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Abstract

We consider a game, called *newsvendor game*, where several retailers, who face a random demand, can pool their resources and build a centralized inventory that stocks a single item on their behalf. Profits have to be allocated in a way that is advantageous to all the retailers. A game in characteristic form is obtained by assigning to each coalition its optimal expected profit. A similar game (modeled in terms of costs) was considered by Müller et al. (2002), who proved that this game is balanced for every possible joint distribution of the random demands.

In this paper we consider newsvendor games with possibly an infinite number of newsvendors. We prove in great generality results about balancedness of the game, and we show that in a game with a continuum of players, under a nonatomic condition on the demand, the core is a singleton. For a particular class of demands we show how the core shrinks to a singleton when the number of players increases.

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1 Introduction

The newsvendor problem is a classic textbook example in optimization. A newsvendor sells a product (newspapers) during a short selling period (a morning) with stochastic demand. The newsvendor can order inventory before the selling period and has no additional replenishment opportunity. If the order quantity is greater than the realized demand, the newsvendor must dispose of the remaining stock at a loss. If the order quantity is lower than realized demand, the newsvendor forgoes some profit. Therefore, in choosing an order quantity the newsvendor must balance the costs of ordering too little against the costs of ordering too much. For a survey of this problem see for instance Petruzzi and Dada (1999).

In recent years strategic versions of the newsvendor problem have been considered by several authors. The reader is referred to Cachon and Netessine (2004) for a nice survey of both cooperative and noncooperative models in supply chain.

Noncooperative versions of the newsvendor game appear in different variations. For instance Parlar (1988), Wang and Parlar (1994), Ernst and Kouvelis (1999), and Netessine and Rudi (2003) study the role of inventory in the competition among retailers and determine uniqueness of the Nash equilibrium. Lippman and McCardle (1997) study competition between firms in a single-period setting, where a consumer may switch among firms to find available inventory. In Cachon and Lariviere (1999a,b) retailers behave strategically when ordering from a supplier with limited capacity.

Cooperative versions have been considered for instance by Eppen (1979), Gerchak and Gupta (1991), Robinson (1993), Hartman and Dror (1996), Hartman et al. (2000), Müller et al. (2002), and Slikker et al. (2005). In a cooperative newsvendor game several retailers can pool their resources and build a centralized inventory that stocks a single item on their behalf. Profits have to be allocated in a way that is advantageous to all the retailers. Otherwise some of them will prefer not to join the centralized inventory.

If we assign to each coalition the expected profit that it obtains if it stocks the optimal number of newspapers (as in the single-newsvendor model), then we have a cooperative game in characteristic form. Every newsvendor will find convenient to build the centralized inventory if the core of the game is nonempty (i.e., if the game is balanced). Müller et al. (2002) model a newsvendor game in terms of costs (rather than profits), and prove that if the costs are linear and homogeneous across newsvendors, then the newsvendor game is balanced for every possible joint distribution of the random demands. Slikker et al. (2005) prove a similar result for games modeled in terms of profits, when transshipment costs are taken into account.

In this paper we consider large newsvendor games. In order to prove our results it is useful to see the newsvendor game as an infinite-dimensional measure game as in Milchtaich (1998). This allows to treat both finite and infinite games. The core of the game is now a set of charges (finitely additive measures) that dominate the game. A section of the paper will be devoted to nonatomic newsvendor games, namely, to games where the influence of each newsvendor is negligible.

A huge literature exists on nonatomic games. The basic framework is Aumann and Shapley (1974), which, although more oriented towards the theory of value, contains the foundations of cooperative games where each single agent is negligible. Some classical forerunners are the papers by Aumann (1964, 1966) on nonatomic competitive economies. The connection between large and nonatomic games has been studied, among others, by Debreu and Scarf (1963), Kannai (1970), and Debreu (1975). Interesting surveys can be found, e.g., in Debreu and Scarf (1972), Hildenbrand (1974), and Anderson (1986, 1992). Major contributions on the core of infinite TU games are due to Schmeidler (1967), Kannai (1969), Schmeidler (1972), and Delbaen (1974). A survey of the area and general proofs of the main results can be found in Marinacci and Montrucchio (2004).

Several results that are in the same spirit as those contained in Müller et al. (2002) are proved in greater generality with different techniques. For instance here, in order to show that the game is balanced, a charge, which is always in the core, is explicitly computed. Conditions for exactness, supermodularity, and positivity of the game are established.

The main result of the paper is that for a nonatomic newsvendor game the core is a singleton, whenever the aggregate demand has a continuous distribution.

A parametric class of games is then considered and conditions are determined, under which the core shrinks to a singleton when the number of newsvendors increases. It is shown that this shrinking does not happen when the random demands are independent.

We want to emphasize that the full force of our results comes from the infinite-dimensional-measure-game approach proposed by Milchtaich (1998). Our paper provides then an application of this approach, and develops techniques that should prove useful in the analysis of other vector measure games. To show this we briefly consider a class of games, that we call Markowitz games, to which some results presented for newsvendor games apply.

The paper is organized as follows. In Section 2 the newsvendor game is presented and some general results are stated. Section 3 deals with nonatomic newsvendor games and provides the main result of the paper. In Section 4 we briefly treat Markowitz games. In Section 5 a class of newsvendor games with many players is considered and some asymptotic results are stated. Section 6 studies some useful properties of an operator that is used in the newsvendor problem. Section 7 contains the proofs of the results.

2 Newsvendor games

2.1 The newsvendor problem

We introduce the newsvendor problem in an abstract setting that will prove suitable for the analysis of the game.

A newsvendor has to decide how many newspapers to stock in order to face an unknown demand, knowing that no replenishment is allowed. If she faces a demand x and orders a quantity y, then she obtains a profit

$$\psi(x,y) = \begin{cases} px - h(y-x) & \text{if} \quad x \le y, \\ py - \pi(x-y) & \text{if} \quad x > y, \end{cases} \tag{2.1}$$

for some constants $p, h > 0$, and $\pi \ge 0$. The constant p is the price of the newspaper (net of the wholeseller's price that the newsvendor has to pay), the constant h represents the holding cost of stocking more newspapers than are actually sold, and the constant π is the penalty cost of not ordering enough newspapers to meet the demand. This cost can be null.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random quantities will be defined, and the space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ of all integrable random variables (in fact with a slight abuse of notation, throughout the paper we identify an integrable random variable Z with its class of equivalence $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$). A central role will be played by the operator $\Pi : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, defined as

$$\Pi(X) = \max_{y \in \mathbb{R}} \mathbb{E}[\psi(X, y)]. \tag{2.2}$$

The operator Π represents the expected profit for a newsvendor who orders the optimal amount of newspapers. Its properties will be studied extensively in Section 6.

Remark 2.1. From an economic viewpoint formula (2.2) should be $\max_{y \geq 0} \mathbb{E}[\psi(X, y)]$. On the other hand the function $y \mapsto \mathbb{E}[\psi(X, y)]$ is concave. If the demand X is positive, we will show that the maximum is attained at a point $y^* > 0$. Therefore, due to well known properties of concave functions, maximizing over the whole real line or only over its positive part are equivalent. Definition (2.2) has the advantage of allowing not necessarily positive demands X. Another way to see this is considering that the operator Π satisfies $\Pi(X + a) = \Pi(X) + pa$ for all scalar $a \in \mathbb{R}$.

2.2 The game

A general newsvendor game is defined on a measurable space of agents $(I, \mathcal{C})$. The set I is a set of players (newsvendors), and $\mathcal{C}$ is a σ-algebra of subsets of I. Elements of $\mathcal{C}$ are then feasible coalitions of newsvendors. Any coalition orders a fixed number of newspapers to face a random demand. As in the newsvendor problem, if a coalition faces a demand x and orders a quantity y, then it obtains a profit $\psi(x, y)$, where ψ is defined as in (2.1).

The random demand will be represented by a function $X : \Omega \times \mathcal{C} \to \mathbb{R}$ that satisfies the following conditions

- for all $A \in \mathcal{C}$, the map $\omega \to X(\omega, A)$ is an integrable random variable,
- for all $\omega \in \Omega$, the map $A \to X(\omega, A)$ is an additive measure on $(I, \mathcal{C})$.

We can define an *additive vector measure* $D : \mathcal{C} \to L^1(\Omega, \mathcal{F}, \mathbb{P})$ as follows: for any coalition A, $D(A) = X(\cdot, A) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, which is interpreted as the joint demand faced by coalition A. Details about vector measures can be found in the next subsection.

The optimal amount ordered by coalition A is

$$y_A^* = \arg\max_{y \in \mathbb{R}} \mathbb{E}[\psi(D(A), y)],$$

that is the amount that maximizes the expected profit for the coalition. We will show that the maximizer y_A^* exists and is a $(\pi + p)/(h + \pi + p)$-quantile of the distribution of $D(A)$. Hence the optimal expected profit for coalition A is

$$\Pi(D(A)) = \max_{y \in \mathbb{R}} \mathbb{E}[\psi(D(A), y)] = \mathbb{E}[\psi(D(A), y_A^*)].$$

The newsvendor game may then be defined as

$$\nu(A) = \Pi(D(A)) \tag{2.3}$$

for all $A \in \mathcal{C}$. The amount $\nu(A)$ is the profit that members of the coalition A jointly obtain.

