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Proposed Running Head: Optimal Growth: An Example

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Abstract

The dynamics of a welfare maximizing, heterogeneous agent, one sector optimal Ramsey model is analyzed assuming two agents, each with a distinct discount factor and log utility. Production is Cobb-Douglas. Explicit time varying policy functions are derived, one for each period. A Twisted Turnpike Property and eventually monotone dynamics are demonstrated to govern the evolution of the economy's aggregate capital stock.

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1 Introduction

The object of this paper is to modify the canonical representative agent discounted optimal growth model with log utility and Cobb-Douglas production function to include many households, each with a different discount factor. The planner's welfare function is taken to be a weighted function of the underlying households' intertemporal utility functions. The weights are predetermined and fixed for all time as the planner solves the optimization problem. This maximization problem is a well-known way to compute a particular Pareto optimal allocation for a many agent Ramsey model.[1] By varying the welfare weights it is possible to trace out the economy's utility possibility frontier and find all the Pareto optimal allocations. This procedure also yields a proof of the existence of a competitive equilibrium by combining the welfare theorems with Negishi's (1960) existence argument.

The qualitative properties of the welfare maximization problem and the detailed development of the Negishi argument can be found for the general one-sector model in the papers by Duran and Le Van (2003) and Le Van and Vailakis (2003). The latter authors prove that the optimal capital sequence is convergent, but not necessarily monotonic, in a general one-sector framework. My example supplements their results by showing the optimal paths starting from different initial stocks come together in the limit. Mitra (1979) calls this the optimal capital sequences' *twisted turnpike property*. A second contribution is to show that each optimal capital sequence *starting from an arbitrary initial capital stock is eventually monotonic*.

That eventual monotonicity of the optimal capital sequence is the best possible convergence property reflects a point made by Le Van and Vailakis (2003). Consider the case where both agents have positive welfare weights, yet the economy starts off with the stationary optimal capital stock for the representative agent economy when the welfare weight is concentrated entirely on the most patient agent. It turns out in the Le Van and Vailakis paper that this capital stock is the attractor for the long-run optimal sequence and that sequence is not a constant one.[2] In fact, the first period's capital stock must be smaller than the initial stock in this situation. Put differently, this starting stock is not a steady state for the heterogeneous agent optimum growth model even though it is the limit point of the optimal accumulation program. This fact is also easily proven in the example given the solution's explicit formulas in terms of the economy's primitive taste and technology parameters.

The two-agent model is setup in Section 3 following a review of the representative agent example in Section 2. The basic two-agent welfare optimization problem is transformed to a representative agent problem with a time varying discount factor in Section 3. The Twisted Turnpike result also appears there along with a demonstration based on the formal properties of the policy function sequences constructed in Section 4. Section 5 develops the qualitative dynamics of the model including results on the optimal path's eventual monotonicity property. Concluding comments appear in Section 6.

2 The Representative Agent Example

Frank Ramsey's (1928) seminal article on optimal capital accumulation concentrated on a single planner's infinite horizon optimization problem. Modern economic theorists interpret this planning problem's solution as a perfect foresight competitive equilibrium for an economy with a representative agent whose preferences coincide with the planner's preferences over future consumption streams.[3]

The *logarithmic utility, Cobb-Douglas production economy* is an important example of Ramsey's one-sector optimal growth problem. Let consumption and capital at time t be denoted by c_t and x_t, respectively. The planner solves the discrete time program

$$\sup \sum_{t=1}^{\infty} \delta^{t-1} \ln c_t$$

by choice of $\{c_t, x_t\}_{t=1}^{\infty}$ subject to the constraints

$$c_t + x_t \leq L x_{t-1}^{\rho}, \text{ for } t = 1, 2, \ldots$$

and $c_t, x_t \geq 0$ for all t with $x_0 \leq k$ the given initial capital stock. Here $0 < \delta, \rho < 1$ are this economy's deep taste and technology parameters; L is a factor which reflects the quantity of fixed labor. For the representative agent case $L = 1$ and for the case of two agents supplying identical labor services analyzed below $L = 2^{1-\rho}$ as the underlying production function in capital and labor is taken to be Cobb-Douglas.

This Ramsey problem is explicitly solved by a variety of methods.[4] The solution is described by the *consumption policy function* $H(k) = (1 - \delta\rho)Lk^\rho$ and the *capital policy function* $h(k) = \delta\rho Lk^\rho$. At each date, the policy functions tell the decision maker how much to consume and how much to save given the current level of the capital stock, k. The optimal decision taken at any date depends only upon the amount of capital the planner starts the period with and not on the particular moment in calender time. This is the *time consistency* property.

The optimal capital and consumption sequences are computed by iterating the policy functions. Carrying out that iteration leads to the explicit solution for the capital sequence:

$$k_t(k) = (\delta\rho L)^{\rho^{t-1}+\cdots+1}\, k^{\rho^t}, \tag{1}$$

where $k_0(k) = k$ is the given initial capital stock.

The optimal capital sequence is monotonic and converges to the unique positive fixed point of the capital policy function. That fixed point, $k(\delta)$, is called the *modified golden-rule level of capital* and satisfies the equation $h(k(\delta)) = k(\delta)$, which implies that

$$k(\delta) = (\delta\rho L)^{\frac{1}{1-\rho}}.$$

If the positive initial capital is below the modified golden-rule, then the economy accumulates capital and the sequence of optimal capital stocks increases and converges to the modified golden-rule capital stock. Similarly, the optimal capital stocks decrease and converge to the modified golden-rule when the starting stock is larger than the positive fixed point. If the initial capital happens to equal the modified golden-rule stocks, then it will be optimal to maintain those stocks in every period. Thus, the modified golden-rule is a steady state of the dynamical system

$$k_{t+1} = h(k_t) = \delta\rho L k_t^{\rho}.$$

The corresponding consumption sequence is also monotonic since the consumption policy function is increasing in capital. The resulting consumption sequence converges to the *modified golden-rule consumption* level defined by

$$c(\delta) = (1 - \delta\rho)(k(\delta))^{\rho}.$$

The convergence of the optimal capital and consumption sequences is known as the *turnpike theorem*. Finally, note that the turnpike property implies that $\mid k_t(k) - k_t(k') \mid \to 0$ as $t \to \infty$ for nonzero initial conditions $k \neq k'$. That is, the optimal capital sequences "come together" as t tends to infinity. This obtains in the two agent example developed below and the optimal capital sequence is shown to be eventually monotonic.

