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Abstract

We define an algebro-topological concept of essential map and we use it to prove several results in the theory of general equilibrium and Nash equilibrium refinement.

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1 Introduction

We use some tools of algebraic topology to give a unified treatment of several results on stable sets and the existence of general equilibria.

1.1 Definition and elementary properties of essential sets

Let $(M, \partial M)$ and $(T, \partial T)$ be compact manifolds with boundary of dimension m and t respectively.

Let π_1 and π_2 be the projections $(M, \partial M) \times (T, \partial T) \to (T, \partial T)$ and $(M, \partial M) \times (T, \partial T) \to (M, \partial M)$ respectively.

All homology coefficients will be assumed to be in a field of characteristic zero.

It is assumed that the manifolds are oriented, ζ_M will denote the orientation class of $(M, \partial M)$.

To avoid trivial pathologies we assume that all spaces are CW complexes (or ANR or subanalytic or your favourite "nice" space).

Definition 1.1 *A closed subset $\eta \subset (M, \partial M) \times (T, \partial T)$ is called* essential *if the induced map on cohomology*

$$H^t(T, \partial T) \to H^t(\eta)$$

is injective.

1.2 Families of solutions of equations and fixed point sets are essential

A parameterized family of maps is a continuous map $f : T \times M \to N$, the space T functions as parameter space. For $t \in T$, by f_t we mean the restriction of f to $\{t\} \times M$. We consider the set

$$\eta := \{(t, m) | f(t, m) = n_0\}$$

of points that map onto n_0 under f.

Theorem 1.2 *Let $f : T \times M \to N$ be a parameterized family of maps such that $m = n$. Further assume that $n_0 \notin f_t(\partial M)$ for all $t \in T$. If $\deg f \neq 0$, then the projection map $\pi : \eta \to T$ is essential.*

Proof. Let $\overline{a}$ be a generator of $H^m(N, N - n_0)$. Denote $f^*(\overline{a}) \in H^m(T \times M, T \times M - \eta)$ by a. Let a also denote the class a as an element of $H^m(T \times M, T \times \partial M)$. Let $\mu \in H_n(M, \partial M)$ be the oriented class. Given $x \in H^k(T)$ consider the map $x \mapsto (a \cup \pi^* x) \setminus \mu$, where $\cup$ is the cupproduct, and $\setminus$ is the slant product. We have

$$(a \cup \pi^* x) \setminus \mu = (a \otimes x) \setminus \mu = (a \setminus \mu) \cup x = (\deg f) \cdot x.$$

So, if $\deg f \neq 0$, the map $x \mapsto \pi^* x$ is injective when restricted to $\text{supp}(a)$, and hence also injective on η. ∎

For a parameterized family of functions $f : T \times M \to M$, let

$$\eta := \{(t, m) | f(t, m) = m\}$$

be the set of fixed points of f. Denote for each $t \in T$ the Lefschetz number of f_t by $L(f_t)$.

Theorem 1.3 *Let $f : T \times M \to M$ be a parameterized family of maps, and assume that there exists an $L(f) \neq 0$ such that $L(f_t) = L$ for all $t \in T$. Then the projection map $\pi : \eta \to T$ is essential.*

Proof. Let $\Delta \in H_k(M \times M, M \times \partial M)$ and $\Gamma \in H_k(M \times M, \partial M \times M)$ be the images of $\mu \in H_k(M, \partial M)$ under the homomorphisms induced by the maps $m \mapsto (m, m)$ and $m \mapsto (m, f(m))$ respectively. Let $\overline{d} \in H^k(M \times M, \partial M \times M)$ and $\overline{g} \in H^k(M \times M, M \times \partial M)$ be their Poincaré duals. The Lefschetz number is, by definition, $L(f) = \langle \overline{d} \cup \overline{g}, \mu \times \mu \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the algebraic duality operator between cohomology and homology.

Write $d = 1 \otimes \overline{d} \in H^k(T \times M \times M, T \times \partial M \times M)$ and $g = 1 \otimes \overline{g} \in H^k(T \times M \times M, T \times M \times \partial M)$. Note that $d \cup g$ has support near $\eta \subset T \times M \hookrightarrow T \times M \times M$ where the second inclusion is the diagonal.