Definition (2.3) of newsvendor game through the vector-valued measure $A \to D(A)$ presents some analytical advantage, since the newsvendor game can be viewed as an *infinite dimensional measure game* (see Milchtaich (1998)).

Example 2.2. If we set $I = \{1, 2, \dots, d\}$, $\mathcal{C} = 2^I$, and $D(\{i\}) = X_i \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, we get a finite newsvendor game where $\nu(A) = \Pi\left(\sum_{i\in A} X_i\right)$. This game is strongly related to the one studied by Müller et al. (2002). The difference is that they defined the game in terms of costs, whereas here we define it in terms of profits. The two formulations are actually closely related, as shown by Slikker et al. (2005) in the finite setting, and by formula (7.4) of Lemma 7.1 in our more general context.

2.3 Notation

Here we introduce some notation and definitions that will be used throughout the paper.

Two random variables $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ are called *comonotone* if for all $\omega' \in \Omega$ we have $\mathbb{P}(\omega : (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0) = 1$.

Given a measurable space $(I, \mathcal{C})$, a coalitional game is a set-function $\nu : \mathcal{C} \to \mathbb{R}$ such that $\nu(\varnothing) = 0$. We list some standard terminology utilized in cooperative games literature.

A game ν is

- *bounded* if $\sup_{A\in\mathcal{C}} |\nu(A)| < +\infty$;
- *superadditive* if $\nu(A \cup B) \geq \nu(A) + \nu(B)$ for all pairwise disjoint A and B;
- *supermodular* if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all A and B;
- *additive* (a *charge*) if $\nu(A \cup B) = \nu(A) + \nu(B)$ for all pairwise disjoint A and B;
- *σ-additive* (a *measure*) if $\nu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$ whenever $A_i \cap A_j = \varnothing$ for $i \neq j$.
- *continuous at* A, if $\lim_{n\to\infty} \nu(A_n) = \nu(A)$ whenever $A_n \downarrow A$ and $A_n \uparrow A$;
- *continuous* if ν is continuous at A for all $A \in \mathcal{C}$.

The set of bounded charges (i.e., additive measures) is denoted by $\mathrm{ba}(\mathcal{C})$, the set of bounded measures (i.e., countably additive measures) is denoted by $\mathrm{ca}(\mathcal{C})$, and the set of positive bounded measures is denoted by $\mathrm{ca}^+(\mathcal{C})$. Given a positive measure λ, the set of all measures which are absolutely continuous with respect to λ is denoted by $\mathrm{ca}(\mathcal{C}, \lambda)$.

An outcome of the game ν is an element of $\mathrm{ba}(\mathcal{C})$. The *core* of a game ν is the set

$$\mathrm{core}(\nu) = \{\mu \in \mathrm{ba}(\mathcal{C}) : \mu(I) = \nu(I) \text{ and } \mu(A) \geq \nu(A) \text{ for all } A \in \mathcal{C}\}.$$

The core is always a weak*-compact subset of $\mathrm{ba}(\mathcal{C})$.

A game is said to be *balanced* if core$(\nu) \neq \varnothing$. Given a game ν and a coalition $A \in \mathcal{C}$, we may consider the game $\nu_A : \mathcal{C}_A \to \mathbb{R}$ which is the restriction of ν to the coalitions in A. A game is *totally balanced* if core$(\nu_A) \neq \varnothing$ for all $A \in \mathcal{C}$. Any totally balanced game is superadditive.

A game ν is *exact* if core$(\nu) \neq \varnothing$ and

$$\nu(A) = \min_{\mu \in \text{core}(\nu)} \mu(A), \quad \forall A \in \mathcal{C}.$$

Clearly, any exact game is totally balanced. Furthermore any supermodular and bounded game is exact. For positive games see Delbaen (1974). As the games under study are not always positive (see Proposition 2.8) we emphasize the fact that the result continues to hold for all bounded games (see e.g. Marinacci and Montrucchio (2004)).

2.4 General results

The result that follows is rather general and almost free of assumptions. It shows that the core of all newsvendor games is nonempty. More importantly, it offers a specific solution in the core. We will see in the sequel that in the atomic case the core may be quite large. On the other hand, under some mild conditions, in the nonatomic setting the core turns out be a singleton, therefore agreeing with the solution (2.4) below.

Theorem 2.3. *If the measure $A \to D(A)$ is bounded (i.e. satisfies* (7.1)*), then any newsvendor game is totally balanced. Moreover, if the aggregate demand $D(I)$ has a continuous distribution, then $\mu \in$* core(ν)*, where μ is a bounded charge defined as*

$$\mu(A) = (p+h) \int_{D(I) \leq y^*} D(A) \ \mathrm{d}\mathbb{P} - \pi \int_{D(I) \geq y^*} D(A) \ \mathrm{d}\mathbb{P} \tag{2.4}$$

for all $A \in \mathcal{C}$, where y^* is a $(\pi + p)/(h + \pi + p)$-quantile of $D(I)$.

The element of the core defined by (2.4) is particularly appealing for some classes of games, as the next proposition shows.

Proposition 2.4. *Consider a finite newsvendor game with $I = I_d \equiv \{1, \ldots, d\}$. If all the marginal distributions of the random demands $X_1, \ldots, X_d$ are equal, then the measure μ defined by* (2.4) *is the unique element in* core(ν) *such that $\mu_i = \mu(\{i\}) = \Pi(D(I_d))/d$ for all $i \in I_d$. Furthermore, μ is the barycenter of* core(ν)*, provided $(X_i)_{i \in I_d}$ are exchangeable.*

Though all newsvendor games are totally balanced, they are not necessarily exact. Next proposition states a sufficient condition that ensures this property.

Proposition 2.5. *If $D(I)$ and $D(A)$ are comonotone for all coalitions $A \in \mathcal{C}$, then the newsvendor game is exact.*

At least an important example is contemplated by this proposition. If there is no aggregate risk, i.e., $D(I)$ is nonrandom, then $D(I)$ and $D(A)$ are comonotone for every $A \in \mathcal{C}$.

The following result provides a strong property for newsvendor games with a particular structure of the demand D. Two random variables X and Y are *of the same type*, provided $F_X(x) = F_Y(ax + b)$, for some $a > 0$ and b.

Proposition 2.6. *Let the following conditions hold:*

(i) *all $D(A)$ have finite variance,*

(ii) *all $D(A)$ such that* $\mathrm{Var}[D(A)] > 0$ *are of the same type.*

Then the newsvendor game is

$$\nu(A) = p\,\mathbb{E}[D(A)] - k\sqrt{\mathrm{Var}[D(A)]} \tag{2.5}$$

for some $0 < k \le \max\{h, p+\pi\}$.

In addition, if the random variables $D(A)$ and $D(B)$ are uncorrelated for all A and B such that $A \cap B = \varnothing$, then the newsvendor game is supermodular.

Specializing Proposition 2.6 to finite games yields a remarkable result when the demands X_i have Gaussian distributions. We obtain another explicit solution in core(ν). Let $e_i \in \mathbb{R}^d$ be the vector whose i-th element is 1 and the others are 0. The vector e_A is defined as $e_A = \sum_{i\in A} e_i$.

Proposition 2.7. *Let ν be a finite newsvendor game with multinormal demands $(X_i)_{i\in I}$. Denoting by $m_X = (m_i)$ its expectation and by $\Sigma = [\sigma_{ij}]$ its covariance matrix, we have*

$$\begin{aligned}\nu(A) &= p\,m_X' e_A - k\,(e_A' \Sigma e_A)^{1/2} \qquad (2.6)\\ &= p\sum_{i\in A} m_i - k\left(\sum_{(i,j)\in A\times A} \sigma_{ij}\right)^{1/2},\end{aligned}$$

for all coalitions $A \in 2^I$. The measure

$$\mu(\{i\}) = p\,m_i - k\left(\sum_{(i,j)\in I\times I} \sigma_{ij}\right)^{-1/2} \sum_{j\in I} \sigma_{ij} \tag{2.7}$$

lies in core(ν)*.*

In addition, if the $(X_i)_{i\in I}$ are exchangeable, with variance σ^2 and correlation coefficient $-1/(d-1) \le \rho \le 1$, then we get the symmetric game

$$\nu(A) = p\,m|A| - k\sigma\sqrt{(1-\rho)\,|A| + \rho\,|A|^2}, \tag{2.8}$$

which is supermodular.

As a by-product, Proposition 2.7 shows that the sufficient condition for supermodularity used in Proposition 2.6 is not necessary.

Solutions (2.7) and (2.4) do not coincide in general. They do when the demands are exchangeable. In this case, solution (2.7) turns out be the barycenter of core(ν). In supermodular games the barycenter is necessarily the Shapley value (see Shapley (1971)).

We close this section with two results on the positivity of the game ν. This implies in turn that its core contains only nonnegative charges. The first result is obvious and holds for all p. Actually if $\pi = 0$, choosing $y^* = 0$ gives a null expected profit. The second result holds for sufficiently large prices.