3 A Two Agent Ramsey Model Example

Assume for simplicity that there are only two households, denoted by $h = 1, 2$, with lifetime utility given by $\sum_{t=1}^{\infty} \delta_h^{t-1} \ln c_t^h$. Here $\{c_t^h\}_{t=1}^{\infty}$ is a given (nonnegative) consumption sequence and $0 < \delta_h < 1$ is the agent's discount factor and $\delta_2 < \delta_1$. Let $\lambda \geq 0$ denote the welfare weight assigned to agent 1 and $(1 - \lambda)$ the welfare weight assigned to the second agent. Assume further that $0 \leq \lambda \leq 1$ until otherwise noted. The cases where λ equals zero or one reduce to representative agent problems of the form found above. The planner's *welfare maximization problem* is

$$\sup \lambda \sum_{t=1}^{\infty} \delta_1^{t-1} \ln c_t^1 + (1 - \lambda) \sum_{t=1}^{\infty} \delta_2^{t-1} \ln c_t^2$$

by choice of nonnegative sequences $\{c_t^1, c_t^2, k_{t-1}\}_{t=1}^{\infty}$subject to

$$c_t^1 + c_t^2 + k_t \leq L k_{t-1}^{\rho}, \text{for } t = 1, 2, \ldots$$

and $k_0 \leq k$, where $k > 0$ is given, $1 > \delta_1 > \delta_2 > 0$, and $L = 2^{1-\rho}$. The parameter ρ satisfies $0 < \rho < 1$. The parameter L is the labor input to the production process. Note that both households inelastically supply one unit of identical labor services at each time to a Cobb-Douglas production function $F(\ell, k) = \ell^{1-\rho} k^{\rho}$ with $L k^{\rho} = F(2, k)$.

The welfare maximization problem's objective can be rewritten as

$$\sup \sum_{t=1}^{\infty} \delta_1^{t-1} \left[\lambda \ln c_t^1 + (1-\lambda) \left(\frac{\delta_2}{\delta_1} \right)^{t-1} \ln c_t^2 \right].$$

Define *aggregate consumption at time t* as c_t with $c_t^1 + c_t^2 = c_t$. The distribution of aggregate consumption within each period can be separated from the problem of calculating the optimal aggregate consumption over time by solving in every period the auxiliary problem

$$u(t, \lambda, c_t) = \sup \lambda \ln c_t^1 + (1-\lambda) \left(\frac{\delta_2}{\delta_1} \right)^{t-1} \ln c_t^2$$

by choice of nonnegative consumption levels c_t^1 and c_t^2 and given $c_t > 0$ subject to

$$c_t^1 + c_t^2 \leq c_t,$$

where λ is given and $0 < \lambda < 1$. The function $u(t, \lambda, c_t)$ is this program's value function. This auxiliary problem's first order conditions imply that

$$\frac{c_t^2}{c_t^1} = \left(\frac{1-\lambda}{\lambda} \right) \left(\frac{\delta_2}{\delta_1} \right)^{t-1}.$$

Using the constraint, it is easy to show that each agent's optimal consumption

share can be written as

$$
\begin{aligned}
\frac{c_t^1}{c_t} &= \left(1 + \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{\delta_2}{\delta_1}\right)^{t-1}\right)^{-1}, \text{ and} \\
\frac{c_t^2}{c_t} &= \left(1 + \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{\delta_2}{\delta_1}\right)^{t-1}\right)^{-1}\left(\frac{1-\lambda}{\lambda}\right)\left(\frac{\delta_2}{\delta_1}\right)^{t-1}.
\end{aligned}
$$

These equations yield an interesting result. Notice that $(\delta_2/\delta_1)^{t-1} \searrow 0$ as $t \to \infty$ implies

$$
\frac{c_t^1}{c_t} \nearrow 1 \text{ and } \frac{c_t^2}{c_t} \searrow 0 \tag{2}
$$

provided the aggregate consumption path is bounded away from zero along a welfare maximizing path. This will hold as long as the initial capital stocks are positive. Hence, the first household emerges as the dominant consumer; its consumption approaches one hundred percent of the economy's aggregate consumption and the second household's consumption shrinks towards zero.[5]

The calculation of each agent's consumption share yields the explicit form of the value function by substitution. That is,

$$
u(t, \lambda, c_t) = \left[\lambda + (1-\lambda)\left(\frac{\delta_2}{\delta_1}\right)^{t-1}\right] \ln c_t + \gamma_t,
$$

where

$$
\begin{aligned}
\gamma_t &= (1-\lambda)\left(\frac{\delta_2}{\delta_1}\right)^{t-1} \ln\left(\left(\frac{1-\lambda}{\lambda}\right)\left(\frac{\delta_2}{\delta_1}\right)^{t-1}\right) \\
&\quad - \left(\lambda + (1-\lambda)\left(\frac{\delta_2}{\delta_1}\right)^{t-1} \ln\left(1 + \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{\delta_2}{\delta_1}\right)^{t-1}\right)\right).
\end{aligned}
$$

From the viewpoint of intertertemporal maximization, a sequence $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ solves the welfare optimization problem if and only if it solves the problem

$$\sup \sum_{t=1}^{\infty} \delta_1^{t-1} u(t, \lambda, c_t)$$

by choice of nonnegative sequences $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ subject to

$$c_t + k_t \leq L k_{t-1}^{\rho}, \text{ for } t = 1, 2, \ldots$$

and $k_0 \leq k$, with $k > 0$ given. The information stored in the value function u is sufficient to decompose the aggregate consumption into the optimal consumption allocations for each agent given the preassigned welfare weights.

The constant γ_t defined at each time does not depend on the aggregate consumption's level or how it is allocated across households. Hence, the $\{\gamma_t\}$ have no influence on the determination of the optimal aggregate consumption or capital accumulation paths and can be neglected when calculating the welfare maximizing optimal program. So, the welfare maximization problem is solved if and only if the Ramsey problem with a *time variable discount factor* defined below is solved. The latter problem is expressed as the **Welfare Optimization Problem:**

$$\sup \sum_{t=1}^{\infty} \Delta_t \ln c_t \qquad (\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{L}))$$

by choice of nonnegative sequences $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ subject to

$$c_t + k_t \leq L k_{t-1}^{\rho} \text{ for } t = 1, 2, \ldots$$

and $k_0 \leq k$, with $k > 0$ given. Here the planner's discount factor at time t, focal date time 0, can also be written as

$$\Delta_t = \left(\lambda \delta_1^{t-1} + (1-\lambda)\, \delta_2^{t-1}\right).$$

The dependence of Δ_t on the choice of the welfare weight λ is suppressed in this notation. This time-varying discount factor is clearly the weighted average of the two agents' discount factors where the weights are given by the preassigned welfare weights. Problem $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{L})$ is an example of the time varying discounted optimal growth model studied by Mitra (1979).The sequence $\boldsymbol{\Delta} \equiv \{\Delta_t\}_{t=1}^{\infty}$ does **not** form a geometric sequence, unlike the sequences $\{\delta_h^{t-1}\}_{t=1}^{\infty}$ for $h = 1, 2$. This implies that the optimal welfare maximizing path of consumption is not time consistent in the manner defined by Strotz (1955), in contrast to the representative agent model discussed above. A direct argument supporting this conclusion is found in Section 4.