As before consider the map A that assigns to an element $x \in H^k(T)$ the element $A(x) := \pi^*(x \cup d \cup g)/\mu \otimes \mu$. As before we that $A(x) = L(f)x$. So, the map π^* is injective from $H^k(T)$ into $H^k(N_\varepsilon)$ where N_ε is any neighborhood of η. This implies that it is injective in Čech cohomology. ∎

2 Existence of Stable Sets

A KM perturbation of the game $\Gamma = (N, u)$ is a vector $\eta = (\eta_i)_{i \in N}$ where $\eta_i = (\eta_i(s_i))_{s_i \in S_i}$ is a vector of non-negative numbers $\eta_i(s_i)$. In the η-perturbed game $\Gamma(\eta)$ each player i is forced to play each pure strategy s_i with probability at least $\eta_i(s_i)$. The set of KM perturbations is denoted by $\mathcal{K}$. By E we denote the set of pairs (η, σ) in $\mathcal{K} \times \Sigma$ for which the mixed strategy profile σ is a Nash equilibrium of the game $\Gamma(\eta)$.

For $\varepsilon > 0$ we use the following notation. A KM perturbation η is of size ε when $\|\eta\|_\infty \leq \varepsilon$. By $\mathcal{K}_\varepsilon$ we denote the set of KM perturbations $\eta \in \mathcal{K}$ of size ε. Let $\partial\mathcal{K}_\varepsilon$ be the set of KM perturbations $\eta \in \mathcal{K}_\varepsilon$ for which there is a player $i \in N$ and a pure strategy $s_i \in S_i$ with

$\eta_i(s_i) \in \{0, \varepsilon\}$. For a set $T \subset E$, we write T_ε for the set of pairs $(\eta, \sigma) \in T$ for which η is of size ε. By ∂T_ε we denote the vertical boundary of T_ε, the set of pairs $(\eta, \sigma) \in T_\varepsilon$ with $\eta \in \partial K_\varepsilon$.

Let $\pi : \mathcal{K} \times \Sigma \to \mathcal{K}$ be the orthogonal projection map that assigns η to the pair $(\eta, \sigma) \in \mathcal{K} \times \Sigma$. A closed set $T \subset E$ is a *germ* if for every sufficiently small size $\varepsilon > 0$,

(1) $T_\varepsilon \setminus \partial T_\varepsilon$ is connected,

(2) T_ε equals the closure of $T_\varepsilon \setminus \partial T_\varepsilon$, and

(3) the map $\pi : (T_\varepsilon, \partial T_\varepsilon) \to (\mathcal{K}_\varepsilon, \partial \mathcal{K}_\varepsilon)$ is essential.

A closed set $S \subset \Sigma$ is *stable* if there exists a germ $T \subset E$ with

$$S = \{\sigma \in \Sigma | (0, \sigma) \in T\}.$$

Theorem 2.1 *Every finite game in strategic form has a stable set.*

Proof. Take a finite game in strategic form $\Gamma = (N, u)$. Define the map $f : \mathcal{K} \times \Sigma \to \Sigma$ as follows. For $\sigma \in \Sigma$, define $r_i(\sigma) = (r_i(\sigma)(s_i))_{s_i \in S_i}$ by, for each $s_i \in S_i$,

$$r_i(\sigma)(s_i) = u_i(\sigma | s_i) - u_i(\sigma).$$

For $(\eta, \sigma) \in \mathcal{K} \times \Sigma$, define $g_i(\eta, \sigma) = (g_i(\eta, \sigma)(s_i))_{s_i \in S_i}$ by

$$g_i(\eta, \sigma)(s_i) = \max\{\sigma(s_i) + r_i(\sigma)(s_i), \eta_i\}.$$

Define $\varepsilon^* > 0$ by

$$\varepsilon^* = \frac{1}{2} \cdot \max_{i \in N} \left\{ \frac{1}{|S_i|} \right\},$$

so that the strategy space of the perturbed game $\Gamma(\eta)$ is of full dimension for each KM perturbation η of size ε^*.

Claim. Suppose η is of size ε^* and σ is feasible for η. Then $\sum_{s_i \in S_i} g_i(\eta, \sigma)(s_i) > 0$.