We recall that a family X_α of nonnegative random variables is uniformly integrable if $\int_{X_\alpha \geq N} X_\alpha \, \mathrm{d}\mathbb{P} \to 0$ as $N \to \infty$ uniformly in α (see e.g. Dunford and Schwartz (1988)).

Proposition 2.8. *Let $\nu(A) = \Pi(D(A))$, with $D(A) \geq 0$ for all $A \in \mathcal{C}$.*

(i) *If $\pi = 0$, then $\nu(A) \geq 0$ for all p.*

(ii) *If the family of random variables $\{D(A)/\mathbb{E}[D(A)]\}$ is uniformly integrable when A varies in $\{A \in \mathcal{C} : D(A) \neq 0\}$, then $\nu(A) \geq 0$ for all p sufficiently large.*

3 Nonatomic newsvendor games

In this section we examine a nonatomic version of the newsvendor game, namely, a version where there is a continuum of players and each on of them has a negligible weight.

First we state a known result about nonatomic vector measures. We recall that a vector measure D is said to be *nonatomic* if $D(A) \neq 0$ implies the existence of some $B \in \mathcal{C}$, with $B \subseteq A$, such that $D(B) \neq 0$ and $D(A \setminus B) \neq 0$.

Proposition 3.1. *Assume that the demand vector measure D has Radom-Nikodym derivative, i.e., there exist $\lambda \in \mathrm{ca}^+(\mathcal{C})$ nonatomic and a λ-measurable, Bochner integrable function $\delta : I \to L^1(\Omega, \mathcal{F}, \mathbb{P})$, such that*

$$D(A) = \int_A \delta \, \mathrm{d}\lambda. \tag{3.1}$$

Then $A \mapsto D(A)$ is nonatomic.

Here $\int_A \delta \, \mathrm{d}\lambda$ denotes a Bochner integral. Some more details can be found in Section 7.

The main result of this section establishes that in the nonatomic setting, when the aggregate demand has a continuous distribution, the core of newsvendor games is a singleton.

Theorem 3.2. *Assume that in the newsvendor game $\nu(A) = \Pi(D(A))$ the demand vector measure satisfies* (3.1)*. Furthermore let the aggregate demand $D(I)$ have a continuous distribution. Then* $\mathrm{core}(\nu) \subset L^1(I, \mathcal{C}, \lambda)$ *is a singleton, given by* (2.4)*.*

It is well known (see Dunford and Schwartz (1988, Theorem 17, p. 198)) that a perfectly equivalent way of giving a λ-Bochner integrable function $\delta : I \to L^1(\Omega, \mathcal{F}, \mathbb{P})$ is to assign a $\lambda \otimes P$-integrable function $\overline{\delta} : I \times \Omega \to \mathbb{R}$ such that $\delta(i) = \overline{\delta}(i, .)$, λ-a.e., and $\left(\int \delta \, \mathrm{d}\lambda\right)(\omega) = \int \overline{\delta}(i, \omega) \, \mathrm{d}\lambda$, P-a.e. This allows to explicitly write the density of the unique element μ of the core as

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = (p + h) \int_{D(I) \leq y^*} \overline{\delta}(i, \omega) \, \mathrm{d}\mathbb{P} - \pi \int_{D(I) \geq y^*} \overline{\delta}(i, \omega) \, \mathrm{d}\mathbb{P}.$$

Theorem 3.2 has an important corollary. The uniqueness of the elements of the core in Theorem 3.2 requires the additional condition that the demand $A \to D(A)$ is representable by a Bochner integral. If the range of $D(A)$ lies in some L^p space with $p \in (1, \infty)$ (this is the case for instance when the demands are normally distributed) we can dispense with this assumption, as the reflexive spaces enjoy the Radon-Nydokim property according to a classical theorem by Phillips (Diestel and Uhl (1977)).

Corollary 3.3. *The results of Theorem 3.2 are true for any σ-additive, nonatomic, and bounded-variation vector measure $D : \mathcal{C} \to L^p(\Omega, \mathcal{F}, \mathbb{P})$, with $1 < p < \infty$.*

Nonatomicity poses restrictions on the distribution of the demands. For instance, the following proposition shows that the last claim of Proposition 2.6 is not effective in the nonatomic framework.

Proposition 3.4. *Assume that $A \mapsto D(A)$ is nonatomic and bounded-variation, and that the distributions of all $D(A) \neq 0$ are continous and of the same type, with finite variance. For each coalition A, for which $D(A) \neq 0$, there exist two disjoint subcoalitions $A_1, A_2 \subseteq A$, such that* $\mathrm{Cov}[D(A_1), D(A_2)] \neq 0$.

4 Markowitz games

As already mentioned in the introduction, the approach undertaken in this paper for studying the newsvendor game may be used for other models. We give a flavor of it by briefly examining a closely related class of games, that we call *Markovitz games*. A mean-variance approach à la Markowitz has been used in the nonstrategic newsvendor problem by Choi et al. (2001).

Let $A \to F(A) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a vector-valued measure defined on $(I, \mathcal{C})$. Given two positive weights α and β, define the Markovitz game

$$\nu(A) = \alpha \mathbb{E}[F(A)] - \beta \sqrt{\mathrm{Var}[F(A)]}, \tag{4.1}$$

which has an obvious interpretation assuming that each coalition A evaluates the random return $F(A)$ according to the mean-variance approach. In view of Proposition 2.6, newsvendor games are Markovitz games, if the demands $F(A)$ are all of the same type. If we replace the operator Π with the Markovitz operator $M(X) = \alpha \mathbb{E}[X] - \beta \sqrt{\mathrm{Var}[X]}$, *mutatis mutandis* all our arguments go through. The following proposition summarizes the main results.

Proposition 4.1. *Assume* $\mathrm{Var}[F(I)] \neq 0$ *and let F be bounded. The Markovitz game* (4.1) *is totally balanced. The charge*

$$\mu(A) = \alpha \mathbb{E}[F(A)] - \frac{\beta \,\mathrm{Cov}[F(A), F(I)]}{\sqrt{\mathrm{Var}[F(I)]}} \tag{4.2}$$

lies in $\mathrm{core}(\nu)$. *In addition, if $F(A)$ is σ-additive, nonatomic and of bounded variation, then* $\mathrm{core}(\nu) = \{\mu\}$.

The charge μ defined in (4.2) coincides with the measure defined in (2.7) whenever the Markowitz game is a newsvendor game with Gaussian demands.

5 Large newsvendor games

In view of the results of Section 3 on the uniqueness of elements in the core for nonatomic newsvendor games, here we study the shrinking of the core of large newsvendor games as the number of players increases. We restrict our analysis to Gaussian case studied in Proposition 2.7.

Let $I_d = \{1, \dots, d\}$ be the set of players. If μ is a measure on I_d, we denote by $\|\mu\| = |\mu|(I) = \sum_{i=1}^{d} |\mu(\{i\})|$ its total variation norm. For any game ν on I_d we define the diameter of the core of ν as

$$\Phi(\nu) = \max_{\mu,\mu' \in \text{core}(\nu)} \|\mu - \mu'\| .$$

We will consider a sequence $\{\nu_d\}_{d \in \mathbb{N}}$ of Gaussian games defined on $I_d = \{1, 2, \dots, d\}$. A standardization is necessary to get significant asymptotic results for the diameter of the core. In view of (2.6) first we notice that the diameter of the core of such games is invariant with respect to changes of the mean m_X of the multinormal demand. Therefore, without any loss of generality we will assume that $m_X = 0$. Second we impose the normalization condition $\nu_d(I_d) = -1$. Therefore we get the negative game

$$\nu_d(A) = -\left(\frac{e'_A \Sigma_d e_A}{e'_{I_d} \Sigma_d e_{I_d}}\right)^{1/2}, \tag{5.1}$$

where Σ_d is the covariance matrix of the multinormal demand.

Proposition 5.1. *Let the demand $(X_1, \dots, X_d)$ of the game ν_d have an exchangeable multinormal distribution with mean 0 and covariance matrix Σ_d, with elements $\sigma_{ii}(d) = \sigma^2$, $\sigma_{ij}(d) = \sigma^2\rho$, for $i \neq j$, where $-1/(d-1) \leq \rho \leq 1$. Then*

$$\lim_{d\to\infty} \Phi(\nu_d) = \begin{cases} 0 & \text{if } \rho > 0, \\ 2\sqrt{2} - 1 & \text{if } \rho = 0. \end{cases}$$

We see therefore that the diameter of the core shrinks only if and only if the correlation among the demands is positive. If ρ is allowed to vary with d and to assume negative values, then it is possible that the core does not shrink to a singleton.

Notice that, for fixed d, the diameter diverges to infinity as $\rho \to -1/(d-1)$ (which corresponds to the case where the aggregate demand is nonrandom).

In the next proposition we will consider a more general case where the correlation coefficients may differ, and we see that the core shrinks to a singleton, provided the covariances are all bounded away from zero, and the row sums of the covariance matrix are equal.