The main result is:

Theorem 1: Twisted Turnpike. *If $\{k_t(y)\}$ and $\left\{k_t\left(y^{\#}\right)\right\}$ are optimal capital sequences starting from the endowments y and $y^{\#}$, respectively, then $\left|k_t(y) - k_t\left(y^{\#}\right)\right| \to 0$ as $t \to \infty$.*

The welfare optimization problem $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{L})$ is explicitly solved, and the theorem proved, using Boyd's symmetry technique (1986, 1990).[6] Let

$$\mu_j = \sum_{t=j}^{\infty} \Delta_t \rho^{t-j}, \text{for } j = 1, 2, \ldots \ .$$

Clearly μ_j depends on the choice of the welfare weight, λ, and

$$\mu_j = \frac{\lambda \delta_1^{j-1}}{1-\delta_1 \rho} + \frac{(1-\lambda)\, \delta_2^{j-1}}{1-\delta_2 \rho} \text{ for } j = 1, 2, \ldots \ .$$

This model's optimal capital stock at time t, denoted by $k_t(y)$, has the explicit form:

$$k_t(y) = \left(1 - \frac{\Delta_t}{\mu_t}\right)\left(1 - \frac{\Delta_{t-1}}{\mu_{t-1}}\right)^{\rho} \cdots \left(1 - \frac{\Delta_1}{\mu_1}\right)^{\rho^{t-1}} y^{\rho^{t-1}}, \tag{3}$$

where $y = Lk^{\rho}$ is the output of goods available at time 1 given the initial stocks k.[7]

Now suppose that $k^{\#}$ is any other initial capital stock, $y^{\#} = L(k^{\#})^{\rho}$ and the corresponding optimal capital sequence is denoted by $\{k_t\left(y^{\#}\right)\}_{t=1}^{\infty}$. Then

$$k_t\left(y^{\#}\right) = \left(1 - \frac{\Delta_t}{\mu_t}\right)\left(1 - \frac{\Delta_{t-1}}{\mu_{t-1}}\right)^{\rho} \cdots \left(1 - \frac{\Delta_1}{\mu_1}\right)^{\rho^{t-1}} (y^{\#})^{\rho^{t-1}}.$$

Following Boyd (1990), define $z_t = k_t\left(y\right)/k_t\left(y^{\#}\right)$ and notice that

$$z_t = \left(\frac{y}{y^{\#}}\right)^{\rho^{t-1}},$$

or

$$\ln z_t = \rho^{t-1} \ln\left(\frac{y}{y^{\#}}\right).$$

As $0 < \rho < 1$, this last equation implies $\ln z_t \to 0$ as $t \to \infty$ and hence $z_t \to 1$ as $t \to \infty$. In particular, this means that $\left|k_t(y) - k_t\left(y^{\#}\right)\right| \to 0$ as $t \to \infty$. Put differently, the optimal capital accumulation sequences starting from different initial capital stocks converge to each other, or come together, in the limit. The optimal capital sequence exhibits the twisted turnpike property.

There is a feasible path of capital accumulation in which aggregate consumption is stationary, over time as is the capital stock, which also satisfies the condition $\delta_1 \rho L \bar{k}^{\rho-1} = 1$. This last equation is the steady state capital stock for the Ramsey optimal growth model when $\lambda = 1$ and $c_t^2 = 0$ — the case where the model collapses to a single agent problem with the first agent's welfare receiving all the planner's weight in the objective. One might think that the stocks $\bar{k}$ so defined would attract, in the limit, the economy's aggregate capital stocks when both agents have positive welfare weights. After all, agent 2's consumption converges to zero as time unfolds, so perhaps it is possible that $\bar{k}$ is the limiting capital stock for those nontrivial welfare weights. This turns out to be true, but the reason is very subtle. The convergence of the optimal capital sequences along the twisted turnpike to $\bar{k}$ is true, but that stock is **not itself** a steady state. This is a fundamental property of the many agent Ramsey model and it shows one way in which the many agent problem differs significantly from the representative agent model. The fact $\bar{k}$ is not a stationary equilibrium stock is a consequence of a general result due to Le Van and Vailakis (2003), but is easily shown for my example. Their theorem states that the constant path defined by $\bar{k} = k_t$ is not an optimal path from initial stocks $k_0 = \bar{k}$ and hence, it is not a stationary equilibrium. I give an independent proof of this fact to illustrate one benefit derived from knowing the example's explicit solution.

Proposition 1. *$\bar{k}$ is not a steady state for the Welfare Optimization Problem* $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{L})$.

Proof. It is shown in Section 4 that this problem is solved by calculating a sequence of policy functions, one for each period of time. The general form of the optimal decision at time t is

$$k_t = \left(1 - \frac{\Delta_t}{\mu_t}\right) y_t,$$

where y_t is the output available at the start of period t. Iterating from time $t = 1$ starting at $y_1 \equiv y$ yields $k_t(y)$ as displayed in (3). In particular,

$$k_1(y) = k_1 = \left(1 - \frac{1}{\mu_1}\right) y$$

in period 1. Set $y = L\bar{k}^{\rho}$; a simple calculation shows $k_1 < \bar{k}$. Hence, $\bar{k}$ is not a steady state for $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{L})$.[8] ■

Le Van and Vailakis' Theorem shows something more: the optimal capital sequence initiated at $\bar{k}$ converges to it in the long-run even though it is not a constant sequence.[9] In this case, $\{k_t(\bar{y})\}$ is not monotonic since $k_1(\bar{y}) < \bar{k}$. The twisted turnpike property implies **all optimal capital sequences converge to the stock** $\bar{k}$ as time tends to infinity. In particular, this implies that if the economy starts with the stocks $\bar{k}$, then it is optimal for the planner to deviate from those stocks and only return to them asymptotically. The resulting optimal capital sequence may not be monotonic, although it turns out to be eventually monotonic.[10] In part, this reflects the fact that the households enjoy time varying consumption along their optimal path. The aggregate consumption levels change over time, but the first household emerges as the dominant consumer in the limit.

4 The Symmetry Solution

This section details the symmetry solution to the Welfare Optimization Problem, the derivation of a sequence of Bellman equations of optimality, and the calculations giving rise to (3). Boyd's (1986, 1990) symmetry technique underlies the calculations that support the following results. Also, see Kamihigashi (2008) for a general treatment of nonstationary deterministic dynamic programs of which this model is one example. A *symmetry* is a one-to-one mapping between feasible sets for the problems $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{L})$ and $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{1})$ in such a manner that if consumption program $\{c_t\}$ is at least as preferred as consumption program $\left\{c_t^{'}\right\}$ in one problem, say $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{1})$, then its image under the symmetry mapping preserves that preference order in the other problem, $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{L})$.

Let $J\left(y|\Delta, L\right)$ be the value function for problem $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{L})$and let $J\left(y|\Delta, 1\right)$ be the value function for problem $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{1})$. Set $\Delta = \{\Delta_t\}_{t=1}^{\infty}$ and put $S\Delta = \{\Delta_t\}_{t=2}^{\infty}$, where S is the corresponding *shift operator* (also known as the *backward shift operator).* Symmetries mapping feasible (and optimal) solutions for problem $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{1})$ to $\mathbf{P}(\mathbf{y}, \boldsymbol{\Delta}, \mathbf{L})$ underlie the computations supporting the

Twisted Turnpike result.

