THE CARLO ALBERTO NOTEBOOKS

Collegio Carlo Alberto

Risk Measures: Rationality and Diversification

Simone Cerreia-Vioglio
Fabio Maccheroni
Massimo Marinacci
Luigi Montrucchio

Working Paper No. 100
December 2008
www.carloalberto.org

Risk Measures: Rationality and Diversification[1]

Simone Cerreia-Vioglio[a] Fabio Maccheroni[b] Massimo Marinacci[c] Luigi Montrucchio[c]

[a]Department of Economics, Columbia University

[b]Department of Decision Sciences, Università Bocconi

[c]Collegio Carlo Alberto, Università di Torino

December 2008

[1]We thank Damir Filipović, Marco Frittelli, Michael Kupper, and Walter Schachermayer for stimulating discussions and helpful comments. Part of this research was done while the first two authors were visiting the Collegio Carlo Alberto, which they thank for its hospitality. Simone Cerreia-Vioglio thanks Università Bocconi for summer support.

Abstract

When there is uncertainty about interest rates (typically due to either illiquidity or defaultability of zero coupon bonds) the cash-additivity assumption on risk measures becomes problematic. When this assumption is weakened, to cash-subadditivity for example, the equivalence between convexity and the diversification principle no longer holds. In fact, this principle only implies (and it is implied by) quasiconvexity.

For this reason, in this paper quasiconvex risk measures are studied. We provide a dual characterization of quasiconvex cash-subadditive risk measures and we establish necessary and sufficient conditions for their law invariance. As a byproduct, we obtain an alternative characterization of the actuarial mean value premium principle.

JEL classification: D81

Keywords: Risk Measures, Diversification, Cash-subadditivity, Quasiconvexity, Law-invariance, Mean Value Premium Principle

1 Introduction

Risk assessment is a fundamental activity for both regulators and agents in financial markets. The problem of a formal definition of a risk measure and of the economic and mathematical properties that it should satisfy has been heating the debate since the seminal papers of Artzner, Delbaen, Eber, and Heath (1997, 1999) on *coherent risk measures*.

In the last ten years there has been a flourishing of methodological proposals, mathematical extensions, and variations on this topic. The *convex monetary risk measures* of Föllmer and Schied (2002, 2004) and Frittelli and Rosazza Gianin (2002) are especially interesting in terms of economic content and mathematical tractability among the generalizations of coherent risk measures. Moreover, these measures naturally appear in pricing and hedging problems in incomplete markets, as shown, for example, by El Karoui and Quenez (1997), Carr, Geman, and Madan (2001), Frittelli and Rosazza Gianin (2004), Staum (2004), Filipović and Kupper (2008), and Jouini, Schachermayer, and Touzi (2008).

A risk measure is a decreasing function ρ that to a future risky position X associates the minimal reserve amount $\rho(X)$ that should be collected today to cover risk X, from the point of view of a supervising agency. Decreasing monotonicity is a minimal *rationality* requirement imposed on the agency: higher losses require higher reserves.

Convex monetary risk measures have the additional requirement of being convex and cash-additive.[1] As pointed out by El Karoui and Ravanelli (2008), cash-additivity fails as soon as there is any form of uncertainty about interest rates; for example when the risk-free asset is illiquid or inexistent.[2] For this reason, they suggest to replace cash-additivity with cash-subadditivity, and, maintaining convexity, they provide a representation result for *convex cash-subadditive risk measures*, together with several examples arising from applications.

This paper starts from the observation that once cash-additivity is replaced with the economically sounder assumption of cash-subadditivity, *convexity* should be replaced by *quasiconvexity* in order to maintain the original interpretation in terms of *diversification*. Although convexity is generally regarded as the mathematical translation of the fundamental principle "diversification cannot increase risk," literally this principle means

"if position X is less risky than Y, so it is any diversified position $\lambda X + (1-\lambda) Y$ with λ in $(0,1)$."

Using a measure of risk ρ, this statement translates into

"$\rho(X) \le \rho(Y)$ implies $\rho(\lambda X + (1-\lambda) Y) \le \rho(Y)$ for all λ in $(0,1)$,"

which is equivalent to convexity under the cash-additivity assumption, while in general (also under cash-subadditivity) it only corresponds to quasiconvexity.[3]

For these reasons, in this paper we study *quasiconvex cash-subadditive risk measures*. We show in Theorem 1 that these measures take the form

$$\rho(X) = \sup_{Q \in \mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q), \tag{1}$$

[1] See Section 2 for details and formal definitions.
[2] Black (1972) is one of the first contributions that casted doubts on liquidity and existence of riskless assets.
[3] See Proposition 1 and Example 1.

where $\mathcal{M}_{1,f}$ is the set of (finitely additive) probabilities and $R : \mathbb{R}\times\mathcal{M}_{1,f} \to [-\infty, \infty]$ is an upper semicontinuous quasiconcave function that is increasing and nonexpansive in the first component and such that $\inf_{t\in\mathbb{R}} R(t, \cdot)$ is constant. The function R is unique.

Convex monetary risk measures correspond to the separable specification

$$R(t, Q) = Dt - \alpha(DQ) \tag{2}$$

for some constant $D \in (0, 1]$, while convex cash-subadditive risk measures correspond to

$$R(t, Q) = \sup_{c\in[0,1]} (ct - \alpha(cQ)), \tag{3}$$

where $\alpha(\cdot)$ is the Fenchel conjugate of $\rho(-\cdot)$.

Representation (1) is not only general enough to capture most of the risk measures introduced in the literature, but it also has a very natural interpretation: $R(t, Q)$ is the reserve amount required today, under the probabilistic scenario Q, to cover an expected loss t in the future. Since there is uncertainty about probabilistic scenarios, the supervising agency follows the most cautious approach, that is, it requires the maximum reserve. The evaluations $R(t, Q)$ keep two factors into account, the expected loss t and the plausibility of scenario Q, assessed by the supervising agency. As the special cases (2) and (3) show (see again the discussion in El Karoui and Ravanelli, 2008), the separability of these two risk factors is lost as soon as risky positions and reserve amounts cannot be expressed in the same numeraire in an unambiguous way.

It is important to observe that, while the results on convex and cash-subadditive measures build on classic convex duality, our results build on the quasiconvex monotone duality developed in Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b). As a result, there is also a substantial difference between the mathematics that underlies our results and that used in the study of convex risk measures.

In view of the importance of law-invariance with respect to a given probability measure P, in Theorem 2 we characterize quasiconvex risk measures that satisfy this property and we show that in this case the quantile representation

$$\rho(X) = \sup_{Q\in\mathcal{M}_1} R\left(\int_0^1 q_{-X}(s)\, q_{\frac{dQ}{dP}}(s)\, ds, Q\right) \tag{4}$$

holds. This result extends those of Chong and Rice (1971), Kusuoka (2001), Föllmer and Schied (2004), Dana (2005), Frittelli and Rosazza Gianin (2005), and Leitner (2005) from the domain of convex analysis to that of quasiconvex analysis.

As a byproduct, in Lemma 2 we characterize the risk measures that agree with the actuarial *mean value premium principle* (see Rotar, 2007), that is, the measures of the form

$$\rho(X) = \ell^{-1}(\mathbb{E}_P(\ell(-X))),$$

where ℓ is a strictly increasing and convex loss function. Though in a static setting, this result is in the spirit of a very recent one of Kupper and Schachermayer (2008) and it builds on the classic Nagumo-Kolmogorov-de Finetti Theorem.[4] Interestingly, Proposition 5 shows that for this class of functions

$$R(t, Q) = t - L(-t; Q, P),$$

where $L(-t; Q, P)$ is the generalized distance between probability measures induced by ℓ, introduced by Bellini and Frittelli (2002) in the context of minimax martingale measures (see also Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2008a).

[4] See, Nagumo (1930), Kolmogorov (1930), de Finetti (1931), as well as Hardy, Littlewood, and Pólya (1934).

2 Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $L^\infty(\Omega, \mathcal{A}, P)$ be the space of bounded random variables.[5] Its topological dual $L^\infty(\Omega, \mathcal{A}, P)^*$ is isometrically isomorphic to the space of all bounded finitely additive set functions on $\mathcal{A}$ that are absolutely continuous with respect to P (e.g., Yosida, 1980, Ch. IV.9).

