UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 31
Business Economic Series 08
September 2011

Departamento de Economía de la Empresa
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34-91) 6249607

"Estimating US persistent and transitory monetary shocks: implications for monetary policy"*

Juan Ángel Lafuente, [1] Rafaela Pérez, [2], Jesús Ruiz [3]

Abstract

This paper proposes an estimation method for persistent and transitory monetary shocks using the monetary policy modeling proposed in Andolfatto et al, [Journal of Monetary Economics, 55 (2008), pp.: 406-422]. The contribution of the paper is threefold: a) to deal with non-Gaussian innovations, we consider a convenient reformulation of the state-space representation that enables us to use the Kalman filter as an optimal estimation algorithm. Now the state equation allows expectations play a significant role in explaining the future time evolution of monetary shocks; b) it offers the possibility to perform maximum likelihood estimation for all the parameters involved in the monetary policy, and c) as a consequence, we can estimate the conditional probability that a regime change has occurred in the current period given an observed monetary shock. Empirical evidence on US monetary policy making is provided through the lens of a Taylor rule, suggesting that the Fed's policy was implemented accordingly with the macroeconomic conditions after the Great Moderation. The use of the particle filter produces similar quantitative and qualitative findings. However, our procedure has much less computational cost.

Keywords: Kalman filter, non-normality, particle filter, monetary policy

JEL Classification:*C4;F3*.

*We would like to thank Oscar Jordà, Alfonso Novales, participants in the econometrics seminar at UC Davis and the 4th International Conference on Computational and Financial Econometrics for their helpful comments and suggestions. Financial support from the Ramón Areces Foundation and the Spanish Ministry of Education through grants ECO2009-10398 and PR2009-0536 is gratefully acknowledged. The usual disclaimer applies.
[1] Dpto. de Finanzas y Contabilidad, Universitat Jaume I. Email: lafuen@cofin.uji.es.
[2] Dpto. de Fundamentos del Análisis Económico I, Universidad Complutense
[3] Dpto. de Economía Cuantitativa, Universidad Complutense.

1. Introduction

State space models are useful for many economic applications. As it is well-known, under normality, the classical Kalman filter provides the minimum-variance estimate of the current state taking into account the most recent signal. This prediction is just the conditional expectation. However, under non-linearity and/or non-normality, the filtering procedure developed by Kalman (1960) becomes non-optimal. Two alternatives has been developed in the literature to deal with this aspect: a) the use of first order Taylor series expansion to get linearized equations (transition and/or observation) and b) the use of simulations techniques based on sequential estimation of conditional densities through lot of replications. The first alternative leads to biased estimators (see, for example, Tanizaki, 1995) while the second one generally requires a great amount of computational burden. Following the second approach, the path-breaking paper of Fernández- Villaverde and Rubio-Ramirez (2007) shows how to deal with the likelihood-based estimation of non-linear DSGE models with non-normal shocks using a sequential Monte-Carlo method (particle filter). This procedure requires a great amount of computational burden.

This paper rethinks about the non-optimality of the Kalman filter. In particular we retake the signal extraction problem proposed in Andolfatto et al. (2008). These authors propose a state-space modelling with non-gaussian innovations to decompose monetary shocks in a Taylor rule into persistent and transitory components. Under their framework, the Kalman filter is not fully optimal, because the persistent disturbance in the monetary rule is not normally distributed. They perform a number of tests to check whether this “quasi-rationality” is quantitatively important in their experiments to reach the conclusion that this restriction is not statistically affecting their results. Our paper contributes to the literature by showing how to perform an optimal decomposition. In particular, we propose an alternative state-space representation to the one used in the above-mentioned paper (*AHM-representation* hereafter). Our reformulation for the measurement and transition equations that we will refer as *LPR-representation* requires the use of state-contingent matrices and, in spirit, joins with the growing body of recent literature on the role of expectations in monetary policy making. Our procedure has two advantages over the use of the standard particle filter considering the *AHM-*

representation (an adequate estimation procedure in this case): i) lower computational cost, and ii) the possibility to estimate the probability that a regime change has occurred in the current period conditional on an observed monetary shock. As an alternative to calibration, the use of confidence intervals should prove useful for research that incorporates such a monetary policy structure into a DSGE model with no analytical solution.

Our estimation procedure is used to provide additional insights on monetary policy modelling through the lens of Taylor's rule. As Assenmacher-Wesche (2006) points out, empirical studies of reaction functions are confronted with the problem not only that the reaction function is a reduced form to present monetary policy making, but also that parameter estimates could be unstable, especially when considering long time periods. One possibility is to consider that regime changes are due to non-observable inflation targeting, with the monetary authority updating the current inflation target in response to changes in economic fundamentals. This approach is just the proposal in Andolfatto et al. (2008) and is similar in spirit to Sims and Zha (2006), who conclude that what changes across regimes are only the variance of the structural disturbances and not the coefficients of the policy rule. In a similar way, the recent work of Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010) consider that uncertainty in their DSGE model is due to time-varying volatility of structural shocks. Our procedure allows the estimation of the unobserved inflation target. The other alternative is the use of Switching Markov techniques to estimate different responses of the monetary authority to output gap and inflation depending on the state of the economy.

The paper provides empirical evidence for the US using quarterly data covering the period 1980–2011 (first quarter). We find that the evidence of a regime change in US monetary policy making during the period 1984 to 1999 is weak. However, after the Great Moderation, September eleven, the recession that started in March 2001 and the subprime crisis are three events clearly affecting inflation targets in terms of the long-term the nominal anchor. These regime shifts are matched with those suggested by the Switching Markov techniques. However, this approach also reveals two additional regime changes during the nineties that are not supported from an inflation flexible targeting point of view. We also compare empirical findings based on our estimation procedure with the results we obtain using the particle filter considering the *AHM-representation*. The point estimates of all the parameters involved are statistically the same. Moreover, Monte Carlo simulations reveal that a) the probability distribution of the discrepancy between the current inflation target and its long-term mean is statistically the same in most of cases (83%) and b) the median of mean squared errors to predict theoretical values of the "inflation gap" is lower when using our estimation procedure.

The rest of the paper is organized as follows: The next section summarizes the specification of the Taylor rule and the monetary policy scheme for monetary innovations proposed in Andolfatto et al. (2008). Section III describes the reformulation of the state-space representation proposed and demonstrates that is equivalent in terms of conditional mean and variance. Section IV presents empirical evidence for the US, together with the comparison of empirical findings based on our estimation procedure with the results we obtain using either the particle filter or the Switching Markov approach. Finally, section V summarizes and provides concluding remarks.

2. The Taylor rule and the monetary setting

Following Andolfatto et al. (2008), let us consider the following Taylor rule:

$$i_t = (1-\rho)\left[r^* + \pi_t^* + \alpha(\pi_t - \pi_t^*) + \beta\left(y_t - y_t^*\right)\right] + \rho i_{t-1} + u_t, \quad (1)$$

where r^* is the equilibrium real rate, π_t^* denotes the inflation target, $y_t - y_t^*$ is the output gap, ρ is the parameter accounting for monetary policy inertia and u_t represents the monetary shocks, which can be interpreted as errors underlying the central bank's control over the policy instrument. Also, it is assumed that the imperfect control of the monetary policy rule expressed in (1) as a reaction to time-varying economic environment does not lead to persistent errors. Therefore, the time evolution of this shock can be represented as follows:

$$u_{t+1} = \phi\, u_t + e_{t+1},\ 0, < \phi \ << 1,\ e_{t+1} \sim N(0, \sigma_e^2). \quad (2)$$

A second disturbance to monetary policy is considered. This noise represents the change in the proper rate of inflation the central bank should pursue as a consequence of changes in the economic outlook. We express these shifts with the variable $z_t \equiv \pi_t^* - \pi^*$, so that z_t represents the deviation of the current target (π_t^*) from its long term (time-invariant) mean π^*. It is expected that these shifts will exhibit significant duration:

$$z_{t+1} = \begin{cases} z_t, & \text{with probability } p \\ g_{t+1}, & \text{with probability } 1-p \end{cases} \tag{3}$$

where $g_{t+1} \sim N(0, \sigma_g^2)$.