Lemma 1. *The symmetry* $\mathbf{S}(c_t, k_t) = e^{\sigma_t}(c_t, k_t)$ *maps* $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{1})$ *to* $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{L})$*. This mapping is* $1-1$ *and onto for each* $L > 0$*. Here*

$$\begin{aligned} \sigma_t &= \ell + \ell\rho + \ell\rho^2 + \cdots + \ell\rho^{t-1}, \\ \ell &= \ln L. \end{aligned}$$

Proof. Suppose $\{c_t, x_t\}$ is a feasible program for problem $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{1})$. A routine computation shows the program defined by

$$(c_t^*, x_t^*) = e^{\sigma_t}(c_t, x_t), \; t = 1, 2, \ldots,$$

is feasible for problem $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{L})$. The planner's discounted utility in the first problem is $\sum_{t=1}^{\infty} \Delta_t \ln c_t$ and it is $\sum_{t=1}^{\infty} \Delta_t \ln c_t^*$ in the second. Clearly,

$$\sum_{t=1}^{\infty} \Delta_t \ln c_t^* = \sum_{t=1}^{\infty} \Delta_t \ln c_t + \sum_{t=1}^{\infty} \Delta_t \sigma_t,$$

where the series $\sum_{t=1}^{\infty} \Delta_t \sigma_t$ is convergent as it is the weighed average of series of the form $\ell \sum_{t=1}^{\infty} (\delta_h \rho)^{t-1}$ $(h = 1, 2)$. Hence, $\{c_t, x_t\}$ is at least as preferred as the feasible program $\{\bar{c}_t, \bar{x}_t\}$ for problem $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{1})$ if and only if the transformed sequences have the property $\{c_t^*, x_t^*\}$ is at least as preferred as $\{\bar{c}_t^*, \bar{x}_t^*\}$ for problem $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{L})$. ■

Corollary 2. *The corresponding value functions are related as follows:*

$$J\left(y|\Delta, L\right) = J\left(y|\Delta, 1\right) + \sum_{t=1}^{\infty} \Delta_t \sigma_t.$$

The symmetry $\mathbf{S}\left(\bullet\right)$ did **not** act on the endowment, y. The symmetry $\mathbf{T}$ defined next maps $\mathbf{P}\left(y, \Delta, 1\right)$ to $\mathbf{P}\left(\theta y, \Delta, 1\right)$ for some positive constant θ. Formally, given the feasible sequence $\{c_t, x_t\}$ for $\mathbf{P}\left(y, \Delta, 1\right)$, let

$$\mathbf{T}\left(c_t, x_t\right) = e^{\tau_t}\left(c_t, x_t\right),$$

where $\tau_t = \rho\tau_{t-1}$ and $\tau_1 = \ln\theta$. Then, $\tau_t = \rho^{t-1}\ln\theta$; follow the same line of reasoning as in Lemma 1 and Corollary 2 to obtain:

Lemma 3.

$$\begin{aligned} J\left(y|\Delta, 1\right) &= J\left(1|\Delta, 1\right) + \ln\theta\sum_{t=1}^{\infty}\Delta_t\rho^{t-1} \\ &= J\left(1|\Delta, 1\right) + \ln\theta\left[\frac{\lambda}{1-\delta_1\rho} + \frac{(1-\lambda)}{1-\delta_2\rho}\right]. \end{aligned}$$

Commentary: The trick here is to let $\bar{y} = 1$ and for $y \neq 1$ set $\theta\bar{y} = y$. That is: $\theta = y$. Then

$$\begin{aligned} \ln\theta + \ln\bar{y} &= \ln y, \text{so,} \\ \ln\theta &= \ln y. \end{aligned}$$

Restating Lemma 3:

Lemma 3*.

$$J\left(y|\Delta,1\right)=J\left(1|\Delta,1\right)+\ln y\left[\frac{\lambda}{1-\delta_1\rho}+\frac{(1-\lambda)}{1-\delta_2\rho}\right].$$

4.1 Period 1 Policy Functions

J is differentiable in y (apply the argument developed by Mirman and Zilcha (1975)). Hence

$$\frac{\partial J}{\partial y}=\left[\frac{\lambda}{1-\delta_1\rho}+\frac{(1-\lambda)}{1-\delta_2\rho}\right]\diagup y.$$

Equivalently,

$$\frac{\partial J}{\partial y}=\frac{\mu_1}{y}.$$

Standard Mirman-Zilcha (1975) arguments yield the envelope relation:

$$\begin{aligned}\frac{\partial J}{\partial y} &= \frac{\Delta_1}{c},\ \text{or, since } \Delta_1=1: \\ \frac{\partial J}{\partial y} &= \frac{1}{c}.\end{aligned}$$

Combining and solving for the first-period's optimal consumption and capital, given the endowment yields:

Lemma 5.

$$\begin{aligned}c_1 &= \frac{\Delta_1}{\mu_1}y,\text{ and} \\ k_1 &= \left(1-\frac{\Delta_1}{\mu_1}\right)y.\end{aligned}$$

The Le Van – Valiakis (2003) result that $\bar{k} \equiv (\delta_1 \rho)^{\frac{1}{1-\rho}}$ is NOT a steady state follows. This result also uses the symmetry structure to prove it for the case $L = 1$ and then know it can be mapped to the case $L = 2^{1-\rho}$. This result also depends crucially on $1 > \lambda > 0$.

Corollary 6. Let $L = 1$; set $\bar{y} = (\delta_1 \rho)^{\frac{\rho}{1-\rho}} = \left(\bar{k}\right)^{\rho}$. Then

$$k_1\left(\bar{k}\right) = \left(1 - \frac{\Delta_1}{\mu_1}\right)\left(\bar{k}\right)^{\rho} \neq \bar{k}.$$

4.2 Policy Functions for $t \geq 2$.

Bellman equations capturing the **Principle of Optimality** are expressed for each time t, with focal date time 0:[11]

$$\begin{aligned} J(y|\Delta, 1) &= \max_{0 \leq c \leq y} \left\{\Delta_1 \ln c + J\left((y-c)^{\rho} |\mathbf{S}\Delta, 1\right)\right\}, \\ J(y|\mathbf{S}\Delta, 1) &= \max_{0 \leq c \leq y} \left\{\Delta_2 \ln c + J\left((y-c)^{\rho} |\mathbf{S}^2\Delta, 1\right)\right\}, \\ &\vdots \\ J\left(y|\mathbf{S}^t\Delta, 1\right) &= \max_{0 \leq c \leq y} \left\{\Delta_t \ln c + J\left((y-c)^{\rho} |\mathbf{S}^{t+1}\Delta, 1\right)\right\}. \end{aligned}$$