The positive unit ball of $L^\infty(\Omega, \mathcal{A}, P)^*$ is denoted by $\mathcal{M}_{1,f}(\Omega, \mathcal{A}, P)$ and coincides with the set of finitely additive probabilities that are absolutely continuous with respect to P; in particular, $\mathcal{M}_1(\Omega, \mathcal{A}, P)$ is the subset of $\mathcal{M}_{1,f}(\Omega, \mathcal{A}, P)$ consisting of all its countably additive elements. For this reason, given $X \in L^\infty(\Omega, \mathcal{A}, P)$ and $\mu \in L^\infty(\Omega, \mathcal{A}, P)^*$, we indifferently write: $\mu(X)$, $\int X d\mu$, or even $\mathbb{E}_\mu(X)$ if $\mu \in \mathcal{M}_{1,f}(\Omega, \mathcal{A}, P)$. The specification of the probability space $(\Omega, \mathcal{A}, P)$ is often omitted and we just write L^∞ and $\mathcal{M}_{1,f}$.

Unless otherwise stated, $L^\infty(\Omega, \mathcal{A}, P)$ is endowed with its norm topology, $L^\infty(\Omega, \mathcal{A}, P)^*$ is endowed with its weak* topology, and its subsets with the relative weak* topology. Product spaces are endowed with the product topology.

We consider one period of uncertainty $\{0, T\}$. The elements of L^∞ represent payoffs at time T of financial positions held at time 0. A *risk measure* is a decreasing function $\rho : L^\infty \to [-\infty, \infty]$. As anticipated in the introduction, $\rho(X)$ is interpreted as the minimal reserve amount that should be collected today to cover future risk X. Decreasing monotonicity is justified by the fact that smaller losses cannot require greater reserves.

Given a (deterministic) discount factor $D \in (0, 1]$, the function ρ is a *monetary* risk measure if, in addition, it satisfies:

Cash-additivity $\rho(X - m) = \rho(X) + Dm$ for all $X \in L^\infty$ and $m \in \mathbb{R}$.

This condition is interpreted in the following way "if m is subtracted to the future position, the present capital requirement is augmented by the same discounted amount Dm." In fact, investing Dm in a risk-free manner offsets the certain future loss m.

Cash-additivity is a controversial assumption, both from a theoretical and practical viewpoint. For, D is the price of a non-defaultable zero coupon bond available on the market at time 0, with maturity T and face value 1: existence and liquidity of such an asset is not an innocuous assumption and, as observed by El Karoui and Ravanelli (2008), any form of uncertainty in interest rates is sufficient to make the cash-additivity assumption too stringent. For example, in case of illiquidity, D may well depend on the amount m of purchased assets.

These considerations lead to the following relaxed version of cash-additivity, which only takes into account the time value of money:

Cash-subadditivity $\rho(X - m) \leq \rho(X) + m$ for all $X \in L^\infty$ and $m \in \mathbb{R}_+$.

The meaning of this condition is "when m dollars are subtracted to a future position the present capital requirement cannot be augmented by more than m dollars." This is a much more compelling assumption relative to cash-additive since it just relies on the fact that an additional reserve of m dollars surely covers the additional loss of the same amount.[6]

As discussed in the introduction, the risk diminishing effect of diversification is usually translated by:

---

[5] Equalities and inequalities among random variables hold almost surely with respect to P.

[6] Notice that cash-subadditivity is equivalent to require $\rho(X + m) \geq \rho(X) - m$ for all $X \in L^\infty$ and all $m \in \mathbb{R}_+$. In fact, it implies $\rho(X) = \rho(X + m - m) \leq \rho(X + m) + m$, and the converse is proved in the same way. In particular, our definition is equivalent to that of El Karoui and Ravanelli (2008).

Convexity $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in L^\infty$ and $\lambda \in (0,1)$.

But, it actually corresponds to the much weaker:

Quasiconvexity $\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}$ for all $X, Y \in L^\infty$ and $\lambda \in (0,1)$.

The next simple proposition shows that convexity is equivalent to quasiconvexity for monetary risk measures. Clearly, this is not the case for cash-subadditive risk measures.[7] In reading the result, recall that a function $\rho : L^\infty \to [-\infty, \infty]$ is *nonexpansive* if $\rho(Y) \leq \rho(X) + \|X - Y\|$ for all $X, Y \in L^\infty$.

Proposition 1 *Let ρ be a risk measure.*

(a) If ρ is cash-additive, then it is convex if and only if it is quasiconvex.

(b) ρ is cash-subadditive if and only if it is nonexpansive.

In both cases, ρ is either finite valued or identically $\pm\infty$.

Proof. (a) is essentially known (e.g., Gilboa and Schmeidler, 1989, Lemma 3.3, or Marinacci and Montrucchio, 2004, Corollary 4.2). Next we prove (b). If $\rho : L^\infty \to \mathbb{R}$ is nonexpansive, then $\rho(X - m) \leq \rho(X) + 1\|X - (X - m)\|$ for all $X \in L^\infty$ and all $m \in \mathbb{R}_+$, that is $\rho(X - m) \leq \rho(X) + m$. Conversely, for all $X, Y \in L^\infty$, $X - Y \leq \|X - Y\|$, then $X - \|X - Y\| \leq Y$, monotonicity and cash-subadditivity deliver $\rho(Y) \leq \rho(X - \|X - Y\|) \leq \rho(X) + \|X - Y\|$, as wanted. ■

Next example shows how the illiquidity of the risk-free asset naturally generates quasiconvex cash-subadditive risk measures that are neither convex nor cash-additive.

Example 1 *Let $\emptyset \subsetneq C \subsetneq L^\infty$ be the set of future positions considered acceptable by the supervising agency, and assume that C is convex and $C + L^\infty_+ \subseteq C$. For all $m \in \mathbb{R}$ denote by $v(m)$ the price at time 0 of m dollars at time T and define, as in Artzner, Delbaen, Eber, and Heath (1999),*

$$\rho_{C,v}(X) = \inf\{v(m) : X + m \in C\} \quad \forall X \in L^\infty.$$

If $v(m) = Dm$ with $D \in (0,1]$, then $\rho_{C,v}$ is a (finite valued) convex monetary risk measure.[8] *The linearity of v is precisely the assumption that fails when zero coupon bonds with maturity T are illiquid. Still it remains sensible to assume that $v : \mathbb{R} \to (-\infty, \infty]$ is increasing and $v(0) = 0$.*

Provided v is also upper semicontinuous, we have

$$\rho_{C,v}(X) = v(\inf\{m \in \mathbb{R} : X + m \in C\}) = v(\rho_{C,\mathrm{id}}(X)) \quad \forall X \in L^\infty,$$

where id $: \mathbb{R} \to (-\infty, \infty]$ *is the identity. Moreover, since $\rho_{C,\mathrm{id}}$ is a convex monetary risk measure, then for any nonexpansive v that is not convex, $\rho_{C,v}$ is a quasiconvex cash-subadditive risk measure that is neither convex nor cash-additive.*

Finally, $\mathcal{R}_0(\mathbb{R}\times\mathcal{M}_{1,f})$ denotes the class of functions $R : \mathbb{R}\times\mathcal{M}_{1,f} \to [-\infty, \infty]$ that are upper semicontinuous, quasiconcave, increasing in the first component, with $\inf_{t\in\mathbb{R}} R(t, Q) = \inf_{t\in\mathbb{R}} R(t, Q')$ for all $Q, Q' \in \mathcal{M}_{1,f}$. Moreover, $\mathcal{R}_1(\mathbb{R}\times\mathcal{M}_{1,f})$ is the subset of $\mathcal{R}_0(\mathbb{R}\times\mathcal{M}_{1,f})$ consisting of functions R that are nonexpansive in the first component, that is, $R(t', Q) \leq R(t, Q) + |t - t'|$ for all $t, t' \in \mathbb{R}$ and all $Q \in \mathcal{M}_{1,f}$.

[7] See Example 1 below.

[8] See, for example, Föllmer and Schied (2004, Ch. 4).

3 Representation

We are now ready to state and prove our first representation result.

Theorem 1 *A function $\rho : L^\infty \to [-\infty, \infty]$ is a quasiconvex cash-subadditive risk measure if and only if there exists $R \in \mathcal{R}_1 (\mathbb{R}\times\mathcal{M}_{1,f})$ such that*

$$\rho(X) = \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty. \tag{5}$$

The function $R \in \mathcal{R}_1 (\mathbb{R}\times\mathcal{M}_{1,f})$ for which (5) holds is unique and satisfies

$$R(t, Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) = t\} \quad \forall (t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f}. \tag{6}$$

Recall that ρ is a quasiconvex cash-subadditive risk measure if and only if it is a quasiconvex and nonexpansive risk measure. The next lemma characterizes quasiconvex and upper semicontinuous risk measures.