Proposition 5.2. *Let the demand $(X_1, \dots, X_d)$ of the game ν_d be multinormal with mean 0 and covariance matrix Σ_d, such that $\sigma_{ij}(d) \geq \eta > 0$ for all $i \neq j \in \{1, \dots, d\}$, and $\sum_{j=1}^{d} \sigma_{ij}(d) = k(d)$ for all $i \in \{1, \dots, d\}$. Then $\lim_{d\to\infty} \Phi(\nu_d) = 0$.*

6 The operator Π

Most of the proofs of the results stated in the previous sections rely on properties of the operator Π defined in (2.2). In this section we study such properties.

Proposition 6.1. *The operator* $\Pi : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *satisfies the following properties*

(a) Π *is positively homogeneous,*

(b) Π *is concave,*

(c) Π *is comonotonically additive, namely,* $\Pi(X+Y) = \Pi(X) + \Pi(Y)$, *whenever* X *and* Y *are comonotone.*

(d) $\Pi(X) = \mathbb{E}[\psi(X, y^*)]$, *where* y^* *is a* $(\pi+p)/(h+\pi+p)$*-quantile of the distribution of* X*.*

The following useful result establishes the Lipschitz continuity of Π.

Proposition 6.2. *For all* $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *we have*

$$|\Pi(X) - \Pi(Y)| \leq \gamma \|X - Y\|, \tag{6.1}$$

with $\gamma = \max\{h+p, \pi+2p\}$*.*

By (6.1), the operator Π is continuous. Hence, Π is a *support function* (see for instance Hörmander (1955)). For $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ we write $\langle X, Y \rangle = \int XY \, \mathrm{d}\mathbb{P} = \mathbb{E}[XY]$. By Hormander's theorem, there exists a unique weak*-compact and convex set $\Pi^* \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\Pi(X) = \min_{Y \in \Pi^*} \langle Y, X \rangle. \tag{6.2}$$

The next proposition, which plays an important role in the proof of many of our results, provides a complete characterization of the set Π^*.

Proposition 6.3. $Y \in \Pi^*$ *if and only if*

(i) $\int Y \, \mathrm{d}\mathbb{P} = p$,

(ii) $-\pi \leq Y \leq p + h \quad \mathbb{P}$*-a.s.*

In addition, if X *has continuous distribution, then there exists a unique* $\overline{Y} \in \Pi^*$ *such that*

$$\Pi(X) = \left\langle \overline{Y}, X \right\rangle, \tag{6.3}$$

whose expression is

$$\overline{Y} = (p+h)1_{\{X \leq y^*\}} - \pi 1_{\{X \geq y^*\}}. \tag{6.4}$$

where y^* *is a* $(\pi+p)/(h+\pi+p)$*-quantile of the distribution of* X*.*

Remark 6.4. The set of $Y \in \Pi^*$ such that $\Pi(X) = \langle Y, X \rangle$ can be studied also when the distribution of X is not continuous. As long as $\mathbb{P}(X = y^*) = 0$ it is clear from the proof of Proposition 6.3 that Π^* remains a singleton.

Remark 6.5. By convex analysis, it turns out that $\Pi^* = \partial\Pi(0)$, where $\partial\Pi(0)$ is the superdifferential of the concave function Π. Likewise, the set of elements $Y \in \Pi^*$ such that $\Pi(X) = \langle X, Y \rangle$ is nothing but $\partial\Pi(X)$. Therefore saying that $\partial\Pi(X)$ is a singleton, when X has continuous distribution, amounts to affirming that Π is Gateaux differentiable at X (some more details are discussed in the proof of Theorem 3.2).

7 Proofs

First we introduce some more notation and known results that will be used throughout the proofs.

Given a game ν, a coalition $N \in \mathcal{C}$ is ν-null, whenever $\nu(A \cup N) = \nu(A)$ for all $A \in \mathcal{C}$. For $\lambda \in \mathrm{ca}^+(\mathcal{C})$, a game ν is called λ-continuous if $\lambda(A) = 0$ implies that A is ν-null.

As well known, $\mathrm{ba}(\mathcal{C})$ is (isometrically isomorphic to) the norm dual of the space $B(\mathcal{C})$ of all bounded and measurable functions (endowed with the supnorm), the duality being $\langle f, \mu \rangle = \int f \, \mathrm{d}\mu$, with $f \in B(\mathcal{C})$ and $\mu \in \mathrm{ba}(\mathcal{C})$. We consider the relevant subset $B_1(\mathcal{C}) = \{f \in B(\mathcal{C}) : 0 \leq f \leq 1\}$, whose members are often called *ideal coalitions* (see Aumann and Shapley (1974)).

The set of ideal coalitions can be endowed with the na-*topology* due to Aumann and Shapley (1974), which is the coarsest topology for which all the functionals $f \mapsto \int f \, \mathrm{d}\mu$, with μ nonatomic, are continuous. By Lyapunov's theorem the indicator functions are na-dense in $B_1(\mathcal{C})$. Therefore, any game ν, when viewed as the function $1_A \mapsto \nu(A)$ defined on the space of indicator functions, has at most one na-continuous extension to $B_1(\mathcal{C})$. We use na-extensions of newsvendor games in our main Theorem 3.2.

The newsvendor game is defined in (2.3) through a vector-valued measure $D : \mathcal{C} \to X$, where X is a Banach space (specifically, $X = L^1(\Omega, \mathcal{F}, \mathbb{P})$, the space of integrable random variables defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$). Diestel and Uhl (1977) is the standard reference for them. We recall just some definitions.

An additive measure $F : \mathcal{C} \to X$ is bounded, if

$$\sup\{\|F(A)\| : A \in \mathcal{C}\} < \infty. \tag{7.1}$$

If F is countably additive then F is necessarily bounded (see Diestel and Uhl (1977, Cor. 19, p. 9)). We recall that we can associate with any $F : \mathcal{C} \to X$, its *semivariation* $\|F\|$ which is a scalar subadditive set-function (see Diestel and Uhl (1977, p. 2)). The measure F is said to be of bounded semivariation if $\|F\|(I) < +\infty$. Any countably additive vector measure F is of bounded semivariation (see Diestel and Uhl (1977, Proposition 11, p. 4)).

Given a vector measure $F : \mathcal{C} \to X$, the variation of F is the extended nonnegative measure $|F|$ defined as $|F|(A) = \sup_\pi \sum_{B \in \pi} \|F(B)\|$, where the supremum is taken over all partitions of A into a finite number of pairwise disjoint members of $\mathcal{C}$. If $|F|(I) < +\infty$, F is then called of *bounded variation,* a more stringent condition than bounded semivariation.

We recall that if $\mu \in \mathrm{ca}(\mathcal{C})$, a μ-measurable function $f : I \to X$, where X is a Banach space, is *Bochner integrable* if $\int \|f\| \, \mathrm{d}\mu < \infty$, where $\|f\|$ is the norm function: $\|f\|(i) = \|f(i)\|$ (see Diestel and Uhl (1977, p. 45)). Given a μ-Bochner integrable function $f : I \to X$, we can define the X-valued measure $F(A) = \int_A f \, \mathrm{d}\mu$, for $A \in \mathcal{C}$.

Section 6

We prove results of Section 6 first because they are used in the proofs of the other results.

Define the function $\varphi : \mathbb{R} \to \mathbb{R}$ as

$$\varphi(t) = \begin{cases} h \cdot t & \text{if} \quad t \geq 0, \\ -(\pi + p) \cdot t & \text{if} \quad t < 0, \end{cases} \tag{7.2}$$

and the operator $\Gamma : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ as

$$\Gamma(X) = \min_{y \in \mathbb{R}} \mathbb{E}[\varphi(y - X)]. \tag{7.3}$$

The following lemma relates the properties of the operators Γ and Π.

Lemma 7.1.

$$\Pi(X) = p\,\mathbb{E}[X] - \Gamma(X). \tag{7.4}$$

Proof. The result follows immediately from the relation $\psi(x, y) = px - \varphi(y - x)$. $\square$

Proof of Proposition 6.1. Müller et al. (2002) proved that the operator Γ is positively homogeneous, convex, and comonotonically additive. Then properties (a), (b), and (c) follow directly from Lemma 7.1. Concerning (d), Müller et al. (2002) proved that $\arg\min_y \mathbb{E}[\varphi(y - X)]$ is a $(\pi + p)/(h + \pi + p)$-quantile of the distribution of X, provided φ is given by (7.2). Therefore the result follows from (7.4). $\square$

Proof of Proposition 6.2. Let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Since

$$-\Pi(X) = \Gamma(X) - p\,\mathbb{E}[X],$$

and $\varphi(t) \leq \gamma_1 |t|$, with $\gamma_1 = \max\{h, \pi + p\}$, we have

$$\begin{aligned} -\Pi(X) &\leq \mathbb{E}[\varphi(-X)] - p\,\mathbb{E}[X] \\ &\leq \gamma_1 \|X\| + p\|X\| \\ &= \gamma \|X\|. \end{aligned}$$

By subadditivity of $-\Pi$ we have

$$\begin{aligned} -\Pi(X) &= -\Pi(X - Y + Y) \\ &\leq -\Pi(X - Y) - \Pi(Y) \\ &\leq \gamma \|X - Y\| - \Pi(Y), \end{aligned}$$

that is, $\Pi(Y) - \Pi(X) \leq \gamma \|X - Y\|$. Interchanging the role of X and Y we obtain (6.1). $\square$