Define

$$f_i(\eta, \sigma)(s_i) = \frac{g_i(\eta, \sigma)(s_i)}{\sum_{s_i \in S_i} g_i(\eta, \sigma)(s_i)}.$$

Claim. The function $f : \mathcal{K} \times \Sigma \to \Sigma$ satisfies the conditions of Theorem 1.3.

3 On Monetary Equilibria

In the next application, we extend the canonical general equilibrium model with monetary exchange of Drèze and Polemarchakis (2001), a model that is compatible with Chapter 2 of Woodford (2003) and can be viewed as the general equilibrium extension of that model. We extend it to deal with the case of general initial endowments of money.

Consider a private ownership monetary economy $\mathcal{E} = (\mathcal{T}, (X^h, \preceq^h, e^h, n^h, \mu^h, \theta^h)_{h \in \mathcal{H}}, r)$. Such an economy consists of H individuals and one central bank.

There is an event tree $\mathcal{T}$ with the set of date-events $\mathcal{S}$ as nodes. The cardinality of $\mathcal{S}$ is S. The set $\mathcal{S}$ is partitioned into subsets $\mathcal{S}_0, \ldots, \mathcal{S}_T$, where $\mathcal{S}_t$ are the date-events at date t. The set of successors of date-event s_t is denoted by s_t^+, a subset of $\mathcal{S}_{t+1}$. The unique predecessor of s_t is denoted by s_t^-, an element of $\mathcal{S}_{t-1}$. Date t represents the starting point of period t. Period t ends at date $t+1$ and is interpreted as the time interval separating a node from its successor, i.e. a time interval of unspecified length between date t and date $t+1$ during which transactions take place. We will refer to date-events s_t and periods s_t to distinguish between points and intervals of time.

At date-event s_t there is trade in L commodities and $|s_t^+|$ one-period Arrow securities.[1] An Arrow security for date-event s_{t+1} pays one nominal unit if and only if date-event s_{t+1} occurs. Because of the availability of Arrow securities, markets are sequentially complete.

For notational convenience we introduce at each terminal date-event $s_T \in \mathcal{S}_T$ an elementary security that pays one unit of money at the end of that date-event. We therefore extend the date-event tree by a set of states $\mathcal{S}_{T+1}$ with the same cardinality as $\mathcal{S}_T$ and use labels s_{T+1} to denote the date-events in $\mathcal{S}_{T+1}$. Every date-event in $\mathcal{S}_T$ has exactly one successor in $\mathcal{S}_{T+1}$.

Commodity prices at date-event s_t are denoted $\widetilde{p}_{s_t}$ and belong to $\mathbb{R}^L_+$. For $s_t \in (\mathcal{S} \cup \mathcal{S}_{T+1}) \setminus \{0\}$, the Arrow security for date-event s_t is traded at date-event s_t^- at price $\widetilde{q}_{s_t}$.

At the beginning of each date-event s_t, the central bank sets the interest rate r_{s_t}. The central bank supplies money balances as demanded by the households. For $\tau \in [t, t+1]$, aggregate money balances issued by the bank at τ are $m^{\mathrm{b}}_{s_t}(\tau)$, a non-negative quantity. Households obtain a bank loan as a counterpart to money borrowed. Aggregate bank loans at time τ, $b^{\mathrm{c}}_{s_t}(\tau)$, are by definition equal to aggregate money balances issued at τ, $m^{\mathrm{b}}_{s_t}(\tau)$. Aggregate money

[1] We have in mind that there is a complete set of security markets at every date-event, but without loss of generality we can restrict attention to the case where only one-period securities are traded.

balances in period s_t equal $m^{\mathrm{b}}_{s_t} = \int_{\tau=t}^{t+1} m^{\mathrm{b}}_{s_t}(\tau)d\tau$, and also equal aggregate bond holdings in period s_t, $b^{\mathrm{c}}_{s_t} = \int_{\tau=t}^{t+1} b^{\mathrm{c}}_{s_t}(\tau)d\tau$. At the end of period s_t, the bank is entitled to $r_{s_t} b^{\mathrm{c}}_{s_t}$ monetary units of interests payments, and makes profits, seignorage, equal to $\check{v}^{\mathrm{c}}_{s_t} = r_{s_t} b^{\mathrm{c}}_{s_t}$. We use ˇ to indicate end-of-period values. The central bank issues the entire seignorage as dividends to its shareholders at the end of the period. Household h receives $\theta^h \check{v}^{\mathrm{c}}_{s_t}$ at the end of period s_t.

