median Problem: Location and Districting of Hierarchical Facilities. Part I

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Abstract

To achieve an efficient and effective hierarchical location system, it is necessary to obtain not only an efficient set of facility locations, but also an efficient districting of the catchment areas, since these areas will be the ones to benefit from the services provided by the located facilities. In seeking an effective relation among the different levels in a hierarchy of facilities, all the cells assigned to a particular facility at one hierarchical level should belong to one and the same district in the next level of the hierarchy. This property leads to a "coherent" districting structure. The research presented here concerns the optimal location and districting of hierarchical facilities in a network. A linear integer model that locates facilities in a two-level hierarchical system is presented in this paper: the pq-median model minimizes average distance to the closest facility in each hierarchy, while coherence among the levels of the hierarchy is enforced. Solution methods and computer times are provided.

1 Introduction

The past three decades have witnessed an explosive growth in the field of network-based facility location modeling. As Krarup and Pruzan (1983) point out, this is not at all surprising since location policy is one of the more profitable areas of applied systems analysis and ample theoretical and applied challenges are offered. Location-allocation models seek the location of facilities and/or services (e.g. schools, hospitals, warehouses) so as to optimize objectives generally related to the efficiency of the system or to the allocation of resources.

Most of the research to date has focussed on facilities that are assumed to be alike or of a single type. Nevertheless, it is widely accepted that many facility systems and institutions are hierarchical in nature, providing several levels of service. More specifically, a hierarchical system is one in which services are organized in a series of levels that are somehow related to one another in the complexity of function/service.

The organizational structure of hierarchical systems may vary considerably. There may be institutional ties between levels, whereby lower levels are administratively subordinate to higher ones (e.g., health care delivery systems, banking systems). On the other hand, there are several hierarchical systems that have no such inter-level linkages, different levels being distinguished solely by the range of goods and/or services they provide (e.g., educational systems, production-distribution systems, waste collection systems) (Hodgson 1986).

A number of objectives and constraints may be stated for an efficient siting of hierarchical facilities. First, facilities may be located to attract the maximum utilization of services or generate the maximum demand for goods (Getis and Getis, 1966); Secondly, the number of hierarchies and facilities are to be the minimum possible. Third, a facility may have a threshold requirement in its location and can be classified into a range of orders on the basis of the size of the threshold requirement.

A fourth consideration in the design of hierarchical location-allocation systems is that the services offered by the facilities are nested; that is, a level k facility offers services of type k, k-1,k-2,...,1. For example, in a health care delivery system, large hospitals supply to the population all the health care services offered by smaller hospitals as well as the less frequently required specialty and superspeciality services that need more sophisticated equipment and skilled personnel.

To achieve an efficient and effective hierarchical system, it is necessary to obtain not only an efficient set of locations but also an effective districting of the catchment areas, since these areas will be the ones to benefit from the services provided by the located facility (Bach 1980, Serra 1988). Pezzella et al. (1981) point out that one of the basic problems in the organization of social services is the division of the territory under study into a series of districts, each of which provides an integrated and possibly complete set of services to its assigned population. Districting can be defined as the process by which a given area (e.g., a state) is partitioned into small areas (districts) each of which is assigned (a) given function(s) or service(s). Alternatively, districting can be defined as the method of dividing a region into sub-regions according to some organizational schema. A district is generally formed by a unitary assignment of smaller subdivisions (e.g., census tracts), to a larger division.

Districts often form the basis of government through election districts and provide for the orderly and efficient availability of services, such as health care, refuse collection, police patrols, and the allocation of children to schools.