Proof of Proposition 6.3. It is well known that

$$Y \in \Pi^* \text{ if and only if } \langle X, Y\rangle \geq \Pi(X) \text{ for all } X \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

First we prove that conditions (i) and (ii) are satisfied by any $Y \in \Pi^*$. Fix $Y \in \Pi^*$. We have

$$\Pi\left(X\right) = p\,\mathbb{E}[X] - \min_y \mathbb{E}\left[\varphi\left(y - X\right)\right] \leq \left\langle X, Y\right\rangle, \quad \text{for all } X \in L^1.$$

Hence

$$\min_y \mathbb{E}\left[\varphi\left(y - X\right)\right] \geq \left\langle p - Y, X\right\rangle,$$

which implies

$$\mathbb{E}\left[\varphi\left(y - X\right)\right] \geq \left\langle p - Y, X\right\rangle, \quad \text{for all } X \in L^1 \text{ and all } y \in \mathbb{R}. \tag{7.5}$$

Fix an element $X \in L^1$ and a scalar $y \neq 0$, and consider the parametrized family of random variables $X_\lambda = (\lambda - 1)\,y + X$ with $\lambda \in \mathbb{R}$. We obtain

$$\begin{aligned} \mathbb{E}\left[\varphi\left(\lambda y - X_\lambda\right)\right] &\geq \left\langle p - Y, X_\lambda\right\rangle \\ \mathbb{E}\left[\varphi\left(y - X\right)\right] &\geq \left(\lambda - 1\right) y \left\langle p - Y, 1\right\rangle + \left\langle p - Y, X\right\rangle \end{aligned}$$

which holds for all $\lambda \in \mathbb{R}$. Clearly, this implies that $\langle p - Y, 1\rangle = 0$ iff $\int Y \; \mathrm{d}\mathbb{P} = p$, namely, condition (i).

By setting $y = 0$ and replacing X with $-X$ in (7.5), we get $\mathbb{E}\left[\varphi\left(X\right)\right] \geq \langle Y - p, X\rangle$. In view of (7.2), we have

$$\int_{X \geq 0} X\left(h + p - Y\right)\; \mathrm{d}\mathbb{P} + \int_{X < 0} X\left(-Y - \pi\right)\; \mathrm{d}\mathbb{P} \geq 0$$

which must hold for all X. In particular, if X is nonnegative, we have $\int X\left(h + p - Y\right)\; \mathrm{d}\mathbb{P} \geq 0$ for all $X \geq 0$. Clearly, this implies that $Y \leq h + p$ almost surely. By using nonpositive random variables, we get $Y \geq -\pi$.

Conversely, we prove that any Y satisfying (i) and (ii) lies in Π^*. Consider the difference

$$\mathbb{E}[\varphi\left(y - X\right)] - \left\langle p - Y, X\right\rangle$$

where $X \in L^1$ and $y \in \mathbb{R}$. In view of condition (i), we have $\langle p - Y, y\rangle \equiv 0$, hence

$$\begin{aligned} \mathbb{E}[\varphi\left(y - X\right)] - \left\langle p - Y, X\right\rangle &= \int_\Omega \varphi\left(y - X\right)\; \mathrm{d}\mathbb{P} - \left\langle p - Y, X - y\right\rangle \\ &= \int_\Omega \varphi\left(Z\right)\; \mathrm{d}\mathbb{P} + \left\langle p - Y, Z\right\rangle, \end{aligned}$$

where $Z = y - X$. On the other hand,

$$\int_\Omega \varphi\left(Z\right)\; \mathrm{d}\mathbb{P} + \left\langle p - Y, Z\right\rangle = \int_{Z \geq 0} Z\left(h + p - Y\right)\; \mathrm{d}\mathbb{P} + \int_{Z < 0} Z\left(-Y - \pi\right)\; \mathrm{d}\mathbb{P} \geq 0,$$

where the two addenda are nonnegative by condition (ii). This proves that $\mathbb{E}[\varphi(y-X)] \geq \langle p-Y, X\rangle$ and, in turn, $\Pi(X) \leq \langle X, Y\rangle$ for all $X \in L^1$ and therefore that $Y \in \Pi^*$.

To prove the last statement, it suffices to calculate $\langle Y, X\rangle - \Pi(X)$. Let y^* be a $(\pi + p)/(h+\pi+p)$-quantile of the distribution of X. Since X has a continuous distribution, it follows that $\mathbb{P}(X = y^*) = 0$. Therefore,

$$\langle Y, X\rangle - \Pi(X) = \int_{X \leq y^*} (h+p-Y)(y^*-X) \ \mathrm{d}\mathbb{P} + \int_{X \geq y^*} (\pi+Y)(X-y^*) \ \mathrm{d}\mathbb{P} \geq 0,$$

for all $Y \in \Pi^*$. Hence, $\Pi(X) - \langle Y, X\rangle = 0$ if and only if $Y = \overline{Y}$ as defined in (6.4). □

Section 2

Lemma 7.2. *Assume that $A \to D(A)$ is σ-additive. Then:*

(i) *ν is bounded,*

(ii) *ν is continuous,*

(iii) $\operatorname{core}(\nu) \subset \operatorname{ca}(\mathcal{C})$,

(iv) *there exists a nonnegative real-valued countably additive measure $\overline{\lambda}$ on $(I, \mathcal{C})$ such that* $\operatorname{core}(\nu) \subset \operatorname{ca}(\mathcal{C}, \overline{\lambda}) \equiv L^1(I, \mathcal{C}, \overline{\lambda})$

Proof. (i) As D is σ-additive, D is bounded (see Diestel and Uhl (1977, Cor. 19, p. 9)). Namely, $\|D(A)\| \leq N$ for all $A \in \mathcal{C}$ and for some scalar N. In view of (6.1), we have $|\nu(A)| \leq \gamma N$ and ν is bounded.

(ii) If, for instance, $A_n \uparrow A$, then $\|D(A_n) - D(A)\| \to 0$. Proposition 6.2 implies that $\nu(A_n) \to \nu(A)$.

(iii) It is well known that the core of games, that are continuous at $\varnothing$ and at the grand coalition I, consists of countably additive measures (see Aumann and Shapley (1974, p. 173) or Marinacci and Montrucchio (2004, Proposition 4.4)).

(iv) By Bartle-Dunford-Schwartz Theorem (see Diestel and Uhl (1977, Cor. 6, p. 14)) there is a positive σ-additive measure $\overline{\lambda}$ such that $\overline{\lambda}(E) \to 0$ iff $\|D\|(E) \to 0$ where $\|D\|$ denotes the semivariation. In particular, we have the implications $\overline{\lambda}(E) = 0 \implies \|D\|(E) = 0 \implies \|D(E)\| = 0$. If $\mu \in \operatorname{core}(\nu)$ and $\overline{\lambda}(E) = 0$, then $\mu(E) \geq \Pi(D(E)) = 0$. Moreover,

$$\begin{aligned}\mu(E) &= \mu(I) - \mu(I \setminus E) \\ &\leq \mu(I) - \nu(I \setminus E) \\ &= \Pi(D(I)) - \Pi(D(I) - D(E)) \\ &= 0.\end{aligned}$$

Hence, $\mu(E) = 0$, which proves that μ is absolutely continuous with respect $\overline{\lambda}$.

□

Proof of Theorem 2.3. By Lemma 7.2(i) the game ν is bounded. Hence, to prove that it is totally balanced it suffices to check that, for any coalition $A \in \mathcal{C}$, if $\sum_i \lambda_i 1_{A_i} = 1_A$, where $\{A_i\}_i$ are finitely many coalitions, then $\sum_i \lambda_i \nu(A_i) \leq \nu(A)$. This classical result, due to Bondareva (1963) and Shapley (1967), holds also for positive infinite games, as proved by Schmeidler (1967) and Kannai (1969). An extension to bounded (not necessarily positive) games can be found in Marinacci and Montrucchio (2004, Theorem 4.1).

Given a simple function $\varphi = \sum_i \mu_i 1_{A_i}$, $D(\varphi)$ denotes $\sum_i \mu_i D(A_i)$. It is well known that the map $\varphi \to D(\varphi)$ is a linear operator on the space of simple functions. Hence, from $\sum_i \lambda_i 1_{A_i} = 1_A$, it follows

$$\begin{aligned}\nu(A) &= \Pi(D(A)) \\ &= \Pi\left(\sum_i \lambda_i D(A_i)\right) \\ &\geq \sum_i \lambda_i \Pi(D(A_i)) \\ &= \sum_i \lambda_i \nu(A_i).\end{aligned}$$

Consequently the game is totally balanced.

Define now the additive measure $\mu(A) = \langle \overline{Y}, D(A) \rangle$, $A \in \mathcal{C}$, where $\overline{Y} \in \Pi^*$ is given by $\overline{Y} = (p+h)1_{\{D \leq y^*\}} - \pi 1_{\{D \geq y^*\}}$.