Combining the definition of z_t with equation (1), the Taylor rule can be rearranged as follows:

$$i_t = (1-\rho)\left[r^* + \pi^* + \alpha(\pi_t - \pi^*) + \beta\left(y_t - y_t^*\right)\right] + \rho i_{t-1} + \underbrace{(1-\rho)(1-\alpha)z_t + u_t}_{\varepsilon_t}\ , \tag{4}$$

where monetary shock ε_t is a combination of both persistent ($(1-\rho)(1-\alpha)z_t$) and transitory (u_t) components.

Assuming that agents consider the above monetary rule as a plausible representation scheme for monetary policy making, they need to learn about the decisions of the central bank in two ways: a) they should solve a signal extraction problem to break down the aggregate shock into the permanent and the transitory components, and b) they should act as econometricians in order to estimate parameters ϕ , σ_e^2 , σ_g^2 and p .

3. State-space representation and maximum likelihood estimation

As in Andolfatto et al. (2008), we consider that agents face the signal extraction problem concerning z_t and u_t based on the observation of the time evolution of the monetary supply. To deal with this aspect, these authors propose the following state-space representation, which will be called *AHM-representation*:

$$\begin{bmatrix} z_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} z_t \\ u_t \end{bmatrix} + \begin{bmatrix} N_{t+1} \\ e_{t+1} \end{bmatrix}, \text{ where } N_{t+1} = \begin{cases} \left(1-p\right) z_t, \text{ with prob. } p \\ g_{t+1} - p z_t, \text{ with prob. } 1-p \end{cases},$$

$$\varepsilon_t = \left[(1-\rho)(1-\alpha) \quad 1\right] \begin{bmatrix} z_t \\ u_t \end{bmatrix},$$

where the observable signal is the OLS estimate of the error term in the reaction function (equation 4).

As pointed out by Andolffato et al. (2008) the use of the Kalman filter is not fully optimal because z_t is a mixture of a Bernoulli process and a Gaussian noise. To overcome the absence of non-normality let us consider an alternative formulation of the time evolution of z_t that requires a state-space representation with state-contingent matrices in the state equation[1], that explicitly incorporates the crucial role of economic agents' expectations in learning about monetary policy-making. This alternative formulation is as follows:

$$\begin{bmatrix} z_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} \varphi & 0 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} z_t \\ u_t \end{bmatrix} + \begin{bmatrix} \varpi_{S_{t+1}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_t z_{t+1} \\ E_t u_{t+1} \end{bmatrix} + \begin{bmatrix} \delta_{S_{t+1}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_{t+1} \\ e_{t+1} \end{bmatrix}, \tag{5}$$

$$\varepsilon_t = \begin{bmatrix} (1-\rho)(1-\alpha) & 1 \end{bmatrix} \begin{bmatrix} z_t \\ u_t \end{bmatrix}, \tag{6}$$

where:

$$|\varphi| \in (0,1), \quad \varpi_{S_{t+1}} = \begin{cases} \dfrac{1-\varphi}{p}, \text{ if } S_{t+1} = 1, \text{ with prob. } p \\ -\dfrac{\varphi}{p}, \text{ if } S_{t+1} = 0, \text{ with prob. } 1-p \end{cases} \tag{7}$$

$$\delta_{S_{t+1}} = \begin{cases} 0, \text{ if } S_{t+1} = 1, \text{ with prob. } p \\ 1, \text{ if } S_{t+1} = 0, \text{ with prob. } 1-p \end{cases},$$

Proposition 1: *If $\varpi_{S_{t+1}}$ and $\delta_{S_{t+1}}$ are defined as in (7), the dynamics of z_t is observationally equivalent to (3) from the perspective of conditional mean.*

Proof. From (5), we have that

$$z_{t+1} = \varphi z_t + \varpi_{S_{t+1}} E_t z_{t+1} + \delta_{S_{t+1}} g_{t+1}, \tag{8}$$

and, therefore, the conditional expectation of z_{t+1} is:

$$E_t z_{t+1} = \frac{\varphi}{1 - p\varpi_{S_{t+1}=1} - (1-p)\varpi_{S_{t+1}=0}} z_t. \tag{9}$$

With probability p, $S_{t+1} = 1$, and, from (3), $z_{t+1} = z_t$; then, substituting in (8),

$$\varphi z_t + \varpi_{S_{t+1}=1} E_t z_{t+1} + \delta_{S_{t+1}=1} g_{t+1} = z_t \Rightarrow$$

$$\varphi z_t + \varpi_{S_{t+1}=1} \frac{\varphi}{1 - p\varpi_{S_{t+1}=1} - (1-p)\varpi_{S_{t+1}=0}} z_t + \delta_{S_{t+1}=1} g_{t+1} = z_t.$$

This equation holds when:

$$\delta_{S_{t+1}=1} = 0,$$

$$\varphi \left[1 + \frac{\varpi_{S_{t+1}=1}}{1 - p\varpi_{S_{t+1}=1} - (1-p)\varpi_{S_{t+1}=0}} \right] = 1. \tag{10}$$

[1] An alternative solution to deal with non-normality is the implementation of the particle filter, which we will use in our robustness exercise in a section bellow.

Now, with probability 1-*p*, $S_{t+1}=0$, and, from (3), $z_{t+1}=g_{t+1}$; therefore, substituting in (8),

$$\varphi z_t + \varpi_{S_{t+1}=0} E_t z_{t+1} + \delta_{S_{t+1}=0} g_{t+1} = g_{t+1} \Rightarrow$$

$$\varphi z_t + \varpi_{S_{t+1}=0} \frac{\varphi}{1-p\varpi_{S_{t+1}=1}-(1-p)\varpi_{S_{t+1}=0}} z_t + \delta_{S_{t+1}=0} g_{t+1} = g_{t+1}.$$

This equation holds when:

$$\delta_{S_{t+1}=0} = 1,$$

$$1 + \frac{\varpi_{S_{t+1}=0}}{1-p\varpi_{S_{t+1}=1}-(1-p)\varpi_{S_{t+1}=0}} = 0 \qquad (11)$$

Equations in (10) and (11) define a system for the variables $\left\{\varpi_{S_{t+1}=1}, \varpi_{S_{t+1}=0}\right\}$, with the following solution:

$$\varpi_{S_{t+1}=1} = \frac{1-\varphi}{p}; \ \varpi_{S_{t+1}=0} = -\frac{\varphi}{p}. \square$$

Note that the representation that we propose is a function of the parameter φ. Next, we prove that there is a unique value of φ in terms of probability *p* that yields the same conditional variance as in (3) for the z_t process.

Proposition 2: *If* $\varphi = 1 - p/2$, *then the LPR-representation yields the same conditional variance as in (3) for the* z_t *.process.*

Proof: In accordance with equation (3), the conditional variance of z_t is as follows:

$$\begin{aligned} \text{var}_t(z_{t+1}) &= E_t\left(z_{t+1} - E_t z_{t+1}\right)^2 \underbrace{=}_{E_t z_{t+1} = p z_t} E_t\left(z_{t+1} - p z_t\right)^2 = \\ &= p(1-p) z_t^2 + (1-p)\sigma_g^2 \end{aligned} \qquad (12)$$

Using our representation we have:

$$\begin{aligned} \text{var}_t(z_{t+1}) &= E_t\left(z_{t+1} - E_t z_{t+1}\right)^2 \underbrace{=}_{E_t z_{t+1} = p z_t} E_t\left(z_{t+1} - p z_t\right)^2 \\ &= E_t\left(\varphi z_t + \varpi_{S_{t+1}} E_t z_{t+1} + \delta_{S_{t+1}} g_{t+1} - p z_t\right)^2 \\ &= E_t\left[\left(\varphi - (1-\varpi_{S_{t+1}}) p\right) z_t + \delta_{S_{t+1}} g_{t+1}\right]^2 \\ &= p(2\varphi - 1)^2 z_t^2 + (1-p) E_t\left[-p z_t + g_{t+1}\right]^2 \\ &= p(2\varphi - 1)^2 z_t^2 + (1-p)\left(p^2 z_t^2 + \sigma_g^2\right) \end{aligned} \qquad (13)$$