Lemma 1 *A function $\rho : L^\infty \to [-\infty, \infty]$ is a quasiconvex upper semicontinuous risk measure if and only if there exists $R \in \mathcal{R}_0 (\mathbb{R}\times\mathcal{M}_{1,f})$ such that*

$$\rho(X) = \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty. \tag{7}$$

The function $R \in \mathcal{R}_0 (\mathbb{R}\times\mathcal{M}_{1,f})$ for which (7) holds is unique and satisfies

$$R(t, Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) = t\} \quad \forall (t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f}. \tag{8}$$

Proof. Notice that L^∞ is a normed Riesz space with unit I_Ω, $\mathcal{M}_{1,f}$ is the positive unit ball of its topological dual, and $-\rho$ is a quasiconcave, lower semicontinuous, and monotone increasing function. The statement then follows from Lemma 8 and Theorem 3 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b). ■

Proof of Theorem 1. It only remains to show that ρ is cash-subadditive if and only if R is nonexpansive in the first component.

Suppose ρ is cash-subadditive, then, for all $(t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f}$ and $m \in \mathbb{R}_+$,

$$\begin{aligned} R(t+m, Q) &= \inf\{\rho(X) : \mathbb{E}_Q(-X) = t+m\} = \inf\{\rho(X) : \mathbb{E}_Q(-(X+m)) = t\} \\ &= \inf\{\rho(Y-m) : \mathbb{E}_Q(-Y) = t\} \leq \inf\{\rho(Y)+m : \mathbb{E}_Q(-Y) = t\} = R(t, Q) + m. \end{aligned}$$

Therefore, for all $t, t' \in \mathbb{R}$ and $Q \in \mathcal{M}_{1,f}$, $t' \leq t + |t-t'|$ and monotonicity of R in the first component imply

$$R(t', Q) \leq R(t + |t-t'|, Q) \leq R(t, Q) + |t-t'|,$$

as wanted.

Conversely, if R is nonexpansive in the first component, then, for all $(t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f}$ and $m \in \mathbb{R}_+$,

$$R(t+m, Q) \leq R(t, Q) + |t-(t+m)| = R(t, Q) + m.$$

Moreover, for all $X \in L^\infty$, there is $Q' \in \mathcal{M}_{1,f}$ such that $\rho(X-m) = R(\mathbb{E}_{Q'}(-(X-m)), Q')$. Therefore,

$$\begin{aligned} \rho(X-m) &= R(\mathbb{E}_{Q'}(-(X-m)), Q') = R(\mathbb{E}_{Q'}(-X)+m, Q') \leq R(\mathbb{E}_{Q'}(-X), Q') + m \\ &\leq \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) + m = \rho(X) + m, \end{aligned}$$

as wanted. ■

In particular, denoting by $\alpha(\cdot)$ the Fenchel conjugate of $\rho(-\cdot)$, a quasiconvex cash-subadditive risk measure ρ is convex if and only if

$$R(t,Q) = \sup_{c\in[0,1]} (ct - \alpha(cQ)) \quad \forall (t,Q) \in \mathbb{R}\times\mathcal{M}_{1,f},$$

thus obtaining the result of El Karoui and Ravanelli (2008). Moreover, ρ is cash-additive if and only if

$$R(t,Q) = Dt - \alpha(DQ) \quad \forall (t,Q) \in \mathbb{R}\times\mathcal{M}_{1,f},$$

which corresponds to the well known characterization of convex monetary risk measures.[9]

Maintaining the interpretation of $R(t,Q)$ as the reserve amount required today, under the probabilistic scenario Q, to cover an expected loss t in the future, the above relations corroborate the claim of El Karoui and Ravanelli (2008) that the passage to cash-subadditivity is the most parsimonious way of taking into account interest rate uncertainty and a supervising agency that is averse to such uncertainty.

Next proposition shows that, as in the two special cases above, the possibility of replacing finitely additive probabilities with countably additive probabilities in the variational representation (5), and indeed in (7), correspond to the following requirement:

Continuity from below $X_n \nearrow X$ implies $\rho(X_n) \to \rho(X)$ for all $X_n, X \in L^\infty$.

Proposition 2 *Let $\rho : L^\infty \to [-\infty,\infty]$ be a quasiconvex upper semicontinuous risk measure. The following conditions are equivalent:*

(i) ρ is continuous from below;

(ii) $R(t,Q) = \inf_{L^\infty} \rho$ for all $(t,Q) \in \mathbb{R}\times(\mathcal{M}_{1,f} \setminus \mathcal{M}_1)$.

In this case,

$$\max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) = \max_{Q\in\mathcal{M}_1} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty. \tag{9}$$

Proof. Consider the following condition:

(iii) $\{Q \in \mathcal{M}_{1,f} : R(t,Q) \geq m\} \subseteq \mathcal{M}_1$ for all $m \in (\inf_{L^\infty} \rho, +\infty]$ and all $t \in \mathbb{R}$.

We show that (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

(i) implies (iii). Let $t \in \mathbb{R}$, $m \in (\inf_{L^\infty} \rho, +\infty]$, and $Q' \in \{Q \in \mathcal{M}_{1,f} : R(t,Q) \geq m\}$. Since $m > \inf_{L^\infty} \rho$, there exist $X \in L^\infty$ such that $\rho(X) < m$ and $x \geq X$ in $\mathbb{R}$ such that $\rho(x) \leq \rho(X) < m$. If $E_n \searrow \emptyset$ in $\mathcal{A}$, then $x - k\mathrm{I}_{E_n} \nearrow x$ in L^∞ for each $k > 0$. Continuity from below guarantees that there exists $N_k \in \mathbb{N}$ such that for all $n \geq N_k$

$$m > \rho(x - k\mathrm{I}_{E_n}) = \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(k\mathrm{I}_{E_n} - x), Q) = \max_{Q\in\mathcal{M}_{1,f}} R(kQ(E_n) - x, Q).$$

If $kQ'(E_{n'}) - x \geq t$ for some $n' \geq N_k$, since R is increasing, it follows that

$$\max_{Q\in\mathcal{M}_{1,f}} R(kQ(E_{n'}) - x, Q) \geq R(kQ'(E_{n'}) - x, Q') \geq R(t, Q') \geq m,$$

[9] For more details on the relations between convex duality and quasiconvex monotone duality, see Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b).

which is absurd. Then $kQ'(E_n) - x < t$ for all $n \geq N_k$, hence

$$Q'(E_n) < \frac{x+t}{k} \quad \forall n \geq N_k,$$

thus $\lim_{n\to\infty} Q'(E_n) \leq k^{-1}(x+t)$. Since this is the case for each $k > 0$, then $\lim_{n\to\infty} Q'(E_n) = 0$ and $Q' \in \mathcal{M}_1$.

(iii) implies (ii). Clearly, for all $(t,Q) \in \mathbb{R}\times\mathcal{M}_{1,f}$, $R(t,Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) = t\} \geq \inf_{L^\infty}\rho$. If, per contra, there exists $(t_0, Q_0) \in \mathbb{R}\times(\mathcal{M}_{1,f} \setminus \mathcal{M}_1)$ such that $R(t_0,Q_0) > \inf_{L^\infty}\rho$, then, setting $m_0 = R(t_0,Q_0)$, by (iii) it follows that

$$Q_0 \in \{Q \in \mathcal{M}_{1,f} : R(t_0,Q) \geq m_0\} \subseteq \mathcal{M}_1,$$

a contradiction.

(ii) implies (i). Let $\{X_n\}_{n\geq1}$ be a sequence in L^∞ such that $X_n \nearrow X_0 \in L^\infty$. For each $n \geq 0$, define $\gamma_n : \mathcal{M}_{1,f} \to [-\infty,+\infty]$ by

$$\gamma_n(Q) = R(\mathbb{E}_Q(-X_n), Q) \quad \forall Q \in \mathbb{R}\times\mathcal{M}_{1,f}.$$

Each γ_n is weak* upper semicontinuous, and the sequence $\{\gamma_n\}$ is decreasing. If $Q \in \mathcal{M}_1$, then $\mathbb{E}_Q(-X_n) \searrow \mathbb{E}_Q(-X_0)$, by the Levi Monotone Converge Theorem, and so, since $R(\cdot,Q)$ is upper semicontinuous and increasing on $\mathbb{R}$, $\lim_{n\to\infty} R(\mathbb{E}_Q(-X_n),Q) = R(\mathbb{E}_Q(-X_0),Q)$; else if $Q \notin \mathcal{M}_1$, then $R(\mathbb{E}_Q(-X_n),Q) = \inf_{L^\infty}\rho$ for all $n \geq 0$. Conclude that $-\gamma_n$ pointwise converges (and so Γ-converges, see, e.g., Dal Maso, 1993, Rem. 5.5) to $-\gamma_0$. By Theorem 7.4 of Dal Maso (1993), $\min_{Q\in\mathcal{M}_{1,f}} -\gamma_n(Q) \to \min_{Q\in\mathcal{M}_{1,f}} -\gamma_0(Q)$, that is $-\rho(X_n) \to -\rho(X_0)$.