A standard no-arbitrage argument implies that at equilibrium the sum of the prices of the Arrow securities must be equal to $1/(1 + r_{s_t})$. At no-arbitrage prices asset demand is indeterminate as any household is indifferent between holding one unit less of the bank loan and one unit more of every Arrow security. To lift this indeterminacy, we will set beginning-of-period bank loan equal to zero for every household.

At the beginning of a date-event $s_t \in \mathcal{S}$, household h has wealth given by the initial endowment of money $n^h_{s_t}$, returns from investments in elementary securities in the previous period, $\eta^h_{s_t}$, minus the bank loan at the end of the previous period, $\check{b}^h_{s_t^-}$. Since the beginning-of-period bank loan has been normalized to zero, this bank loan equals net expenditures on commodities in the previous period plus interest payments minus dividends received,

$$\check{b}^h_{s_t^-} = \widetilde{p}_{s_t^-}(x^h_{s_t^-} - e^h_{s_t^-}) + r_{s_t^-} b^h_{s_t^-} - \check{v}^{\mathrm{h}}_{s_t^-},$$

where $b^h_{s_t^-} = \int_{\tau=t-1}^{t} b^h_{s_t^-}(\tau)d\tau$ is the bank loan of household h in period s_t^- and $\check{v}^{\mathrm{h}}_{s_t^-} = \theta^h \check{v}^{\mathrm{c}}_{s_t^-}$. Although $m^{\mathrm{b}}_{s_t}(\tau) = b^{\mathrm{c}}_{s_t}(\tau)$ is a non-negative quantity, for some households h it may be the case that $b^h_{s_t}(\tau) < 0$, in particular for those household with negative excess demands in period s_t.

Household h invests its wealth in Arrow securities $\eta^h_{s_{t+1}}$, where $s_{t+1} \in s_t^+$. The no-arbitrage constraint specifies

$$\sum_{s_{t+1} \in s_t^+} \widetilde{q}_{s_{t+1}} = \frac{1}{1 + r_{s_t}}.$$

Under this condition, uniform holdings of Arrow securities are perfect substitutes for bank loans, and household demands are indeterminate. Since we have lifted this indeterminacy by setting beginning-of-period bank loans equal to zero for every household we have implicitly imposed that the household invests its entire wealth in elementary securities.

Household h faces the following sequence of budget constraints

$$\begin{array}{rcll} \sum_{s_1 \in s_0^+} \widetilde{q}_{s_1} \eta^h_{s_1} & = & n^h_{s_0}, & \\ \sum_{s_{t+1} \in s_t^+} \widetilde{q}_{s_{t+1}} \eta^h_{s_{t+1}} & = & n^h_{s_t} + \eta^h_{s_t} - \check{b}^h_{s_t^-}, & s_t \in \mathcal{S} \setminus \{0\}, \\ 0 & = & \eta^h_{s_{T+1}} - \check{b}^h_{s_{T+1}^-}, & s_{T+1} \in \mathcal{S}_{T+1}, \end{array}$$

and the accounting identities

$$\begin{array}{rcll} \check{b}^h_{s_t} & = & \widetilde{p}_{s_t}(x^h_{s_t} - e^h_{s_t}) + r_{s_t} b^h_{s_t} - \check{v}^h_{s_t}, & s_t \in \mathcal{S}, \\ m^h_{s_t} & = & b^h_{s_t}, & s_t \in \mathcal{S}. \end{array}$$

The correspondence $\mu^h : \mathbb{R}^{SL}_+ \times X^h \to \mathbb{R}^S$ defines the transaction technology of household h. It assigns to each non-negative price system $\widetilde{p}$ and consumption bundle x^h a set of vectors of amounts of money withdrawn at periods $s_t \in \mathcal{S}$ that are needed to carry out purchases and sales involved in consumption x^h at prices $\widetilde{p}$.