Substituting $\varphi = 1 - p/2$ into equation (13), is straightforward to get expression in (12).$\square$

Our state-space formulation, which is characterized by having Gaussian innovations, is:

$$\xi_{t+1} = F\xi_t + B_{(S_{t+1})}E_t\xi_{t+1} + U_{(S_{t+1})}\upsilon_{t+1}, \tag{14}$$

$$\varepsilon_t = H'\xi_t, \tag{15}$$

where:

$$\xi_{t+1} = \left[z_{t+1} \;\; u_{t+1} \right]', \qquad \upsilon_{t+1} = \left[g_{t+1} \;\; e_{t+1} \right]', \qquad F = \begin{bmatrix} 1-\dfrac{p}{2} & 0 \\ 0 & \phi \end{bmatrix}, \qquad B_{(S_{t+1})} = \begin{bmatrix} \varpi_{S_{t+1}} & 0 \\ 0 & 1 \end{bmatrix},$$

$$U_{(S_{t+1})} = \begin{bmatrix} \delta_{S_{t+1}} & 0 \\ 0 & 1 \end{bmatrix}, E\left[\upsilon_t \upsilon_t'\right] = \begin{bmatrix} \sigma_g^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}, H' = \left[(1-\rho)(1-\alpha) \quad 0 \quad 1 \right]$$

and $\varpi_{S_{t+1}}$ and $\delta_{S_{t+1}}$ are defined as in (7).

Equations (14) and (15) define a state-space system (see Hamilton (1994), chapter 13), where (14) is the state equation and (15) is the observation equation.

For each of the two relevant histories, $S_t = k$ ($k = 0, 1$), the equations for the Kalman filter are[2]:

$$\begin{aligned} K_t^{(k)} &= P_{t|t-1}^{(k)} H \left(H' P_{t|t-1}^{(k)} H \right)^{-1}, \\ \hat{\xi}_{t+1|t}^{(k)} &= (I - B_{(S_t=k)})^{-1} F \left[\hat{\xi}_{t|t-1}^{(k)} + K_t^{(k)} \left(\hat{\varepsilon}_t - H' \hat{\xi}_{t|t-1}^{(k)} \right) \right], \\ P_{t+1|t}^{(k)} &= F P_{t|t-1}^{(k)} F' - F K_t^{(k)} P_{t|t-1}^{(k)} F' + U_{(S_t=k)} Q U'_{(S_t=k)}. \end{aligned}$$

Next, we describe how to get the log-likelihood function to be maximized with respect to the parameters ϕ , $p, \sigma_g^2, \sigma_e^2$:

Step 1: Computing the density functions for each history:

The conditional density function of $\hat{\varepsilon}_t$ to $Y_{t-1} \equiv \left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, ..., \hat{\varepsilon}_{t-1} \right)'$ is:

$$\begin{aligned} f\left(\hat{\varepsilon}_t \mid Y_{t-1}, S_t; \theta \right) &= \left(2\pi \right)^{-1/2} \left| \omega_t^{(k)} \right|^{-1/2} \exp\left(-\frac{1}{2} \hat{\mu}_t^{(k)\prime} \left[\omega_t^{(k)} \right]^{-1} \hat{\mu}_t^{(k)} \right) \\ &= \left(2\pi \right)^{-1/2} \left| \omega_t^{(k)} \right|^{-1/2} \exp\left(-\frac{1}{2} \left[\hat{\mu}_t^{(k)} \right]^2 / \omega_t^{(k)} \right), \end{aligned}$$

where: $\omega_t^{(k)} = H' P_{t|t-1}^{(k)} H$; $\hat{\mu}_t^{(k)} = \hat{\varepsilon}_t - H' \hat{\xi}_{t|t-1}^{(k)}$. and $\theta \equiv (\phi , p, \sigma_g^2, \sigma_e^2)'$

Step 2: Computing the marginal density function of $\hat{\varepsilon}_t$ conditional to Y_{t-1}:

$$f\left(\hat{\varepsilon}_t \mid Y_{t-1}; \theta \right) = \sum_{k=0}^{1} f\left(\hat{\varepsilon}_t \mid Y_{t-1}, S_t = k; \theta \right) P\left[S_t = k \right].$$

[2] In Appendix 1 we derive the equations for the Kalman filter using our state-space representation.

Step 3: Computing and maximizing the log-likelihood function of $\hat{\varepsilon}$:

$$\underset{\theta}{Max}\ \ln L(\theta) = \underset{\theta}{Max} \sum_{t=1}^{T} \ln f\left(\hat{\varepsilon}_t \mid Y_{t-1};\theta\right)$$

Once the parameters have been estimated, the probability of a regime change in the current period conditional on a given shock can be estimated as follows:

$$\Pr\left[S_t = 0 \mid \hat{\varepsilon}_t\right] = \frac{\Pr\left[S_t = 0\right] \cdot f\left(\hat{\varepsilon}_t \mid Y_{t-1}, S_t = 0; \hat{\theta}\right)}{f\left(\hat{\varepsilon}_t \mid Y_{t-1}; \hat{\theta}\right)},$$

where $\hat{\theta}$ denotes the vector of estimated parameters.

4. Empirical evidence

In this section we provide empirical evidence on monetary policy making for the US. Using quarterly data from the EcoWin Economic & Financial database, we collect information on interest rates, inflation and GDP for the sample period covering 1980:Q1-2011:Q1. We estimate the output gap by subtracting a non-linear trend from real GDP using the Hoddrick-Prescott filter. Figure 1 depicts the time evolution for inflation, output-gap and interest rates. Colours highlight the time periods corresponding to the tenure of the three chairmen of the Fed involved in the sample: Volcker (green), Greenspan (yellow) and Bernanke (orange).

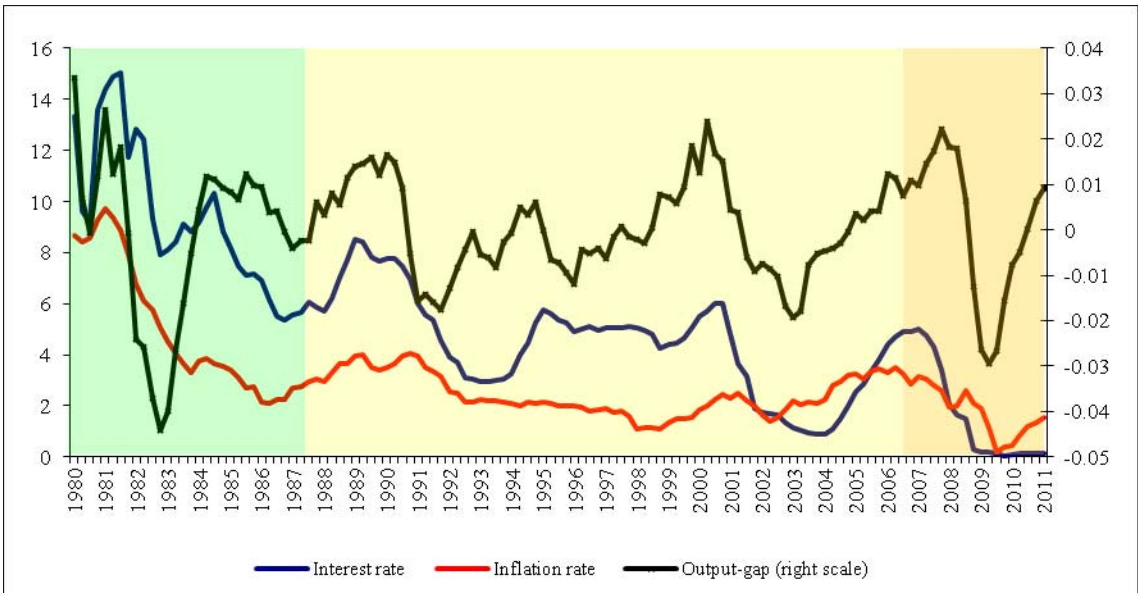

Figure 1: Time series for Inflation, Output-Gap abd Interest Rate for US economy.