Finally, we show that (ii) implies (9). If $X \in \arg\inf_{L^\infty}\rho$, then for all $Q \in \mathcal{M}_{1,f}$, by Lemma 1,

$$\rho(X) \geq R(\mathbb{E}_Q(-X),Q) = \inf\{\rho(Y) : \mathbb{E}_Q(-Y) = \mathbb{E}_Q(-X)\} \geq \inf_{L^\infty}\rho = \rho(X).$$

Therefore the maximum in (7) is attained at each Q in $\mathcal{M}_{1,f}$, thus at each Q in $\mathcal{M}_1$. Else if $\rho(X) > \inf_{L^\infty}\rho$, by (ii), the maximum in (7) cannot be attained on $\mathcal{M}_{1,f} \setminus \mathcal{M}_1$, thus it is attained on $\mathcal{M}_1$. ■

3.1 Remarks on Continuity

First notice that continuity from below implies norm upper semicontinuity for a risk measure ρ.

Proposition 3 *A risk measure ρ is continuous from below (resp., above) if and only if it is upper (resp., lower) semicontinuous with respect to bounded pointwise convergence.*

Proof. Let X_n be a bounded sequence in L^∞ that pointwise converges to X. Set $Y_n = \inf_{k\geq n} X_k$ for all $n \in \mathbb{N}$. Then $X_n \geq Y_n \nearrow X$, and monotonicity and continuity from below deliver

$$\limsup_n \rho(X_n) \leq \lim_n \rho(Y_n) = \rho(X).$$

Conversely, if $X_n \nearrow X$, then monotonicity of ρ delivers $\rho(X) \leq \liminf_n \rho(X_n)$, while upper semicontinuity with respect to bounded pointwise convergence delivers $\limsup_n \rho(X_n) \leq \rho(X)$. ■

Moreover, continuity from below and norm lower semicontinuity imply continuity with respect to bounded pointwise convergence, provided ρ is quasiconvex. Formally:

Proposition 4 *Let $\rho : L^\infty \to [-\infty,\infty]$ be a quasiconvex risk measure. The following conditions are equivalent:*

(i) ρ is continuous from below and norm lower semicontinuous;

(ii) ρ is continuous with respect to bounded pointwise convergence.

Proof. Clearly, (ii) and Proposition 3 deliver (i). By Proposition 3, to prove the converse it is sufficient to show that ρ is continuous from above. Let $X_n \searrow X$. By monotonicity, $\rho(X_n)$ is increasing and $\lim_n \rho(X_n) \leq \rho(X)$. Assume, per contra, strict inequality holds. Then $\{X_n\}_{n\in\mathbb{N}}$ is contained in $\{\rho < c\}$ for some $c < \rho(X)$. The assumptions on ρ guarantee that $\{\rho \leq c\}$ is not empty, convex, norm closed, and

$$\{\rho \leq c\} \subseteq \bigcap_{i\in I} [Q_i \geq b_i],$$

where $\{(b_i, Q_i) : i \in I\} = \{(b, Q) \in \mathbb{R} \times \mathcal{M}_1 : [Q \geq b] \supseteq \{\rho \leq c\}\}$. As to the converse inclusion, let $Y \notin \{\rho \leq c\}$. By a Separating Hyperplane Theorem, there exist $b \in \mathbb{R}$, $\varepsilon > 0$, and $Q \in L^\infty(\Omega, \mathcal{A}, P)^* \setminus \{0\}$ such that

$$\{\rho \leq c\} \subseteq [Q \geq b] \text{ and } Y \in [Q < b - \varepsilon].$$

Monotonicity allows to assume $Q \in \mathcal{M}_{1,f}$.[10] If $Q \notin \mathcal{M}_1$, then $R(t, Q) = \inf_{L^\infty} \rho \leq \rho(X_1) < c$ for all $t \in \mathbb{R}$. For $t = -b + \varepsilon$, this implies

$$c > R(-b + \varepsilon, Q) = \inf\{\rho(Z) : \mathbb{E}_Q(Z) = b - \varepsilon\}.$$

Then $\rho(Z') < c$ for some $Z' \in [Q = b - \varepsilon]$, which is absurd since $\{\rho \leq c\} \subseteq [Q \geq b]$. Summing up, if $Y \notin \{\rho \leq c\}$ there are $b \in \mathbb{R}$ and $Q \in \mathcal{M}_1$ such that $[Q \geq b] \supseteq \{\rho \leq c\}$ and $Y \notin [Q \geq b]$. Thus, $\{\rho \leq c\}^c \subseteq \left(\bigcap_{i\in I} [Q_i \geq b_i]\right)^c$.

Finally, $\{X_n\}_{n\in\mathbb{N}} \subseteq \{\rho \leq c\}$ implies $\mathbb{E}_{Q_i}(X_n) \geq b_i$ for all $n \in \mathbb{N}$ and $i \in I$. By the Monotone Convergence Theorem, $\mathbb{E}_{Q_i}(X) \geq b_i$ for all $i \in I$, then $\rho(X) \leq c$ which contradicts $c < \rho(X)$. ■

4 Law-invariance

In this section we consider a continuous from below quasiconvex risk measure

$$\rho(X) = \max_{Q\in\mathcal{M}_1} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty.$$

In the study of law-invariance it is useful to consider some important stochastic orders. The *convex order* $\succsim_{cx}$ is defined on L^1 by

$$X \succsim_{cx} Y \text{ if and only if } \mathbb{E}_P(\phi(X)) \geq \mathbb{E}_P(\phi(Y))$$

for all convex $\phi : \mathbb{R} \to \mathbb{R}$. The *increasing convex order* $\succsim_{icx}$ and *second order stochastic dominance* $\succsim_{ssd}$ are defined analogously by replacing convex functions with increasing convex functions and increasing concave functions, respectively. Notice that $X \succsim_{icx} Y$ if and only if $-X \precsim_{ssd} -Y$ and that the three preorders share the same symmetric part $\sim$, which is the *identical distribution* with respect to P relation.[11]

As widely discussed in the literature (see, e.g., the classic Rothschild and Stiglitz, 1970, and Marshall and Olkin 1979), $X \succsim_{cx} Y$ intuitively means that the values of X are more dispersed than

[10] If $Z \in L^\infty_+$ then $X_1 + nZ \in \{\rho \leq c\}$ for all $n \in \mathbb{N}$, and $Q(X_1) + nQ(Z) \geq b$ delivers $Q(Z) \geq 0$. Then Q is a non-zero positive linear functional, and if $Q \notin \mathcal{M}_{1,f}$ it is sufficient to normalize it.

[11] See Chong (1974) for this fact and for altenative characterizations of these orders.

those of Y, while $X \succsim_{ssd} Y$ is the standard formalization of the statement "X is less risky than Y," provided P is the unanimously accepted model for uncertainty.

The convex order naturally induces a relation on $\mathcal{M}_1$ by

$$Q \succsim_{cx} Q' \text{ if and only if } \frac{dQ}{dP} \succsim_{cx} \frac{dQ'}{dP}.$$

The intuition is the same: the probability masses $dQ(\omega)$ are more scattered with respect to $dP(\omega)$ than the masses $dQ'(\omega)$.

An extended real valued function γ defined on a subset of L^1 is *law-invariant* (or *rearrangement invariant*) if and only if

$$X \sim Y \text{ implies } \gamma(X) = \gamma(Y);$$

while γ is *Schur concave* if and only if

$$X \succsim_{cx} Y \text{ implies } \gamma(X) \leq \gamma(Y).$$

Finally, γ *preserves second order stochastic dominance* if and only if

$$X \succsim_{ssd} Y \text{ implies } \gamma(X) \leq \gamma(Y).$$

Clearly the latter property is desirable for a risk measure, under the assumption that all the agents agree on P. If X is recognized to be less risky than Y, it is difficult for the supervising agency to require a higher reserve amount for X than for Y.

Theorem 2 *Let ρ be a quasiconvex and continuous from below risk measure. The following conditions are equivalent:*

(i) ρ preserves second order stochastic dominance;

(ii) $R(t,\cdot)$ is Shur concave on $\mathcal{M}_1$ for all $t \in \mathbb{R}$.