A household takes prices $(\widetilde{p}, \widetilde{q})$, interest rates r, and dividends $\check{v}^h$ as given and chooses a maximal element (x^h, η^h, m^h) for $\preceq^h$, the preference relation of household h defined on X^h, subject to the constraints imposed by the consumption set, $x^h \in X^h$, the transaction technology, $m^h \in \mu^h(\widetilde{p}, x^h)$, the sequence of budget constraints

$$\begin{array}{rcll} \sum_{s_1 \in s_1^+} \widetilde{q}_{s_1} \eta^h_{s_1} & = & n^h_{s_0}, & \\ \sum_{s_{t+1} \in s_t^+} \widetilde{q}_{s_{t+1}} \eta^h_{s_{t+1}} & = & n^h_{s_t} + \eta^h_{s_t} - \widetilde{p}_{s_t^-}(x^h_{s_t^-} - e^h_{s_t^-}) - r_{s_t^-} m^h_{s_t^-} + \check{v}^h_{s_t^-}, & s_t \in \mathcal{S} \setminus \{0\}, \\ 0 & = & \eta^h_{s_{T+1}} - \widetilde{p}_{s_{T+1}^-}(x^h_{s_{T+1}^-} - e^h_{s_{T+1}^-}) - r_{s_{T+1}^-} m^h_{s_{T+1}^-} + \check{v}^h_{s_{T+1}^-}, & s_{T+1} \in \mathcal{S}_{T+1}. \end{array}$$

The budget set $B^h(\widetilde{p}, \widetilde{q}, \check{v}^h)$ consists of all tuples (x^h, η^h, m^h) satisfying the restrictions specified above.

Definition 3.1 *A competitive equilibrium for the monetary economy* $(\mathcal{T}, (X^h, \preceq^h, e^h, n^h, \mu^h, \theta^h)_{h \in \mathcal{H}}, r)$ *is a tuple* $(\widetilde{p}^*, \widetilde{q}^*, x^*, \eta^*, m^*)$ *such that*

(a) *dividends satisfy*

$$\begin{array}{rcll} \check{v}^{*\mathrm{c}}_{s_t} & = & r_{s_t} m^{*\mathrm{c}}_{s_t}, & s_t \in \mathcal{S}, \\ \check{v}^{*h}_{s_t} & = & \theta^h \check{v}^{*\mathrm{c}}_{s_t}, & s_t \in \mathcal{S}, \end{array}$$

(b) *the no-arbitrage conditions hold,*

$$\sum_{s_{t+1} \in s_t^+} \widetilde{q}_{s_{t+1}} = \frac{1}{1 + r_{s_t}}, \quad s_t \in \mathcal{S},$$

(c) *for each h, $(x^{*h}, \eta^{*h}, m^{*h})$ is $\preceq^h$-maximal on $B^h(\widetilde{p}^*, \widetilde{q}^*, \check{v}^{*h})$,*

(d) *commodity markets clear, $\sum_h x^{*h} = \sum_h e^h$,*

(e) *Arrow security markets clear, $\sum_h \eta^{*h} = 0$,*

(f) *banks supply money demanded, $m^{*\mathrm{b}} = \sum_h m^{*h}$.*

On top of A1–A3, we make the following assumptions.

A4 Aggregate monetary endowments are zero: $\sum_h n^h = 0$.

A5 For every h, μ^h, is lower hemi-continuous and closed, is convex-valued, for every $\widetilde{p} \in \mathbb{R}^{SL}_+$ there exists $(x^h, m^h) \in X^h \times -\mathbb{R}^S_+$ such that $x^h \ll e^h$ and $m^h \in \mu^h(\widetilde{p}, x^h)$, monetary needs are not positively affected by commodities with zero prices: if $m^h \in \mu^h(\widetilde{p}, x^h)$ and $\overline{x}^h \in X^h$ satisfies $\overline{x}^h \geq x^h$ while $p_{s_t l} = 0$ for $\bar{x}^h_{s_t l} > x^h_{s_t l}$ implies $m^h \in \mu^h(\widetilde{p}, \overline{x}^h)$, monetary needs are bounded: there are continuous functions $\underline{n}^h, \overline{n}^h : \mathbb{R}^{SL}_+ \times X^h \to \mathbb{R}^S$ such that $m^h \in \mu^h(\widetilde{p}, x^h)$ implies $m^h \geq \underline{n}^h(\widetilde{p}, x^h)$ and $(\min\{m^h_{s_t}, \overline{n}^h_{s_t}(\widetilde{p}, x^h)\})_{s_t \in \mathcal{S}} \in \mu^h(\widetilde{p}, x^h)$.