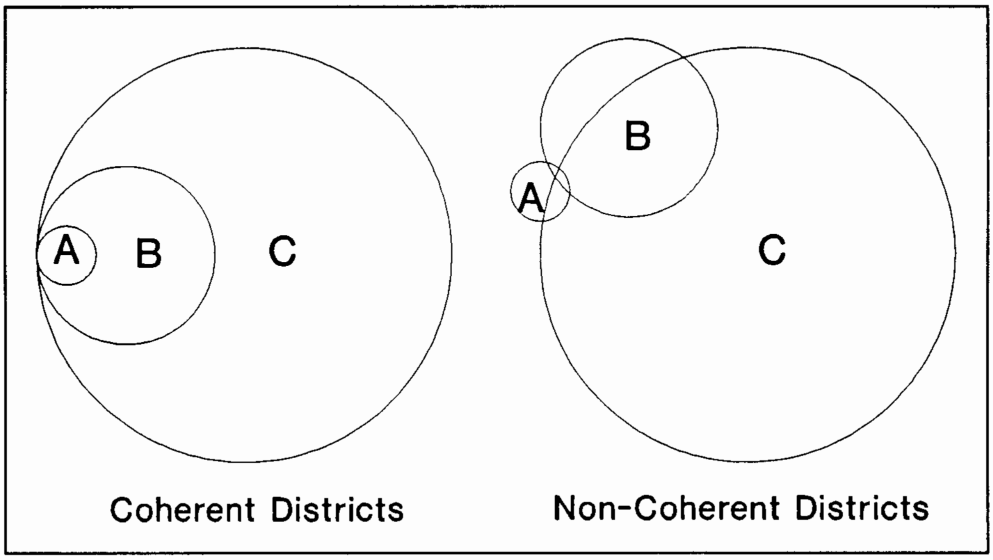
Most districting plans are hierarchical in nature. Districts created at one level of the hierarchy may be the building blocks for larger districts at the following level. For example in a health care districting plan, a catchment area corresponding to a large hospital may be disaggregated into smaller areas, each one of those belonging to a basic health care center. The criteria of contiguity, compactness and relative population equality are important across levels, i.e., they are expected to occur at each level. A fourth criterion, specific to hierarchies is "coherence".

The coherence property, which creates an effective relation between levels, may be thought in the following way: If an area X is assigned to a facility A at level k, and this facility A is assigned to another facility B in the next level (k+1) then the area X is also assigned to the same facility B in level k+1. This "transitive" property of some hierarchical systems leads to a "coherent" districting structure. All areas assigned to a particular lower level facility are assigned to one and the same higher level facility. Figure 1 presents two districting situations. The top figure represents a region with three districts, represented by circles. The first district is completely included in the second district, and the second one is entirely included in the third district. This region is districted with coherence.

The bottom picture in Figure 1 presents a non-coherent districting pattern: neither district A nor district B are included in their next higher hierarchical district. If coherence is not observed, then there may be regions that are not continuous, that is, the areas of a region assigned to a facility at level k may be associated with more than one facility at level k+1.

Consider the health care delivery system where there is generally a referral between levels, that is, users of a given level are sent to another level for more specialized treatment. At each successive level, the amount of demand is at least partially determined by the number of patients referred from lower levels in the hierarchy. The patient, once directed to a larger hospital for specialized treatment, will in a non-emergency situation generally go back to his home, make an appointment, and then go to the hospital. In such a system, the most important feature from a location viewpoint is the distance or travel time between the demand area and the nearest facility at each level of the hierarchy, not the distance or time between the facilities themselves.

Figure 1: Non-Coherent Versus Coherent Districts



This paper proposes a new formulation to address the issue of coherence in a locational hierarchy, a task not previously undertaken to the authors' knowledge. While location and districting models have traditionally been studied separately, their integration promises improvements in the optimal allocation of resources in space. This research seeks to link coherence and efficiency within a single hierarchical model.

Two hierarchical levels are considered in this paper: the first level corresponds to facilities that offer basic services and cover a smaller amount of area (or population) relative to the second level. Hence it is assumed that there will be a larger number of first level facilities than second level facilities. For ease of notation, facilities belonging to the first level will be called type A facilities, and those pertaining to the second level will be designated as type B facilities. The services offered by type B facilities are nested, that is, they also include the services offered by type A facilities. Following the classification presented by Narula (1983), the model locates coherent facilities with successively inclusive services.

The pq-median model locates two types of facilities by combining two p-median formulations. Each level has the objective of minimizing the average distance or travel time from the demand areas to the nearest facility whilst ensuring coherence. Hence, a trade off between access to each hierarchical level is expected.

2 Literature review

The p-median model locates p facilities such that the average distance from the users to their closest facility is minimized. In a hierarchical setting it has been generally used to located a given number of facilities for each level, one at a time.

Calvo and Marks (1973) constructed a multiobjective integer linear model to locate multi-level health care facilities: the model minimized distance (travel time), minimized user costs, maximized demand or utilization, and maximized utility. It was based on assumptions that (1) users go to the closest appropriate level; (2) there is no referral to higher levels; and (3) all facilities offer lower level services.