In this case,

$$\rho(X) = \max_{Q\in\mathcal{M}_1} R\left(\int_0^1 q_{-X}(s)\, q_{\frac{dQ}{dP}}(s)\, ds, Q\right) \quad \forall X \in L^\infty \tag{10}$$

and

$$R(t,Q) = \inf\left\{\rho(Y) : \int_0^1 q_{\frac{dQ}{dP}}(s)\, q_Y(1-s)\, ds = -t\right\} \quad \forall (t,Q) \in \mathbb{R}\times\mathcal{M}_1. \tag{11}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then (i) and (ii) are equivalent to:

(iii) ρ is law-invariant;

(iv) $R(t,\cdot)$ is rearrangement invariant on $\mathcal{M}_1$ for all $t \in \mathbb{R}$.

Recall that q_Z denotes any *quantile* of $Z \in L^1$ (see, e.g., Föllmer and Schied, 2004), and a probability space is *adequate* if and only if it is either finite and endowed with the uniform probability or non-atomic. We used the term "rearrangement invariant" rather than the equivalent "law-invariant" in (iv) since it gives a better intuition of what happens in the finite case: $R(t,Q) = R(t, Q \circ \sigma)$ for all permutations σ of Ω and all $(t,Q) \in \mathbb{R}\times\mathcal{M}_1$.

Proof. The proof heavily relies on the theory of rearrangement invariant Banach spaces developed by Luxemburg (1967) and Chong and Rice (1971). For convenience, the latter reference is denoted from now on by CR. Following its notation, if X is measurable, set

$$\begin{aligned}\delta_X(s) &\equiv \inf\{x \in \mathbb{R} : P(\{\omega \in \Omega : X(\omega) > x\}) \leq s\} \\ &= \inf\{x \in \mathbb{R} : F_X(x) \geq 1-s\} \equiv F_X^{-1}(1-s) \equiv q_X^-(1-s)\end{aligned}$$

for all $s \in [0,1]$.

Step 1. If $Y \in L^{\infty}$ and $Q \in \mathcal{M}_1$, then

$$\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \precsim_{cx} Q\} = \left[\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds\right]. \tag{12}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(1-s)\, ds = \min\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \sim Q\} \text{ and} \tag{13}$$

$$\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds = \max\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \sim Q\}. \tag{14}$$

Proof of Step 1. [CR 13.4] and [CR 13.8] guarantee that, if Y and X belong to the set $M(\Omega, \mathcal{A}, P)$ of measurable functions and $\delta_{|Y|}\delta_{|X|} \in L^1([0,1])$, then

$$\left\{\int YX'dP : M(\Omega, \mathcal{A}, P) \ni X' \precsim_{cx} X\right\} = \left[\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_X(s)\, ds\right].$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds = \min\left\{\int YX'dP : M(\Omega, \mathcal{A}, P) \ni X' \sim X\right\} \text{ and}$$

$$\int_0^1 \delta_Y(s)\, \delta_X(s)\, ds = \max\left\{\int YX'dP : M(\Omega, \mathcal{A}, P) \ni X' \sim X\right\}.$$

The condition $\delta_{|Y|}\delta_{|X|} \in L^1([0,1])$ is implied by $\delta_{|Y|} \in L^p([0,1])$ and $\delta_{|X|} \in L^q([0,1])$, where either $p = \infty$ and $q = 1$ or $p = 1$ and $q = \infty$, which is equivalent to $Y \in L^p(\Omega)$ and $X \in L^q(\Omega)$ [CR 4.3]. In this case,

$$\{X' \in M(\Omega, \mathcal{A}, P) : X' \precsim_{cx} X\} = \{X' \in L^q : X' \precsim_{cx} X\}.$$

In fact, $X \in L^q$ and $X' \precsim_{cx} X$ imply $X' \in L^q$ [CR 10.2]. Therefore, if $Y \in L^p(\Omega)$ and $X \in L^q(\Omega)$, then

$$\left\{\int YX'dP : L^q \ni X' \precsim_{cx} X\right\} = \left[\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_X(s)\, ds\right]. \tag{15}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds = \min\left\{\int YX'dP : L^q \ni X' \sim X\right\} \text{ and} \tag{16}$$

$$\int_0^1 \delta_Y(s)\, \delta_X(s)\, ds = \max\left\{\int YX'dP : L^q \ni X' \sim X\right\}. \tag{17}$$

If, in addition, X is a probability density (p.d.) and $X' \precsim_{cx} X$, then $X' \geq 0$ [CR 10.2] and $\mathbb{E}(X') = \mathbb{E}(X) = 1$, that is X' is a probability density. Finally, if $Y \in L^{\infty}$ and $Q \in \mathcal{M}_1$, then

$$\begin{aligned}
\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \precsim_{cx} Q\} &= \left\{\int YX'dP : X' \text{ is a p.d. and } X' \precsim_{cx} \frac{dQ}{dP}\right\} \\
&= \left\{\int YX'dP : L^1 \ni X' \precsim_{cx} \frac{dQ}{dP}\right\} \\
&= \left[\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds\right].
\end{aligned}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\begin{aligned}\int_0^1 \delta_Y(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds &= \max\left\{\int YX'dP : L^1 \ni X' \sim \frac{dQ}{dP}\right\}\\ &= \max\left\{\int YX'dP : X' \text{ is a p.d. and } X' \sim \frac{dQ}{dP}\right\}\\ &= \max\left\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \sim Q\right\}.\end{aligned}$$

The formula for the minimum is proved in the same way. □

The next step is essentially due to Hardy:

Step 2. Let $p = \infty$ and $q = 1$ or viceversa, $X, X' \in L^p$ and $Y \in L^q$.

(a) $X \precsim_{cx} X'$ implies $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$.

(b) $X \precsim_{cx} X'$ implies $\int_0^1 \delta_X(s)\,\delta_Y(1-s)\,ds \ge \int_0^1 \delta_{X'}(s)\,\delta_Y(1-s)\,ds$.

(c) $X \precsim_{icx} X'$ and $Y \ge 0$ implies $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$.

Proof of Step 2. $X, X' \in L^p$ and $Y \in L^q$ is equivalent to $\delta_X, \delta_{X'} \in L^p([0,1])$ and $\delta_Y \in L^q([0,1])$ [CR 4.3]. In particular, $\delta_X\delta_Y, \delta_{X'}\delta_Y \in L^1([0,1])$. Also notice that $f(s) \in L^q([0,1])$ if and only if $f(1-s) \in L^q([0,1])$, and

$$\int_0^1 f(s)\,ds = \int_0^1 f(1-s)\,ds.$$

(a) (resp., (b)) If $X \precsim_{cx} X'$, then $\int_0^w \delta_X(s)\,ds \le \int_0^w \delta_{X'}(s)\,ds$ for all $w \in [0,1]$ and $\int_0^1 \delta_X(s)\,ds = \int_0^1 \delta_{X'}(s)\,ds$, since $\delta_Y(s)$ is decreasing (resp., $\delta_Y(1-s)$ is increasing), then $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$ (resp., $\int_0^1 \delta_X(s)\,\delta_Y(1-s)\,ds \ge \int_0^1 \delta_{X'}(s)\,\delta_Y(1-s)\,ds$) [CR 9.1].

(c) If $X \precsim_{icx} X'$ and $Y \ge 0$, then $\int_0^w \delta_X(s)\,ds \le \int_0^w \delta_{X'}(s)\,ds$ for all $w \in [0,1]$ and δ_Y is decreasing and non-negative [CR 2.8], then $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$ [CR 9.1]. □

Step 3. If either $R(t,\cdot)$ is Shur concave on $\mathcal{M}_1$ for all $t \in \mathbb{R}$, or $(\Omega, \mathcal{A}, P)$ is adequate and $R(t,\cdot)$ is rearrangement invariant on $\mathcal{M}_1$ for all $t \in \mathbb{R}$, then

$$\rho(X) = \max_{Q \in \mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right) \quad \forall X \in L^\infty. \tag{18}$$

Proof of Step 3. Let $X \in L^\infty$. Then, $\mathbb{E}_Q(-X) \le \int_0^1 \delta_{-X}(s)\,\delta_{dQ/dP}(s)\,ds$ for all $Q \in \mathcal{M}_1$, by (12), thus monotonicity of R in the first component implies

$$\rho(X) = \max_{Q \in \mathcal{M}_1} R(\mathbb{E}_Q(-X), Q) \le \sup_{Q \in \mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right).$$

Conversely, for any $Q \in \mathcal{M}_1$, by (12) there exists $Q' \succsim_{cx} Q$ (resp., by (14) there exists $Q' \sim Q$) such that

$$\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds = \mathbb{E}_{Q'}(-X).$$

Thus,

$$R\left(\int_0^1 \delta_{-X}(s)\,\delta_{dQ/dP}(s)\,ds, Q\right) = R(\mathbb{E}_{Q'}(-X), Q) \le R(\mathbb{E}_{Q'}(-X), Q') \le \rho(X)$$

by Shur concavity (resp., rearrangement invariance). Therefore,

$$\sup_{Q \in \mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right) \le \rho(X)$$

and the supremum is attained. $\square$

Step 4. (ii) implies (i) and (10), also (iv) implies (i) provided $(\Omega, \mathcal{A}, P)$ is adequate.

