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1/N and Long Run Optimal Portfolios: Results for Mixed Asset Menus.*

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Abstract

Recent research [e.g., DeMiguel, Garlappi and Uppal, (2009a), *Rev. Fin. Studies*] has cast doubts on the out-of-sample performance of optimizing portfolio strategies relative to a naive, equally-weighted ones. However, most of the existing results concern the simple case in which an investor has a one-month horizon and mean-variance preferences. In this paper, we examine whether this finding holds for longer investment horizons, when the asset menu includes bonds and real estate beyond stocks and cash, and when the investor is characterized by constant relative risk aversion preferences which are not locally mean-variance for long horizons. Our experiments indicates that power utility investors with horizons of one year and longer would have on average benefited, *ex-post*, from an optimizing strategy that exploits simple linear predictability in asset returns over the period January 1995 - December 2007. This result is insensitive to the degree of risk aversion, to the number of predictors being included in the forecasting model, and to the deduction of transaction costs from measured portfolio performance.

JEL Classification Codes: G11, L85.

Keywords: equally weighted portfolios, long investment horizon, real-time strategic asset allocation, public real estate vehicles, ex post performance, predictability, parameter uncertainty.

1. **Introduction**

Individual investors tend to allocate their pension wealth across different asset classes by equally weighting them (see, e.g., Benartzi and Thaler, 2001, Huberman and Jiang, 2006, Liang and Weisbenner, 2002). This

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behavior does not align with the prescriptions of optimal asset allocation models, which suggest attributing more weight to those assets that contribute to a higher expected return-to-risk ratio. Yet, this observed behavior might still be consistent with higher levels of *ex-post* utility if the typical implications of the portfolio choice literature contain biases and suffer from severe misspecifications, i.e., investors may be savvy enough to recognize that ignoring prescriptions that may be optimal only in an *ex-ante* sense, may reward them with higher *ex-post* welfare levels. A number of papers, starting with Jorion's (1985) pioneering study, document that the *out-of-sample* performance of *ex-ante* optimal portfolios may be worse than that of simpler strategies such as equally weighting all available asset classes (a strategy that we call here "$1/N$"). Recently, DeMiguel, Garlappi, and Uppal (2009a, henceforth DGU) have reported that the $1/N$ strategy consistently outperforms almost every optimizing model they scrutinize for simple stock portfolio selection problems.

However, the results in DGU cannot be brought to bear on the described individual behavior in allocating resources with retirement planning goals for several reasons. First, the results in DGU refer to a one-month investment horizon only, whereas pension-oriented portfolios of individual investors are likely to be targeted to much longer horizons. As a result of their very short-term focus, DGU consider T-Bills as a riskless asset class, which is inappropriate from the point of view of a longer-term investor who ignores the future level of the short term rate and suffers from obvious inflation risks (see e.g., Brennan and Xia, 2001, and Campbell and Viceira, 2001). Moreover, DGU only briefly touch upon the possibility of time-varying, predictable risk premia which is a fundamental issue in long-term portfolio choice problems (see Campbell and Viceira, 2002, for a review of the main issues). Second, DGU's asset menu is narrower than the one usually available to individuals, as pension plans members can invest in bonds and – at least since the early 1990s, with the increasing availability of publicly traded real estate vehicles (REITs) – in real estate, besides bills and equity. More generally, we still do not know whether the startling performance of the equally weighted strategy in DGU extends to long term portfolio problems with multiple, heterogeneous risky assets. This is the question we address in the paper. We tackle this important question by comparing the *ex-post* performance of $1/N$ to that of optimal portfolios that realistically include several assets with changing risk premia, and among them public real estate (i.e., equity REITs), allowing the investors' horizon to range from one to sixty months. The answer to our main question, while relevant to portfolio choice in general, may suggest a rationale for the puzzling investors' obsession over simplistic, equally-weighted portfolio strategies that we have cited above.

Our experiment uses standard US monthly data on returns on stocks, eREITs, long-term government bonds, and T-bills for the sample period 1972-2007. Our main empirical finding is that a constant relative risk aversion investor with an horizon of one year or more obtains a higher realized, ex-post welfare from portfolio strategies that are derived from optimization that accounts for predictability of real returns. This means that in mixed asset menus that include public real estate investment vehicles, the application of explicit optimized portfolio strategies pays off over time in actual, ex-post terms. Such superior performance holds with respect to both naive strategies avoiding all calculations and portfolios of intermediate complexity, deriving from an optimization of the long-run risk-return trade-off which ignores predictability. Therefore the observed tendency of investors to equally weight all available assets in their retirement plans is sub-optimal and may represent puzzling evidence of irrationality in decision-making under conditions of uncertainty. This conclusion is robust to changes in parameters affecting optimal portfolio choices, such as the coefficient of risk aversion imputed to the investor, the econometric model used to capture any evidence of predictable risk premia, and the inclusion of plausible levels of transaction costs that penalize trading induced by the attempt to time market conditions.

Another finding of our paper concerns short-term portfolio choices. Our results confirm earlier ones (in particular, DGU's) when we compare $1/N$ to the optimal portfolios obtained under constant risk premia: equal weighting provides higher ex-post, realized welfare, especially to moderately risk-averse investors.[1] It is thus the prediction of the dynamics of real returns over time that may allow an investor – with horizons exceeding one year – to achieve a higher, ex-post realized utility. Thus, households preference for equal weighting appears irrational unless they – or their advisors – are unable to predict the risk premium.

Our paper is part of a growing literature that emphasizes how portfolio decisions and resulting welfare may strongly depend on the investor's horizon when predictability is taken into account (see Barberis, 2000, and Brandt, 1999, for seminal papers on the empirical effects of predictability; and MacKinnon and Al-Zaman, 2009, for evidence on mixed asset menus). While with short-horizons it may be irrelevant whether predictability is modeled, considering longer horizons affects the composition of optimal portfolios if a changing opportunity set and/or parameter uncertainty are accounted for. Indeed, the annualized conditional means of asset returns are constant in the absence of predictability, whereas they can be increasing or decreasing with the investment horizon depending on the intertemporal features of the return generating process. Similarly, the conditional variances and covariances of asset returns will depend on the investment horizon, as a result of the presence of correlation in the shocks to the vector autoregressive relationships that are used to capture linear predictability patterns. However, these forecasts of the relevant conditional moments of asset returns will also be subject to increasing uncertainty as the investment horizon grows. Accordingly, a power utility investor changes her portfolio as her investment horizon increases when she is aware of the growing uncertainty of her forecasts. In this paper we therefore investigate the *out-of-sample* realized performance achieved by optimal portfolios that alternatively consider return predictability or IID (independently and identically distributed) real asset returns, and when the investor alternatively overlooks or accounts for parameter uncertainty using Bayesian methods (as in Barberis, 2000).

Besides our key result, we have three empirical findings that represent novel contributions to our understanding of the ex-post realized performance of dynamic asset allocation methods in mixed asset portfolios. First, when a Bayesian investor with no access to public real estate investments considers parameter uncertainty without predictability, her realized utility is always lower than for a $1/N$ investor, no matter what her horizon is, when she overlooks real estate. These findings extend DGU's results to a longer horizon, supporting the *ex post* rationality of individual investors in their use of $1/N$, under the counter-factual assumption that their choice menus exclude public real estate. However this pattern appears to be the exception rather

[1] With monthly data, the optimal portfolio for a 1-month horizon power utility investor is likely to be well approximated by a simpler mean-variance objective, as in DGU. Equivalently, over short-horizons, power utility is locally mean-variance.

than the rule. If we allow for real estate in the asset menu, $1/N$ becomes dominated for horizons equal to two years or longer, even if the investor continues to overlook predictability. For instance, the CER (certainty equivalent return) *differential* at $T = 60$ months is equal to -0.32%, shifting to $+0.20\%$ – hence in favour of optimizing models – when eREITs are included.[2]

Second, we assess the interplay of longer investment horizons and predictability, allowing for up to four predictors – the inflation rate, dividend-price ratio, riskless term premium and default spread – to forecast future returns along with their lagged real returns, in a typical vector autoregression à la Campbell et al. (2003). Equal weighting turns out to be a losing long-horizon portfolio strategy, even if we omit real estate from the asset menu and even if the investor overlooks parameter uncertainty. For instance, allowing all predictors to enter our VAR, produces an optimizing strategy that beats $1/N$ for horizons equal or longer than one year. In this case, the welfare (CER) differential for a Bayesian investor at $T = 60$ shifts to a hefty $+1.16\%$. This result is obtained even if allowing for predictability increases the number of parameters to be estimated, thus making optimizing models less likely to beat $1/N$. We also estimate parsimonious forecasting models with one predictor at a time, so as to reduce the potential for estimation error. Interestingly, the model with all predictors usually delivers higher investor's welfare than the more parsimonious ones. Moreover, an investor with horizons equal or longer than one year is better off than a $1/N$ investor even if she uses the worst predictive model, which is frequently the one based on the default spread. For instance, the increase in CER when using the best (worst) predictive model instead of $1/N$ is equal to $+0.07\%$ ($+0.02\%$) for a Classical investor with a five year horizon, and to $+0.17\%$ ($+0.07\%$) for a Bayesian investor.

Of course, the roots of these findings can be found in the structure of optimized portfolios and in its differences vs. the equally-weighted benchmark. For instance, a key driving force is that (as we show in Section 3.3) in our data set, long-term government bonds are riskier for an investor with longer horizon than for an investor with a short term horizon. For a standard level of risk aversion, a short term investor has an optimal portfolio that contains on average 26% of bonds, whereas a longer horizon one would rather hold almost no bonds (3%). Ex post, it turns out that this strategy would have paid out, at least in our sample. Such differences in desired holdings derive from differences in conditional moments of short and longer term returns. Bonds are a good hedge against shocks to real estate returns for a one-period investor, the correlation with eREITs – the asset with the highest Sharpe ratio – is a modest 0.14. However, this increases to 0.41 for a two-year horizon, while T-Bills remain a good hedge of real estate risk with correlation coefficient below 0.07. Ex post, this improves all moments of the return distribution of optimized portfolios.

Third, we find that the inclusion of real estate in the asset menu plays a key role in our main finding. Benartzi and Thaler (2001) argue that the fund menu offered by pension plans exert a strong influence on the assets that participants end up owning. We therefore consider two different asset menus, respectively including

[2] These results and most of the ones that are discussed in this Introduction refer to a power utility investor with coefficient of relative risk aversion of 5. This result echoes the general awarenes in the literature that the ex post performance of optimal portfolios improves when they are based on forecasts that account for estimation errors (see e.g., Jorion, 1985). Moreover, it has already been observed that public real estate investment vehicles enhance the *ex-post* performance of optimal portfolios when parameter uncertainty is taken into account (see, e.g., Fugazza, Guidolin, and Nicodano, 2009).

and excluding equity REITs, because several pension plans still fail to offer securitized real estate. We can thus assess whether the *out-of-sample* performance of naive diversification relative to optimizing strategies changes with the introduction of real estate in the asset menu. In principle, following DGU, increasing the number of assets should widen the performance gap between $1/N$ and optimal portfolios However, this can be reversed if the additional asset has a high Sharpe ratio, as eREITs do over our sample period. Empirically, we find that it is especially with asset menus including real estate that naive $1/N$ has the potential to reduce the ex-post realized welfare of risk-averse investors.