In view of Proposition 6.3, we have $\nu(I) = \Pi(D(I)) = \langle \overline{Y}, D(I) \rangle = \mu(I)$. Moreover, for all $A \in \mathcal{C}$, $\nu(A) = \Pi(D(A)) \leq \langle \overline{Y}, D(A) \rangle = \mu(A)$. Therefore $\mu \in \text{core}(\nu)$. Clearly any charge in core(ν) is bounded, provided ν is bounded. □

Proof of Proposition 2.4. If all the random variables X_i have the same distributions, then

$$\mu_i = (p+h) \int_{D(I) \leq y^*} X_i \, d\mathbb{P} - \pi \int_{D(I) \geq y^*} X_i \, d\mathbb{P}$$

is independent of i. Suppose now that the X_i are exchangeable. Let $\xi : I_d \to I_d$ be any permutation. If $\lambda \in \text{core}(\nu)$, then $\lambda_\xi \in \text{core}(\nu)$, where $\lambda_\xi(A) = \lambda(\xi A)$. If λ is an extremal point of core(ν), then λ_ξ is, too. Hence μ agrees with $\sum_\xi (d!)^{-1} \lambda_\xi$, since $\sum_\xi (d!)^{-1} \lambda_\xi$ is uniform over I_d. Note that the elements λ_ξ are not necessarily all different but they can be regrouped into distinct classes of the same cardinality. Therefore $\mu = \sum_\xi (d!)^{-1} \lambda_\xi$ is the barycenter of core(ν). □

Proof of Proposition 2.5. Fix $A \in \mathcal{C}$. Set $X_1 = D(A)$ and $X_2 = D(I)$. By assumption, X_1 and X_2 are comonotone. Hence, $\Pi(X_1 + X_2) = \Pi(X_1) + \Pi(X_2)$. Set

$$\begin{aligned}\Pi_1^* &= \{Y \in \Pi^* : \langle Y, X_1 \rangle = \Pi(X_1)\} \\ \Pi_2^* &= \{Y \in \Pi^* : \langle Y, X_2 \rangle = \Pi(X_2)\} \\ \Pi_3^* &= \{Y \in \Pi^* : \langle Y, X_1 + X_2 \rangle = \Pi(X_1 + X_2)\}.\end{aligned}$$

Clearly $\Pi_1^* \cap \Pi_2^* = \Pi_3^*$. For, if $Y \in \Pi_1^* \cap \Pi_2^*$,

$$\begin{aligned}\Pi(X_1 + X_2) &= \Pi(X_1) + \Pi(X_2) = \langle Y, X_1 \rangle + \langle Y, X_2 \rangle \\ &= \langle Y, X_1 + X_2 \rangle\end{aligned}$$

and $Y \in \Pi_3^*$. The converse can be proved in a similar way.

As Π_3^* is nonempty, there exist some $\overline{Y} \in \Pi_1^* \cap \Pi_2^*$. The measure $\mu(E) = \langle \overline{Y}, D(E) \rangle$ lies in the core by construction. Further, $\mu(A) = \langle \overline{Y}, D(A) \rangle = \langle \overline{Y}, X_1 \rangle = \Pi(X_1) = \Pi(D(A)) = \nu(A)$. This proves that the game is exact. □

Proof of Proposition 2.6. Set $\sigma = \sqrt{\mathrm{Var}[D(A)]}$. The random variable $Z = \sigma^{-1}(D(A) - \mathbb{E}[D(A)])$ is of the same type as $D(A)$, and has zero mean and unit variance. Hence $D(A) = \sigma Z + \mathbb{E}[D(A)]$, which remains valid also when $\mathrm{Var}[D(A)] = 0$. Therefore

$$\begin{aligned}\nu(A) &= p\,\mathbb{E}[D(A)] - \Gamma(D(A)) \\ &= p\,\mathbb{E}[D(A)] - \Gamma(Z)\sigma \\ &= p\,\mathbb{E}[D(A)] - k\sqrt{\mathrm{Var}[D(A)]},\end{aligned}$$

where $k = \Gamma(Z)$. Clearly $\Gamma(Z) \le \gamma \|Z\|_1 \le \gamma \|Z\|_2 = \gamma$, by Hölder's inequality.

As far as the last statement is concerned, it suffices to observe that in this case the set function $A \to \mathrm{Var}[D(A)]$ is additive. Actually, $A \cap B = \varnothing$ implies

$$\begin{aligned}\mathrm{Var}[D(A \cup B)] &= \mathrm{Var}[D(A) + D(B)] \\ &= \mathrm{Var}[D(A)] + \mathrm{Var}[D(B)] + 2\,\mathrm{Cov}[D(A), D(B)] \\ &= \mathrm{Var}[D(A)] + \mathrm{Var}[D(B)].\end{aligned}$$

Since $t \to -t^{1/2}$ is convex, it is well known that $-k\sqrt{\mathrm{Var}[D(A)]}$ is supermodular. Therefore $\nu(A)$ is supermodular, too, since $A \mapsto p\,\mathbb{E}[D(A)]$ is additive. □

Proof of Proposition 2.7. Representation (2.6) follows easily from (2.5). If the demands are exchangeable, then $m_i \equiv m$ for all i, $\sigma_{ii} = \sigma^2$, and $\sigma_{ij} = \sigma^2 \rho$ for $i \neq j$. This leads to (2.8). The function $t \mapsto p\,mt - k\sqrt{(1-\rho)t + \rho t^2}$ is convex over $\mathbb{R}_+$, hence these games are supermodular, provided the X_i are exchangeable. We need to prove that (2.7) gives an element in the core. Observe that the game $\nu(A) = p\,m_X' e_A - k\,(e_A' \Sigma e_A)^{1/2}$ has a natural extension to $[0,1]^n$, given by the function $\widetilde{\nu}(x) = p\,m_X' x - k\,(x' \Sigma x)^{1/2}$, with $x \in [0,1]^n$ and where a coalitions A is identified with the extremal points e_A of $[0,1]^n$. The function $\widetilde{\nu}(x)$ is concave and linearly homogeneous. Furthermore $\widetilde{\nu}(x)$ is differentiable, consequently the derivative $\mathrm{D}\,\widetilde{\nu}(x)$ at the diagonal point $2^{-1}e$, with $e = e_I$, is a superdifferential. By a standard argument (see the proof of Theorem 3.2), the derivative belongs to core(ν). Straightforward computations lead to $\mathrm{D}\,\widetilde{\nu}(2^{-1}e) = p\,m_X - k\,(e'\Sigma e)^{-1/2}\,\Sigma e$, which is the desired result. □

Proof of Proposition 2.8. (i): The profit $\psi(x,y)$ in (2.1) can be written as $\psi(x,y) = p(x \wedge y) - \phi(y - x)$. Hence

$$\Pi(X) = \max_y p\,\mathbb{E}[X \wedge y] - \mathbb{E}[\phi(y - X)].$$

If $\pi = 0$, then we have $\mathbb{E}[\psi(X, 0) = p\,\mathbb{E}[X \wedge 0] - \mathbb{E}[\phi(-X)] = 0$, provided $X \geq 0$. We infer that $\Pi(X) \geq 0$, and (i) is proved.

(ii): Consider first a uniformly integrable family X_α with $\mathbb{E}[X_\alpha] = 1$. Straightforward algebra leads to

$$\begin{aligned}(p+\pi+h)&\mathbb{E}[\psi(X_\alpha, y)] \\ &= y\,\mathbb{P}(X_\alpha \geq y) + \frac{\pi}{p+h+\pi}\left(\int_{X_\alpha<y} X_\alpha \,\mathrm{d}\mathbb{P} - 1\right) \\ &\qquad + \frac{1}{p+h+\pi}\left((p+h)\int_{X_\alpha<y} X_\alpha \,\mathrm{d}\mathbb{P} - hy\right). \end{aligned} \tag{7.6}$$

Evaluate the right hand side of (7.6) at $\overline{y} = tp/h$, where t is a fixed element of $(0,1)$. Notice that $\overline{y} \to \infty$, as $p \to \infty$. The first two addenda of (7.6) go to zero as $p \to \infty$, uniformly in α (notice that $\int_{X_\alpha \geq y} X_\alpha \,\mathrm{d}\mathbb{P} \geq y\,\mathbb{P}(X_\alpha \geq y)$). The last addendum approaches $1 - t > 0$, as $p \to \infty$, uniformly in α. We deduce that $\Pi(X_\alpha) \geq \mathbb{E}[\psi(X_\alpha, \overline{y})] \geq 0$ for $p \geq p_0$ independently of α. The result is easily proved by using the family of uniformly integrable random variables $X_A = D(A)/\mathbb{E}[D(A)]$ and observing that

$$\nu(A) = \Pi(D(A)) = \mathbb{E}[D(A)]\;\Pi\left(\frac{D(A)}{\mathbb{E}[D(A)]}\right).$$

□

Section 3

The following technical lemmata are crucial to prove our main theorem. Notice that the functional Π is clearly weakly upper semicontinuous, since Π is concave, but it may fail to be weakly continuous over $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 7.3. *The function Π is weakly continuous when restricted to any relatively norm compact subset of $L^1(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. Let $K \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ be relatively norm compact and B^* be the unit ball of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

First we prove that the bilinear map $(X, Y) \to \langle X, Y\rangle$ is jointly continuous over $K \times B^*$ where K is endowed with the weak topology and B^* with the weak* topology.