Proof of Step 4. By Step 3, (ii) guarantees that (18) holds, and the same is true for (iv) if $(\Omega, \mathcal{A}, P)$ is adequate. But (18) is equivalent to (10) since $\delta_Y(s) = q_Y^-(1-s)$ for $s \in [0,1]$.

Moreover, $X \succsim_{ssd} Y$ if and only if $-X \precsim_{icx} -Y$. Thus, Step 2.c implies $\int_0^1 \delta_{-X}(s)\,\delta_{dQ/dP}(s)\,ds \leq \int_0^1 \delta_{-Y}(s)\,\delta_{dQ/dP}(s)\,ds$ for all $Q \in \mathcal{M}_1$, and monotonicity of R allows to conclude that

$$\rho(X) = \max_{Q \in \mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right) \leq \max_{Q \in \mathcal{M}_1} R\left(\int_0^1 \delta_{-Y}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right) = \rho(Y).$$

Therefore, ρ preserves second order stochastic dominance and, in particular, it is law-invariant. $\square$

Step 5. If either ρ preserves second order stochastic dominance or $(\Omega, \mathcal{A}, P)$ is adequate and ρ is law-invariant, then, for all $(t, Q) \in \mathbb{R}\times\mathcal{M}_1$,

$$R(t,Q) = \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Y(1-s)\,ds \leq -t\right\} = \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Y(1-s)\,ds = -t\right\}. \tag{19}$$

Proof of Step 5. Notice that if ρ preserves second order stochastic dominance, then it is Shur convex, that is, $X \precsim_{cx} Y$ implies $\rho(X) \leq \rho(Y)$. Let ρ be Shur convex (resp. law-invariant). First observe that

$$R(t,Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) \geq t\} = \inf\{\rho(X) : \mathbb{E}_Q(X) \leq -t\}$$

for all $(t,Q) \in \mathbb{R}\times\mathcal{M}_1$.[12] Since ρ is Shur convex (resp., rearrangement invariant), then

$$\begin{aligned}\inf\{\rho(X) : \mathbb{E}_Q(X) \leq -t\} &= \inf\{\rho(Y) : \text{there exists } X \precsim_{cx} Y \text{ such that } \mathbb{E}_Q(X) \leq -t\}\\ (\text{resp. } &= \inf\{\rho(Y) : \text{there exists } X \sim_{cx} Y \text{ such that } \mathbb{E}_Q(X) \leq -t\}),\end{aligned}$$

but,

$$\inf\{\rho(Y) : \mathbb{E}_Q(X) \leq -t \text{ for some } X \precsim_{cx} Y\} = \inf\left\{\rho(Y) : \min\left\{\int \frac{dQ}{dP} X\,dP : L^\infty \ni X \precsim_{cx} Y\right\} \leq -t\right\}$$

$$(\text{resp., } \inf\{\rho(Y) : \mathbb{E}_Q(X) \leq -t \text{ for some } X \sim Y\} = \inf\left\{\rho(Y) : \min\left\{\int \frac{dQ}{dP} X\,dP : L^\infty \ni X \sim Y\right\} \leq -t\right\}).$$

By (15) and (16), for all $(t,Q) \in \mathbb{R}\times\mathcal{M}_1$,

$$R(t,Q) = \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Y(1-s)\,ds \leq -t\right\} \leq \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Y(1-s)\,ds = -t\right\}.$$

Finally, assume per contra that $R(t,Q) < \inf\left\{\rho(Y) : \int_0^1 \delta_{dQ/dP}(s)\,\delta_Y(1-s)\,ds = -t\right\}$ for some $(t,Q) \in \mathbb{R}\times\mathcal{M}_1$. This implies the existence of $Z \in L^\infty$ for which $\int_0^1 \delta_{dQ/dP}(s)\,\delta_Z(1-s)\,ds \leq -t$ and

$$\rho(Z) < \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Y(1-s)\,ds = -t\right\}.$$

Since $\delta_{Z+m} = \delta_Z + m$ for all $m \in \mathbb{R}$, then

$$\int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_{Z+m}(1-s)\,ds = \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Z(1-s)\,ds + m.$$

[12] Clearly, $R(t,Q) \geq \inf\{\rho(Y) : \mathbb{E}_Q(-Y) \geq t\}$. Conversely, assume per contra that $R(t,Q) > \inf\{\rho(Y) : \mathbb{E}_Q(-Y) \geq t\}$ for some $(t,Q) \in \mathbb{R}\times\mathcal{M}_1$. This implies the existence of $Z \in L^\infty$ for which $\mathbb{E}_Q(-Z) \geq t$ and $\rho(Z) < R(t,Q)$. Set $m = \mathbb{E}_Q(-Z) - t \geq 0$, then $Z + m \geq Z$, $\mathbb{E}_Q(-(Z+m)) = t$ and $R(t,Q) \leq \rho(Z+m) \leq \rho(Z) < R(t,Q)$, a contradiction. The second equality is trivial.

Choose $m \geq 0$ so that

$$\int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_{Z+m}(1-s)\, ds = -t,$$

then $Z + m \geq Z$, and $\rho(Z+m) \leq \rho(Z) < \inf\left\{\rho(Y) : \int_0^1 \delta_{dQ/dP}(s)\, \delta_Y(1-s)\, ds = -t\right\}$, a contradiction. $\square$

Step 6. (i) implies (ii) and (11), also (iii) implies (ii) provided $(\Omega, \mathcal{A}, P)$ is adequate.

Proof of Step 6. By Step 5, (i) guarantees that (19) holds, and the same is true for (iii) if $(\Omega, \mathcal{A}, P)$ is adequate. But, the second part of (19) is equivalent to (11) since $\delta_Y(s) = q_Y^-(1-s)$ for $s \in [0,1]$. While the first part, together with Step 2.b, yields the following chain of implications

$$\begin{aligned} Q \precsim_{cx} Q' &\Longrightarrow \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Y(1-s)\, ds \geq \int_0^1 \delta_{\frac{dQ'}{dP}}(s)\, \delta_Y(1-s)\, ds \text{ for all } Y \in L^\infty \\ &\Longrightarrow \left\{Y : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Y(1-s)\, ds \leq -t\right\} \subseteq \left\{Y : \int_0^1 \delta_{\frac{dQ'}{dP}}(s)\, \delta_Y(1-s)\, ds \leq -t\right\} \quad \forall t \in \mathbb{R} \\ &\Longrightarrow R(t,Q) \geq R(t,Q') \quad \forall t \in \mathbb{R}. \end{aligned}$$

Hence, $R(t,\cdot)$ is Shur concave for all $t \in \mathbb{R}$. $\square$

Finally, Steps 4 and 6 guarantee that (i)$\Longleftrightarrow$(ii), and in this case (10) and (11) hold. Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, the same steps deliver (iv)$\Longrightarrow$(i)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(ii)$\Longrightarrow$(iv). $\blacksquare$

Theorem 2 considers law-invariant quasiconvex risk measures that are upper semicontinuous with respect to bounded pointwise convergence (see Proposition 3). Jouini, Schachermayer, and Touzi (2006) show that law-invariant convex monetary risk measures are automatically lower semicontinuous with respect to bounded pointwise convergence, provided $(\Omega, \mathcal{A}, P)$ is standard. Whether this remains true for quasiconvex risk measures is left for future research (but see Proposition 4).

4.1 Mean Value Premium Principle

We conclude by studying an interesting class of law-invariant quasiconvex risk measures that are continuous from below, that is, those of the form

$$\rho(X) = \ell^{-1}\left(\mathbb{E}_P\left(\ell(-X)\right)\right) \quad \forall X \in L^\infty, \tag{20}$$

where ℓ is a strictly increasing and convex loss function. The characterization of these measures is a version of the classic Nagumo-Kolmogorov-de Finetti Theorem and relies on two additional properties:

Constancy $\rho(m) = -m$ for all $m \in \mathbb{R}$.