Our paper relates to an extensive literature in empirical finance that examines methods of optimal asset allocation and their implications (see Brandt, 2004, for a survey). Since an exhaustive review would take too much space, let us mention only a few closely related papers. Diris et al. (2008) find that the equally weighted strategy may be dominated by fully dynamic strategies over a five-year time horizon – when the Bayesian investor has an informative prior.[3] Our study focusses on simpler strategies that individual investors appear to adopt in practice, thus comparing $1/N$ with buy-and-hold strategies. Our focus on buy-and-hold derives from the observed inertia in rebalancing decisions: investors appear to never rebalance their initial allocations in pension schemes (see e.g., Choi et al., 2004). Despite its simplicity and our use of uninformative priors, this strategy may still dominate equal weighting. Tu and Zhou (2008a) modify the standard mean-variance framework to account for estimation errors and show this has the potential to produce strategies that perform as well as the $1/N$ strategy under a variety of frameworks, such as multi-factor models with and without mispricing. Kirby and Ostdiek (2009) stress that DGU's results rests on the implicit use of unusually high expected return targets in the optimization, which greatly magnifies the impact of estimation errors. Within a static framework, Kirby and Ostdiek find that mean-variance efficient portfolios often perform better than naïve diversification if transactions costs are negligible, provided the investor has adequate market timing skills. Also our paper magnifies the effects of predictability and hence of market timing, but it emphasizes instead the effects of long horizons on realized performances from optimizing strategies, while generally recovering DGU's results for short horizons and under standard uninformative priors. Fugazza, Guidolin, and Nicodano (2009) investigate the ex-post performance of time-diversification (i.e., deriving from predictability captured by a VAR model) and across asset diversification using a 1972-2004 sample that includes real estate. They find that time diversification is less important than static diversification across assets. Differently from their paper, we use a longer sample period, a wider range of predictability models and put special emphasis on whether, how, and why optimizing strategies that exploit time or static diversification (or both) may outperform naive equally-weighted strategies.

The rest of the paper is organized as follows. Section 2 presents details on the research design. It deals with the structure of the buy-and-hold optimizing problems, the econometric models describing the predictability patterns for asset returns, the difference between classical and Bayesian strategies, and explains the way in which *ex-post* performances are computed. Sections 3-5 report the results of the empirical analysis. Section

[3] Tu and Zhou (2008b) explore the related idea that economic objectives may be reflected by Bayesian priors and find that investment performance under the objective-based priors can be significantly improved.

3 describes the data set and presents estimates of a full-scale VAR(1) that includes all predictors and the implied forecasts of risk premia, volatilities and covariances. Section 4 reports a number of summary statistics concerning the properties and realized performances for the full range of 478 different strategies tested in the paper. Section 5 performs a number of robustness checks to reassure a Reader that our key finding do not depend on specific choices concerning preferences, predictors, or details of the optimization problem (e.g., transaction costs). Section 6 discusses the implications of our main results and concludes.

2. **Research Design**

This Section presents the basic blocks of our research strategy. Section 2.1 starts by introducing the portfolio problem solved throughout for alternative asset menus, preferences, econometric models of predictability and of parameter uncertainty. Sections 2.2 and 2.3 elaborate on this last aspect and briefly explain what is the difference between classical and Bayesian approaches to portfolio selection. Section 2.4 lists the optimizing portfolio strategies that we examine. Section 2.5 contrasts such strategies with the simple, equally weighted criterion. Section 2.6 concludes by describing the recursive structure of the portfolio exercise and by reviewing a few criteria for measuring and comparing portfolio performance.

2.1. **The Buy-and-Hold Portfolio Problem**

Let N be the number of asset classes available to the investor. For every strategy, we consider the asset menu usually analyzed in the empirical finance literature (see e.g., Campbell et al., 2003), i.e., stocks, long-term government bonds, and T-bills ($N = 3$), as well as a more realistic asset menu that includes public real estate, in the form of equity REITs ($N = 4$). For instance, in the case of $N = 4$, the investor's terminal real wealth, letting her initial wealth W_t be unity, is given by:

$$W_{t+T} = \omega^s_{t,T} \exp(R^s_{t,T}) + \omega^b_{t,T} \exp(R^b_{t,T}) + \omega^r_{t,T} \exp(R^r_{t,T}) + (1 - \omega^s_{t,T} - \omega^b_{t,T} - \omega^r_{t,T}) \exp(R^f_{t,T}), \qquad (1)$$

where $\omega^j_{t,T}$ is the fraction of wealth invested in the j-th asset class when the horizon is $T \geq 1$ months, and $R^j_{t,T}$ denotes the continuously compounded *real* cumulative returns on asset $j = 1, ..., N$ between t and T:

$$R^j_{t,T} \equiv \sum_{k=1}^{T} r^j_{t+k}, \quad j = s, b, r \quad R^f_{t,T} \equiv \sum_{k=1}^{T} r^f_{t+k}, \qquad (2)$$

with r^s_t, r^b_t, r^r_t being the *real* continuously compounded returns on stocks, bonds and public real estate, and r^f_t is the *real* real return on T-bills. By construction, for all times t, $\sum_{j=1}^{N} \omega^j_t = 1$ which explains the residual structure of the weight assigned to T-bills.

We allow for investment horizons ranging from $T = 1$ to $T = 60$ months. Given the observed investor's inertia, we focus on buy-and-hold strategies – i.e. the investor determines the asset allocation at the beginning of the investment horizon and never rebalances afterwards.[4] Her final expected utility is

$$\max_{\boldsymbol{\omega}_{t,T}} E_t \left[\frac{W_{t+T}^{1-\gamma}}{1-\gamma} \right] \quad \gamma > 1 \tag{3}$$

where $\gamma = 2,\ 5,\ 10$ is her coefficient of constant relative risk aversion. The maximization is solved subject to (1) and any other relevant constraints, such as no short sales ($\omega_{t,T}^{j} \in [0,1]$ for $j = s,\ b,\ r$).[5] The optimal asset allocation is computed by maximizing the expectation taken with respect to a joint predictive density of future, T-horizon asset returns for the appropriate asset menu. In our paper, the joint predictive densities are obtained using both Classical and Bayesian approaches to estimate the relationship among asset returns and predictors. The following sub-sections describe these methods in detail.

2.2. **Classical Portfolio Strategies**

Under the classical method, we estimate the parameters that characterize a set of simultaneous linear relationships (i.e., a VAR(1)) and then apply a simple "plug-in approach", by which the conditional predictive moments and density of future asset returns is computed by replacing the unknown parameter values with their least-squares estimates, and naively ignoring the fact that the latter are not simply coefficients but random variables (estimators) with a random distribution. In the presence of short-sale constraints, the program in (3) is solved using numerical Monte Carlo methods, as in Barberis (2000):

$$\max_{\boldsymbol{\omega}_{t,T}} \frac{1}{S} \sum_{i=1}^{S} \left[\frac{\{\omega_{t,T}^{s} \exp(R_{t,T}^{s,i}) + \omega_{t,T}^{b} \exp(R_{t,T}^{b,i}) + \omega_{t,T}^{r} \exp(R_{t,T}^{r,i}) + (1 - \omega_{t,T}^{s} - \omega_{t,T}^{b} - \omega_{t,T}^{r}) \exp(R_{t,T}^{f,i})\}^{1-\gamma}}{1-\gamma} \right], \tag{4}$$

subject to $\omega_{t,T}^{j} \in [0,1]$ for $j = s,\ b,\ r,\ f$. Here S is a large number of draws from the T-month ahead joint predictive density of real asset returns.

In this paper we focus on a range of simple linear predictability models in which the risk premia can be forecast using their own past and – more importantly – the values of a number of predictor variables. In particular, a simple Gaussian VAR(1) model for asset returns that allows for time-varying risk premia, as in Barberis (2000) or Campbell, Chan and Viceira (2003) is:

$$\mathbf{z}_{t+1} = \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{z}_t + \boldsymbol{\epsilon}_{t+1}, \tag{5}$$

where $\mathbf{z}_t \equiv [r_t^s\ r_t^b\ r_t^r\ r_t^f\ \mathbf{x}_t']'$, $\boldsymbol{\epsilon}_t \sim NIID(\mathbf{0}, \boldsymbol{\Sigma})$ and $\mathbf{x}_t$ represents a vector of M predictors (see below for specific comments). $\boldsymbol{\mu}$ is the $(N+M) \times 1$ vector of intercepts and $\boldsymbol{\Phi}$ is the $(N+M) \times (N+M)$ matrix of (own- and cross-) autoregressive coefficients.[6] The model in (5) implies an assumption of constant variances and covariances of the shocks to the system, as captured by the $(N+M) \times (N+M)$ covariance matrix $\boldsymbol{\Sigma}$.

[4] Although only continuous rebalancing is rational, a substantial fraction of the empirical asset allocation literature has also entertained buy-and-hold portfolio problems, e.g., Brennan et al. (1997), Barberis (2000), and Avramov (2002).

[5] Constraining portfolio weights to remain nonnegative is equivalent to using the sample covariance matrix after having reduced its large elements and then choosing unconstrained weights (see Jagannathan and Ma, 2003). Despite such shrinkage, which improves the ex-post performance of optimizing portfolios, in DGU (2009a) $1/N$ still outperforms the optimizing strategies.

[6] NIID means normally and identically independently distributed. Notice that $\boldsymbol{\mu}$ is not the mean of $\mathbf{z}_t$ and therefore does

Notice that the real return on 1-month T-bills, r_t^f, is never actually risk-free in this set up, because of the uncertainty on the inflation rate already over $[t, t+1]$. Moreover, over a longer horizon $[t, T]$ with $T \geq 2$, the future nominal T-bill yields become risky as well, implying that a T-month investor who only buys T-bills will have to roll-over 1-month T-bills $T-1$ times which is a risky strategy in the face of interest rate and inflation variability. The Appendix provides further details on the solution of the problem.