Tien et al. (1983) argued that the approach taken by Calvo and Marks resulted in a locally inclusive location pattern. In order to resolve this deficiency, they presented models derived from Calvo and Mark's formulation: (1) a successively exclusive model (i.e., non-nested) and a (2) successfully inclusive model. They also introduce a new feature whereby a demand cannot assign to a place more than once even if additional service levels may available at that point. Both models, unlike Calvo and Marks', can be solved by standard integer programming solution procedures. Mirchandani (1987) extended the hierarchical p-median formulation of Tien et al. model, allowing various allocation schemes by redefining the cost parameter in the objective.

Harvey et al. (1974) used a p-median formulation to determine the number and optimal locations of intermediate level facilities in a central place hierarchy. The p-median model was used in one-level problem, but consideration was given on the interaction among lower and higher levels.

Narula et al. (1975) developed a nested hierarchical health care facility location model that located on a network a fixed number of facilities. At each level, the objective was to minimize patients' total travel. They considered referrals between levels, based on the proportion of patients treated at each level. Narula and Ogbu (1979) gave some heuristic procedures for the solution of the problem. Narula and Ogbu (1985) solved a two-level successively inclusive mixed-integer p-median problem using the same objective and referral pattern.

Berlin et al. (1976) studied two hospital and ambulance location problems. The first one focused on patient needs by minimizing (1) average ambulance response time from ambulance bases to demand areas and (2) average distance to hospitals from demand areas. The second model added a new objective to take into account the efficiency of the system: minimization of (3) distance from ambulance bases to hospitals. It was named the "dual-facility" location problem: the locations of both hospitals and ambulance depots were basic to determine response times. It is interesting to note that although two levels are defined (stations or depots where ambulances sit, and hospitals), the formulation can be decomposed into independent hospital and ambulance location problems and solved optimally. It is not a clear hierarchical model since relations among levels differ from the traditional regionalized models.

Fisher and Rushton (1979) and Rushton (1984) used the average and maximum distance from any demand area to its closest health care center to study and compare actual and optimal hierarchical location patterns in India. The Teitz and Bart heuristic was used in three ways to determine hierarchies: constructing top-down hierarchical procedure (same as Banergi and Fisher 1974), constructing a bottom-up hierarchical procedure (opposite of top-down), and constructing a hierarchical procedure where the first step was to locate a middle-level of the hierarchy optimally, and then proceed as the bottom heuristic for upper levels, and use the top-down heuristic for lower levels.

Tien and El-Tell (1984) defined a two-level hierarchical LP model consisting of village and regional clinics. It is a top-down formulation in the sense that the flow patterns start at the hospitals. That is, health professionals go from hospitals to village centers. Both village and regional clinics are located using a criterion of minimizing the weighted distance of assigning villages to clinics and village clinics to regional clinics. The model has been applied to 31 villages in Jordan.

Hodgson (1984) demonstrated that the use of top-down or bottom-up techniques to locate hierarchical systems generally leads to suboptimal locational patterns. By a top-down (bottom-up) technique is meant the location first of the highest (lowest) level of the hierarchy and then successive location of facilities in the following level. Hodgson used both the p-median model and a formulation based on Reilly's gravitational law (Reilly, 1929) to compare both techniques with the simultaneous location of all hierarchies.

3 The PQ-Median Model

Coherent hierarchies have two main characteristics that distinguish them from conventional hierarchies. The first one is related to the relationship between the levels of a hierarchy. Basically, a hierarchy is coherent if all population areas assigned to a particular k level facility are also assigned to one and the same (k + 1) level facility. The second characteristic is that the access time between the population and each of the hierarchical levels is accounted. Both these features must be reflected in the formulation of a coherent hierarchical location model.

The p-median problem seeks to select p facility sites from among n eligible locations to minimize the average distance from the population locations to their closest facility. An interesting characteristic of this formulation is that its solution determines the catchment areas of each facility located, since it uses assignment variables that express the linkages between population areas and designated facilities. This feature is very useful in the construction of hierarchical coherent models.

Suppose one solves a p-median model and a q-median model, one for each level of the hierarchy, but with no connections whatsoever between the models. The outcomes will be that all population areas are assigned to their closest type A facility and also to their closest type B facility; average distance is minimum in both cases. However, it is very likely that the population areas that are assigned to any particular type A facility will not all be assigned to the same type B facility, i.e. the solution is not coherent. In addition, in this case, type B facilities cannot be considered as offering type A services since they were not taken into account when solving the p-median problem for the type A facility. The pq-median model will attempt to resolve these two issues.