A least square regression of the following Taylor rule:

$$i_t = \beta_0 + \beta_1 i_{t-1} + \beta_2 \pi_t + \beta_3 \left(y_t - y_t^* \right) + \varepsilon_t \tag{16}$$

yields the following parameter estimates (standard deviations in brackets):

$$\begin{aligned} i_t = {} & 0.0007 \; + \; 0.8722 i_{t-1} \; + \; 0.1710 \pi_t + \; 0.1503 \left(y_t - y_t^* \right) + \hat{\varepsilon}_t \\ & [0.0014] \quad [0.0346] \quad [0.0635] \quad [0.0534] \end{aligned} \tag{17}$$

$$\bar{R}^2 = 0.9463, \;\; \hat{\sigma}_\varepsilon^2 = 0.0001.$$

$$\hat{\alpha} = g_1(\hat{\boldsymbol{\beta}}) = \frac{\hat{\beta}_2}{1-\hat{\beta}_1} = 1.3376; \; std(\hat{\alpha}) = \left(\nabla g_1 \cdot \text{cov}(\hat{\boldsymbol{\beta}}) \cdot \nabla g_1' \right)^{1/2} = 0.3046,$$

$$\hat{\beta} = g_2(\hat{\boldsymbol{\beta}}) = \frac{\hat{\beta}_3}{1-\hat{\beta}_1} = 1.1763; std(\hat{\beta}) = \left(\nabla g_2 \cdot \text{cov}(\hat{\boldsymbol{\beta}}) \cdot \nabla g_2' \right)^{1/2} = 0.4923,$$

where $\boldsymbol{\beta} = \left[\beta_0, \beta_1, \beta_2, \beta_3 \right]'$, and $\nabla g_i, i = 1,2$ denotes the gradient of the function $g_i(\cdot), \; i = 1,2$.

Consistent with previous empirical research, a significant point estimate of the lagged policy rate is detected, suggesting very slow partial adjustment in US monetary policy making. Also, the estimated response for the "inflation gap" is consistent with the Taylor principle, that is, the nominal interest rate raises more than point-for-point when inflation exceeds the target inflation rate[3].

As for the nature of regime switching detected from the estimated monetary shocks, the maximization of the likelihood function yields the following point estimates (standard deviations are in brackets): $\hat{p} = 0.8626[0.0247]$, $\hat{\sigma}_g = 0.4416[0.1024]$, $\hat{\sigma}_e = 0.0030[0.0004]$ and $\hat{\phi} = 0.5636[0.0962]$. These parameters are the estimated probability of regime change ($1-p$), the estimated volatility of permanent and transitory shocks (σ_e^2 and σ_g^2, respectively) and the AR(1) parameter that corresponds to the time evolution of the transitory shock (ϕ). The unconditional probability of regime change for the US is around 13%, which implies a mean duration of shifts of around seven quarters. Also, as expected, the volatility of the shocks in the two regimes differs significantly. In particular, the volatility of transitory shocks is clearly lower than that of corresponding to permanent shocks. Moreover, the estimated coefficient ϕ is positive, statistically significant at the 1% significance level and clearly lower than one,

[3] It is well known that $\alpha > 1$ is a sufficient condition for equilibrium determinacy in the context of DSGE models (see for example, Woodford, 2003 or Galí, 2008).

a finding that is consistent with the assumptions made.

Figure 2 depicts the time evolution for the probability of regime change conditional to a given monetary shock, as well as the permanent component of the monetary shock, that is, the deviation of the current inflation target from its long-term mean ($\hat{z}_{t|t-1}$).

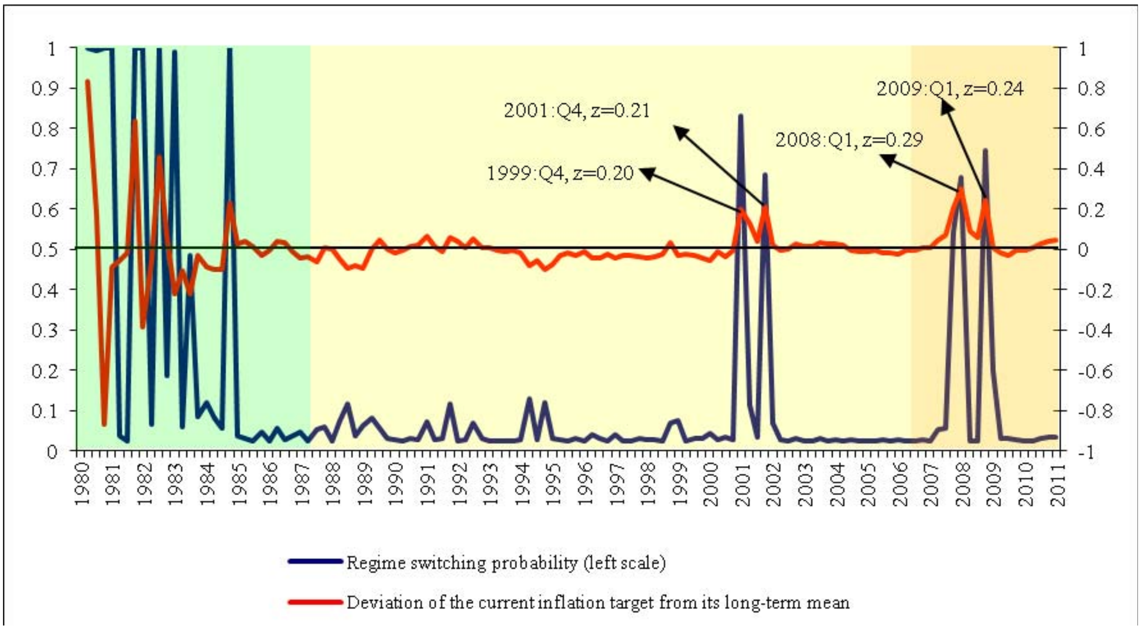

Figure 2. Monetary policy with time-varying inflation target during the Volcker-Greenspan-Bernanke period.

Our empirical findings show an extremely high probability of regime change at the beginning of the eighties. This is consistent with historical monetary policy making in the US[4]: in the period following the Great Inflation, Fed operating procedures were modified. On October 1979, targeting of non-borrowed reserves directly replaced Fed funds rate targeting, but after the meeting of the Federal Open Market Committee in October 1982, the Fed abandoned non-borrowed targeting and concluded that short-run control of monetary aggregates was less strict than interest control. After the Great Moderation, the probability of regime change approaches unity in March and December 2001. On 26 November 2001 the National Bureau of Economic Research announced that the US economy had been in recession since 1 March 2001. However, as Mostaghimi (2004) notes, there was some speculation that even though US monetary authorities had anticipated the severity of the problems in the US economy in 2000, they

[4]See Orphanides (2003) for a detailed analysis of US monetary policy and the usefulness of the Taylor-rule framework to interpret it.

hesitated to act promptly because of the prolonged US presidential election process. Another probable regime change detected is immediately after the unexpected shock of 9/11 event, which undoubtedly accelerated the decline in consumer confidence first noted in August 2001. After the terrorist attack, the Fed took up the challenge of maintaining and managing countercyclical policy in a stable price environment. To face the crisis, target federal funds rates was lowered quickly, and US monetary policy was easy during the period 2002 to 2006.

It is also observed two potential regime changes in the first quarter of 2008 and 2009, which are both related to the subprime mortgage crisis. The initial signals for the crisis in financial markets can be dated in June-July 2007 (problems at the Bear Stearns hedge fund); next, economic growth weakened and the recession officially started in December 2007. In March 2008 Bear Stearns collapsed, while Lehman Brothers followed in September 2008. By late 2008, nominal interest rates were close to the zero bound, but financial markets were not responding as expected. The Fed took additional measures. On march 18, 2009 the press release made by the Fed stated: “to provide greater support to mortgage lending and housing markets, the Committee decided today to increase the size of the Federal Reserve’s balance sheet further by purchasing up to an additional $750 billion of agency mortgage-backed securities, bringing its total purchases of these securities to up to $1.25 trillion this year, and to increase its purchases of agency debt this year by up to $100 billion to a total of up to $200 billion. Moreover, to help improve conditions in private credit markets, the Committee decided to purchase up to $300 billion of longer-term Treasury securities over the next six months”.