With each $X \in K$, we associate the continuous function $\widehat{X} \in C(B^*)$, defined by $\widehat{X}(Y) = \langle X, Y\rangle$. Observe that the linear map $X \to \widehat{X}$ is an isometry. Actually, by Hanh-Banach theorem

$$\|X\|_{L_1} = \max_{Y \in B^*} |\langle X, Y\rangle| = \max_{Y \in B^*} \left|\widehat{X}(Y)\right| = \left\|\widehat{X}\right\|_\infty,$$

where $\|\cdot\|_\infty$ is the supnorm of $C(B^*)$. Since K is relatively norm compact, so is the image set $\widehat{K}$. By Ascoli-Arzelà's theorem, the family of functions $\widehat{X}$ in $\widehat{K}$ is equicontinuous. Fix $(X_0, Y_0) \in K \times B^*$. We have

$$\begin{aligned}|\langle X, Y\rangle - \langle X_0, Y_0\rangle| &= \left|\widehat{X}(Y) - \widehat{X}_0(Y_0)\right| \\ &\leq \left|\widehat{X}(Y) - \widehat{X}(Y_0)\right| + \left|\widehat{X}(Y_0) - \widehat{X}_0(Y_0)\right|.\end{aligned}$$

Given an $\varepsilon > 0$, there exists a weak* neighborhood $U(Y_0)$ of Y_0 such that $\left|\widehat{X}(Y) - \widehat{X}(Y_0)\right| \leq \varepsilon/2$ for all $X \in K$, due to the equicontinuity of $\widehat{K}$. Further, there is a weak neighborhood $U\left(\widehat{X}_0\right)$ of $\widehat{X}_0$ such that $X \in U\left(\widehat{X}_0\right) \cap K$ implies $\left|\widehat{X}(Y_0) - \widehat{X}_0(Y_0)\right| \leq \varepsilon/2$. Therefore, $|\langle X, Y\rangle - \langle X_0, Y_0\rangle| \leq \varepsilon$, for all $(X, Y) \in U\left(\widehat{X}_0\right) \times U(Y_0)$. This proves the continuity of the bilinear function $\langle \cdot, \cdot \rangle$.

Consider now our functional

$$\Pi(X) = \min_{Y \in \Pi^*} \langle X, Y \rangle, \qquad X \in K.$$

Clearly the property shown above still holds if one replaces $K \times B^*$ by $K \times B^*_\rho$, where $B^*_\rho = \rho B^*$ is the ball with radius ρ. Chose B^*_ρ such that $\Pi^* \subseteq B^*_\rho$. Therefore the function Π turns out to be weakly continuous over K by Berge's maximum theorem (see Aliprantis and Border (1994, Theorem 16.31)). □

Lemma 7.4. *Assume that in the newsvendor game $\nu(A) = \Pi(D(A))$ the demand vector measure satisfies* (3.1). *Then the game ν admits an* na*-continuous extension to the set of the ideal coalitions $B_1(\mathcal{C})$, which is concave and positively homogeneous.*

Proof. Consider the map $T : L^\infty(I, \mathcal{C}, \lambda) \to \mathbb{R}$, given by $f \to \int f \, \mathrm{d}D$. It is well defined, as $\lambda(A) = 0$ implies $D(A) = 0$. By a consequence of Bartle-Dunford-Schwartz's theorem, the map T is a weak*-to-weak continuous linear operator (see Diestel and Uhl (1977, Corollary 7, p. 14)). Restrict this operator to the subset

$$\mathcal{I}^\infty(\lambda) = \{f \in L^\infty(I, \mathcal{C}, \lambda) : 0 \leq f \leq 1, \ \lambda \text{-a.e.}\}. \tag{7.7}$$

Clearly, $T(\mathcal{I}^\infty(\lambda))$ is the extended range of the vector measure D. By Uhl's theorem (see Diestel and Uhl (1977, Theorem 10, p. 206)), the extended range in norm compact. By invoking Lemma 7.3, we deduce that the functional $f \to \Pi\left(\int f \, \mathrm{d}D\right)$ is weak* continuous over $\mathcal{I}^\infty(\lambda)$. Consider the space $B_1(\mathcal{C})$ of the ideal coalitions. As λ is nonatomic, the map $f \to [f]$ from $B_1(\mathcal{C})$ to $\mathcal{I}^\infty(\lambda)$ is na-to-weak* continuous. As a consequence, the functional $\nu^*(f) = \Pi\left(\int f \, \mathrm{d}D\right)$ is the na-continuous extension of the game ν to the ideal coalitions, and is concave and linearly homogeneous. □

Proof of Theorem 3.2. The proof is somewhat related to Einy et al. (1999, Theorem A), although they use dna-continuous extensions and here we exploit the na-extension $\nu^*(f)$ defined over $\mathcal{I}^\infty(\lambda)$, as defined in (7.7) of Lemma 7.4. We think of $\mathcal{I}^\infty(\lambda) \subset L^\infty(I, \mathcal{C}, \lambda)$, endowed with two topologies. The first one is the strong topology of the uniform convergence. The second one is the weak* topology. Consider the superdifferential $\partial\nu^*(2^{-1}1_I)$ of the concave function $\nu^* : \mathcal{I}^\infty(\lambda) \to \mathbb{R}$ at the point $2^{-1}1_I$. The elements of $\partial\nu^*(2^{-1}1_I)$ lie in $(L^\infty(I, \mathcal{C}, \lambda))' = \mathrm{ba}(I, \mathcal{C}, \lambda)$.

If $p \in \partial\nu^*(2^{-1}1_I)$, we have

$$\nu^*(f) \leq 2^{-1}\nu(I) + \langle p, f\rangle - 2^{-1}p(I)$$

for all $f \in \mathcal{I}^{\infty}(\lambda)$. Setting $f = 0$ and $f = 1_I$, we deduce that $p(I) = \nu(I)$. Setting $f = 1_A$, for any coalition A, we obtain $p(A) \geq \nu(A)$. Consequently, $p \in \text{core}(\nu)$. By Lemma 7.2, $p \in L^1(I, \mathcal{C}, \lambda)$. Hence, $\partial\nu^*(2^{-1}1_I) \subseteq \text{core}(\nu) \subset L^1(I, \mathcal{C}, \lambda)$.

We now prove that $\text{core}(\nu) = \partial\nu^*(2^{-1}1_I)$. Let $m \in \text{core}(\nu)$. We know that $m \in L^1(I, \mathcal{C}, \lambda)$ and $m(A) \geq \nu(A)$ for all $A \in \mathcal{C}$. Namely, $\langle m, 1_A \rangle \geq \nu^*(1_A)$. Both ν^* and $\langle m, . \rangle$ are w^*-continuous. By Lyapunov theorem (see Kingman and Robertson (1968)), the indicator functions are weak* dense. Hence $\langle m, f \rangle \leq \nu^*(f)$ holds for all $f \in \mathcal{I}^{\infty}(\lambda)$. Therefore

$$\nu^*(f) \leq \nu^*\left(2^{-1}1_I\right) + \left\langle m, f - 2^{-1}1_I \right\rangle$$

and $m \in \partial\nu^*(2^{-1}1_I)$.

As a last step, we prove that $\partial\nu^*(2^{-1}1_I)$ is a singleton, namely that ν^* is (Gateaux) differentiable at $2^{-1}1_I$. First observe that the superdifferential of the concave functional Π is given by

$$\partial\Pi(X) = \{Y \in \Pi^* : \langle Y, X \rangle = \Pi(X)\}.$$

Consequently, if $D(I)$ has a continuous distribution, then Proposition 6.3 implies $\partial\Pi(2^{-1}1_I) = \partial\Pi(1_I) = \{\overline{Y}\}$, where $\overline{Y}$ is given by (6.4).

Now compute the directional derivative of ν^* at $2^{-1}1_I$, that is

$$\mathrm{D}\,\nu^*\left(2^{-1}1_I; h\right) = \lim_{t \to 0^+} \frac{\nu^*(2^{-1}1_I + th) - \nu^*(2^{-1}1_I)}{t}$$

with $h \in L^{\infty}(I, \mathcal{C}, \lambda)$. Denoting $Tf = \int f \, \mathrm{d}D$ and T^* its transpose, we obtain

$$\begin{aligned} \mathrm{D}\,\nu^*\left(2^{-1}1_I; h\right) &= \lim_{t \to 0^+} \frac{\Pi(D(I) + 2tTh) - \Pi(D(I))}{2t} \\ &= \left\langle \overline{Y}, Th \right\rangle = \left\langle T^*\overline{Y}, h \right\rangle, \end{aligned}$$

where $\overline{Y} = \mathrm{D}\,\Pi(D(I))$. Since the directional derivative is linear, ν^* is differentiable at $2^{-1}1_I$. As a consequence, $\partial\nu^*(2^{-1}1_I) = \text{core}(\nu)$ is a singleton. In view of Theorem 2.3 the element in the core is given by (2.4). □

Proof of Corollary 3.3. Notice the following facts:

(i) If we restrict the operator Π to some $L^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p \in (1, \infty)$, we get again the representation

$$\Pi(X) = \min_{Y \in \Pi^*} \langle Y, X \rangle$$

for all $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$, where Π^* is characterized by Proposition 6.3. This can be quickly checked, as Π^* is weakly compact in any $L^p(\Omega, \mathcal{F}, \mathbb{P})$ with $1 \leq p < \infty$. For a direct proof it suffices to remark that the proof of Proposition 6.3 remains unchanged if we set the restriction $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$. This representation implies that, provided $D(I)$ has continuous distribution, the equality $\partial\Pi(2^{-1}D(I)) = \left\{\overline{Y}\right\}$ holds for the functionals $\Pi : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, as well.