The pq-median model combines two p-median formulations, one for each hierarchical level. The relation between each level is expressed by two factors: (1) that type B facilities also offer type A services; hence, a given area can be assigned to a type B facility for type A services (successively inclusive services), and (2), that all areas assigned to a type A facility must be assigned to one and the same type B facility (coherence condition). The number of facilities to locate is known: p type A facilities and q type B facilities are located where p > q . The mathematical formulation of the pq-median model is as follows:

$$Min\ A = \sum_{i \in I} \sum_{j \in J} a_i d_{ij} x_{ij} \qquad Min\ B = \sum_{i \in I} \sum_{k \in K} a_i d_{ik} y_{ik}$$

Subject to:

$$\sum_{j \in J} x_{ij} = 1 \qquad \forall i \in I \qquad (1)$$

$$\sum_{k \in K} y_{ik} = 1 \qquad \forall i \in I \qquad (2)$$

$$\sum_{k \in K} z_{jk} = 1 \qquad \forall j \in J \qquad (3)$$

$$x_{ij} \leq u_j \qquad \forall j \in J \qquad (4)$$

$$x_{ij} \leq u_j + v_j \qquad \forall i \in I \quad \forall j \in J \qquad (5)$$

$$x_{ij} \leq v_j \qquad \forall i \in I \quad \forall j \in J \qquad (6)$$

$$y_{ik} \leq v_k \qquad \forall i \in I \quad \forall k \in K \qquad (7)$$

$$x_{ij} + z_{jk} \leq y_{ik} + 1 \qquad \forall i \in I \quad \forall j \in J \quad \forall k \in K \qquad (8)$$

$$\sum_{j \in J} u_j = p \qquad (9)$$

$$\sum_{k \in K} v_k = q \qquad (10)$$

$$x_{ij}, y_{ik}, z_{jk}, u_j, v_k = (0,1) \qquad \forall i \in I \quad \forall j \in J \quad \forall k \in K \qquad (11)$$

Where:

i, I = index and set of demand areas;

j, J = index and set of possible locations for type A facilities;

k, K = index and set of possible locations for type B facilities;

J_A = set of possible locations for type A facilities only; it is $J - (J \cap K)$;

J_{AB} = set of possible locations for type A facilities and type B facilities; it is $J \cap K$;

K_B = set of possible locations for type B facilities only; it is $K - (J \cap K)$;

a_i = population at demand area i;

d_{ij} = distance between i and j;

p = number of type A facilities to be located;

q = number of type B facilities to be located;

x_{ij} = 1, if demand node i is assigned to a facility at j for type A services; 0, otherwise;

y_{ik} = 1, if demand node i is assigned to a facility at k for type B services; 0, otherwise;

z_{jk} = 1, if type A facility at j is assigned in the hierarchy to a type B facility at k; 0, otherwise;

u_j = 1, if there is a type A facility at j; 0, otherwise; and

v_k = 1, if there is a type B facility at k; 0, otherwise.

Both of the objectives correspond to the objective used in the classical p-median formulation. The first objective minimizes the total population weighted distance from demand areas to type A services (whether found at type A or type B facilities). The second objective minimizes the total population weighted distance from demand areas to services which are only found at type B facilities.

Demand areas are assigned to one and only one facility for type A services and to one and only one type B facility for type B services by virtue of constraints (1) and (2) respectively. Similarly, constraint group (3) forces a type A facility to assign to one and only one type B facility. This assignment creates the functional link in the hierarchy. Constraint set (4) says that if demand area i is assigned for level A services to node j ($j \in J_A$), then this node must a have a type A facility. Observe that this node is not a candidate for a type B facility as it is capable of housing only a type A facility. If node j is candidate for both type A and type B facilities, then constraint type (5) is used; this constraint allows a node i to assign either to a type A or to a type B facility for its type A services. That is, demand nodes may assign to a type B facility for type A services if this assignment gives a better overall objective than assigning to a type A facility for the same services. If node j can only have type B facilities ($j \in K_B$) then node i will be free to assign to it for type A services only if a type B facility is sited there (constraint group (6)). Observe that constraints (4), (5) and (6) together account for all possible assignments to type A services.