We set M up to a value of 4. Section 3 provides details on the choice and statistical properties of the predictors, but these are the CPI inflation rate, the trailing, moving window stock dividend yield, the term spread between long- and short-term riskless yields, and the default spread between investment and speculative grade corporate bonds. Additionally, we entertain 5 types of predictability models: portfolio strategy *ALL* implies that all predictors are used at the same time ($M = 4$); the strategies named *CPI*, *DY*, *TERM*, and *DEF*, respectively, use one predictor at the time and therefore set $M = 1$. An additional – no-predictability – benchmark is given by the case in which $\mathbf{\Phi}$ is constrained to a matrix of zeros ($\mathbf{\Phi} = \mathbf{O}$) so that (5) simplifies to a Gaussian IID model,

$$\mathbf{z}_{t+1} = \boldsymbol{\mu} + \boldsymbol{\epsilon}_{t+1} \quad NIID(\mathbf{0}, \mathbf{\Sigma}), \tag{6}$$

in which both risk premia, variances, and covariances are constant over time. Obviously, the collection of marginal densities that only concern real asset returns, $[r_t^s \; r_t^b \; r_t^r \; r_t^f]'$, follows a similar Gaussian IID process, with mean $\boldsymbol{\mu}_r$ (the first N elements of $\boldsymbol{\mu}$) and covariance matrix Σ_{rr} (the $N \times N$ north-western block of Σ). As shown in the seminal paper by Samuelson (1969), the portfolio implied by a Gaussian IID benchmark is insensitive to the investment horizon and, if portfolio returns were lognormally distributed, it would coincide with the sample-based mean variance portfolio analyzed by DGU.[7]

2.3. **Bayesian Portfolio Strategies**

Although typical in the literature (see e.g., Campbell et al., 2003), it is well-known that the predictive densities obtained from the Classical approach in Section 2.1 ignore that the parameter estimators are themselves random variables, thus leaving out an important source of uncertainty (known as "estimation risk" or parameter uncertainty).[8] In the Bayesian case, we follow Barberis (2000) and specify an uninformative set of prior beliefs as to the parameters characterizing the linear relationships among asset returns and predictors. A posterior distribution of such parameters is then obtained, by an application of Bayes' rule, which depends on the actual data observed for returns and predictors, as summarized by the likelihood function. The resulting

not estimate the (unconditional) risk premia. Instead, the vector $(\mathbf{I}_{N+M} - \mathbf{\Phi})^{-1}\boldsymbol{\mu}$ contains expected real returns. A VAR(1) specification is without loss of generality because a VAR(p) can be re-written as a VAR(1) by augmenting the set of state variables by re-labelling various lags of the same predictors as if they were different predictors (see Hamilton, 1994).

[7]Under a buy-and-hold strategy, the portfolios implied by the no predictability benchmark are only approximately insensitive to the investment horizon, as Samuelson's result holds only under continuous rebalancing. However, starting from Barberis (2000), many papers using data similar to ours have shown that even under buy-and-hold the Gaussian IID weights fail to depend on T for all practical purposes.

[8]Investor's welfare can substantially increase if she takes into account the uncertainty in forecasts by using Bayesian updating (see Jorion, 1985, and Kandel and Stambaugh, 1996), especially when return predictability is statistically weak.

joint posterior distribution is then used to generate a conditional, predictive density of returns and, therefore, a predictive distribution of future utility levels, from which the expectation in (3) can be computed as a functional of portfolio weights $\boldsymbol{\omega}_{t,T}$.

Call θ the vector collecting all the parameters entering the generic VAR(1) model in (5), i.e., $\boldsymbol{\theta} \equiv [\boldsymbol{\mu}' \; vec(\boldsymbol{\Phi})' \; vech(\boldsymbol{\Sigma})']'$. The joint predictive distribution for $\mathbf{z}_t$ obtains then by integrating the joint distribution of $\boldsymbol{\theta}$ and returns, $p(\mathbf{z}_{t,T}, \boldsymbol{\theta}|\ddot{\mathbf{Z}}_t)$ with respect to the posterior distribution of $\boldsymbol{\theta}$, $p(\boldsymbol{\theta}|\ddot{\mathbf{Z}}_t)$:

$$p(\mathbf{z}_{t,T}) = \int p(\mathbf{z}_{t,T}, \boldsymbol{\theta}|\ddot{\mathbf{Z}}_t)d\boldsymbol{\theta} = \int p(\mathbf{z}_{t,T}|\ddot{\mathbf{Z}}_t, \boldsymbol{\theta})p(\boldsymbol{\theta}|\ddot{\mathbf{Z}}_t)d\boldsymbol{\theta}, \tag{7}$$

where $\ddot{\mathbf{Z}}_t$ collects the time series of asset returns and predictors up to time t, $\ddot{\mathbf{Z}}_t \equiv \{\mathbf{z}_i\}_{i=1}^{t}$. In turn, the posterior obtains from a standard uninformative prior. The portfolio optimization problem becomes:

$$\max_{\boldsymbol{\omega}_t} \int \frac{W_{t+T}^{1-\gamma}}{1-\gamma} p(\mathbf{z}_{t,T}) \cdot d\mathbf{z}_{t,T}, \tag{8}$$

subject to constraints. The Appendix provides details on solution methods. Also in this case, while the portfolio strategy *ALL* obtains from the "full" ($M = 4$) VAR(1), *CPI*, *DY*, *TERM*, and *DEF* are simpler $M = 1$ cases that can be seen as obtained from imposing restrictions on $\boldsymbol{\Phi}$. When $\boldsymbol{\Phi} = \mathbf{O}$, the no predictability benchmark emerges; if portfolio returns were lognormally distributed, our Bayesian IID case would coincide with the Bayesian diffuse-prior mean variance portfolio analyzed by DGU with constant risk premia.[9]

2.4. Optimal Asset Allocation Strategies

The optimizing strategies that we plan to compare to the naive, equal-weighting strategy originate from combinations of five distinct (sets of) parameters. These are:

I. the econometric relationship linking real asset returns to lagged asset returns and lagged values of the selected predictors,
II. the asset menu (i.e., with or without real estate investment vehicles),
III. the treatment of parameter uncertainty (Classical vs. Bayesian methods),
IV. the investment horizon (T), and
V. the curvature of the utility function as captured by the coefficient of relative risk aversion (γ).

In particular, we entertain 6 alternative econometric models, i.e.,

1. VAR(1) in which all predictors forecast subsequent real asset returns (the strategies called *ALL* in what follows),
2. VAR(1) in which only lagged inflation forecasts asset returns (*CPI*),
3. VAR(1) in which only the lagged dividend yield forecasts asset returns (*DY*),

[9] Barberis (2000) has shown that when parameter uncertainty is accounted for by using typical Bayesian technology, then optimal weights under the Gaussian IID model are no longer insensitive to the investment horizon: as the coefficients that relate to alternative risky assets are exposed to time-varying relative intensity of parameter uncertainty, an investor may optimal change her portfolio weights as a function of T, even in the absence of predictability.

4. VAR(1) in which only the lagged riskless term spread forecasts asset returns ($TERM$),
5. VAR(1) in which only the lagged default spread forecasts asset returns (DEF),
6. a Gaussian IID model in which there is no risk premia predictability.

We also consider 2 asset menus, 2 alternative ways to compute portfolio weights (Classical and Bayesian), 6 different investment horizons (1, 3, 6, 12, , 24, and 60 months), and 3 alternative relative risk aversion coefficients (2, 5, and 10) which span the typical values used in the asset allocation literature. This combination of $6 \times 2 \times 2 \times 6 \times 3$ parameter values yields a total of 432 alternative optimizing portfolio strategies on which we report in Sections 4 and 5.

2.5. **The 1/N Strategy**

The equally-weighted portfolio rule allocates a weight of $1/N$ to each of the N assets available in the asset menu. Obviously, this strategy is not optimizing, in the sense that at least in general (ruling out some odd configurations of the parameters) optimal asset allocation will fail to deliver $1/N$ as the utility-maximizing choice. However, by following the $1/N$ rule, investors enjoy the benefits of "naive" diversification, in the sense that they spread risk over a set of assets with different risk-return trade-off. Clearly, the benefits of this naive, cross-sectional diversification grow large early on when N goes from 1 or 2 to intermediate values and when new asset classes which are substantially different in terms of their risk-return trade-off are added; as N increases to infinity, it is well known from elementary finance that these benefits will decline rapidly. However, *in-sample*, any optimizing portfolio strategy outperforms $1/N$ by construction: the equally-weighted portfolio can always be seen as an attempt to impose artificial constraints on the control vector of (3) and this can only reduce the optimal value of the problem. Importantly, this obvious result holds in-sample, only. There is in fact no guarantee that optimized portfolios will always (or ever!) deliver *out-of-sample* performance results superior to those yielded by naive portfolios. The reason is that, as we have discussed in Sections 2.2 and 2.3, solving (3) to find an optimizing portfolio requires an econometric framework capturing the dynamics of asset returns and their predictive density. However, any econometric model – even if sensible ex-ante – may turn out to be either misspecified or plagued by large parameter estimation errors.[10] As a result, portfolio strategies that were ex-ante optimal may gravely disappoint ex-post and produce realized portfolio outcomes that are inferior to those of simple benchmarks.

In practice, a recent literature has explored exactly these issues and we now know that, despite its simplicity, $1/N$ turns out to be a welfare-enhancing strategy in several *out-of-sample* experiments involving from three to twenty-four stock portfolios and a one-period horizon (see e.g., DeMiguel, Garlappi, and Uppal, 2009a). This obtains because the gains deriving from optimal diversification are often smaller than the loss

[10]Misspecification means that whilst an unknown econometric model with a different (implicit or explicit) functional form generates the data of interest, the portfolio manager uses a functional form that is incorrect and cannot provide an accurate approximation to the true process. Parameter uncertainty arises when a model is possibly correctly specified but – because of its features or the absence of sharp information in the data (or simply, the lack of sufficiently long data samples) – some or most parameters cannot be estimated with adequate accuracy to inform portfolio selection.

due to the use, in classical mean-variance optimization, of inputs estimated with large errors.[11] In fact, DGU also prove that the in-sample based mean-variance strategy implies a higher expected utility than the $1/N$ strategy if the sample size τ exceeds a critical value τ^* which increases in N and falls in the difference between the ex-ante Sharpe ratio of the optimal and the naive strategy.

In our experiment, we have $N = 3$ when real estate is excluded from the asset menu, and $N = 4$ otherwise. Differently from DGU, we include cash among the N risky assets. Since N in our experiment is smaller than in most of theirs and – as we will see below – our sample size for estimation is longer, it is less likely that the naive diversification dominates over the Classical IID strategy in one-month horizon experiments. Of course, in this paper we also entertain a range of predictability models of VAR-type and examine performances at (long) investment horizons that are not considered by DGU. These extension over the baseline research design in DGU offer the opportunity for optimizing strategies to outperform $1/N$ and are important occasions to advance our understanding of the properties and limitations of equally-weighted portfolios.

2.6. **Measuring Ex-Post Performance**

We use a recursive scheme of model estimation and portfolio optimization. We initialize our experiment using data from January 1972 up to December 1994 (that is, 276 monthly observations) to estimate the parameters of our 6 alternative econometric models and to produce forecasts of T-month ahead means, variances, and covariances of returns on all asset classes. Additionally, we compute predictions of the T-month ahead joint density of real returns and use this density to determine optimal portfolio weights in the classical and Bayesian frameworks.[12] After recording predicted moments, densities, and the corresponding optimal portfolio weights under alternative specifications of γ and horizons, we proceed to expand the recursive estimation window by adding one additional month, which transforms the original sample into a 1972:01-1995:01 one.[13] At this point, predictions and *ex-ante* optimal portfolio weights are re-computed and saved. Iterating this recursive scheme until December 2007 yields a sequence of 156 sets of optimal portfolio shares – one for each of the 432 optimal strategies listed in Section 2.4 – as well as realized portfolio returns from such ex-ante optimal choices, from which we calculate *ex-post* performance measures for our alternative portfolio strategies.