A6 Only the bank can create money: if $x \in \prod_h X^h$ satisfies $\sum_h x^h = \sum_h e^h$ and, for some $\widetilde{p} \in \mathbb{R}^{SL}_+$, for $h \in \mathcal{H}$, $m^h \in \mu^h(\widetilde{p}, x^h)$, then $\sum_h m^h \geq 0$.

A7 The bank is owned by the households: for every h, $\theta^h \geq 0$, and $\sum_{h \in \mathcal{H}} \theta^h = 1$.

Notice that A5 implies A3.

A natural assumption, but not needed for equilibrium existence, is that μ^h be 0-homogeneous.

A8 $m^h \in \mu^h(\widetilde{p}, x^h)$ implies, for every $\overline{s}_t \in \mathcal{S}$, for every $c > 0$, $\overline{m}^h \in \mu^h(\overline{p}, x)$, where $\overline{m}^h_{\overline{s}_t} = c m^h_{\overline{s}_t}$ and $\overline{m}^h_{s_t} = m^h_{s_t}$, $s_t \neq \overline{s}_t$, and $\overline{p}_{\overline{s}_t} = c\widetilde{p}_{\overline{s}_t}$ and $\overline{p}_{s_t} = \widetilde{p}_{s_t}$, $s_t \neq \overline{s}_t$.

Spot prices of Arrow securities, $\widetilde{q}_{s_t}$, $s_t \in (\mathcal{S} \cup \mathcal{S}_{T+1}) \setminus \{0\}$, define present-value prices q_{s_t}, $s_t \in (\mathcal{S} \cup \mathcal{S}_{T+1}) \setminus \{0\}$ of units of money at date-events in $\mathcal{S} \cup \mathcal{S}_{T+1}$ by setting

$$\begin{array}{rcl} q_{s_0} & = & 1, \\ q_{s_t} & = & \widetilde{q}_{s_1(s_t)} \times \cdots \times \widetilde{q}_{s_{t-1}(s_t)} \times \widetilde{q}_{s_t}, \quad s_t \in (\mathcal{S} \cup \mathcal{S}_{T+1}) \setminus \{0\}, \end{array} \tag{1}$$

where $s_\tau(s_t)$ denotes the unique predecessor of s_t at date $\tau < t$.

The sequence of budget constraints of household h can be consolidated into a single present-value constraint. Indeed,

$$\begin{aligned} &\sum_{s_t \in \mathcal{S}} q_{s_t} \sum_{s_{t+1} \in s_t^+} \widetilde{q}_{s_{t+1}} \eta^h_{s_{t+1}} \\ &\qquad = n^h_{s_0} + \sum_{s_t \in \mathcal{S} \setminus \{s_0\}} q_{s_t} (n^h_{s_t} + \eta^h_{s_t} - \widetilde{p}_{s_t^-}(x^h_{s_t^-} - e^h_{s_t^-}) - r_{s_t^-} m^h_{s_t^-} + \check{v}^h_{s_t^-}) \\ &\qquad\quad + \sum_{s_{T+1} \in \mathcal{S}_{T+1}} q_{s_{T+1}} (\eta^h_{s_{T+1}} - \widetilde{p}_{s_{T+1}^-}(x^h_{s_{T+1}^-} - e^h_{s_{T+1}^-}) - r_{s_{T+1}^-} m^h_{s_{T+1}^-} + \check{v}^h_{s_{T+1}^-}), \end{aligned}$$

or equivalently, after cancelling the η-terms which appear on both sides with identical multiplicands, and rearranging terms, we obtain

$$\sum_{s_t \in (\mathcal{S} \cup \mathcal{S}_{T+1}) \setminus \{0\}} q_{s_t} (\widetilde{p}_{s_t^-} x^h_{s_t^-} + r_{s_t^-} m^h_{s_t^-}) = \sum_{s_t \in \mathcal{S}} q_{s_t} n^h_{s_t} + \sum_{s_t \in (\mathcal{S} \cup \mathcal{S}_{T+1}) \setminus \{0\}} q_{s_t} (\widetilde{p}_{s_t^-} e^h_{s_t^-} + \check{v}^h_{s_t^-}).$$