Constraint (7) says that if demand area i is assigned to node k, k in K, for type B services, then there must be a type B facility located at node k. Constraints (8) enforce "coherence" for the location pattern. All centers assigned to the same type A facility are also assigned to the same type B facility. Mathematically this constraint works as follows: if x_{ij} is 1 (i is assigned to node j for type A services) and z_{jk} is 1 (type A facility located at node j is assigned to a type B facility located at k) then y_{ik} must be one (demand area i is assigned for type B services to the type B facility located at k).If x_{ij} and/or z_{jk} are equal to 0, then y_{ik} will not be equal to 1, since there is only one assignment allowed by constraints (1), (2) and (3), and this one will occur only when x_{ij} and z_{jk} equal 1. Finally, constraints (9) and (10) determine the number of type A and type B facilities to be located respectively.

This formulation is specific for a coherent facility hierarchy with successively inclusive services, which is the more standard type of coherent hierarchy. Nevertheless, the model can be transformed to accomodate other types of service hierarchies.

If the facilities of the hierarchy present successively exclusive services, that is, if type B facilities do not offer type A services, constraints (5), (6) and (7) can be replaced by the following constraint set:

$$x_{ij} \le u_j \qquad \forall i \in I \quad \forall j \in J \qquad (12)$$

Now demand area i has to assign to a type A facility for type A services, since type B facilities are not offering this class of service.

The formulation can also be modified to consider locally inclusive services, that is, a type B facility offers type A services only to the node where the facility is located. In this case, constraints (5) and (6) are replaced by the following constraints:

$$x_{ij} \leq u_j + v_j \qquad \forall i \in I \cap J_{AB} \quad \forall j \in J_{AB} \qquad (13)$$

$$x_{ij} \leq v_j \qquad \forall i \in I \cap K_B, \ \forall j \in K_B \qquad (14)$$

Constraint groups (13) and (14) allow demand area i to assign to a type B facility for type A services if and only if this facility is located at the same demand area i. In any other case the demand area will be necessarily assigned to a type A facility for type A services.

Note that the pq-median model can be transformed into a hierarchical pq-median with maximum distance constraints by substituting h_{ij} in both objectives for the parameter d_{ij}, where h_{ij} is equal to a large number M if the distance between areas i and j is greater than a given distance standard, and d_{ij} otherwise.

The censuses of most cities have their census tracks defined with homogeneous areas in population size[2]. Each grid cell was originally designed to have the same population. Due to changes in morphology and migration between these areas, the total population of each area may generally vary from the original constant by relatively small amounts. The following constraints can be added to obtain districts relatively equal in population.

$$c^{A_j} \leq \sum_{i \in I} x_{ij} \leq f_j^A \qquad \forall j \in J \qquad (15)$$

$$c_k^B \leq \sum_{j \in J} z_{jk} \leq f_k^B \qquad \forall k \in K \qquad (16)$$

where x_{ij} is the number of population units assigned to j, and c^A and f^A are the lower and upper limits, given in population units, of census areas to be assigned to type A facilities. The z_{jk} are the number of type A facilities in population units assigned to k, and c^B and f^B are the lower and upper limits of type A facilities to be assigned to type B facilities. Observe that this is equivalent to setting capacity constraints on the facilities, since the census tracks are quite homogeneous in population. This would not be the case if there were substantial differences in the population size of each area.

[2]Baltimore and Barcelona are examples

The constraints on the number of facilities to be located can be replaced by a budget constraint, and trade-offs between average distance to both types of facilities and budget can be examined. Solution methods would differ from those discussed next, which are specific to problems with limited numbers of facilities.

4 Solution Method

Formulations of the p-median and plant location problems generally require large storage and computational time for reasonably sized problems, when solved using the revised simplex method with branch and bound when necessary. The pq-median model does not escape this problem. The linear programming formulation of the pq-median presents a very large number of constraints and variables. The Balinski type constraints (4) - (7), together with the large number of integer variables involved in the model, makes traditional solution methods such as linear programming with branch and bound when needed too expensive. As the size of the network increases the number of variables and constraints increases exponentially. For example, in a 25-node network, there are 1925 integer variables and 16,952 constraints. If the network had 107 nodes, there would be 34,668 variables and more than 1.2 million constraints!