Conditional consistency Let $A \in \mathcal{A}$ and $X, Y, Z \in L^\infty$,

$$\rho(X\mathrm{I}_A) > \rho(Y\mathrm{I}_A) \Longleftrightarrow \rho(X\mathrm{I}_A + Z\mathrm{I}_{A^c}) > \rho(Y\mathrm{I}_A + Z\mathrm{I}_{A^c}).$$

The latter property is inspired by Savage (1954)'s "sure thing principle" and clearly hints at dynamic consistency (see, e.g., Ghirardato, 2002). The seminal paper of Ellsberg (1961) shows how this assumption is non-controversial only if agents think that P is a reliable model of the uncertainty they face.[13]

[13] See, e.g., Maccheroni, Marinacci, and Rustichini (2006) for a recent discussion of this issue.

Lemma 2 *Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space. A law-invariant risk measure ρ satisfies constancy, conditional consistency, and continuity with respect to bounded pointwise convergence if and only if there exists a strictly increasing and continuous $\ell : \mathbb{R} \to \mathbb{R}$ such that*

$$\rho(X) = \ell^{-1}(\mathbb{E}_P(\ell(-X))) \quad \forall X \in L^\infty.$$

The function ℓ is unique up to strictly increasing affine transformations, and it is convex if and only if ρ is quasiconvex.

Proof. Sufficiency is trivial. Necessity reduces to check that the function $\mathfrak{M} : \mathcal{D}^\infty \to \mathbb{R}$ defined for each distribution with bounded support $F = F_X$ by

$$\mathfrak{M}(F) = -\rho(X)$$

satisfies the assumptions of the Nagumo-Kolmogorov-de Finetti Theorem.

For the sake of completeness we include such check. Let $[a, b]$ be any closed interval in the real line and $\mathcal{D}(a, b)$ be the set of all simple probability distributions supported in $[a, b]$. The Dirac distribution concentrated in x is denoted by D_x.

Constancy guarantees that:

Step 1. For all $x \in [a, b]$, $\mathfrak{M}(D_x) = x$.

Step 2. If $F, G \in \mathcal{D}(a, b)$, $F \geq G$, and $F \neq G$, then $\mathfrak{M}(F) < \mathfrak{M}(G)$.

Proof of Step 2. Since $(\Omega, \mathcal{A}, P)$ is non-atomic there are two simple measurable functions $X \leq Y$ such that $F = F_X$ and $G = F_Y$. By monotonicity of ρ, $\mathfrak{M}(F) \leq \mathfrak{M}(G)$. Assume per contra that $\mathfrak{M}(F) = \mathfrak{M}(G)$, that is $\rho(X) = \rho(Y)$. If $X = Y$, then $F = G$, which is absurd. Thus (again by non-atomicity) there exist $n \in \mathbb{N}$, $A_1 \in \mathcal{A}$ with $P(A_1) = 1/n$, and $x, y \in \mathbb{R}$ such that

$$X(\omega) < x < y < Y(\omega) \quad \forall \omega \in A_1.$$

Therefore,

$$X \leq x\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c} \leq y\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c} \leq Y.$$

By monotonicity of ρ,

$$\rho\left(x\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c}\right) = \rho\left(y\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c}\right),$$

and so, by conditional consistency,

$$\rho(x\mathrm{I}_{A_1}) = \rho(y\mathrm{I}_{A_1}).$$

Let $A_2, ..., A_n$ be such that $\{A_i\}_{i=1}^n$ form a partition of Ω with $P(A_i) = 1/n$ for all i. By law-invariance

$$\rho(x\mathrm{I}_{A_i}) = \rho(y\mathrm{I}_{A_i}) \quad \forall i = 1, ..., n.$$

Repeated application of conditional consistency then delivers

$$\begin{aligned}\rho(x\mathrm{I}_\Omega) &= \rho(x\mathrm{I}_{A_1} + x\mathrm{I}_{A_2} + x\mathrm{I}_{A_3} + ... + x\mathrm{I}_{A_n}) = \rho(y\mathrm{I}_{A_1} + x\mathrm{I}_{A_2} + x\mathrm{I}_{A_3} + ... + x\mathrm{I}_{A_n}) \\ &= \rho(y\mathrm{I}_{A_1} + y\mathrm{I}_{A_2} + x\mathrm{I}_{A_3} + ... + x\mathrm{I}_{A_n}) = ... = \rho(y\mathrm{I}_\Omega),\end{aligned}$$

which is absurd by constancy. □

Step 3. If $F, G, H \in \mathcal{D}(a, b)$, $\lambda \in (0, 1)$, and $\mathfrak{M}(F) = \mathfrak{M}(G)$, then $\mathfrak{M}(\lambda F + (1-\lambda) H) = \mathfrak{M}(\lambda G + (1-\lambda) H)$.

Proof of Step 3. For every $\lambda \in [0, 1]$, since $(\Omega, \mathcal{A}, P)$ is non-atomic, there are $X, Y, Z \in L^\infty$ and $A = A_\lambda \in \mathcal{A}$ that are independent and such that $F = F_X$, $G = F_Y$, $H = F_Z$, and $P(A) = \lambda$ (see, e.g., Billingsley, 1995, Theorem 5.3). Independence guarantees that $F_{W\mathrm{I}_A+W'\mathrm{I}_{A^c}} = \lambda F_W + (1-\lambda) F_{W'}$ if $W, W' \in \{X, Y, Z\}$.

If $\lambda = 1/2$, then $F_{W\mathrm{I}_A+W'\mathrm{I}_{A^c}} = 2^{-1}F_W + 2^{-1}F_{W'} = 2^{-1}F_{W'} + 2^{-1}F_W = F_{W'\mathrm{I}_A+W\mathrm{I}_{A^c}}$. Assume, per contra, $\mathfrak{M}\left(2^{-1}F + 2^{-1}H\right) \neq \mathfrak{M}\left(2^{-1}G + 2^{-1}H\right)$, then $\rho\left(X\mathrm{I}_A + Z\mathrm{I}_{A^c}\right) \gtrless \rho\left(Y\mathrm{I}_A + Z\mathrm{I}_{A^c}\right)$, by conditional consistency and law-invariance

$$\rho(X) = \rho\left(X\mathrm{I}_A + X\mathrm{I}_{A^c}\right) \gtrless \rho\left(Y\mathrm{I}_A + X\mathrm{I}_{A^c}\right) = \rho\left(X\mathrm{I}_A + Y\mathrm{I}_{A^c}\right) \gtrless \rho\left(Y\mathrm{I}_A + Y\mathrm{I}_{A^c}\right) = \rho(Y),$$

which contradicts $\mathfrak{M}(F) = \mathfrak{M}(G)$. Thus the statement is true for $\lambda = 2^{-1}$. Induction guarantees that it is true for any dyadic rational. Continuity with respect to bounded pointwise convergence of ρ and the Skorohod Theorem (see, e.g., Billingsley, 1995, Theorem 25.6) guarantee that the statement is true for any λ. □

Let $[a, b] = [-n, n]$. The Nagumo-Kolmogorov-de Finetti Theorem guarantees that for all $n \in \mathbb{N}$ there exists a unique strictly increasing function $\phi_n : [-n, n] \to \mathbb{R}$ such that $\phi_n(0) = 0 = \phi_n(1) - 1$ and

$$\mathfrak{M}(F) = \phi_n^{-1}\left(\int_{\mathbb{R}} \phi_n(x)\, dF(x)\right) \quad \forall F \in \mathcal{D}(-n, n).$$

Define $\phi(x) = \phi_n(x)$ if $|x| \leq n$ to obtain $\mathfrak{M}(F) = \phi^{-1}\left(\int_{\mathbb{R}} \phi(x)\, dF(x)\right)$ for each simple probability distribution. Then,

$$\rho(X) = -\phi^{-1}\left(\mathbb{E}_P\left(\phi(X)\right)\right)$$

for all simple and measurable $X : \Omega \to \mathbb{R}$. Continuity with respect to bounded pointwise convergence yields the result for $\ell(\cdot) = -\phi(-\cdot)$.

Finally, if ρ is quasiconvex, Theorem 2 guarantees that ρ preserves second order stochastic dominance. Hence ℓ is convex. The converse is trivial. ■

Our final result builds on Rockafellar (1971) and explicitly evaluates R for risk measures that admit an expected loss representation.

Proposition 5 *If $\ell : \mathbb{R} \to \mathbb{R}$ is a strictly increasing convex function and $\rho(X) = \ell^{-1}\left(\mathbb{E}_P\left(\ell(-X)\right)\right)$ for all $X \in L^\infty$, then*

$$R(t, Q) = \ell^{-1}\left(\max_{x \geq 0}\left[xt - \mathbb{E}_P\left(\ell^*\left(x\frac{dQ}{dP}\right)\right)\right]\right) \quad \forall (t, Q) \in \mathbb{R} \times \mathcal{M}_1.$$

Observe that this amounts to say that $R(t, Q) = t - L(-t; Q, P)$, where $L(w; Q, P)$ is the generalized distance between probability measures considered by Bellini and Frittelli (2002) and corresponding to an initial endowment w and a utility $-\ell(-\cdot)$.