Here,

$$\begin{aligned} J\left(y|\mathbf{S}^t\Delta, 1\right) &= J\left(1|\mathbf{S}^t\Delta, 1\right) + \mu_t \ln y; \\ \mu_t &= \sum_{s=t}^{\infty} \Delta_s \rho^{s-t} = \frac{\lambda \delta_1^{t-1}}{(1-\delta_1\rho)} + \frac{(1-\lambda)\delta_2^{t-1}}{(1-\delta_2\rho)}. \end{aligned}$$

Lemma 7. *For each t:*

$$\begin{aligned}\frac{\partial J}{\partial y} &= \frac{\mu_t}{y} = \frac{\Delta_t}{c};\\ c &= \frac{\mu_t}{\Delta_t} y;\\ k &= \left(1 - \frac{\mu_t}{\Delta_t}\right) y.\end{aligned}$$

Now iterate: start at $t = 1$ with the endowment, y. Then (recalling $L = 1$):

$$\begin{aligned}k_1(y) &= \left(1 - \frac{\Delta_1}{\mu_1}\right) y;\ y_1 = k_1(y)^{\rho},\\ k_2(y) &= \left(1 - \frac{\Delta_2}{\mu_2}\right) y_1\\ &= \left(1 - \frac{\Delta_2}{\mu_2}\right) (k_1(y))^{\rho}\\ &= \left(1 - \frac{\Delta_2}{\mu_2}\right) \left(1 - \frac{\Delta_1}{\mu_1}\right)^{\rho} y^{\rho},\end{aligned}$$

and so on. Note: It is easy to verify that $0 < \left(1 - \frac{\mu_t}{\Delta_t}\right) < 1$ for each t.

Lemma 8. *The optimal capital sequence up to period t is found by the formula:*

$$k_t(y) = \left(1 - \frac{\Delta_t}{\mu_t}\right) \left(1 - \frac{\Delta_{t-1}}{\mu_{t-1}}\right)^{\rho} \cdots \left(1 - \frac{\Delta_1}{\mu_1}\right)^{\rho^{t-1}} y^{\rho^{t-1}}.$$

The Twisted Turnpike property follows.

NOTE: When $1 > \lambda > 0$, the terms

$$\left(1 - \frac{\Delta_t}{\mu_t}\right) \neq \left(1 - \frac{\Delta_{t-1}}{\mu_{t-1}}\right).$$

This means that the optimal policies at each time t depend on calendar time — this is the manifestation of the **time inconsistency**, or, the **Strotz effect.**

Everything is proven for $L = 1$. Use the symmetry maps to find the $L = 2^{1-\rho}$ optimum.

5 The Convergence Theorem

The Twisted Turnpike property is a sweeping characterization of optimal solutions starting from different initial conditions. It does not fully exploit the information contained in the sequence of policy functions derived in the symmetry procedure. The following **Convergence Theorem** provides additional qualitative properties of each optimal path and refines the Twisted Turnpike property. The Convergence Theorem implies all sequences of optimal capital stocks converge to $\bar{k}$ independently of the initial conditions. The Convergence Theorem embodies the same conclusions as reached by Le Van and Vailakis (2003, Propositions 4 and 7). The policy function based proof, exploiting all available information, is considerably shorter, and more elementary, than theirs which considers a broader class of utility and production functions. The basic logic of the argument is analogous to one pursued by Becker and Foias (1987) in a different heterogeneous agent model.

Theorem 2: Convergence. *If* $\{k_t(y)\}$ *is the optimal capital sequence for the problem* $\mathbf{P}(y, \Delta, 1)$*, then*

$$\lim_{t \to \infty} k_t(y) = \bar{k}. \tag{4}$$

Moreover, the sequence $\{k_t(y)\}$ *is eventually monotonic.*

For each $y, y^{\#} > 0$ the optimal capital sequences $k_t(y) \to \bar{k}$ and $k_t\left(y^{\#}\right) \to \bar{k}$ as $t \to \infty$ implies the Twisted Turnpike property: $|k_t(y) - k_t\left(y^{\#}\right)| \to 0$. The Convergence Theorem contains more information on the mode of convergence to $\bar{k}$ and refines the interpretation of "all optimal paths coming together."

The Convergence Theorem's proof follows from a series of lemmas developed below that are constructed on the basis of properties enjoyed by the policy functions. The results are demonstrated for the case $L = 1$; by symmetry the qualitative results carry over to the case $L = 2^{1-\rho}$. Note that it is convenient to let the initial capital stock be $k = f^{-1}(y)$, $y = f(k) = k^{\rho}$. Denote the optimal sequence by $\{k_{t-1}\}_{t=1}^{\infty}, k_0 = k$, and define the sequence of policy functions in terms of k:

$$\theta_t^{\lambda}(k) = \left(1 - \frac{\Delta_t}{\mu_t}\right) k^{\rho}, t = 1, 2 \ldots. \tag{5}$$

where

$$\Delta_t(\lambda) = \lambda\delta_1^{t-1} + (1-\lambda)\,\delta_2^{t-1}; \tag{6}$$

$$\mu_t(\lambda) = \lambda\left(\frac{\delta_1^{t-1}}{1-\delta_1\rho}\right) + (1-\lambda)\left(\frac{\delta_2^{t-1}}{1-\delta_2\rho}\right). \tag{7}$$

Optimal capital sequences are computed by the recursion: $k_t = \theta_t^{\lambda}(k_{t-1})$ for each t assuming $k_0 = k$. Choose λ, $0 \leq \lambda \leq 1$ and assume $k > 0$. Notice:

$$\theta_t^{1}(k) = \delta_1\rho k^{\rho} \text{ and } \theta_t^{0}(k) = \delta_2\rho k^{\rho}.$$

It will be convenient to define $g(k) = \delta_2 \rho k^\rho = \theta_t^0(k)$ and $h(k) = \delta_1 \rho k^\rho = \theta_t^1(k)$ and observe that $h(k) \geq g(k)$ with equality when $k = 0$ as $\delta_1 > \delta_2$.

When the welfare weights are concentrated on one agent alone ($\lambda = 1$ or $\lambda = 0$), the corresponding policy function sequences are constant sequences for each k. In addition, each of the functions g and h has a unique positive fixed point. Clearly $\bar{k} = h\left(\bar{k}\right)$ and there is a unique $\underline{k} > 0$ such that $\underline{k} = g\left(\underline{k}\right)$. Evidently, $\underline{k} < \bar{k}$. Moreover, $k > \bar{k}$ implies $h(k) > k$ and $0 < k < \bar{k}$ implies $k < h(k)$. A similar property holds for g and $\underline{k}$. Each element of the sequence of policy functions when $0 < \lambda < 1$ also has a positive fixed point and related inequalities as seen in the next lemma.

Policy functions have a number of important properties expressed in the following list. The indexing of the policy function at time t by the weight λ is suppressed below to ease notation when the meaning is clear.