The number of variables in a p-median model can be reduced by the following method. In the p-median problem, nodes assign to their closest facility. If there are p facilities to locate, then a node can assign to all nodes except for the (p - 1) furthest potential candidate nodes from it. The corresponding (p - 1) x_{ij} variables will always equal 0 in the optimal solution and therefore can be discarded. That is, for each node, p - 1 of the x_{ij} variables can be eliminated. Therefore, the total number of assignment variables in the traditional p-median problem that can be eliminated with certainty that the final solution will not be affected is equal to np - n), where n is the number of nodes in the network (see Rosing et al. 1979).

In the case of the pq-median there are 3 types of assignment variables: x_{ij}, y_{ik}, and z_{jk}. The first group of variables, x_{ij}, can be reduced, as in the p-median model, from n^2 to $n(n - p - q + 1)$ variables, since there are p + q facilities that offer type A services. The second group, y_{ik}, can be reduced from n^2 to $n(n - q + 1)$. The number of z_{jk} variables cannot be reduced with certainty that no alteration of the solution will occur if all nodes are candidate for a type A or type B facility. In this case there are n^2 of the z_{jk} variables. The total reduction in the number of variables is np - 2nq - 2n.

The number of constraints can also be reduced in the same way. Each assignment variable has a corresponding Balinski type constraint and a coherence constraint. By eliminating some of these variables, some constraints are redundant and therefore can be eliminated. For the pq-median, the full blown problem has $3n + 3n^2 + n^3 + 2$ constraints. The number of constraints can be reduced without changing the formulation to $3n + n(n - p + 1) + n(n - q + 1) + n^2 + (n - p - q + 1)(n - q + 1)n$. For example, if n = 25, p = 5 and q = 2, the number of variables will go from 1,925 to 1,700 (-11.6%) and the number of constraints will go from 17,577 to 13,177 (-25.0%). The larger the number of facilities to locate, the larger the amount of the reduction.

The Balinski type constraints (4) to (7) could be replaced by Effroymson and Ray (1966) constraints, but although the number of rows in the problem would be reduced slightly, it is virtually certain that the continous solution of the linear program will have fractional variables.

A solution of the pq-median problem must have all its variables equal to 1 or 0. Since the coefficient matrix of the formulation is not unimodular[3], it is likely that some non-integer solutions will be found and that branch and bound will be needed. Nevertheless, it may not be necessary to declare all variables integer when re-solving the problem using branch and bound as a post-simplex resolution method. Actually, in most cases, only the locational variables u_j and v_k need to be declared integer. The assignment variables x_{ij}, y_{ik}, and z_{jk} can be allowed to be continous, and most of the time the solution will be integer. Since this is a minimization problem, the program will try to set to one (the upper bound of the continous variables) the assignment variable that has the smallest coefficient in the objective. Constraints (1) (2) and (3) will not affect the integer properties, since by setting only one assignment variable equal to one they are met. Since the locational variables are integer, the Balinski type constraints (4) to (7) will not force the assignment variables to be a fraction. They will still be free to be 0 or 1. Coherence constraint (8) can cause some values to be non-integer, since variables z_{jk} are not in the objective. Nevertheless, none of the problems solved in our research did require a declaration of the assignment variables as integer.

5 Computational Experience

The pq-median formulation has been solved in a 25-node network using relaxed linear programming; branch and bound was used when necessary to resolve fractional solutions. A commercial software package, MPSX/MIP[4] was used on an IBM 3090-E600 mainframe computer to obtain optimal solutions. The full specification of the 25-node problem involved 1,925 integer variables and more than 17,500 constraints. The constraint set on coherence alone had 25^3 elements. In order to reduce the problem size the number of assignment variables x_{ij}, y_{ik}, and

[3]For a discussion of unimodularity, see Garfinkel and Nemhauser (1972) p. p. 66-70, Taha (1975) p. 5.

[4]MPSX: Mathematical Programming System EXtended/370. MIP: Mixed Integer Programming/370. This software was developed by IBM.

z_{jk}, was cut down by using the Rosing et al. approach described in section 4. The weighting method was used to solve the multiobjective problem. In order to identify the non-inferior set in objective space, several runs were made with a different set of weights for each combination of type A and B facilities.