As to the estimates of “inflation gap” we can observe in Figure 2 that, after the Great Moderation, the regime changes detected in monetary policy making are matched with substantial updates in the current inflation target. It is also remarkable that our empirical evidence suggests that, during the period 1994-2000, the monetary policy implemented by the Federal Reserve was, in general, based on short-run inflation targets below the long-term target. This path for flexible inflation targeting is consistent with no accommodative monetary policy, in line with the Fed’s policy during this period.

The economic environment at the beginning of the past decade was sharply affected by the terrorist attack of September 11, 2001. During the period covering 1999-

2001 our estimates reveal two significant updates of inflation target, in the fourth quarter of 1999 and 2001, respectively. This two "regime shifts" are motivated not only by geopolitical uncertainties derived from the terrorist attack, but also by the weak recovery of US economy after the moderate recession between March and November 2001. For the period 2001-2004, the estimated discrepancy between the current inflation target and the long-term inflation target is, on average, positive, revealing that inflation did not appear as a serious concern in the short-run for the Federal Open Market Committee during this period. Therefore, the maximum sustainable employment arises now as the only relevant goal in this period.

Both aspects explain the aggressive response of the Fed in 2002 and 2003. As pointed out by Bernanke (2010), the discrepancy between the actual federal funds rates and the values implied by the Taylor rule during this time period is the most commonly cited evidence that monetary policy was too easy in order to prevent further bubbles in financial markets. However, our empirical findings suggest that the Fed managed the federal fund rates in accordance with short and long-run inflation targets. Furthermore, we can observe that the period 2004-2006 is characterized by negative differences between current inflation targets and the long-term inflation target. This suggests that, as a difference with the previous period (2001-2004), the Fed should now face the classical trade-off between employment and inflation in monetary policy making. And to prevent for inflationary pressures that might cause US economic growth, especially encouraged by the aggressive response of the Fed after 2001, just in June 2004 the Federal Market Committee began to raise the target rate, reaching 5.25% in June 2006. In 2008 and 2009 two clear changes in inflation targeting are detected, in a similar way as described for 1999 and 2000. After 2008, the estimated departures of current inflation targets are positive, on average, suggesting that employment becomes again the key short-run objective for the Fed. We can conclude that our empirical evidence on flexible inflation targeting suggests that US monetary policy was implemented accordingly with the macroeconomic conditions after the Great Moderation.

4.1. Alternative approaches

As previously showed, our estimation procedure is based on a convenient reformulation of the state-space model representation that allows us the optimal use of the Kalman filter to estimate not only the conditional probability of regime change, but also the non-observable departures of the current inflation target from its long-term mean. This relevant information could be also estimated, but not jointly, using Switching Markov techniques and the standard particle filter. The first procedure enables researchers to estimate probabilities of regime change, while the particle filter allows estimating inflation departures. The next two subsections show why our estimation procedure should be an interesting alternative to the combined use of these two techniques, at least for the US.

4.1.1 Swicthing Markov

We now check whether our estimation procedure produce similar empirical findings to those using a Taylor rule with time-varying parameters. It is widely accepted that well designed monetary policy should smooth cyclical fluctuations in prices and employment, thereby improving overall economic stability and investor confidence. Usually, when economic growth is lower than expected, accommodative monetary policy can stimulate aggregate demand to encourage employment. Likewise, when inflationary pressures arise as a consequence of systematic and intensive economic growth, a restrictive monetary policy can restore the ability of the central bank to achieve the inflation target.

In accordance with this principle, we consider the following specification of the Taylor rule:

$$i_t = c_{0,S_t} + c_{1,S_t} i_{t-1} + c_{2,S_t} \pi_t + c_{3,S_t} \left(y_t - y_t^* \right) + u_t, \quad u_t \sim N\left(0, \sigma_{S_t}^2\right), \tag{11}$$

where S_t $\left(S_t = 1, 2\right)$ denotes the state of the economy at date t, and c_{j,S_t} $(j = 0,1,2,3)$ and $\sigma_{S_t}^2$ are changing parameters depending on the regime of the economy. Specifically, we assume that S_t evolves according a two state first-order Markov process with the following transition probabilities:

$$P = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{pmatrix},$$

where $p_{ij} = p\left(S_t = j \mid S_{t-1} = i\right)$.

Table 1 presents the maximum likelihood parameter estimates (standard deviations in brackets) under both regimes:

	c_{0,S_t}	c_{1,S_t}	c_{2,S_t}	c_{3,S_t}	$\sigma^2_{S_t}$	α_{S_t}	β_{S_t}
$S_t = 1$	.0027 [.0008]	.8989 [.0149]	.0560 [.0373]	.1802 [.0309]	.0026 [.0003]	.5536 [.3793]	1.7824 [.3413]
$S_t = 2$	.0002 [.0048]	.7742 [.1003]	.3359 [.1633]	.0446 [.1228]	.0128 [.0017]	1.4877 [.3914]	.1975 [.5757]

Table 1. Parameters estimated of the Taylor Rule using Switching Markov.

As for the transition probabilities we obtain $p_{11} = .9272\,[.0262]$ and $p_{22} = .8491\,[.0599]$. Two regimes are clearly identified. In state 1, the Fed's Response is only significant to economic growth, while in state 2 the response of the monetary authority is caused by inflation departures from the target. It is also remarkable that state 2 is characterized by an extremely higher volatility of monetary shocks. This reveals that when US economy is experiencing inflationary pressures with weak economic growth, monetary policy is subject to higher uncertainty, as it currently takes place as a consequence of price shocks from raw materials.

Figure 3 depicts the probability of the first regime (accommodative monetary policy) conditional on the past history of the economy.

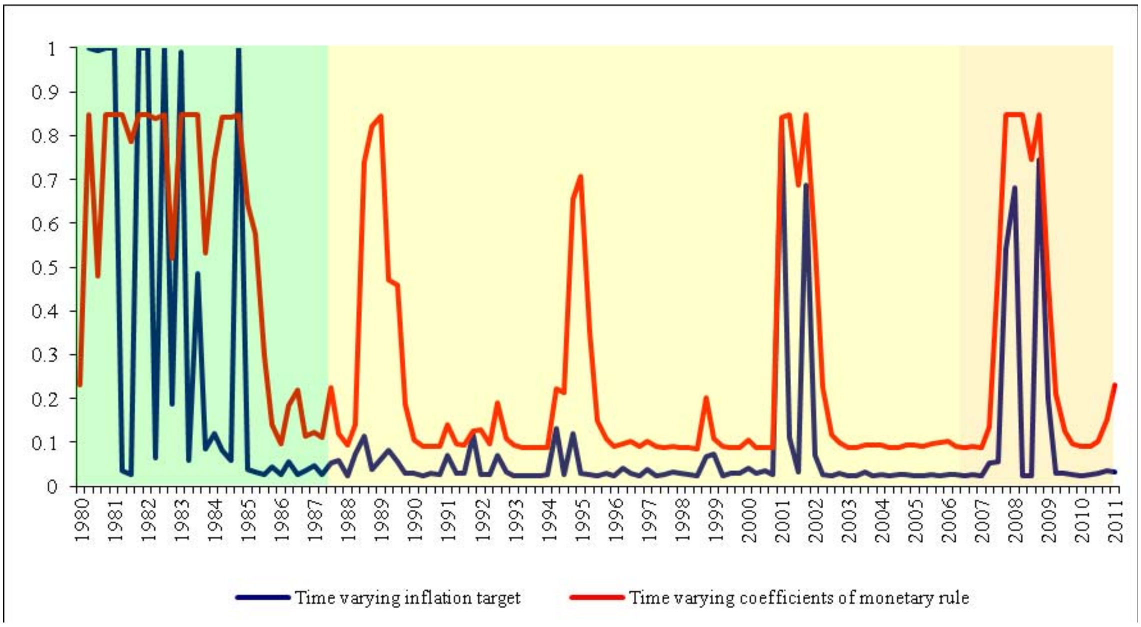

Figure 3. Regime switching probabilities using the LPR-representation and the Switching Markov, respectively.