Although the literature on applied portfolio management offers a wide choice of performance measurement criteria, in this paper we focus on only two key indicators: the realized Sharpe ratio and the certainty equivalent return. The Sharpe ratio (SR) of strategy (as defined by T, γ, the econometric model i, the asset menu, and a classical vs. Bayesian approach at computing the joint predictive density) has a standard

[11] The $1/N$ strategy has the empirically important feature of imposing the highest degree of shrinking, because it completely disregards the data and shrinks all the asset moments to common values.

[12] In the classical case, T-step ahead real returns have a multivariate normal predictive density with means, variances, and covariances that are those which are predicted in our recursive exercise, so that the numerical characterization of such densities is not required. However, in the Bayesian case, Monte Carlo methods are required.

[13] We do not use a “rolling-window” of a fixed length, but an “expanding-window”, i.e. we do not drop the data for the earliest period when adding new data. One reason is that only an expanding window guarantees acceptable saturation ratios (i.e., the ratio between available observations and number of parameters implied by the model) in estimation, which is generally identified with an index of approximately 20 observations per parameters.

definition:

$$SR_i(\gamma, T) \equiv \frac{\frac{1}{156-T}\sum_{\tau=1}^{156-T}[R_{i\tau}(\gamma, T) - r_\tau^f]}{\sqrt{\frac{1}{156-T}\sum_{\tau=1}^{156-T}\{[R_{i\tau}(\gamma, T) - r_\tau^f] - \frac{1}{156-T}\sum_{\tau=1}^{156-T}[R_{i\tau}(\gamma, T) - r_\tau^f]\}^2}}, \quad (9)$$

where $R_{i\tau}(\gamma, T)$ is the realized real return from model i when portfolio weights are computed for horizon T and risk aversion γ. Notice that although quite popular, the Sharpe ratio is an appropriate criterion only for a truly mean-variance investor, which is not the preference specification employed in this paper.[14] In particular, an increase in $SR_i(\gamma, T)$ is not necessarily associated with higher welfare, if it is achieved at the cost of worse higher-order moment properties of portfolio returns. This is because investors are commonly averse to negative skewness and excess kurtosis (see e.g., Dittmar, 2002), and these preferences are fully captured only by utility functions more general than a simple mean-variance objective.

These obvious drawbacks of Sharpe ratios, lead us to conclude that comparing realized power utility across different portfolio strategies is the only remedy to the presence of non-normalities in realized portfolio returns. This performance measure aligns the ex-ante preferences of investors driving the selection of optimal portfolios to their ex-post evaluation. However, realized optimal power utility usually lacks of any insightful interpretation. Moreover, realized power utility cannot be compared across different horizons and values for γ. As a result, the other – in some sense, the only completely consistent – performance criterion we are computing is the annualized certainty equivalent return (CER) which is the solution of the implicit equation $u(W_t(1 + (T/12) \times CER(\hat{\boldsymbol{\omega}}_{i,t}(\gamma, T))) = E[u(W_{t+T}(\hat{\boldsymbol{\omega}}_{i,t}(\gamma, T)))]$, where $W_{t+T}(\hat{\boldsymbol{\omega}}_{i,t}(\gamma, T))$ is optimal terminal wealth associated with a given optimal strategy and $u(\cdot)$ is the utility function of the investor. Under the power utility function in (3), the overall, recursive out-of-sample CER is defined as:

$$CER_i(\gamma, T) \equiv \frac{12}{T}\left\{\frac{1}{W_t}\left[\frac{1}{156-T}\sum_{\tau=1}^{156-T}[W_{\tau+T}(\hat{\boldsymbol{\omega}}_{i,\tau}(\gamma, T))]^{1-\gamma}\right]^{\frac{1}{1-\gamma}} - 1\right\}. \quad (10)$$

$CER_i(\gamma, T)$ is not only consistent with the power utility criterion specified in Section 2.1, but also allows us to compare the (utility-weighted) value of each strategy across heterogeneous set-ups.

3. Data and Empirical Evidence on Predictability

After presenting summary statistics for our data in Section 3.1, Section 3.2 presents estimates of the VAR(1) that includes all predictors and the implied forecasts of risk premia, volatilities and covariances. These will help us to develop an understanding of the welfare rankings involving portfolio strategies in Section 4. Section 3.3 uses the estimates from Section 3.2 to produce forecasts of means, variances, and correlations of asset returns as a function of the investment horizon, which helps to develop intuition for portfolio results.

[14] In fact, mean-variance may derive in our case from a quadratic utility function in final wealth W_{t+T}, which has obvious disadvantages. Otherwise, power utility is consistent with a mean-variance approximation only for short investment horizons (see, e.g., Campbell and Viceira, 2002). Furthermore, it is well-known that the Sharpe ratio is highly sensitive to non-normally distributed returns (see, e.g., Ingersoll and Welch, 2007).

3.1. **Data and Summary Statistics**

Our sample spans the period January 1972 - December 2007 for a total of 432 monthly observations. The initial date is determined by the availability of prices and realized total returns on public real estate. Stock returns are computed applying standard continuous compounding to the value-weighted CRSP index covering all listings on the NYSE, NASDAQ and the AMEX. The 10-Year constant maturity portfolio returns on US government bonds as well as the 1-month T-bill returns come from the Federal Reserve Bank of St. Louis database (FREDII®). The NAREIT web site (www.NaReit.com) provides monthly returns on US equity REITs. We use continuously compounded total return market-capitalization indices, including both capital gains and income return components. Real returns are calculated by deducting the realized monthly rate of change in the consumer price index for urban consumers provided by FREDII® from total returns on assets.

We follow a large literature and use the dividend yield computed on the CRSP index along with the term and default spreads as predictors of asset returns.[15] As customary, the dividend yield is computed as the ratio between the moving average of the 12 most recent monthly cash dividends paid out by companies in the CRSP universe, divided by the $t-12$ value-weighted CRSP price index. The term spread is the difference between the yield on a portfolio of long-term US government bonds (10 year maturity) and the yield on 1-month Treasury Bills. The default spread is measured as the yield difference of Baa corporate bonds and the 10-year constant-maturity Treasury bond yield yield series are annualized. It is commonly thought (see Fama and French, 1989) that both term and default spreads are leading indicators of the business cycle. Since much literature allows for a relationship between real estate returns and the rate of inflation (see e.g., Karolyi and Sanders, 1998, Ling, Naranjo and Ryngaert, 2000), we also augment the space of predictor variables by the inflation rate, measured as the continuously compounded rate of change of the CPI Index for All Urban Consumers. Finally, also the real short-term rate (difference between 1-month T-bill returns and the inflation rate) is used as a predictor (see e.g., Campbell, 1987, and Detemple et al., 2003).

Descriptive statistics of the asset returns and predictor variables are reported in Table 1. Mean real stock returns are close to 0.35% per month with mean real long-term bond returns around 0.12% implying annualized returns of 4.2% and 1.40% respectively. Estimates of volatility imply annualized values of around 15.6% for real stock returns and 7.6% for real bond returns, yielding unconditional (annual) Sharpe ratios of 0.04 and -0.02 respectively. The latter value is a bit exceptional and cannot be taken as representative of equilibrium conditions, but it simply reflects the long period of rising short-term real rates in the 1970s and early 1980s, which caused realized bond returns to be negative and large. It is interesting to notice the excellent risk-return trade-off-characterizing equity REITs, with an annualized real mean return of 5.9% and an annualized volatility of 14.1%, both slightly better than (but statistically indistinguishable from) mean and volatility for stocks. However, the resulting unconditional Sharpe ratio for eREITs is relatively high, 0.08

[15]The dividend yield is widely used in the literature as a predictor of future excess asset returns, see, e.g., Campbell and Shiller (1988), Fama and French (1989), and Kandel and Stambaugh (1996). Karolyi and Sanders (1998) and Liu and Mei (1992) find that the dividend yield also helps predicting REIT returns. Previous examples of the use of term and default spreads are Brandt (1999) and Campbell et al. (2003).

which is practically double the ratio of stocks. This plays a key role in the analysis that follows. Real asset returns are characterized by significant skewness (the only exception is long-term bonds) and excess kurtosis.

3.2. Implied Forecasts of Risk Premia and Second Moments

With reference to a standard VAR(1) that includes all predictors, Table 2 displays MLE estimates of conditional mean coefficients (upper panel) along with robust t-statistics and estimates of the residuals' variance-covariance matrix (lower panel). These estimates refer to the entire sample, i.e., 1972:01-2007:12.[16] The table shows that future stock returns are positively (and reliably, in a statistical sense) predicted by the dividend yield, as known from Fama and French (1989). They are also negatively predicted by the term spread, the real short rate, and the inflation rate. These are the only statistically significant (at a 5% size) links between real stock returns and lagged predictor values. Interestingly, none of the lagged real asset returns forecasts subsequent real stock returns and the link to the lagged default spread is also not statistically significant.

Fewer predictors help forecasting subsequent real bond returns, namely inflation (which predicts lower subsequent bond premia) and – as first indicated by Chen, Roll and Ross (1986) – especially the default spread, which forecasts higher future bond premia. In principle, also lagged real stock returns forecast subsequent real bond returns, but the corresponding coefficient is economically small. The positive relation between equity REIT returns and the dividend yield is similar (also in terms of the associated coefficients) to the one uncovered for stocks and may capture a link between commercial real estate and the business cycle. Also lagged real returns on long-term bonds are good predictors for eREIT returns, as if financial wealth poured into (out of) securitized real estate after bond market booms (busts). On the contrary, an increase in the real short term rate – even after controlling for any term structure effects – predicts a reduction in future REITs returns, possibly because of the anticipated increase in mortgage rates. A negative association between REITs and lagged inflation has also long been observed before (e.g., Liu, Hartzell and Hoesli, 1997), suggesting that public real estate is not a good inflation hedge in the short-run.

Finally, the equation for the real short-term rate illustrates the typical autoregressive dynamics followed by the short rate, with a high AR(1) (partial) coefficient estimated at 1.02.[17] Also lagged term spreads, default spreads, and inflation exercise a rather large and statistically significant effect on subsequent real T-bill returns, and all forecast higher future real short term rates. Interestingly, also lagged real returns on long-term bonds and eREITs forecast real short term rates, but these linkages are economically small in the sense that the relevant coefficients are puny. Overall, the full VAR(1) model in Table 2 explains a relatively large share of total eREITs variance ($R^2 \simeq 8\%$), higher than the proportion of variance explained

[16] Although results are similar across the 144 estimations performed by expanding the sample by one month at a time, the R^2 and the statistical significance of the predictors are slightly decreasing over time. This is in line with several studies (among others, Goyal and Welch, 2008, and Aït-Sahalia and Brandt, 2001) documenting a reduction in the predictability of (stock) returns after the 1990s. We have estimated and analyzed recursive estimates also for the remaining 4 VARs in which predictors enter one at the time, and results are qualitatively similar.