Since optimal solutions are known (by solving the problem using linear programming and branch and bound when needed) the optimal trade-off curve is also known. It is possible, however, that some of the non-inferior points were not found, even though more than ten runs were done with different weights to try to obtain all of them. The use of the weighting method for an integer programming problem implies that one cannot be certain that all of the non-inferior solutions are found (Cohon 1978). Indeed, the use of the weighting method for 0,1 problems is unlikely to generate all non-inferior points because of the occurrence of gap points.

Three non-inferior points (labeled A, B and C in Figure 2) were found when locating three type A facilities and two type B facilities. There is a clear trade-off between both OBJA and OBJB. For example, if one moves from point A to point B in Figure 2, the average distance to type B facilities increases by only 3 percent, while the average distance to type A facilities is reduced by almost 4 percent (more or less 600 meters).

Two solutions are presented in Figure 3. These solutions correspond P and Q in Figure 2, obtained when wA = 1 and wA = 0 were used. The discontinuous arrows show that all nodes assigned to a type A facility are also assigned to the indicated type B facility. For example, in the top picture of Figure 3 all nodes assigned to facilities at nodes 9, 10 and 17 for type A services are assigned exclusively to node 17 for type B services. Therefore, coherence is satisfied.

Table 1 presents the results of the run times of the linear relaxation / branch and bound solutions. The total number of runs in the second column correspond to the number of different weights used. The third column gives the number of times that the solution to the relaxation was integer. The fourth column presents the average CPU time per run that MPSX took to find the continuous solution. The fifth column gives the additional time that it took to find the integer solution after the continous solution was found. Therefore, the total average run time is the sum of the average continuous solution time plus the additional time spent to find the integer solutions. These times are shown in the last column of Table 2. Since each branching requires solution of two new problems, it ca be seen from the average additional run time the very few iterations of Branch and Bound that were actually needed.

It is interesting to note that as the number of type A facilities to locate increases, the average run time for the continuous solution decreases considerably. On the other hand, the additional average run time to obtain integer solutions does not show this behavior.

Table 1: Average CPU Time, pq-Median LP+B&B, 25 nodes

# of facilities located (p-q)	Total number of runs	Optimal int. sol. using linear relax. only	Avge. run time linear solution (CPU sec.)	Avge. additional run time using B&B (CPU sec.)	Total avg. run time (CPU sec.)
2-1	11	3	147.6	106.7	254.3
3-2	10	1	165.1	165.1	227.7
4-2	12	3	101.4	101.4	180.6
5-2	13	2	92.3	92.3	134.7
6-2	11	3	68.5	68.5	148.7
6-3	12	3	139.0	139.0	166.1