Lemma 9. *For each given t :*

A. $\theta_t(0) = 0$, $\theta_t^{'}(k) > 0$, $\theta_t^{''}(k) < 0$, $\theta_t^{'}(k) \to +\infty$ *as* $k \to 0, k > 0$, *and there is a unique* $k(t) > 0$ *such that*

$$\theta_t(k\,(t)) = k\,(t)\,. \tag{8}$$

B. *If* $k > k(t)$, *then* $\theta_t(k) < k$; *if* $k < k(t)$, *then* $\theta_t(k) > k$.

The proof is omitted as it is an easy consequence of the assumed properties of the production function. Suppose $k = \bar{k}$, then $\bar{k} > k(1)$. Hence, along an optimal path starting from $\bar{k}$, in period 1: $k_1\left(\bar{k}\right) = \theta_1\left(\bar{k}\right) < \bar{k}$, as promised earlier.

The next results list properties of sequences of policy functions. The first is the crucial **Fence Property**.

Proposition 2. *For each t, and for each $\lambda, 0 \leq \lambda \leq 1$, and for each k,*

$$g(k) \leq \theta_t(k) \leq h(k), \text{ with strict inequality if } k > 0. \tag{9}$$

Proof. The Proposition is valid if either $\lambda = 0$ or $\lambda = 1$. Assume $0 < \lambda < 1$.

The result follows by showing $\theta_t(k) - h(k) \leq 0$ and $\theta_t(k) - g(k) \geq 0$ with strict inequalities whenever $k > 0$. Notice that it suffices to show for each t :

$$\delta_2\rho \leq \left(1 - \frac{\Delta_t}{\mu_t}\right) \leq \delta_1\rho.$$

The sign of $\theta_t(k) - g(k) = \left(1 - \frac{\Delta_t}{\mu_t}\right) - \delta_2\rho \equiv \mathbb{G}$ is found by computing:

$$\begin{aligned}
\mathbb{G} &= 1 - \delta_2\rho - \frac{\lambda\delta_1^{t-1} + (1-\lambda)\,\delta_2^{t-1}}{\lambda\left(\frac{\delta_1^{t-1}}{1-\delta_1\rho}\right) + (1-\lambda)\left(\frac{\delta_2^{t-1}}{1-\delta_2\rho}\right)} \\
&= \frac{(1-\delta_2\rho)\left[\lambda\left(\frac{\delta_1^{t-1}}{1-\delta_1\rho}\right) + (1-\lambda)\left(\frac{\delta_2^{t-1}}{1-\delta_2\rho}\right)\right] - \left(\lambda\delta_1^{t-1} + (1-\lambda)\,\delta_2^{t-1}\right)}{\lambda\left(\frac{\delta_1^{t-1}}{1-\delta_1\rho}\right) + (1-\lambda)\left(\frac{\delta_2^{t-1}}{1-\delta_2\rho}\right)} \\
&= \frac{\lambda\delta_1^{t-1}\left[\frac{1-\delta_2\rho}{1-\delta_1\rho} - 1\right]}{\left[\lambda\left(\frac{\delta_1^{t-1}}{1-\delta_1\rho}\right) + (1-\lambda)\left(\frac{\delta_2^{t-1}}{1-\delta_2\rho}\right)\right]} \\
&= \frac{\lambda\delta_1^{t-1}\left[\frac{\rho(\delta_1-\delta_2)}{1-\delta_1\rho}\right]}{\left[\lambda\left(\frac{\delta_1^{t-1}}{1-\delta_1\rho}\right) + (1-\lambda)\left(\frac{\delta_2^{t-1}}{1-\delta_2\rho}\right)\right]} > 0
\end{aligned}$$

as $(\delta_1 - \delta_2) > 0$.

A similar argument shows $\left(1-\frac{\Delta_t}{\mu_t}\right)-\delta_1\rho<0$ and $\theta_t(k)-h(k)\leq 0$.

■

Assume that $0<\lambda<1$ in the sequel unless expressly noted otherwise.

One immediate application of the Lemma 9B and Proposition 2 is the observation:

$$\underline{k}<k(t)<\bar{k} \text{ for each } t. \tag{10}$$

That is, the sequence of fixed points $\{k(t)\}$ derived from the policy function sequence inherits the Fence Property in the form of inequalities given in (10). These relations play an important role in proving the eventual monotonicity of the optimal capital sequence.

Another application of (10) occurs if $k=\bar{k}$ and $t=1$. Then $\bar{k}>k(1)$ and $k_1=\theta_1\left(\bar{k}\right)<\bar{k}$, as promised earlier in Proposition 1.

The following results apply to sequences of policy functions, $\{\theta_t\}$, and the optimal capital sequence, $\{k_{t-1}\}$ with $k_t=\theta_t\left(k_{t-1}\right)$ and $k_0=k$.

Lemma 10. *The sequence of policy functions $\{\theta_t\}$ converges pointwise to h. That is, for each $k>0$*

$$\lim_{t\to\infty}\theta_t\left(k\right)=h(k). \tag{11}$$

Moreover, the convergence is uniform on each compact interval $[0,b], b>0$.

Proof.

Observe

$$\left(1-\frac{\Delta_t}{\mu_t}\right)=\left(1-\frac{1+\left(\frac{1-\lambda}{\lambda}\right)(\delta_2/\delta_1)^{t-1}}{\left(\frac{1}{1-\delta_1\rho}\right)+\left(\frac{1-\lambda}{\lambda}\right)(\delta_2/\delta_1)^{t-1}\left(\frac{1}{1-\delta_2\rho}\right)}\right). \qquad (12)$$

Since $(\delta_2/\delta_1)<1$, it follows that

$$\lim_{t\to\infty}\left(1-\frac{\Delta_t}{\mu_t}\right)=\delta_1\rho. \qquad (13)$$

This implies (11).

Fix b, $0<b<+\infty$. For each t, set

$$M_t=\sup_{k\in[0,b]}|\theta_t(k)-h(k)|.$$

Clearly $f(k)=k^\rho$ increasing in k implies

$$M_t=b^\rho\left|\left(1-\frac{\Delta_t}{\mu_t}\right)-\delta_1\rho\right|\to 0 \text{ as } t\to\infty.$$

This proves $\theta_t\to h$ uniformly on each non-empty compact interval.

■

The upper limit b used to define a particular compact interval $[0,b]$ may be chosen as the maximum sustainable stock, but this interpretation is not required for the lemma's validity. Evidently the choice $b=1$ is the maximum sustainable stock for the specification $f(k)=k^\rho$. Hence, the policy function sequence converges to h uniformly on any compact interval containing the maximum sustainable stock. As typical of optimal growth models, the optimal accumulation sequence will eventually reside in that particular compact interval.

Lemma 10 tells us that the function h is well-approximated by the sequence of policy functions $\{\theta_t\}$ even though h is NOT a policy function when $0 < \lambda < 1$. The uniform convergence property suggests that the long-run qualitative properties of the aggregate capital stock should follow from the monotonicity properties of optimal paths when $\lambda = 1$ and h is the optimal policy function. In this sense, it should not be a surprise that monotonicity over all time periods might fail for finitely many periods when $0 < \lambda < 1$. Over a sufficiently long time period agent 1 emerges as the dominant household and the welfare problem's solution emulates the case where ALL the welfare weight is placed on that person. The Convergence Theorem formalizes this intuition and is modeled by way of the following lemmas and corollaries.