[17] However, the VAR(1) is globally stationary: with a vector-autoregressive process covariance stationarity is not only determined by whether the (partial) AR(1) coefficient for each equation is less than one in modulus, see Hamilton (1994).

in the case of stocks (approximately 6%), long-term bonds (7%), but also smaller than the one of T-Bills (34%), although the latter high R^2 is strongly affected by the autoregressive nature of the real short rate. Table 2 also reports conditional mean coefficient estimates for the 4 predictors investigated, but these are less important to develop our intuition on the the predictability patterns involving assets.[18]

The lower panel of Table 2 shows instead the correlation matrix of the shocks characterizing (5), which represent the portion of real asset returns *not* explained by the values of the predictors. Even though these estimates do not simply correspond to the MLE estimate of $\mathbf{\Sigma}$, the reported correlations are implied by $\hat{\mathbf{\Sigma}}$ in obvious ways. Interestingly, a few of these implied correlation coefficients for shocks are large and economically important. In particular, shocks to real stock returns tend to positively correlate with shocks to real REIT returns; shocks to real stock returns tend instead to be accompanied by shocks which are large and of opposite magnitude vs. the dividend yield. Moreover, shocks to real eREITs are also negatively correlated with shocks to the dividend yield – in this sense one may advance an hypothesis that eREITs tend to share many common dynamic properties with real stock returns. Finally, real 1-month T-bill rates display negative and large correlations with shocks to the term spread and the inflation rate.

As explained in Barberis (2000) and Campbell, Chan, and Viceira (2003), when these residual correlations are high in absolute value, they may have powerful effects on long-run optimal portfolio weights. For instance, the negative correlations between shocks to real stock and eREIT returns and shocks to the dividend yield make both asset classes *decreasingly* risky as the investment horizon grows. The reason is simple: a high real stock or eREIT return today is accompanied by a low(er than otherwise) value of the dividend yield, because positive and large shocks to real asset returns come with negative and large shocks to the dividend yield; but future, low dividend yields forecast future low real stock and eREIT returns. Therefore, we should expect that – at least on average – real stock and eREIT real returns should be mean-reverting. Mean reversion implies that these real asset returns become decreasingly risky (their predictability grows) as the horizon grows. Interestingly, by a similar logic, the negative and large correlations of the real short rate with shocks to the term spread and the inflation rate imply the real 1-month T-bills become riskier as the horizon grows: Because current high real short term rates come with positive and large shocks to both the term spread and the inflation rate and these two predictors forecast future, higher real short term rates, the real short rate becomes mean-averting, i.e., it tends to strongly depart from the mean (or to even stochastically drift away from it) to an increasing degree, the longer is the horizon.

We have also obtained Bayesian estimates – under non-informative priors – of the VAR(1) that includes all predictors, which delivers another table similar to Table 2 in terms of contents and economic implications and is therefore omitted to save space.[19]

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[18] However, the dynamic links involving the predictors (e.g., the fact that lagged inflation forecasts future higher default spreads with a significant coeffcient) as well as real asset returns forecasting the predictors (e.g., that high real bond returns predict lower realized inflation) are crucial in shaping the $T \geq 2$ month ahead forecasts of the joint density of asset returns.

[19] The table is available upon request from the Authors. As one would expect, given the uninformative nature of our priors, the "point" estimates (technically, these are means of posterior densities) of all the conditional mean coefficients are practically identical to those in Table 2.

3.3. **Term Structures of Risk and Mean Returns**

Predictability of future stock returns imply that conditional moments – means, variance, and covariances – vary with the investment horizon in "interesting" and not necessarily linear ways (see Campbell and Viceira, 2005), while under the benchmark Gaussian IID case multi-period expected real returns and covariances grow linearly with the investment horizon. Although this point is well-known, most evidence on these issue in the literature refers to ex-ante, in-sample analysis. In Table 3, we report (the averages of) predicted means (upper panel), variances and correlations (lower panel) when estimation risk is not accounted for, as implied by our recursive classical VAR(1) estimates in Table 2. The table focusses on the $T = 1$ and $T = 24$ cases only, as representative of short and medium-term horizons comparable to those that have appeared in the literature (in any event, for $T \geq 24$ the plots of predicted moments as a function of T are flat). All predictions are computed as of the end of our sample, to derive some qualitative insights on the risk-return trade-off effects in the data. The predictions of mean returns are slightly higher for a two-year than for a 1-month investor for all asset classes: predicted mean real stock returns increases from 3.7 to 3.8 percent per year, while in the case of public real estate from an already remarkable 4.7 to 5 percent per year. Real bond returns range from 1.1 to 1.3 percent, and are thus dominated by T-Bills whose predicted mean real returns are 2 percent at $T = 1$ month and 2.1 percent at 24 months. The predicted volatility (annual standard deviation) is approximately constant at 8.4% per year in the case of stocks and 1-month T-bills, somewhat increasing (from 8.4 to 9.6 percent per year) in the case of public real estate, and strongly increasing in the case of long-term bond real returns, from 2.4 percent at a 1-month horizon to 3.6 percent at a 12-month one.

These results on the behavior of predictive volatility as a function of the investment horizon characterize long-term real bond returns and eREIT returns as mean-averting, in the sense that the predictability affecting these two asset classes make them increasingly risky as the horizon grows. In particular, the mean-aversion in real bond returns is driven by the combination between predictability and residual correlation coefficients involving bonds, the default spread and the inflation rate. The residual correlation of 0.35 with the default spread and of -0.09 with inflation shocks along with the sizeable and statistically significant VAR loadings of real bond returns on the lagged default spread (6.9 with 2.9 robust t-statistic) and inflation rate (-1.6 with robust t-statistic of -2) leads bond returns to drift away from their unconditional mean at a rate that increases with the horizon. On the contrary, stocks are moderately mean-reverting and this is mostly due to the fact that shocks to real stock returns have negative (positive) correlation with shocks to dividend yields (inflation), although lagged dividend yield (inflation) forecasts future higher (lower) real stock returns. Real REIT returns occupy instead an intermediate position and turn out – in overall terms, when all predictors are taken into account – to be mildly mean-averting, so that their risk slowly increases with the horizon.

On balance, bonds appear to have negative predicted Sharpe ratios for longer horizon investments, ranging from -0.38 (in annualized terms) to -0.22. This increase is mostly due to the fact that – given their negative risk premium – increasing volatility over expanding investment horizons is actually good news to their risk-trade-off index. On the opposite, the Sharpe ratios for both stocks and public real estate are positive and essentially insensitive to the investment horizon: an invariant 0.20 per year in the case of stocks, and 0.30-0.32

in the case of real estate, which is therefore the dominant asset in risk-return terms.

In a multivariate asset allocation set up it will never be only the Sharpe ratios to drive the optimal portfolio weights, because also correlation patterns matter. With one exception, also in this respect real long-term bonds have deteriorating properties as the horizon grows: the predictive correlation between bonds and stocks grows from 0.20 to 0.42, while the correlation between bonds and REITs increases from 0.14 to 0.41. This confirms the results in Campbell and Viceira (2005) on the worsening of the long-term properties of bond returns as the horizon grows. The exception is that the correlation between bonds and T-bills declines from 0.25 to 0.05. On the contrary (apart from what we have reported for stocks and bonds), the correlations involving stocks and real estate and stocks and 1-month T-bills hardly change as a function of the horizon. The same applies to the correlation between real T-bill returns and public real estate. This makes us predict asset allocation results by which eREITs and stocks progressively come to dominate long-term portfolios because of their positive and stable Sharpe ratios and moderate correlations with other asset classes.[20]

4. **Ex-Post Realized Performance**

In this section, we report a number of summary statistics concerning the properties and performance for the full range of 478 (432 plus 36 "versions" of $1/N$) portfolio strategies.[21] Section 4.1 starts with a comparison of the main, average features of alternative portfolio strategies under different assumptions. It documents that – even though differences across weights are never massive (this is especially the case when real estate belongs to the asset menu) – alternative optimal portfolios are sensitive to how predictability is modelled. Section 4.2 describes our our key findings. Section 4.3 asks whether such findings may be simply understood within a mean-variance framework and concludes that instead higher-order moments play a key role.

4.1. **Comparing Portfolio Weights Across Strategies**

In Table 4 we report means of optimal portfolio weights over the recursive sample period 1994:12-2007:12, which is a way to start appreciating the differences across portfolio strategies. To limit the amount of information provided, the table only reports means for three sets of strategies: the classical optimal VAR(1)-ALL, which covers a total of 36 alternative optimizing strategies, i.e., 6 horizons $\times$ 3 coefficients of relative risk aversion $\times$ 2 different asset menus; the Gaussian IID model with no predictability, implying a total of 6 strategies, since under no predictability the investment horizon does not matter; the $1/N$ benchmark.

The table shows that both risk aversion and, under predictability, the horizon do matter a lot. Moreover, while in the absence of real estate, the VAR(1)-ALL and no-predictability portfolios are very different at all

[20]We repeated this analysis with reference to the Bayesian estimates of the VAR(1), which implies that parameter uncertainty will be accounted for. In general, the result is that the risk of stocks and e-REITs no longer declines (or slowly increases) with the investment horizon, and this is to be imputed to the uncertainty caused by estimation uncertainty (see Barberis, 2000).Complete results are available upon request from the Authors.

[21]The distinction among the 36 alternative $1/N$-type strategies is artificial as far as the portfolio weights are concerned, but important in terms of performances: even if two portfolio display identical weights over time, when they are applied to/under different (i) asset menus, (ii) horizons, (iii) preferences, they will originate different realized, ex-post performance measures.

investment horizons, when real estate enters the asset menu, such differences are modest when the horizons are short and grow larger for long-horizon strategies.[22] This is sensible because it is for long-horizon investors that the interaction between predictability and the existence of correlation structure in the shocks across assets has the maximum potential to affect portfolio weights. In general, we notice that the presence of predictability tends to favor stock investments at long horizons in comparison to the Gaussian IID case (e.g., for $T = 60$ and $\gamma = 5$ the mean investment in stocks is 25% under the VAR model vs. 10% for the no predictability case) and government bond investments at shorter horizons (e.g., for $T = 1$ and $\gamma = 5$ the mean investment in bonds is 27% under the VAR model vs. 17% in the absence of predictability). However, at long horizons – as expected on the basis of Section 3.3 – the effects on bonds reverse and under predictability their demand declines vs. the Gaussian IID case. Finally, the effects of predictability on the optimal eREIT weight as well as on long-term stock weights is modest. In any event, it is clear that for all combinations of horizons, risk aversion coefficients, and asset menus, the implied departures of optimal portfolio weights from the $1/N$ strategy are always major.

Table 5 reports instead recursive mean of portfolio weights for the Bayesian case in which parameter uncertainty is taken into account. In this case, we oppose to the Gaussian IID set of portfolio strategies two different sets of VAR strategies (it is of course redundant to report twice the trivial $1/N$): besides VAR(1)-ALL, we now show average weights for the VAR(1)-DY model, selected because there is widespread evidence in the literature that (especially as far as stocks and bonds are concerned), the dividend yield is the most important among the predictors we are working with (see e.g., Brandt, 1999). The differences between Tables 4 and 5 can be traced back to the "overall " parameter uncertainty that affects optimal weights. Even though the Gaussian IID model is in principle the least exposed to parameter uncertainty because of the lower number of estimated parameters, it is for this model that we have the maximum differences between portfolio weights with and without accounting for parameter uncertainty. For both short- and long-investment horizons, the importance of public real estate is drastically reduced when estimation uncertainty is accounted for; on the contrary the importance of stocks, bonds, and especially bills is increased in the Bayesian framework when predictability is disregarded. The intuition is that parameter uncertainty hits more heavily the asset classes characterized by the highest Sharpe ratios because the ratio structure exposes it to very high perceived uncertainty when the denominator is characterized by enough estimation uncertainty to span small numbers.