Figure 2: Trade-off Curve, PQ-Median Problem
25-node network

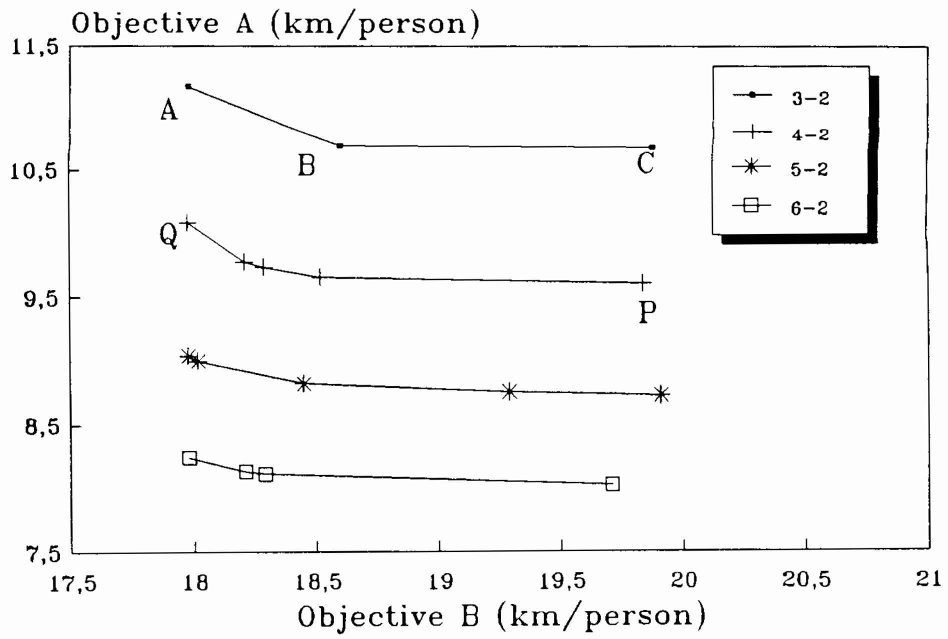
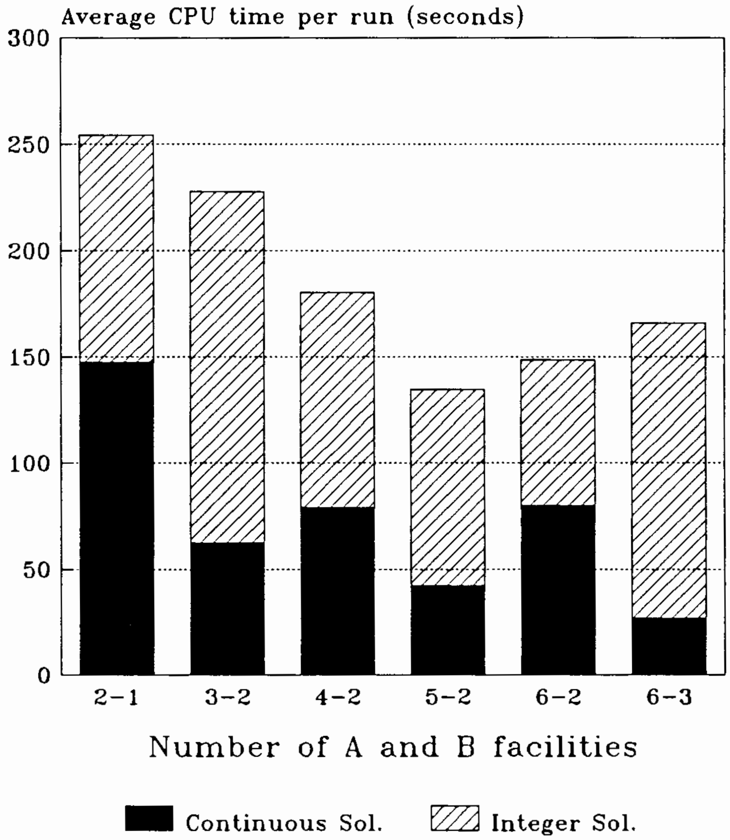


Figure 4: Average CPU Time, MPSX
PQ-Median Problem (25-node network)
IBM 3090-E600



6 Conclusions and Further Research

The discipline of location-allocation theory is rich in formulations designed to model hierarchical systems. Although location models and districting models are studied by the same discipline and are closely related to the design of an optimal spatial allocation of resources, they have, up till now, been studied separately and without connection. The definition of coherence in the context of hierarchical systems, which states that the entire area assigned to a facility at a given level has to be assigned to one and the same facility at the next higher level, joins together these two branches of location-allocation modelling. Coherence also plays an extremely important role in defining administrative regions for better management and in providing integrated planning units for policy studies.

A linear integer multiobjective formulation has been presented to address the issue of location and districting of hierarchical facilities with coherence. The pq-median model locates facilities in a two-level hierarchical system by minimizing the average distance to the closest facility at each level, subject to mandatory coherence. In a two level hierarchy, a trade-off is seen to exist between the average distance to each of the two levels. This trade-off suggests that a decision-maker must focus on the relative importance of the two levels of service in arriving at a final configuration of facilities.

The p-median models tend to embody large formulations due to their rigid assignment structure which requires the use of assignment variables. Most hierarchical models based on p-median formulations to date are expensive to solve if traditional optimal methods such as linear programming and branch and bound (LP+B&B) are used.The pq-median model does not avoid this problem, but the methodology does provide globally optimal solutions. The combination of two p-median models, one for each level, together with the constraint on coherence makes the pq-median model very large even with a relatively small network, and the size of the model increases very fast with the number of nodes in the network. Thus, alternative solutions methods need to be found to obtain optimal or near optimal solutions to the pq-median problem. The value of the research presented here is two-fold. first, we have formally defined coherence and indicated its role in district formation. Second, we have created a formulation with coherence enforced, and we have implemented a solution method which provides optimal answers and tradeoffs. The methodology suggested here can be used to provide bench mark solutions for use in evaluating heuristic approaches to the pq-median problem. Since heuristic algorithms will undoubtedly be needed for larger problems, a standard against which they can be compared is needed. This work provides that standard. We expect that further research in this area will be very rich and fruitful.

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