(ii) Lemma 7.2 is unchanged. Therefore $\Pi : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is weakly continuous over a relatively norm-compact subset of $L^p(\Omega, \mathcal{F}, \mathbb{P})$.

(iii) If $D : \mathcal{C} \to L^p(\Omega, \mathcal{F}, \mathbb{P})$ is σ-additive and of bounded variation, set $\lambda = |D|$, which is the bounded variation measure. Clearly, D is $|D|$-continuous. By Phillips's theorem (see Diestel and Uhl (1977, p. 76))) any $L^p(\Omega, \mathcal{F}, \mathbb{P})$, with $p \in (1, \infty)$ has the Radon-Nikodym property. Consequently, the measure D admits the representation $D(A) = \int_A \delta d\lambda$. Notice that $\lambda = |D|$ is non-atomic, as long as D is.

In view of these facts, the proof goes through like the one of Theorem 3.2.

□

Proof of Proposition 3.4. Assume by contradiction the existence of a coalition A, such that $D(A) \neq 0$ and for all $A_1, A_2 \subseteq A$, with $A_1 \cap A_2 = \varnothing$, $D(A_1)$ and $D(A_2)$ are uncorrelated. Consider the restriction ν_A to the coalition A of the game ν. ν_A turns out to be nonatomic and of bounded variations. As D takes values on L^2, we can invoke Corollary 3.3 and so core(ν_A) is a singleton. On the other hand, by Proposition 2.6, ν_A is supermodular. If the core of a supermodular game is a singleton, then the game is additive. In view of Proposition 2.6, $\nu_A(B) = p\,\mathbb{E}[D(B)] - k\sqrt{\mathrm{Var}[D(B)]}$, for all $B \subseteq A$. Taking any two coalitions $B \subseteq A$ and $A \setminus B$, we have $\sqrt{\mathrm{Var}[D(A)]} = \sqrt{\mathrm{Var}[D(B)]} + \sqrt{\mathrm{Var}[D(A \setminus B)]}$, which implies either $\mathrm{Var}[D(B)] = 0$ or $\mathrm{Var}[D(A \setminus B)] = 0$. Namely, either $D(B) = 0$ or $D(A \setminus B) = 0$. The coalition A would be an atom, a contradiction. □

Section 4

Proof of Proposition 4.1. It is easy to see that the operator $M(X) = \alpha\mathbb{E}[X] - \beta\sqrt{\mathrm{Var}[X]}$ is the support functional

$$M(X) = \min_{Y \in M^*} \langle Y, X \rangle$$

where $M^* = \{Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}) : \mathbb{E}[Y] = \alpha \text{ and } \mathrm{Var}[Y] \leq \beta^2\}$. For any X such that $\mathrm{Var}[X] \neq 0$ there is a unique minimizer $\overline{Y} \in \arg\min_{Y \in M^*} \langle Y, X \rangle$ given by

$$\overline{Y} = \alpha - \beta \frac{X - \mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}}.$$

The same arguments used in the proof of Theorem 2.3 lead to the first two claims. Clearly the core is singleton as a consequence of Corollary 3.3. □

Section 5

Proof of Proposition 5.1. In view of (5.1) we have

$$\nu_d(A) = -\left(\frac{a^2\rho + a(1-\rho)}{d^2\rho + d(1-\rho)}\right)^{1/2}, \tag{7.8}$$

where $|A| = a$. By Proposition 2.7 this game is supermodular. The imputations which are the extreme points of core(ν_d) can be easily computed by using Shapley's theorem (Shapley (1971)). We recall that for a finite supermodular game the extreme points of the core are one-to-one with the so-called marginal worth associated with the maximal chains. More specifically, if $\varnothing = C_0 \subset C_1 \subset \cdots \subset C_{d-1} \subset C_d = I_d$ is a maximal chain, then there exists one and only one measure μ that replicates the game ν_d on this chain, namely, such that $\mu(C_i) = \nu_d(C_i)$. Clearly, for all $j \in I_d$, there is an index i such that $\{j\} = C_i \setminus C_{i-1}$. Hence $\mu_j := \mu(\{j\}) = \nu_d(C_i) - \nu_d(C_{i-1})$. By taking the chain $\overline{C}_i = \{1, 2, \ldots, i\}$ we get the extreme point in the core $\mu_i = \nu_d\left(\overline{C}_i\right) - \nu\left(\overline{C}_{i-1}\right)$. Clearly μ_i is increasing in i. Further, as the game is symmetric, all the extreme points are obtained by permuting the sequence (μ_i).

Given an element ξ in the set Ξ of all permutations of $\{1, 2, \ldots, d\}$, denote by μ_ξ any such measure obtained by permuting the sequence (μ_i). As a consequence of Bauer maximum principle we have

$$\Phi(\nu_d) = \max_{\xi \in \Xi} \|\mu - \mu_\xi\|.$$

Assume first that the number d of players is even. It is easy to see that the maximum is achieved by taking $\mu_\xi = (\mu_d, \mu_{d-1}, \ldots, \mu_1)$. Hence,

$$\begin{aligned}\Phi(\nu_d) &= 2\sum_{i=1}^{d/2}(\mu_{d-i} - \mu_i) \\ &= 2\mu(I_d \setminus \overline{C}_{d/2}) - 2\mu(\overline{C}_{d/2}) \\ &= 2[\mu(I_d) - 2\mu(\overline{C}_{d/2})] \\ &= -2[1 + 2\nu(\overline{C}_{d/2})].\end{aligned} \tag{7.9}$$

Using (7.8), we obtain

$$\frac{1}{2}\Phi(\nu_d) = \left(\frac{d^2\rho + 2d(1-\rho)}{d^2\rho + d(1-\rho)}\right)^{1/2} - 1.$$

The diameter of the core is decreasing in ρ. Its value is zero when $\rho = 1$, which corresponds to the case of comonotone demands. It is equal to $2\sqrt{2} - 2$ for all d, when $\rho = 0$. If ρ is fixed and $\rho > 0$, then $\Phi(\nu_d) \to 0$, as $d \to \infty$. Actually, we have

$$\Phi(\nu_d) = \frac{1-\rho}{\rho}\left[d^{-1} - \frac{5}{4}\left(\frac{1-\rho}{\rho}\right)d^{-2}\right] + o\left(d^{-2}\right)$$

which shows that the diameter of the core shrinks with rate $1/d$.

If the number of players is odd, we get a similar result, where (7.9) is replaced by

$$\Phi(\nu_d) = -2[1 + \nu_d(\overline{C}_{(d-1)/2}) + \nu_d(\overline{C}_{(d+1)/2})].$$

□

Proof of Proposition 5.2. In view of (2.6) the games ν_d are defined as

$$\nu_d(A) = -\left(\frac{e'_A \Sigma_d e_A}{e'_{I_d} \Sigma_d e_{I_d}}\right)^{1/2}$$

for $A \subseteq I_d$. We construct a new sequence of games $\widetilde{\nu}_d$ having multinormal exchangeable demands with mean 0 and covariance matrix $\widetilde{\Sigma}_d$, with elements $\widetilde{\sigma}_{ii}(d) = \sum_{j \in I} \sigma_{ij}(d) - (d-1)\eta \equiv \widetilde{\sigma}^2$, for all $i \in \{1, \ldots, d\}$, and $\widetilde{\sigma}_{ij}(d) = \eta$ for $i \neq j$.

Therefore

$$\sum_{(i,j) \in i_d \times I_d} \widetilde{\sigma}_{ij}(d) = \sum_{(i,j) \in i_d \times I_d} \sigma_{ij}(d),$$

and, for all A,

$$\sum_{(i,j) \in A \times A} \widetilde{\sigma}_{ij}(d) \geq \sum_{(i,j) \in A \times A} \sigma_{ij}(d).$$

Therefore $\nu_d(A) \geq \widetilde{\nu}_d(A)$ and $\nu_d(I_d) = \widetilde{\nu}_d(I_d) = -1$. Clearly, $\operatorname{core}(\nu_d) \subseteq \operatorname{core}(\widetilde{\nu}_d)$.

The game $\widetilde{\nu}_d$ is of the type examined in Proposition 5.1. Since $\operatorname{core}(\widetilde{\nu}_d)$ shrinks to a singleton, so does $\operatorname{core}(\nu_d)$. □

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