When we compare classical and Bayesian recursive portfolios under VAR(1)-ALL predictability, the differences are generally minor for stocks and bonds, although it is clear that the demands for the most risky assets keep being penalized (eREITs and stocks) in favor of a slightly higher demand of T-bills. However, the decline in the optimal weights of public real estate cannot be easily disregarded, similarly to what we had found in the absence of predictability. Such effects are stronger at long-term horizons than at short-term, as one would expect. Finally, Table 5 points out that the specific predictability models fitted to the

[22] In what follows, we concentrate on comparisons for the RE case, which seems to be the most meaningful. However, Table 3 also shows a very simple pattern when VAR(1)-ALL and Gaussian IID allocations are compared in the no-RE case: under predictability, an investor should allocate less to stocks and more to bonds for short horizons, and less to stocks and more to T-bills for long horizons. Section 5.3 reports detailed comments for the case with no real estate investments.

data have rather moderate effects on the resulting optimal portfolio weights, especially when the asset menu includes real estate. All of these findings confirm our ex-ante expectations that different investment horizons and econometric models may imply optimal portfolio strategies that differ substantially. Additionally, the implied portfolios generally depart strongly from simple $1/N$ benchmarks.

4.2. **Baseline Performance Results**

Tables 6 and 7 provide details on the realized, ex-post performances of classical and Bayesian investment strategies, along with results for the $1/N$ benchmark. Performance figures are boldfaced when they are the best in our (pseudo-) out-of-sample recursive experiment, which means the highest realized mean portfolio return, Sharpe ratio or CER and the lowest realized portfolio volatility, for each combination of risk aversion, investment horizon, and asset menu.

In each of the two tables, there are six panels – each for a separate investment horizon, between 1 and 60 months – and 7 sets of columns, the first devoted to the performance of $1/N$, the second to the no-predictability optimizing strategy, and columns 3-7 to alternative VAR models of predictability. Columns 3-6 refer each to VAR(1) models including only one predictor, while column 7 to VAR(1)-ALL. We assign VAR(1)-ALL to the right-most column on purpose: if the "concentration" of boldfaced numbers moves – when reading the tables from top to bottom – from the left to the right, then it means that as the horizon grows, the best performances are obtained not from simple models, such as $1/N$ or the Gaussian IID model, but from predictability models of increasing complexity. In each panel, performances are reported for the cases of $\gamma = 2$, 5, and 10, which are typical values in the empirical asset allocation literature (with $\gamma = 5$ being a focal risk aversion coefficient). Each column contains information on the realized performance for two alternative asset menus, with and without real estate.

First, Table 6 shows that for all horizons $T \leq 12$ months and intermediate and high risk aversion coefficients ($\gamma = 5$ and 10), the CER is systematically higher under $1/N$ than in the simple Gaussian IID model with no predictability. For instance, for $T = 1$ month, the CER is 5.2% per year under $1/N$ vs. 3.3% under the no-predictability model (these figures are obtained for the asset menu that includes real estate; otherwise, they are 4.4% vs. 2.2%). This means that the same key result in DGU also holds with our data and irrespective of whether real estate data are used or not. Importantly, this result also applies to short-investment horizons (up to 6 months) and $\gamma = 5$ and 10 when we compare $1/N$ and the VAR models (or at least most of them, including VAR-ALL), in the sense that the equally-weighted CER is usually not inferior to the classical predictability strategies. For instance, for $T = 1$ month and $\gamma = 5$, the CER yielded by $1/N$ is 5.2% vs. 4% under VAR-ALL, while the highest possible CER from a VAR (-CPI) is anyway only 4.2%.[23]

However, with longer horizons the distance between equal weighting and optimizing narrows down and

[23] This is also consistent with the conclusions in Section 6.3 of DGU, insofar as they state to also have examined the performance of short-horizon portfolios with weight computed using the approximation methods proposed by Campbell et al. (2003). The only marginal discountinuity between out results and DGU's occurs with reference to the case of $\gamma = 2$: in this case, the Gaussian IID case sligthly out-performs $1/N$.

eventually reverses. This is already visible from the fact that the location of the boldfaced cells gradually moves from left to right – in particular towards the column that refers to VAR-ALL – as we move down Table 6 and to longer investment horizons. For instance, at $T = 60$ months the best CER is achieved by the complete VAR for both asset menus (at 5.5 and 4.2% in annualized terms, for the case $\gamma = 5$); interestingly, these high CERs result from high Sharpe ratios (0.34 and 0.15 in annualized terms) which are themselves generated by high realized mean portfolio returns (9.0 and 5.7% in annualized terms). $1/N$ remains (as it is generally true for most configurations we have examined) the strategy that yields the lowest annualized volatility, but this seems to be insufficient to maximize an investor's welfare (the resulting CERs are 5 and 3.2 percent, for the two asset menus).

Table 7 concerns instead the case of Bayesian portfolio strategies. The general qualitative remarks are identical to those reported for Table 6 with reference to classical experiments.[24] Visually, Table 7 makes it evident that the location of the boldfaced cells moves from left to right as the investment horizon grows. For instance, taking again as a reference the case of $\gamma = 5$, under a 1-month horizon, the highest CERs and Sharpe ratios are given by the equally-weighted strategy (for instance, with values of 5.2% and 0.33 in the case of the asset menu including real estate, exceeding the corresponding values obtained from the best VAR model, 4.5% and 0.25, respectively) which outperforms all competing models. The ranking is reversed when long-horizons are investigated: for instance, in the case of $T = 60$ months (again for $\gamma = 5$), the highest CERs and Sharpe ratios are given by the VAR-ALL model.

Three additional comments involve a comparison of Tables 6 and 7. First, it is clear that the location of the boldfaced cells tends to concentrate either in columns 1-2 (in correspondence to $1/N$ and the no-predictability model) or in column 7, with very few VAR(1) models containing only one predictor coming up as best performing models. The only exception concerns classical VAR-CPI strategies which – especially for intermediate horizons and when the asset menu includes real estate – are often strongly performing in the Sharpe ratios and CERs metrics. Section 5.1 contains additional remarks on the case of single-predictor models. Second, a Bayesian investor who ignores predictability, with intermediate or high risk aversion and with short-to-medium horizons always achieves a higher ex-post realized welfare than a classical investor with identical preferences and horizon and who also ignores predictability. For instance, for $T = 1$ month, $\gamma = 5$ and for an asset menu that includes real estate, the CER of the Bayesian investor is 4.2% vs. 3.3% for a classical investor. This is powerful evidence in favor of the ex-post, out-of-sample usefulness of strategies that take parameter uncertainty into account. However, for long-run investors there is no precise pattern and in fact the Bayesian investor is often worse-off vs. the classical one. Interestingly, this finding also tends to hold for the comparison between Bayesian and classical investors that model and use predictability for asset allocation purposes. Finally, it is comforting to report that (especially in the classical case) an investor that exploits the best predictability model always out-perform an identical (i.e., with the same preferences, horizon, and asset menu) long-horizon investor who ignores predictability and uses a naive Gaussian IID

[24] However, there is no precise ranking between $1/N$ and the no-predictability benchmark, which shows the usefulness of taking into account parameter uncertainty to improve realized welfare from portfolio selection.

econometric framework.[25] For instance, for $T = 1$ and $\gamma = 10$ the classical CER increases from 1.1% to 4.2% when the ability of past inflation to forecast asset returns is taken into account; the corresponding results for the case of $T = 60$ are 4.9% and 5.3% for VAR-ALL. For $T = 1$ and $\gamma = 10$ the Bayesian CER increases from 4.2% to 4.4% when the ability of past dividend yields to forecast asset returns is taken into account; the corresponding figures for the case of $T = 60$ are 4.9% and 5.3% for VAR-ALL.

4.3. **Understanding the Results**

In this subsection we examine two sources of intuition for our key result that it is possible to out-perform $1/N$ over long investment horizons by exploiting portfolio strategies that account for the existence of predictability in real asset returns. First, there is evidence of non-normal portfolio returns in our sample. For instance, with a one-month investment horizon, the skewness of realized portfolio returns is -0.88 in the Classical VAR-ALL framework while in the Bayesian it is equal to -1.18. The corresponding estimates for the excess kurtosis of one-month realized portfolio returns are 4.82 and 5.64. As we move to a longer horizon of 24 months, skewness and excess kurtosis respectively decline to 0.29 (0.09 for the Bayesian VAR-ALL model) and 0.09 (1.68). These compare to values of -0.78 and 3.83 for $T = 1$ and -0.15 and 1.82 for $T = 24$ in realized returns from the naive $1/N$ portfolio. We may therefore wonder whether the higher welfare achieved under the optimizing models with predictability arise from non-normalities: it might be that Sharpe ratios are still higher for long-horizon $1/N$ portfolios, yet a power utility investor prefers optimizing strategies because of the relative improvement in higher order moments of realized wealth.

To address this possibility, Tables 6 and 7 have also systematically presented realized out-of-sample Sharpe ratios for all of our 478 alternative strategies. A closer inspection of Table 6-7 suggests however that changing performance criterion hardly affects the qualitative conclusions reported in Section 4.2. Even visually, it is obvious that while for short horizons $1/N$ is always among the top 2-3 Sharpe ratio performers, as T grows larger the boldfaced Sharpe ratios tend to move towards the right of the tables, which means that *some* VAR models exist that are able to produce the highest achievable Sharpe ratio. For instance, for a one-month investor with $\gamma = 5$, the Sharpe ratio of a $1/N$ portfolio strategy (0.33) exceeds the one of an optimizing Bayesian investors that considers either the Gaussian IID (0.18) or VAR-ALL (0.23) strategies. The same pattern emerges for a Classical investor and the Sharpe ratios are numerically close. However, a $T = 60$ Bayesian investor with $\gamma = 5$, derives a Sharpe ratio of 0.27 from the equally-weighted strategy vs. 0.28 from the Gaussian IID model and 0.43 from VAR-ALL. Thus, it appears that *all the properties of the realized portfolio return distribution for the* $1/N$ *strategy worsen as the investment horizon lengthens*. It follows that we cannot attribute the change in the CER rankings in Table 6 and 7 to higher moments only.

[25] $\gamma = 2$ provides some exceptions to these general findings. It is not surprising that $\gamma = 2$ may have implications that differ qualitatively from those obtained for $\gamma = 5$ and 10: our casual observation is that in the case of $\gamma = 2$ the no-short sale constraints are binding most of the time in our sample and for most of our optimizing strategies. This causes a number of consequences that are difficult to control for, such as the inability to completely and correctly exploit any predictability patterns, or the possibility to adequately penalize asset classes because of their excessive "exposure" to estimation risk. Section 5.2 also relates the case of low risk aversion to the role played by higher-order moments in determining portfolio CERs.