Corollary 11. *If the optimal capital sequence $\{k_{t-1}\}$ has a limit, $k^* > 0$, then $k^* = \bar{k}$.*

Proof.

Suppose $k_{t-1} \to k^* > 0$. Then, Lemma 10 implies $\theta_t(k_{t-1}) = k_t \to h(k^*) = k^*$ by Ash (1970, Problem 4, p. 133). But $h(k^*) = k^* > 0$ if and only if $k^* = \bar{k}$.

■

The next result based on the Fence Property says there is a time such that $k_{t-1} \geq \underline{k}$. Its proof turns out to imply the optimal capital sequence cannot converge to zero.

Lemma 12. *Given the optimal capital sequence $\{k_{t-1}\}$, there is a finite* *time T such that $k_{T-1} \geq \underline{k}$.*

Proof.

Step I. Suppose there is a time T such that

$$k_{T-1} < \underline{k} \text{ and } k_T < k_{T-1} < \underline{k} \text{ implies } k_{T+1} < k_T.$$

Notice $k(T+1) = \theta_{T+1}(k(T+1)) > \underline{k} = g(\underline{k})$ by the Fence Property (10) and $k_T < \underline{k} < k(T+1)$. Hence, by Lemma 9B:

$$k_{T+1} = \theta_{T+1}(k_T) > k_T,$$

contradicting the assumption $k_{T+1} < k_T$. Hence, it must be the case that

$$k_{T-1} < \underline{k} \text{ and } k_T < k_{T-1} < \underline{k} \text{ implies } k_{T+1} \geq k_T.$$

Step II. Assume that $k_t < \underline{k}$ for all t. Then $\underline{k} \geq k_{t+1}$ and $\underline{k} \geq k_t$ by the assumed condition. As $k_{t-1} < \underline{k}$, either $k_t < k_{t-1}$, or $k_t \geq k_{t-1}$ hold as well. In the first alternative, Step I implies $k_{t+1} \geq k_t$. In the second alternative $k_{t-1} \leq k_t < \underline{k}$ implies $k_t < \underline{k} < k(t+1)$ by (10). Hence $k_{t+1} > k_t$ as well. Hence, $k_{t+1} \geq k_t$ obtains in either situation.

The Fence Property (10) once again yields : $k(t+2) > \underline{k} > k_{t+1}$, so $k_{t+2} = \theta_{t+2}(k_{t+1}) > k_{t+1}$. Hence, $k_{t+2} > k_{t+1}$ and $\underline{k} \geq k_{t+2}$ (by assumption). The previous paragraph's argument can be repeated starting from any t, in particular at $t = 1$, to yield:

$$k_1 \leq k_2 \leq \cdots \leq \underline{k}.$$

Thus, the sequence $\{k_{t-1}\}$ is bounded above by $\underline{k}$ and it is eventually non-decreasing.[12] Therefore, the limit of this sequence exists, is smaller than or equal to $\underline{k}$, and must be positive. Corollary 11 implies the limit should be $\bar{k}$, which is larger than $\underline{k}$, which is impossible. Therefore, there must exist some T, $0 < T < +\infty$ such that $k_{T-1} \geq \underline{k}$.

■

The proof of Lemma 12 shows $k_{t-1} \nrightarrow 0$ as $t \to \infty$. If $k_{t-1} \to 0$, then there is a time T such that $k_t < \underline{k}$ for all $t \geq T$. But then repeating the argument in Step II of Lemma 12 from time T onwards shows $\{k_{t-1}\}$ is eventually nondecreasing and convergent to $\bar{k}$, which is impossible. Thus,

Corollary 13. $\{k_{t-1}\}$ *optimal implies* $\limsup_{t\to\infty} k_t > 0$.

The Fence Property (10) tells us that if $k_t > \bar{k}$, then $k(t+1) < \bar{k} < k_t$, so $\theta_{t+1}(k_t) = k_{t+1} < k_t$. There are two possibilities: either $k_{t+1} > \bar{k}$ or $k_{t+1} \leq \bar{k}$. The next result addresses the case where $k_t > \bar{k}$ for all but a finite number of periods.

Lemma 14. *Let* $\{k_{t-1}\}$ *be the optimal capital sequence and suppose there is a* T *such that* $k_t > \bar{k}$ *for all* $t \geq T$. *Then* $\lim_{t\to\infty} k_{t-1} = \bar{k}$ *and the convergence is eventually monotonic. In particular,*

$$k_T > k_{T+1} > \cdots > \bar{k}. \tag{14}$$

Proof.

Repeated application of the Fence Property (10) and Lemma 9B yields (14). Hence, the sequence $\{k_{t-1}\}$ is eventually decreasing and bounded from below by $\bar{k}$. Thus, $k_{t-1} \to \bar{k}$ by Corollary 11.

■

The next result is concerned with situations in which there is a first time in which $k_t \leq \bar{k}$. This can arise either at the start, $t = 1$, or at some finite date T in the future. One quick application of Lemma 15, in combination with the previous result, is that $T = 1$ in Lemma 14.

Lemma 15. *Let $\{k_{t-1}\}$ be the optimal capital sequence and suppose there is a T such that $k_T \leq \bar{k}$. Then $k_t \leq \bar{k}$ for all $t \geq T$.*

Proof.

Let T be the first time $k_T \leq \bar{k}$. The Fence Property (10) insures $k(T+1) < \bar{k}$. Then either case (A) or (B) occurs, where

Case (A) $k_T \leq k(T+1) < \bar{k}$, or

Case (B) $k(T+1) < k_T < \bar{k}$.

In Case (A) Lemma 9B implies $\theta_{T+1}(k_T) = k_{T+1} \geq k_T$ with equality if and only if $k(T+1) = k_T$. Moreover, θ_{T+1} increasing implies

$$k_{T+1} = \theta_{T+1}(k_T) \leq \theta_{T+1}(k(T+1)) = k(T+1) < \bar{k}.$$

So, $k_{T+1} < \bar{k}$.

In Case (B), Lemma 9B, once again, implies $\theta_{T+1}(k_T) = k_{T+1} < k_T < \bar{k}$. Therefore, $k_{T+1} < \bar{k}$ holds in either case. Repeat the argument at $T+2$, and so on. The conclusion follows.

■

The last step in proving the Convergence Theorem is to show that once the optimal capital stock falls below $\bar{k}$ it yields a monotonically nondecreasing capital sequence thereafter.

Lemma 16. *Let $\{k_{t-1}\}$ be the optimal capital sequence and suppose there is a T such that $k_T \leq \bar{k}$. Then*

$$k_T \leq k_{T+1} \leq \cdots \leq \bar{k}. \tag{15}$$

Hence, the optimal capital sequence is eventually nondecreasing and converges to $\bar{k}$.

Proof.