Since $\sum_{s_t \in s_t^+} \widetilde{q}_{s_t} = 1/(1+r_{s_t})$, we find

$$\sum_{s_t \in \mathcal{S}} \left(\frac{p_{s_t}}{1+r_{s_t}} x^h_{s_t} + \frac{q_{s_t} r_{s_t}}{1+r_{s_t}} m^h_{s_t} \right) = \sum_{s_t \in \mathcal{S}} \left(q_{s_t} n^h_{s_t} + \frac{p_{s_t}}{1+r_{s_t}} e^h_{s_t} + \frac{q_{s_t}}{1+r_{s_t}} \breve{v}^h_{s_t} \right), \tag{2}$$

where, by definition, $p_{s_t} = q_{s_t} \widetilde{p}_{s_t},\ s_t \in \mathcal{S}$. The set Q of strictly positive state prices that do not admit arbitrage equals

$$Q = \{ q \in \mathbb{R}^{S+S_T}_{++} | q_{s_0} = 1,\ \forall s_t \in \mathcal{S},\ \sum_{s_{t+1} \in s_t^+} q_{s_{t+1}} = \frac{q_{s_t}}{1+r_{s_t}} \}.$$

Given $(p, q, \breve{v}^h) \in \mathbb{R}^{SL}_+ \times Q \times \mathbb{R}^S$, household h chooses a maximal element (x^h, m^h) with $x^h \in X^h$ and $m^h \in \mu^h((p_{s_t}/q_{s_t})_{s_t \in \mathcal{S}}, x^h)$ subject to the constraint (2).

Intuition: Counting equations and unknowns, we have $SL-1$ independent market clearing equations for commodities, in $SL-1+S_T$ unknowns, the SL prices $p_{s_t l}$, the S_T-1 independent prices q_{s_t}. Indeed, there are $S+S_T$ prices q_{s_t}, $q_{s_0} = 1$ by definition, and S no-arbitrage constraints, which leaves us with S_T-1 independent prices q_{s_t}. One therefore expects a set of equilibria with dimension S_T.

Definition 3.2 *Let $\eta \subset T \times M$ be a set of equilibria. Then η is essential with respect to T if the projection map $\pi : \eta \to T$ is essential.*

We will show that the set of monetary equilibria is essential in the price index and in state prices, where the price index is simply defined as the sum of all nominal prices. In terms of notation, we therefore now write an equilibrium as a tuple $(\widetilde{p}, \widetilde{q}, x, \eta, m, I(p), q(\widetilde{q}))$, where $I(\widetilde{p})$ is determined by the formula

$$I(\widetilde{p}) = \sum_{(s_t,l) \in \mathcal{S} \times \mathcal{L}} \widetilde{p}_{s_t l}$$

and $q(\widetilde{q})$ by (1).

Let $Q_\varepsilon = \{ q \in Q | \forall s_t \in \mathcal{S},\ q_{s_t} \geq \varepsilon \}$ be the set of state prices where each state price is at least equal to ε. Clearly, if ε is taken sufficiently small, the set Q_ε is non-empty and has dimension $S_T - 1$. Fix such an ε for the remainder of this section.

Since e^h belongs to the interior of X^h and Q_ε is compact, there is a lowerbound $\underline{I}$ such that for all $\widetilde{p} \in \mathbb{R}^{LS}_+$ with $I(\widetilde{p}) \geq \underline{I}$ for any household h, its budget set has a non-empty interior whenever seignorage $\breve{v}^h$ is non-negative. Fix such a lowerbound $\underline{I}$ as well as some $\overline{I} > \underline{I}$. Let P be the set of prices $\widetilde{p}$ such that $\underline{I} \leq I(\widetilde{p}) \leq \overline{I}$ and let E be the set of monetary equilibria with state prices in Q_ε and commodity prices in P.