Proof. Observe that $\ell(\mathbb{R})$ is an open half line (l, ∞), with $l = \inf_{x \in \mathbb{R}} \ell(x)$. Then ℓ^{-1} can be extended to an extended-valued continuous and monotone function from $[-\infty, \infty]$ to $[-\infty, \infty]$ by setting $\ell^{-1}(x) = -\infty$ if $x < l$ and $\ell^{-1}(\infty) = \infty$. For all $(t, Q) \in \mathbb{R} \times \mathcal{M}_1$,

$$R(t, Q) = \inf\left\{\ell^{-1}\left(\mathbb{E}_P\left(\ell(-X)\right)\right) : \mathbb{E}_Q(-X) = t\right\} = \ell^{-1}\left(\inf\left\{\mathbb{E}_P\left(\ell(-X)\right) : \mathbb{E}_Q(-X) = t\right\}\right).$$

Set $\phi(\cdot) = -\ell(-\cdot)$. Then

$$\begin{aligned} \inf\left\{\mathbb{E}_P\left(\ell(-X)\right) : \mathbb{E}_Q(-X) = t\right\} &= \inf\left\{-\mathbb{E}_P\left(-\ell(-X)\right) : \mathbb{E}_Q(-X) = t\right\} \\ &= -\sup\left\{\mathbb{E}_P\left(\phi(X)\right) : \mathbb{E}_Q(X) = -t\right\}. \end{aligned}$$

But, the function $\Phi(X) = \mathbb{E}_P(\phi(X))$ for all $X \in L^\infty$ is concave, continuous, and monotone. Then, it follows immediately from Lemma 19 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b) and Corollary 2A of Rockafellar (1971) that

$$\sup\{\mathbb{E}_P(\phi(X)) : \mathbb{E}_Q(X) = -t\} = \min_{x\geq 0}[x(-t) - \Phi^*(xQ)] = \min_{x\geq 0}\left[x(-t) - \mathbb{E}_P\left(\phi^*\left(x\frac{dQ}{dP}\right)\right)\right].$$

Thus,

$$R(t,Q) = -\phi^{-1}\left(\min_{x\geq 0}\left[x(-t) - \mathbb{E}_P\left(\phi^*\left(x\frac{dQ}{dP}\right)\right)\right]\right) = \ell^{-1}\left(\max_{x\geq 0}\left[xt - \mathbb{E}_P\left(\ell^*\left(x\frac{dQ}{dP}\right)\right)\right]\right),$$

as wanted. ■

5 A Final Remark

For mathematical convenience we considered risk measures defined on $L^\infty(\Omega, \mathcal{A}, P)$. A parallel analysis can be carried out in any function space with unit,[14] like for example the space $B(\Omega, \mathcal{A})$ of bounded and measurable functions and the space $C_b(\Omega)$ of bounded and continuous functions (provided Ω is a topological space).

References

[1] P. Artzner, F. Delbaen, J. Eber, and D. Heath, Thinking coherently, *RISK*, 10, 68–71, 1997.

[2] P. Artzner, F. Delbaen, J. Eber, and D. Heath, Coherent measures of risk, *Mathematical Finance*, 9, 203–228, 1999.

[3] F. Bellini and M. Frittelli, On the existence of minimax martingale measures, *Mathematical Finance*, 12, 1–21, 2002.

[4] F. Black, Capital market equilibrium with restricted borrowing, *The Journal of Business*, 45, 444–455, 1972.

[5] P. Billingsley, *Probability and Measure*, Wiley, New York, 1995.

[6] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, Carlo Alberto Notebook 77, 2008a.

[7] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Complete monotone quasiconcave duality, Carlo Alberto Notebook 80, 2008b.

[8] P. Carr, H. Geman, and D. B. Madan, Pricing and hedging in incomplete markets, *Journal of Financial Economics*, 62, 131–167, 2001.

[9] K. M. Chong and N. M. Rice, Equimeasurable rearrangements of functions, *Queen's Papers in Pure and Applied Mathematics*, 28, 1971.

[10] K. M. Chong, Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications, *Canadian Journal of Mathematics*, 26, 1321–1340, 1974.

[11] G. Dal Maso, *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[14] That is, in any Riesz space of functions with order unit and endowed with the supnorm.

[12] R. A. Dana, A representation result for concave Schur concave functions, *Mathematical Finance*, 15, 613-634, 2005.

[13] B. de Finetti, Sul concetto di media, *Giornale dell'Istituto Italiano degli Attuari*, 2, 369–396, 1931.

[14] N. El Karoui and M. C. Quenez, Non-linear pricing theory and backward stochastic differential equations in financial mathematics, in *Lecture Notes in Mathematics*, 1656, ed. by W. Runggaldier, Springer, New York, 1997.

[15] N. El Karoui and C. Ravanelli, Cash sub-additive risk measures under interest rate ambiguity, *Mathematical Finance*, forthcoming, 2008.

[16] D. Ellsberg, Risk, ambiguity, and the Savage axioms, *Quarterly Journal of Economics*, 75, 643–669, 1961.

[17] D. Filipović and M. Kupper, 2008, Equilibrium prices for monetary utility functions, *International Journal of Theoretical and Applied Finance*, 11, 325–343, 2008.

[18] H. Föllmer and A. Schied, Convex measures of risk and trading constraints, *Finance and Stochastics*, 6, 429–447, 2006.

[19] H. Föllmer and A. Schied, *Stochastic Finance: An Introduction in Discrete Time*, De Gruyter, Berlin, 2004.

[20] M. Frittelli and E. Rosazza Gianin, Putting order in risk measures, *Journal of Banking and Finance*, 26, 1473–1486, 2002.

[21] M. Frittelli and E. Rosazza Gianin, Dynamic convex risk measures, in *Risk Measures for the 21st Century*, ed. by G. Szegö, Wiley, New York, 2004.

[22] M. Frittelli and E. Rosazza Gianin, Law-invariant convex risk measures, *Advances in Mathematical Economics*, 7, 33–46, 2005.

[23] P. Ghirardato, Revisiting Savage in a conditional world, *Economic Theory*, 20, 83-92, 2002.

[24] I. Gilboa and D. Schmeidler, Maxmin expected utility with a non-unique prior, *Journal of Mathematical Economics*, 18, 141–153, 1989.

[25] G. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[26] E. Jouini, W. Schachermayer, and N. Touzi, Law invariant risk measures have the Fatou property, *Advances in Mathematical Economics*, 9, 49–71, 2006.

[27] E. Jouini, W. Schachermayer, and N. Touzi, Optimal risk sharing for law invariant monetary utility functions, *Mathematical Finance*, 18, 269–292, 2008.

[28] M. Kupper and W. Schachermayer, Representation results for law invariant time consistent functions, preprint, Vienna Institute of Finance, Vienna, 2008.

[29] A. N. Kolmogorov, Sur la notion de la moyenne, *Atti della R. Accademia Nazionale dei Lincei*, 12, 388–391, 1930.

[30] S. Kusuoka, On law-invariant coherent risk measures, *Advances in Mathematical Economics*, 3, 83–95, 2001.

[31] J. Leitner, A short note on second-order stochastic dominance preserving coherent risk measures, *Mathematical Finance*, 15, 649–651, 2005.

[32] W. A. J. Luxemburg, Rearrangement-invariant Banach function spaces, *Queen's Papers in Pure and Applied Mathematics*, 10, 83–144, 1967.

[33] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 1447–1498, 2006.

[34] M. Marinacci and L. Montrucchio, Introduction to the mathematics of ambiguity, in *Uncertainty in Economic Theory*, ed. by I. Gilboa, Routledge, New York, 2004.

[35] A.W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York, 1979.

[36] M. Nagumo, Über eine klasse der mittelwerte, *Japanese Journal of Mathematics*, 7, 71–79, 1930.

[37] R. T. Rockafellar, Integrals which are convex functionals II, *Pacific Journal of Mathematics*, 39, 439–469, 1971.

[38] V. I. Rotar, *Actuarial Models: the Mathematics of Insurance*, CRC Press, Boca Raton, 2007.

[39] M. Rothschild and J.E. Stiglitz, Increasing risk: I. A definition, *Journal of Economic Theory*, 2, 225–243, 1970.

[40] L. J. Savage, *The Foundations of Statistics*, John Wiley and Sons, New York, 1954.

[41] J. Staum, Fundamental theorems of asset pricing for good deal bounds, *Mathematical Finance*, 14, 141–161, 2004.

[42] K. Yosida, *Functional Analysis*, Springer, New York, 1980.