Simplify notation in the proof by dropping the T notation and understand the argument applies for all t taken sufficiently large. The previous lemma implies that once $k_t \leq \bar{k}$ then $k_{t+1} \leq \bar{k}, k_{t+2} \leq \bar{k}$, and so on.

Either

$$k_{t-1} < k_t < k_{t+1} < \cdots < \bar{k}, \tag{16}$$

or there is a first time, t such that

$$k_{t-1} < k_t \text{ and } k_{t+1} \leq k_t. \tag{17}$$

Suppose that the optimal capital sequence satisfies (17). Now consider $\theta_{t+1}(k_t) = k_{t+1}$. There are two possibilities:

Case (A) $k_t \geq k(t+1)$;

Case (B) $k_t < k(t+1)$.

Start with Case (B) where $\theta_{t+1}(k_t) = k_{t+1} > k_t$ holds by Lemma 9B as $k_t < k(t+1)$. This violates the assumed satisfaction of inequality (17). Therefore, Case (A) must hold.

Assume now Case (A) obtains and $k_{t+1} \leq k_t$ from (17). Thus, $k_{t+1} = \theta_{t+1}(k_t) \leq k_t$ by assumption.

Case (A) has two subcases analogous to those defining Cases (A) and (B).

Subcase (a) $k_{t+1} \geq k(t+2)$;

Subcase (b) $k_{t+1} < k(t+2)$.

If subcase (b) occurs, then $k_{t+2} > k_{t+1}$ holds as $\theta_{t+2}(k_{t+1}) > k_{t+1}$ by Lemma 9B.

Suppose Subcase (a) occurs. Then either

$$k_{t+2} \leq k_{t+1} \leq k_t \text{ or } k_{t+2} > k_{t+1}.$$

In this situation, new subcases analogous to (a) and (b) arise for k_{t+2} taking the place of k_{t+1} and $k(t+3)$ in place of $k(t+2)$. In these situations we once again have two possibilities:

$$\begin{aligned} k_{t+3} &\leq k_{t+2} \leq k_{t+1} \leq k_t, \text{ or} \\ k_{t+3} &> k_{t+2} > k_{t+1}. \end{aligned}$$

These alternatives are also the only ones that can arise whenever Subcase (b) occurs. Thus, we may continue in this manner to produce the alternatives:

$$\begin{aligned} \bar{k} &\geq k_t \geq k_{t+1} \geq \cdots, \text{ or} \\ k_{t+1} &< k_{t+2} < \cdots < \bar{k}. \end{aligned}$$

In the first situation, $k_t \searrow k^*$ for some k^*. Clearly $k^* > 0$ by Corollary 13. Thus, $k^* = \bar{k}$ by Corollary 11. This can only occur if $k_t = \bar{k}$ for all t. Hence, $\bar{k}$ must be a steady state, which is impossible as $\left(1 - \frac{\Delta_{t+1}}{\mu_{t+1}}\right) \neq \left(1 - \frac{\Delta_t}{\mu_t}\right)$ for any t when $0 < \lambda < 1$. Thus, the only remaining possibility is for all t sufficiently large,

$$k_{t+1} < k_{t+2} < \cdots < \bar{k}.$$

In this event, $k_t \nearrow \bar{k}$ as well.

■

The **proof of Theorem 2** follows from Lemmas (and Corollaries) 9-16. Just note that the optimal capital sequence $\{k_{t-1}\}$ must satisfy the hypotheses of either Lemma 14 or Lemma 15 (and, hence Lemma 16) and those cases are mutually exclusive.

The need for qualifying statements such as eventually nondecreasing or eventually increasing is necessary to accommodate special cases of optimal solutions. The easiest example occurs in the now familiar situation where the economy starts at $\bar{k}$ with $k_1 = \theta_1 \left(\bar{k}\right) < \bar{k}$. The first period's stocks decline from the ones given in the initial condition, but the optimal sequence of capital eventually increases over time and converge to $\bar{k}$.

6 .Concluding Comments

The monotonicity of the optimal capital sequence is a fundamental property of the representative agent model. It implies that the shadow prices supporting the optimal path are also monotonic. In particular, if the capital sequence is increasing, then capital's rental price (its marginal product) declines and the wage rises over time. Bliss (1975, 1999) calls this feature the "Orthodox Vision" of capital theory. This property is easily seen in the familiar log utility, Cobb-Douglas production, representative agent example.

The heterogeneous agent extension of this example cannot exhibit the Orthodox Vision as a result of the Le Van and Vailakis (2003) theorem. The twisted turnpike property in the example implies $\mid f'(x_{t-1}(k) - f'(x_{t-1}(k') \mid \rightarrow 0$ as $t \rightarrow \infty$, where $f'(x) = \rho L x^{1-\rho}$ is capital's marginal product. That is, the rental prices of capital come together from two different initial conditions. However, the sequence of rental prices from a given initial condition are not generally monotonic as the sequence $\{f'(x_{t-1}(\bar{k})\}_{t=1}^{\infty}$ fails to be monotonic. However, the eventual monotonicity property exhibited along the turnpike shows that *eventually the Orthodox Vision obtains* provided the initial stocks start at or below $\bar{k}$ or fall below $\bar{k}$ in finite time. The turnpike's "twists" occur early over an initial segment of finitely many periods before settling down to monotonic capital accumulation.

Notes

[1]The example corresponds to the model in Le Van and Vailakis (2003) with 100 percent depreciation. The welfare maximization approach to optimal growth was developed in Bewley (1972), Coles (1985, 1986), Kehoe, (1989), Kehoe and Levine (1985), and Kehoe, Levine and Romer (1989, 1990).

[2]Note that they assume each agent's felicity function is bounded below, whereas the log felicity assumed here is not. This fact is demonstrated in my log utility example.

[3]See Becker and Boyd (1997) for more on this interpretation of the planner's problem.

[4]See Becker and Boyd (1997), and Boyd (1986, 1990) for the symmetry technique solution that underlies the example developed below. Other techniques are based on value function iteration using Bellman's optimality equation and Howard's policy improvement algorithm.

[5]Rader formalized this result for exchange economies in Rader (1971, 1972, and 1981). The latter paper emphasizes the class of Bernoulli (iso-elastic) one-period return functions, which include the logartithmic case. A similar result is found in Kehoe (1989) for the two-person exchange economy when agents have log felicity functions. Capital theoretic versions are found in Bewley (1982), Coles (1985, 1986), and Le Van and Vailakis (2003).

[6]The detailed development of this solution is deferred to the next section.

[7]See Boyd ([8] p.253) as well as the discussion in Section 4 for details.

[8]See the comments following Proposition 2 in Section 5.

[9]Recall, they assume that the planner's felicity function is bounded below.

[10]See Le Van and Vailakis (2003) for details as well as my arguments for the Convergence Theorem in the next section.

[11]See Kamihigashi (2008) for related presentations of Bellman equations in non-stationary models.

[12]In fact, it is increasing as the capital stocks are non-zero at each time, so a strict inequality actually obtains.

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