Theorem 3.3 *Let the monetary economy* $(\mathcal{T}, (X^h, \preceq^h, e^h, n^h, \mu^h, \theta^h)_{h\in\mathcal{H}}, r)$ *satisfy A1–A7. Then the set* E *is essential with respect to* $Q_\varepsilon \times [\underline{I}, \overline{I}]$.

Proof. By a standard proof, following Debreu (1959), the set of attainable allocations of commodities, i.e. the set of $x \in \prod_h X^h$ such $\sum_h x^h = \sum_h e^h$, is compact. Let B be such that, for every h, $x^h < B1^{LS}$. We compactify the economy by replacing consumption sets X^h by $\widehat{X}^h$, the subset of elements of X^h for which $x^h \leq B1^{LS}$. We define $M^h = [\underline{m}^h, \overline{m}^h]$, where, for $s_t \in \mathcal{S}$,

$$\underline{m}^h_{s_t} = \min_{(\widetilde{p}, x^h)\in P\times\widehat{X}^h} \underline{n}^h_{s_t}(\widetilde{p}, x^h) \text{ and } \overline{m}^h_{s_t} = \max_{(\widetilde{p}, x^h)\in P\times\widehat{X}^h} \overline{n}^h_{s_t}(\widetilde{p}, x^h),$$

and replace transactions technology μ^h by $\widehat{\mu}^h$, defined by

$$\widehat{\mu}^h(\widetilde{p}, x^h) = \mu^h(\widetilde{p}, x^h) \cap M^h.$$

Given $(\widetilde{p}, q, \check{v}^h) \in P \times Q \times \mathbb{R}^{LS}_+$, household h chooses a maximal element (x^h, m^h) with $x^h \in \widehat{X}^h$ and $m^h \in \widehat{\mu}^h(\widetilde{p}, x^h)$ subject to the constraint (2). We denote the set of maximizers by $\delta^h(\widetilde{p}, q, \check{v}^h)$.

A standard proof, which follows Debreu (1959) since the constraint (2) is equivalent to the usual budget constraint, shows that δ^h is upper hemi-continuous on $P \times Q \times \mathbb{R}^{LS}_+$.

At any $s_t \in \mathcal{S}$, in equilibrium the bank will issue a non-negative amount of money that is bounded above by $\overline{m}^{\mathrm{b}}_{s_t} = \sum_{h\in\mathcal{H}} \overline{m}^h_{s_t}$.

We define the aggregate excess demand correspondence $\zeta : P \times Q \times [0, \overline{m}^{\mathrm{b}}] \to \mathbb{R}^{LS} \times \mathbb{R}^S$ by

$$\zeta(\widetilde{p}, q, m^{\mathrm{b}}) = \sum_{h\in\mathcal{H}} \delta^h(\widetilde{p}, q, \theta^h(m^{\mathrm{b}}_{s_t})_{s_t\in\mathcal{S}}) - \sum_{h\in\mathcal{H}} e^h.$$

Let Z be a compact, convex set containing $\zeta(P \times Q \times [0, \overline{m}^{\mathrm{b}}])$. We define the simplex $\Delta = \{d \in \mathbb{R}^L_+ \mid \sum_{s_t,l\in\mathcal{S}\times\mathcal{L}} d_{s_t,l} = 1\}$. We define the correspondence

$$\varphi : [\underline{I}, \overline{I}] \times Q_\varepsilon \times \Delta \times Z \to \Delta \times Z$$

by

$$\varphi^1(z) \times \varphi^2(I, q, d, m^{\mathrm{b}}),$$

where

$$\varphi^1(z) = \{\bar{d} \in \Delta \mid \bar{d} \cdot z \geq d \cdot z, \ \forall d \in \Delta\}$$

and

$$\varphi^2(I, q, d, m^{\mathrm{b}}) = \zeta(Id, q, m^{\mathrm{b}}).$$

Debreu's proof applies in this case to show that fixed points correspond to equilibria, it is then easy to show that that the set of equilibria is essential. ■

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