After the Great Moderation, the time evolution of these probabilities basically replicates the pattern that we previously showed using our estimation procedure under flexible inflation targeting. However, the Switching Markov technique detect two probably regime-changes that are not suggested by our approach. This likely reflects that the Fed changed substantially the federal fund rate without significant updating of inflation target.

Let us make a brief summary of what happened. The Fed raised the discount rate from 5.5 percent to 6 percent in September 1987, after Alan Greenspan replaced Paul Volcker as Fed Chairman. However, the stock market crash in October 1987 prevented the Fed for additional tightening. By the spring of 1988 the Fed felt it was room to tighten monetary policy and increased the funds rate from the 6 to nearly 10 percent in March 1989. During this time period, core CPI inflation was running at about 4.5 percent. The outcome of this aggressive policy was that Real GDP growth slowed from about 4 percent in 1988 to 2.5 percent in 1989. Then, the Fed reduced the funds rate to around 7 percent by late 1990, but core CPI inflation was running at 5.3 percent. In sum, while tight monetary policy was implemented during 1987-1990 the credibility for low inflation was not restored. The other regime change suggested by Switching Markov is just related to the preventive campaign against inflation developed in 1994. The Fed began to raise the federal funds rate in February 1994, and after seven updates reached the 6 percent by February 1995. In spite of the policy tightening, real GDP grew by 4 percent in 1994, and again, the Fed's credibility for low inflation was far from conclusive[5]. Only after 1996, bond rates reveal that the mission to contain inflation was accomplished.

4.1.2. The Particle Filter

We now explore the empirical results based on the use of the particle filter. Appendix 2 describes the implementation of the particle filter to our particular estimation problem and the next table shows the estimated parameters using the standard particle filter[6] with the *AHM-representation* and the Kalman filter with the *LPR- representation*.

[5] It should be also remembered that in 1994 the Fed started to announce its current intended federal funds rate target just after each meeting of the Federal Open Market Committee.

[6] The results are based 1,000 replications. Empirical estimates are robust to the use of 2,000 and 3,000 replications.

	$\hat{p}$	$\hat{\phi}$	$\hat{\sigma}_g$	$\hat{\sigma}_e$
Kalman filter, (*LPR-representation*),	0.8626 (0.0247)	0.5636 (0.0962)	0.4416 (0.1024)	0.0030 (0.0004)
Particle Filter (*AHM-representation*),	0.8656 (0.0131)	0.5613 (0.0096)	0.4573 (0.0392)	0.0031 (0.0001)

Table 2. Estimates of structural parameters using the Particle Filter with the AHM-representation and the Kalman Filter with the LPR-representation, respectively.

It is readily apparent that parameter estimates using a sequential Monte Carlo filter are statistically equal to those obtained under our *LPR-representation.* For each estimation procedure, the confidence interval at conventional significance levels contains the point estimate obtained with the alternative approach. However, the particle filter exhibits higher accuracy, although with a remarkable computational cost. The interest of the particle filter is the possibility of implementing the decomposition of monetary shocks using a flexible targeting point of view. Figure 4 shows the time evolution of the estimated "inflation gap" using both procedures.

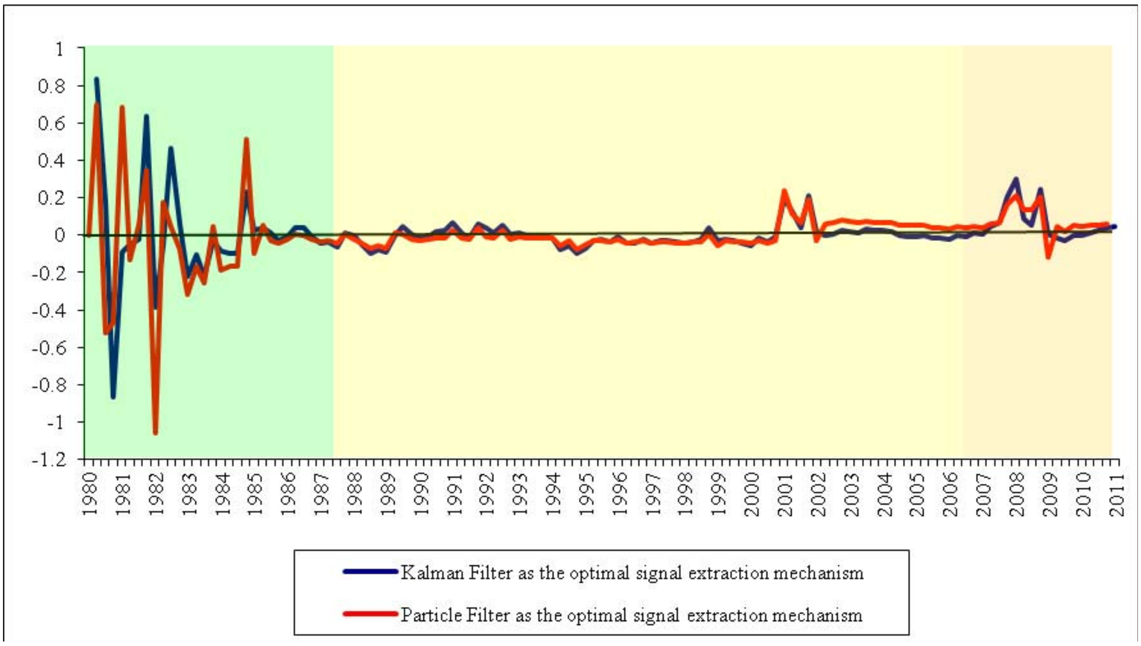

Figure 4. Deviation estimated of the current inflation target from its long-term mean

Interestingly enough, the time evolution of $\hat{z}_{t|t-1}$ is quite similar under both methodologies. To statistically assess whether both procedures lead to the same probability distribution of the "inflation gap" we perform a Monte Carlo simulation in order not only to test null hypothesis of equality between the two distributions but also to compute the Mean Squared Error in forecasting theoretical z_t values[7]. In particular, we proceed as follows: considering a sample size equal to 125 (the same number of observations as in the data sample), we simulated $\{N_{t+1}, e_{t+1}, z_{t+1}, u_{t+1}, \varepsilon_t\}_{t=1}^{125}$ with initial conditions $z_0 = u_0 = 0$. Using these simulated time series we generate shocks $\{\varepsilon_t\}_{t=1}^{125}$ in accordance with equation (6). Now we consider the estimated parameters using the Kalman filter with the *LPR-representation* to generate theoretical values of z_t. After that, we make simulation exercises for shocks to generate conditional estimates of z_t either using the Particle Filter for *AHM-representation* or the Kalman Filter for the *LPR-representation*. Let us denote each of these time series as $\{\hat{z}_{t|t-1}^{PF}\}_{t=2}^{125}$ and $\{\hat{z}_{t|t-1}^{KF}\}_{t=2}^{125}$, respectively. We can now perform a Kolmogorov-Smirnov test for the null hypothesis of equality of distributions between $\{\hat{z}_{t|t-1}^{PF}\}_{t=2}^{125}$ and $\{\hat{z}_{t|t-1}^{KF}\}_{t=2}^{125}$ at the 5% significance level. The percentage of rejections with 1,000 replications is about 17%, neither so high nor negligible, as expected from the visual inspection of Figure 4.

However, as to the mean squared error to fit the theoretical "inflation gap", we obtain the following median values:

	$MSE^{(PF)}$	$MSE^{(KF)}$
Median(MSE)	0.0260	0.0132

Table 4. Testing the fit of each methodology:

$$MSE^{(j)} = \left(\frac{1}{124}\right)\sum_{t=2}^{125}(z_t - \hat{z}_{t|t-1}^{(j)})^2,\ j \equiv PF, KF$$

Therefore, our simulation experiment shows that our estimation procedure has a better predictive ability to forecast the discrepancy between the short and long-run inflation targets.