We may similarly wonder whether non-normalities drive the improved performance of the Bayesian Gaussian IID model for longer horizon investors. Usually, the uncertainty surrounding parameter estimates compounds over time, making an optimizing investor more cautious and ultimately reducing the ex-ante Sharpe ratio. In our experiment, we already saw that ex-post Sharpe ratios are higher at $T = 24$ than at $T = 1$, with almost unchanged optimal portfolio weights, exceeding the level of the Sharpe ratio associated with equal weighting. The skewness and excess kurtosis of realized returns are then equal to -1.19 and 5.64 at $T = 1$, 0.088 and 1.684 at $T = 24$, so that they considerably improve (increase in the case of skewness and decline in the case of excess kurtosis) to "out-perform" the higher-order moments of the naive equally-weighted strategy. Thus, non-normalities play an important role in explaining the ex-post performance of Bayesian Gaussian IID portfolios.

5. **Robustness Checks**

Given our finding that out-performing $1/N$ in recursive out-of-sample experiments is possible, we perform a range of additional tests. Section 5.1 inspects with greater care results obtained from optimizing models in which one single predictor at the time is used to forecast real asset returns. Section 5.2 isolates patterns related to the risk aversion coefficient γ. Section 5.3 looks again at the results in Table 6 and 7 considering the case in which the asset menu is restricted to exclude real estate vehicles. Section 5.4 concludes by examining how performance changes when we take transaction costs into account. Tabulated results are available from the Authors upon request on the different aspects of the exercises performed here.

5.1. **Single-Predictor Optimizing Strategies**

The optimizing strategies based on the VAR-ALL model have used four predictors in addition to lags of real asset returns on stocks, bonds, eREITs and bills. Typically, augmenting the number of predictors increases the in-sample accuracy and this occurs in our VAR estimates as well. Some of these advantages follow from the implicit sequential conditioning that is applied when estimating a VAR model. For instance, the dividend yield helps predicting future returns on stocks, bonds and the real short rate in the VAR-ALL specification, whereas it cannot predict bond or real T-bill returns in a simpler VAR-DY model. Similarly, the inflation rate helps predicting all future returns in the general specification, but only the future real short-term rate in the VAR-CPI model. The term spread and the default spread, respectively, predict stock and bond returns as well as the short term rate when together, but only the latter when considered in isolation. Therefore the VAR-ALL not only allows the use of a stronger "fire power" by using all predictors together: it also teases out many partial effects that each predictor may exercise only after conditioning on different (opposite) predictive strengths of the remaining predictors. However, it is not always guaranteed that these gains may translate in additional predictive accuracy. As it is well known in the forecasting literature, it is possible that increasing the number of predictors may enhance the fit to the data provided by a model but also inflate the number of parameters to be estimated to such a point that the associated overall estimation error grows so large to effectively damage the forecasting performance. Therefore a check on whether the rankings of $1/N$ vs. VAR-ALL may carry over to single predictor optimizing strategies should be performed not only for completeness, but also because it may reveal out-of-sample realized performances which are actually superior to those reported for VAR-ALL in Section 4.

Although detailed information on realized average performance on single-predictor strategies has appeared already in Tables 6 and 7, Table 8 specifically reports in a synthetic way on the difference in CERs (upper panel) and Sharpe ratios (bottom panel) between the best predictive model and $1/N$ for the baseline case of $\gamma = 5$. Here the exercise consists of opposing to $1/N$ not VAR-ALL as we have systematically done in Section 4.2, but to compare $1/N$ with the *best* among all optimizing portfolios in the light of the natural thought that in reality an portfolio-optimizing investor will not simply pick VAR-ALL but probably adopt the best performing among all VARs. Consistently with our general remarks in Section 4, Table 8 shows that, for shorter investment horizons, the best performing model is either VAR-DY or VAR-CPI in both the CER and Sharpe ratio metrics and whether or not the investor accounts for parameter uncertainty. Nevertheless, for $T \leq 12$ months, it is still the case that $1/N$ has a better out-of-sample portfolio performance. On the opposite, for investment horizons in excess of one year, the best performing model is VAR-ALL most of the times, even though VAR-CPI retains good properties in terms of portfolio performances. Consistently with what we have reported in Section 4, the ex-post performance for horizons longer than 12 months is worse for $1/N$ than for the optimizing portfolio based on the best optimizing VAR model.

5.2. **Risk Aversion**

As observed in Section 4.2, the low risk aversion case of $\gamma = 2$ represents an exception for a few of the performance patterns we have isolated. However, a number of our qualitative findings hold intact also in the case of $\gamma = 2$: for instance, it is still the case that the longer is the investment horizon, the better the performance of optimizing Bayesian strategies in general and of Bayesian strategy that accounts for predictability in particular. Surprisingly, this is the case also for shorter horizons. In fact, our key result is even strengthened when applied to highly risk-averse investor: Tables 6 and 7 show that in this case even for horizons as short as 1 month – and regardless of the fact that parameter uncertainty has been taken into account – optimizing strategies such as VAR-ALL may outperform the equally-weighted naive benchmark; in fact, the realized performance of $1/N$ ranks last among portfolio strategies for $T = 1$.

We may be tempted to ascribe this result to the larger distance between optimized and naive Sharpe ratios: DGU observe that optimizing portfolio strategies are expected to outperform $1/N$ if the ex ante Sharpe ratio of the selected mean-variance efficient portfolio is substantially higher than that of the $1/N$ portfolio. In our experiment, under low risk aversion, the Gaussian IID portfolio is already heavily tilted towards stocks and real estate, i.e., towards assets that display very high Sharpe ratios over our sample period. For instance, the optimal shares invested in stocks and eREITs are equal to 18 and 82 percent in the classical no predictability case and to 22 and 75 percent in the Bayesian no predictability case. Such highly risky portfolio compositions yield Sharpe ratios of 0.27 and 0.32 in the two cases, which are considerably higher than what the optimizing predictability portfolios deliver in the $\gamma = 5$ case (0.21 and 0.18, respectively). Although this would confirm DGU's intuition, such high Sharpe ratios are always lower than the ratios guaranteed – thanks to its low volatility (less than half the volatility typical of optimizing portfolios) – by $1/N$, in this case 0.33. Therefore, simple differences in Sharpe ratios cannot explain by themselves the welfare ranking reversals associated with the case of low risk aversion.

Further investigations reveals that a solid explanation for such divergent CER-Sharpe ratio patterns for short horizon investments and low risk aversion may be ascribed to the role played by higher order moments. Indeed, the naive strategy has slightly worse portfolio return skewness (-0.78) and excess kurtosis (3.83) than Classical (-0.76 and 3.60) or Bayesian no-predictability strategies (-0.58 and 3.83). Importantly, the third and fourth moments of Classical (-0.88 and 4.82) and Bayesian (-0.78 and 3.83) are much worse for an intermediate-risk-averse investor with a one-month horizon. This explains why a large T is needed for optimizing strategies to out-perform $1/N$ in the case of $\gamma = 5$ and 10, while this is not the case for $\gamma = 2$.

As already reported, when the investor is instead highly risk averse ($\gamma = 10$), most results uncovered for the case of $\gamma = 5$ hold. In this case, the Sharpe ratio of the equally-weighted portfolio exceeds the one of optimizing strategies for one-month classical portfolios (see Tables 6 and 7, especially with reference to the asset menu including real estate) which explains why it is only for long-horizons that the optimal portfolios out-perform $1/N$ in terms of CER. With investment horizons of two or more years, we confirm that optimizing strategies dominate naive ones according to both the Sharpe ratio and welfare – reaching best results when parameter uncertainty is accounted for.

5.3. **Smaller Asset Menus: The Role of Real Estate**

Benartzi and Thaler (2001) have argued that the fund menu offered to 401-k plan participants exert a strong influence on the assets they end up owning. In particular, the allocation to stocks increases as the number of stock funds increases. Since the early 1990s (also owing to changes in the tax treatment of REITs), US pension plans have been increasingly offering funds including real estate assets, even though there is still a substantial number of plans that do not. Therefore, it seems of some importance to also analyze how the gain (or loss) in *out-of-sample* welfare associated with optimal portfolio strategies changes with the introduction of real estate in an asset menu comprising only cash, bonds and stocks. In principle, following DGU, increasing the number of assets should widen the performance gap between $1/N$ and optimal portfolios. However, their analysis also reveals that this effect may be reversed if the additional asset to be included in the choice menu has a high (above the mean-variance efficient portfolio before the menu expansion) Sharpe ratio, as eREITs do over our sample period. Therefore an empirical analysis of the issue seems relevant.

When public real estate is excluded from our research design (see appropriate columns in Tables 6-7), it is still the case that $1/N$ outperforms optimizing strategies that capture predictability in terms of both Sharpe ratios and CERs for a one-month horizon and especially for intermediate risk aversion ($\gamma = 5$). Results confirm that such ranking reverses for a long enough horizon. The best strategy, and the one that outperforms $1/N$ for horizons equal to or longer than a year, is the Classical optimizing VAR-based one, with a predominant role played by VAR-ALL. On the opposite, without the availability of the favorable risk-return trade-off offered by real estate, the allocation to cash and bonds for a Bayesian investor is so high – well in excess of 70% – that its net effect is to depresses portfolio performance for $T = 12$ and $T = 24$. However, at long horizons the best performing framework tends to be the VAR-ALL one.

More importantly, whatever the predictive model and the strategy considered, adding real estate to the opportunity set always leads to an increase of CERs and Sharpe ratios, a phenomenon already documented in Fugazza, Guidolin and Nicodano (2009). This occurs because both realized means and volatilities of optimizing portfolios increase when eREITs are included, but the means increase at a pace that is roughly double the increase in volatilities. In general terms, many of the empirical findings on out-of-sample performance reported in Fugazza, Guidolin and Nicodano (2009) extend to the application and sample in this paper.

As for any "time-diversification" gains, the reduction in realized portfolio volatilities due to the fact that predictability may be exploited by the VAR-based strategies, Tables 6 and 7 shows that – by comparing the annualized portfolio volatilities for $T = 1$ and $T = 60$ – these are generally present only when real estate belongs to the asset menu. This conclusion also applies to the four VAR models including a single predictor. For instance, the volatility of the VAR-ALL portfolio for a Bayesian investor with $\gamma = 5$, drops from 8.0 percent per year in the case $T = 1$ to 6.6 percent for $T = 60$ when eREITs are included, whilst it increases from 5.3% to 6.6% when excluded. This is another appealing reason why including real estate in the investment set is important and for why its inclusion may affect our understanding of the performance of simple, equally-weighted benchmarks in key ways.