5. Conclusions

This paper proposes an estimation procedure to decompose monetary shocks

[7] The Mean Square Error computed is $\left((1/T)\sum_{t=1}^{T}(z_t - \hat{z}_{t|t-1})^2\right)$, where z_t is the theoretical value of inflation-target and $\hat{z}_{t|t-1}$ is the estimated value either using either the Kalman-filter with the *LPR-representation* or the Particle-Filter considering the *AHM-representation*.

into permanent and transitory components using an inertial Taylor rule and the monetary innovations scheme proposed in Andolfatto et al. (2008). The contribution of the paper is threefold: a) under our state-space representation, the Kalman filter becomes fully optimal as the signal extraction mechanism because all the relevant noises are Gaussian; b) we provide a direct way to the exact maximum likelihood estimation of the parameters involved in the time evolution of persistent and transitory monetary shocks, and c) in each time period we provide the probability that a regime change has occurred in the current period conditional on an observed monetary shock. Researchers interested in the macroeconomic learning literature could take advantages of our estimation procedure in order to incorporate the monetary signal extraction problem proposed into a DSGE model.

Empirical evidence on US historical monetary policy making through the lens of a Taylor rule is provided using quarterly data during the period 1980:Q1-2011-Q1. Consistent with previous findings, the evidence for a regime change in the inflation target during the nineties is extremely weak. However, September eleven, the recession that started in March 2001 and the subprime crisis were significant events that affected US monetary policy making in the last decade. The use of a Taylor rule with time-varying responses to output gap and deviations from the inflation target suggests the same regime changes after the Great Moderation.

We compare our findings based on a flexible targeting point of view with those suggested by the estimation of a Taylor rule with time-varying responses to output gap using Switching Markov. The evidence found suggests that during the nineties there has been though two regime changes in US monetary policy making in terms of the most relevant objective (unemployment-inflation trade-off). However, during this period, the Fed's inflation targeting was not significantly updated.

Finally, we use Monte Carlo exercises to compare the time evolution of the inflation gap that we estimate using our procedure with the estimates based on the use of the standard particle filter considering the state-space representation proposed in Andolfatto et al. (2008). Simulation exercises reveal that the percentage of rejection at conventional significance levels for the null hypothesis of equality of distribution is about 17%. Though this percentage is not particularly high, our procedure leads to lower mean squared error to forecast theoretical levels of inflation gap.

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Appendix 1

This appendix describes how to get equations for the Kalman filter using our state-space representation with Gaussian innovations.

Following Hamilton (1994), we consider the following state-space system:

$$\underbrace{\xi_{t+1}}_{r\times 1} = \underbrace{F}_{r\times r}\underbrace{\xi_t}_{r\times 1} + \underbrace{B}_{r\times r}\underbrace{E_t\xi_{t+1}}_{r\times 1} + \underbrace{U}_{r\times r}\underbrace{\upsilon_{t+1}}_{r\times 1}, \tag{A.1}$$

$$\underbrace{\varepsilon_t}_{n\times 1} = \underbrace{H'}_{n\times r}\underbrace{\xi_t}_{r\times 1} + \underbrace{w_t}_{n\times 1}, \tag{A.2}$$

with

$$E(\upsilon_t \upsilon_\tau') = \begin{cases} Q, & \text{for } t=\tau \\ 0, & \text{otherwise} \end{cases} \tag{A.3}$$

$$E(w_t w_\tau') = \begin{cases} R, & \text{for } t=\tau \\ 0, & \text{otherwise.} \end{cases} \tag{A.4}$$

We assume that $\{y_1, y_2, ..., y_T\}$ are observable variables and that, *B, U, H, Q* and *R* are known with certainty.

The Kalman Filter calculates the forecasts $\hat{\xi}_{t+1|t}$ recursively, and, associated with each of these forecasts, the Kalman Filter computes the Mean Squared Error matrix: $P_{t+1|t} \equiv E\left[(\xi_{t+1} - \hat{\xi}_{t+1|t})(\xi_{t+1} - \hat{\xi}_{t+1|t})'\right]$. The recursion begins with $\hat{\xi}_{1|0}$ and we assume that $\hat{\xi}_{1|0} = E(\xi_t) = 0$. In a similar way, $P_{1|0} \equiv E\left[\left(\xi_t - E(\xi_t)\right)\left(\xi_t - E(\xi_t)\right)'\right]$. We also assume that eigenvalues of F are all inside the unit circle, that is, the process for ξ_t is covariance-stationary. Thus, $P_{1|0}$ can be computed as follows:

$$\Sigma \equiv E(\xi_{t+1}\xi_{t+1}') = E\left[\left(F\xi_t + BE_t(\xi_{t+1}) + U\upsilon_{t+1}\right)\left(F\xi_t + BE_t(\xi_{t+1}) + U\upsilon_{t+1}\right)'\right]$$

$$\underbrace{=}_{\substack{E_t(\xi_{t+1}) = (I-B)^{-1}F\xi_t \Rightarrow \\ \xi_{t+1} = [I+B(I-B)^{-1}]F\xi_t}} E\left\{\left[\underbrace{\left(I + B(I-B)^{-1}\right)}_{\tilde{B}}F\xi_t + U\upsilon_{t+1}\right]\left[\underbrace{\left(I + B(I-B)^{-1}\right)}_{\tilde{B}}F\xi_t + U\upsilon_{t+1}\right]'\right\}$$

$$= \tilde{B}F\Sigma F'\tilde{B} + \underbrace{UQU'}_{\tilde{Q}}.$$

Therefore, $vec(P_{1|0}) = \left[I_{r^2} - (\tilde{B}F \otimes \tilde{B}F)\right]^{-1} vec(\tilde{Q})$.

The forecasting of y_t is as follows:

$$\hat{y}_{t|t-1} \equiv E(y_t \,|\, \Upsilon_{t-1}) = H'E(\xi_t \,|\, \Upsilon_{t-1}) = H'\hat{\xi}_{t|t-1}, \text{ where } \Upsilon_{t-1} = \left(y_t', y_{t-1}', ..., y_1'\right)'.$$

The associated Mean Squared Error is:

$$E\left[(y_t - \hat{y}_{t|t-1})(y_t - \hat{y}_{t|t-1})'\right] = H'P_{t|t-1}H + R.$$

Next we update ξ_t taking into account the information set available at time *t* as follows:

$$\begin{aligned}\hat{\xi}_{t|t} \equiv \hat{E}(\xi_t \,|\, \Upsilon_t) = \hat{\xi}_{t|t-1} + \left\{E\left[(\xi_t - \hat{\xi}_{t|t-1})(y_t - \hat{y}_{t|t-1})'\right]\right\}\times \\ \left\{E\left[(y_t - \hat{y}_{t|t-1})(y_t - \hat{y}_{t|t-1})'\right]\right\}^{-1}(y_t - \hat{y}_{t|t-1}) \\ = \hat{\xi}_{t|t-1} + P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}(y_t - H'\hat{\xi}_{t|t-1}).\end{aligned} \tag{A.5}$$

with Mean Squared Error:

$$\begin{aligned}P_{t|t} \equiv E\left[(\xi_t - \hat{\xi}_{t|t})(\xi_t - \hat{\xi}_{t|t})'\right] = E\left[(\xi_t - \hat{\xi}_{t|t-1})(\xi_t - \hat{\xi}_{t|t-1})'\right] - \left\{E\left[(\xi_t - \hat{\xi}_{t|t-1})(y_t - \hat{y}_{t|t-1})'\right]\right\}\times \\ \left\{E\left[(y_t - \hat{y}_{t|t-1})(y_t - \hat{y}_{t|t-1})'\right]\right\}^{-1}\times \\ \left\{E\left[(y_t - \hat{y}_{t|t-1})(\xi_t - \hat{\xi}_{t|t-1})'\right]\right\} \\ = P_{t|t-1} - P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}H'P_{t|t-1}.\end{aligned} \tag{A.6}$$

Next, we forecast ξ_{t+1} given the current set of available information as follows:

$$\hat{\xi}_{t+1|t} \equiv \hat{E}(\xi_{t+1} \,|\, \Upsilon_t) = F\,\hat{E}(\xi_t \,|\, \Upsilon_t) + B\,\hat{E}(E_t(\xi_{t+1}) \,|\, \Upsilon_t) + U\,\hat{E}(\upsilon_{t+1} \,|\, \Upsilon_t) = F\hat{\xi}_{t|t} + B\hat{\xi}_{t+1|t}$$

where, given that υ_{t+1} and w_t are Gaussian, we use that $\hat{\xi}_{t+1|t} = E_t(\xi_{t+1})$.

Rearranging the above equation we have

$$\hat{\xi}_{t+1|t} = (I - B)^{-1}F\hat{\xi}_{t|t}. \tag{A.7}$$

Substituting (A.5) into (A.7):

$$\hat{\xi}_{t+1|t} = (I - B)^{-1}F\hat{\xi}_{t|t-1} + (I - B)^{-1}F\,K_t(y_t - H'\hat{\xi}_{t|t-1}), \tag{A.8}$$

where

$$K_t = P_{t|t-1}H(H'\,P_{t|t-1}H + R)^{-1} \tag{A.9}$$

Taking into account not only that $\xi_{t+1} = F\xi_t + B\,E_t(\xi_{t+1}) + U\,\upsilon_{t+1}$, but also that $E_t(\xi_{t+1}) = \hat{\xi}_{t+1|t} = F\hat{\xi}_{t|t} + B\hat{\xi}_{t+1|t}$, we obtain the expression for the forecasting error: $\xi_{t+1} - \hat{\xi}_{t+1|t} = F\hat{\xi}_{t|t} + U\,\upsilon_{t+1}$.

Thus, the Mean Squared Error associated to $\hat{\xi}_{t+1|t}$ can be obtained as follows:

$$P_{t+1|t} = E\left[(F\hat{\xi}_{t|t} + U\,\upsilon_{t+1})(F\hat{\xi}_{t|t} + U\,\upsilon_{t+1})'\right] = F\,P_{t|t}\,F' + \tilde{Q}. \tag{A.10}$$

Substituting (A.6) into (A.8):

$$P_{t+1|t} = F\, P_{t|t-1}\, F' - F\, K_t\, H'\, P_{t|t-1} F' + \tilde{Q}\,. \qquad \text{(A.11)}$$

Summarizing, given $\hat{\xi}_{1|0}$ and $P_{1|0}$, the Kalman Filter computes recursively $\hat{\xi}_{t+1|t}$ and $P_{t+1|t}$ using the equations (A.8), (A.9) and (A.11).

Appendix 2

In this appendix, we describe how to evaluate the likelihood function of monetary innovations using a Sequential Monte Carlo Filter (Fernández-Villaverde and Rubio-Ramírez (2004)) when the *AHM-representation* is considered.

The Andolfatto et al. (2008) specification is:

$$z_{t+1} = p\, z_t + N_{t+1} \qquad \text{(A.12)}$$

$$\varepsilon_t = (1-\rho)(1-\alpha) z_t + u_t \qquad \text{(A.13)}$$

where $\begin{cases} N_{t+1} = \begin{cases} (1-p) z_t, \text{ with prob. } p \\ g_{t+1} - p\, z_t, \text{with prob. } 1-p, \text{ where } g_{t+1} \sim N(0,\sigma_g^2) \end{cases} \\ u_{t+1} = \phi\, u_t + e_{t+1}, \text{ where } e_{t+1} \sim N(0,\sigma_e^2) \end{cases}$

The particle filter is an alternative to overcome non-normality. Assuming that $z_0 = 0$, we proceed as follows:

Step 1: Evaluate the probability of $\hat{u}_{t|t-1}$**:**

i) We draw a random sample of size I =1000 from the uniform distribution in (0,1) and from a Normal distribution with zero mean and σ_g^2 variance. We call each observation of these two initial samples as $U^{i,1} \sim U(0,1)$, and $x^{i,1} \sim N(0,\sigma_g^2)$ for *i*=1,…,*I*.. Now, we use these two samples to generate a new sample the we denote $N^{1|0,i}$ as follows:

$$\begin{cases} \text{If } U^{i,1} \le p \text{ then } N^{1|0,i} = 0 \\ \text{If } U^{i,1} > p \text{ then } N^{1|0,i} = x^{i,1} \end{cases} \quad i = 1,\ldots,I$$

Where 1-*p* is the probability of a regime change. We use the *I* values of $N^{1|0,i}$ to generate an additional sample of *I* values that we denote $z^{1|0,i}$ as follows:

$$z^{1|0,i} = p z_0 + N^{1|0,i} = N^{1|0,i}, \quad i = 1,\ldots,I.$$

ii) Next, we use the estimated value for the first element of the noise vector ε_t, that we denote as $\hat{\varepsilon}_1$, to generate a random sample for the innovation u_t as follows:

$$u^{1|0,i} = \hat{\varepsilon}_1 - (1-\rho)(1-\alpha) z^{1|0,i}, \quad i = 1,...,I .$$

iii) We evaluate the relative weight for each observation $u^{1|0,i}$:

$$q_1^i = \frac{p(u^{1|0,i})}{\sum_{i=1}^{I} p(u^{1|0,i})}, \quad i = 1,...,I, \text{ taking into account that } u_1 \sim N\left(0, \frac{\sigma_e^2}{1-\phi}\right),$$

that is, the marginal distribution of AR(1) process for the first observation.

iv) We update the initial sample $z^{1|0,i}$ by performing a weighted sampling with replacement in accordance with the above-mentioned weights.

v) As in ii), we draw I samples from a uniform probability distribution (U(0,1)) and from a Normal probability distribution (N(0,σ_g^2)). We call each of these samples: $U^{i,2} \sim U(0,1)$, $x^{i,2} \sim N(0,\sigma_g^2)$ for i=1,…,I. Now, we use these values to generate I values of $N^{2|1,i}$ as follows:

$$\begin{cases} \text{If } U^{i,2} \le p \text{ then } N^{2|1,i} = 0 \\ \text{If } U^{i,2} > p \text{ then } N^{2|1,i} = x^{i,2} \end{cases} \quad i = 1,...,I$$

We use the I values of $N^{2|1,i}$ to generate the I values of $z^{2|1,i}$ using (A.12):

$$z^{2|1,i} = p z^{1,i} + N^{2|1,i}, \quad i = 1,...,I.$$

vi) Next, we use the observation ε_2 and each of the I values of $z^{2|1,i}$ obtained in ii) to find I values of $u^{2|1,i}$:

$$u^{2|1,i} = \varepsilon_2 - (1-\rho)(1-\alpha) z^{2|1,i}, \quad i = 1,...,I .$$

vii) We evaluate the relative probability of each of the I values de $u^{2|1,i}$:

$$q_2^i = \frac{p(u^{2|1,i})}{\sum_{i=1}^{I} p(u^{2|1,i})}, \quad i = 1,...,I$$

taking into account that $u_2 | u_1 \sim N\left(\phi u_1, \sigma_e^2\right)$,

that is, the distribution u_2 conditional to u_1 of a AR(1) process for the second observation. In particular, $u^{2|1,i} | u^{1|0,i} \sim N\left(\phi u^{1|0,i}, \sigma_e^2\right)$, $i = 1,...,I$.

viii) Go to v) and proceed as in steps vi)-vii) and iterate until the end of the sample.

Step 2: Using the Law of the Large Numbers:

$$p(\varepsilon_t | \varepsilon^{t-1}) \simeq \frac{1}{I}\sum_{i=1}^{I} p\left(u^{t|t-1,i}\right), \text{ where } u^{t|t-1,i} | u^{t-1|t-2,i} \sim N(\phi u^{t-1|t-2,i}, \sigma_e^2), i = 1,...,I \text{ and}$$

$$\varepsilon^t \equiv \left(\varepsilon_1, \varepsilon_2, ..., \varepsilon_t\right)'.$$

Once the conditional probabilities for monetary innovations are computed, we can evaluate the likelihood function as:

$$p(\varepsilon^T) = \prod_{t=1}^{T} \left[\frac{1}{I} \sum_{i=1}^{I} p(u^{t|t-1,i}) \right]$$

where T denotes the sample size.

Step 3: We maximize the likelihood with respect to the parameters ϕ, σ_e^2, σ_g^2 and p.