5.4. **Turnover Effects**

Up to now the evaluation of optimal buy-and-hold portfolio strategies has disregarded the costs of trading which are incurred in keeping portfolio weights at the desired level over time. In this subsection we discuss how ex-post performance results change when we account for transaction costs. Previous rankings for long-run investors may indeed fail to hold if optimizing strategies require substantially higher turnover than the one needed to keep the portfolio equally invested in the assets under consideration. Equivalently, even if Tables 6-8 have reported that especially at long horizons, the optimizing portfolios tend to out-perform $1/N$ in terms of CERs, such a difference may turn out to be insufficient to cover the whole amount of differential, higher transaction costs that the optimizing strategies – which time the markets by linking weights to the value of one or more state variables – impose on an investor.[26]

[26] As explained by DGU, it is not correct to state that naive, equally-weighted strategies imply no trading costs: for any couple of asset classes i and j, unless the gross returns (inclusive of any dividends or distributions) between t and $t+1$ happen to be realized in the same ratio as the time t weights assigned to assets i and j, respectively (i.e., unless $exp(r^i_{t+1} - r^j_{t+1}) = \omega^i_t/\omega^j_t$ $\forall t$), also a $1/N$ strategy will require rebalancing of portfolio shares over time, just because different gross return realizations endogeneously change the composition of the portfolio that therefore may depart from the desired $1/N$ structure. However, unless asset returns take extreme values, it is true that equally-weighted strategies generally imply modest effective transaction costs.

We compute the turnover as the average percentage of wealth traded in each period measured as:

$$Turnover_i(\gamma, T) \equiv \frac{1}{156 - T} \sum_{t=1}^{156-T} \sum_{i=1}^{N} |\omega_{t+1}^i - \omega_t^i|, \tag{11}$$

where $(156 - T)$ is the number of rebalancing periods over the evaluation sample, N is the number of assets included in the portfolio, ω_t^i is the portfolio weight for asset i before rebalancing, and ω_{t+1}^i is the desired portfolio weight restored by implementing a trade of $\omega_{t+1}^i - \omega_t^i$. For each strategy, we compute the turnover measure after adjusting for returns occurring between rebalancing points. Notice that computing turnover is in a sense a more robust and general approach vs. imposing a direct structure (e.g., variable and fixed) on transaction costs, which we pursue later in this Section.

To save space, we omit reporting versions of Tables 6-8 augmented by turnover results and we limit ourselves to comment on the main findings.[27] Averaging over classical and Bayesian investors with $\gamma = 5$, the average, recursive monthly turnover implied by optimizing investment strategies is on average 3.5 times higher than the one associated with the naive, equally-weigthed strategy, with a standard deviation of 2.5 and a wide range of 0.9-24.0, which means that the lowest turnover implied by an optimizing strategy is 10% below the turnover implied by $1/N$, but the highest turnover is 2,400% higher than the (modest) turnover that $1/N$ induces. Such turnover ratio (scaled by the turnover generated by $1/N$) is decreasing with the investment horizon, from an average of about 5.3 (for a 1-month horizon) to 2.4 (for $T = 60$), which is a sensible finding as longer horizons imply less frequent rebalancing while in general the long-term optimizing positions tend to be remarkably less extreme than the shorter-term ones, resulting in lower rebalancing needs. For example, the optimal weight assigned to eREITs – which is in general large in the Bayesian VAR-ALL portfolios – decreases with the investment horizon (for instance, from 46% in the case of $T = 1$ to 35% when $T = 60$, for $\gamma = 5$). Even in the Bayesian case – when the absence of predictability does not necessarily implies that optimal weights stop depending from T – this does not happen even for the Gaussian IID model.

We also find evidence of considerable variation in turnover ratios across optimizing models and asset menus for every given investment horizon and preference type. For instance, the adjusted turnover (as always, scaled by the adjusted turnover of $1/N$) is much higher for Gaussian IID models (5.1 with a range of 1.8 -10.4) than for models that capture VAR predictability (e.g., 3.1, with a range of 1.1-9.9 in the case of VAR-ALL), when real estate is included in the asset menu. This ranking among optimizing strategies is however reversed when real estate is not included in the asset menu: 3.1 on average (range 0.9-13.2) for the Gaussian IID case vs. 6.1 (with range 0.9-24 for predictability models).

These results on relative adjusted turnover ratios are bad news for the ability of optimizing strategies to out-perform $1/N$, because higher turnover should translate in performance-reducing trading costs to be paid. However, it is interesting to notice that a portion of the optimizing portfolio strategies imply – especially for high risk aversion and long horizons – adjusted turnover ratios between 1 and 3 and in general this ratio declines as T grows, exactly when we have found that optimizing portfolios can and do out-perform $1/N$. One way to break this logical impasse consists of imposing a precise structure for transaction costs and to proceed

[27] Complete and tabulated results are available from the Authors upon request.

to compare actual realized performance, *after transaction costs are accounted for*. Of course, this step causes a loss of generality but it is required to proceed from turnover statistics to actual, realized performances. We follow DUG's baseline case and assume that each trade gives rise to proportional cost of 50 basis points, and compute afresh all realized performance measures net of these variable transaction costs.

In general, we observe that this structure for transaction costs mainly affects mean ex-post portfolio returns leaving their higher moments (in particular, realized recursive volatilities) substantially unchanged. Importantly, it is still the case that $1/N$ outperforms, at least in terms of realized Sharpe ratios, all optimizing models at short horizons, for all risk aversions and irrespective of the estimation method. At longer horizons, $1/N$ continues to produce performances that are superior to Gaussian IID models: in particular, the Bayesian IID strategies have lower Sharpe ratios than those implied by the equally-weighted strategy, even when they previously ranked higher than $1/N$ in terms of gross Sharpe ratio, ignoring transaction costs. Of course, this ranking reversal is associated with the high turnover for no-predictability models relative to $1/N$, which lowers their mean ex post return differential.

However, especially in terms of realized CERs at long investment horizons the optimizing strategies based on predictable returns still outperform $1/N$, when eREITs belong to the asset menu. This means that the positive transaction cost differential, which always impacts negatively onto the optimizing strategies, is unable to counter the worsening in the higher-moment properties of the realized portfolio return distribution induced by $1/N$ as the investment horizon widens. For instance, a moderately risk adverse investor ($\gamma = 5$) with a long horizon ($T = 60$) scores a CER of 5.2% per year (5.9%) when following the portfolio prescriptions from the VAR-ALL classical (Bayesian) model, which exceeds what she obtains by equally weighting her portfolio (4.2% per year, in both cases).[28] These results confirm that our earlier qualitative findings that when real estate belongs to the asset menu, long-horizon ($T \geq 12$) optimizing strategies that account for the presence of predictability may out-perform naive, equally-weighted strategies for all risk aversion levels, does not critically depend on the omission of transaction costs from the analysis in Section 4.[29]

6. **Discussion and Conclusions**

Our results indicate that investors with horizons of one year and longer would on average have benefited, *ex-post*, from an optimizing strategy that exploits return predictability over the period January 1995 - December 2007. Importantly, this result holds even when the realized, recursive portfolio performances are opposed to those derived from a traditionally tough benchmark to beat, i.e., a naive equally-weighted portfolio that simply invests a $1/N$ percentage in each of the available assets. Importantly, this basic empirical finding is:

[28] When risk aversion is low ($\gamma = 2$), predictability-based strategies display CERs higher than $1/N$ for all horizons. The properties of the higher moments of their gross-return distributions, which were found to be critical to the outperformance of the strategy in Section 4.3, are essentially invariant to accounting for transaction costs.

[29] However, when real estate is omitted from the asset menu, this result breaks down. The higher turnover rates implied by portfolios based on VAR-ALL prevents the less risk averse investor ($\gamma = 2$) from outperforming the naive investor at long horizons – contrary to the zero-transaction cost case – and enables the more risk averse ($\gamma = 5$) investor to beat $1/N$ only at the longest allowed horizon ($T = 60$). These findings further align our results to DGU's, Section 6.3.

- robust to a number of variations of our baseline research design, for instance the degree of risk aversion imputed to the decision-maker, the number of predictors included in the analysis, whether portfolios are computed accounting or not for parameter uncertainty, and adjusting realized performances for the presence of transaction costs;

- essentially in-line with the experiments performed by De Miguel et al. (2009a,b): when the horizon is as short as in their papers (1-month), predictability is ignored, and the asset menu excludes real estate assets, our empirical results are qualitatively consistent with theirs, in spite of some data differences and the longer sample periods and estimation windows;

- sensitive to only one crucial aspect, namely the presence of public real estate investment vehicles (here eREITs) in the asset menu: we only fail to find the possibility to out-perform $1/N$ in the very specific case (aligned to one of experiments in De Miguel et al., 2009a) in which the opportunity sets excludes real estate assets.

It is also notable that for investors displaying low risk aversion ($\gamma = 2$), optimizing strategies perform better than naive, equally-weighted diversification even for shorter investment horizons. Such favorable welfare rankings for optimizing strategies is associated with a relative improvement in all the moments – including indices of asymmetries (skewness) and of tail-thickness (excess kurtosis) – of the distribution of realized portfolio returns.These important results that support the usefulness of applied portfolio management techniques (especially whenever asset menus are rich enough) hold in spite of the fact that some features of our research design were selected to avoid maximizing the potential performance of active portfolio strategies. For instance, all the portfolios computed are of a buy-and-hold type and rule out the fact that an investor correctly accounts for future rebalancing choices (see Barberis, 2000). Additionally, we stress that in this paper predictability has been simply captured through simple vector autoregressive models, which are robust but possibly sub-optimal tools in density forecasting applications, such as those underlying our optimal portfolio calculations (see Guidolin and Timmermann, 2008), because of their inability to capture predictability in higher-order moments.

One final issue is the stability of realized performances. It is well known – see, e.g., Goyal and Welch (2008) – that the extent of out-of-sample predictability of the equity risk premium in linear forecasting models is not stable across samples. Although this instability may also be more revealing of the misspecification of linear forecasting models, in this paper we have used a linear framework to try and capture any patterns of predictability. Therefore, we may be potentially exposed to the dangers of unstable performances. As a way to check for the extent of the problem, we have re-applied our entire research design to a shorter sample period – January 1995 through December 2004 – selected because this is the period analyzed in Fugazza, Guidolin, and Nicodano (2009). Performance results (unreported) confirm the presence of some instability. While $1/N$ still outperforms optimizing strategies for $T = 1$, this is the case for most longer horizons as well. Moreover, welfare falls as a function of the investment horizon whereas it increases, albeit non-monotonically, in the longer sample examined in this paper. This comparison thus reveals that optimizing buy-and-hold investors with longer horizons face a higher performance volatility than the naive $1/N$ ones.

One may wonder why the relative performance of long-run optimizing strategies is so different in the two subsamples. Let us focus on the intermediate case of $T = 24$ to get a tentative answer: the optimal average portfolio shares in the shorter subsample are equal to 33 (53) percent for T-bills, 19 (12) percent for stocks, 2 (2) percent for bonds and 54 (37) percent for eREITs, for a Classical (Bayesian) investor with $\gamma = 5$. They are not very dissimilar from those of our longer sub-sample, as the investor remains heavily invested in the most risky assets, especially eREITs, and cash. Thus, asset allocation cannot explain the divergent performance across the 1995-2004 vs. the 1995-2007 samples. If asset allocation is not responsible for the difference, then the 2005-2007 performance of the asset classes involved must have undergone a break that has favored the optimizing, long-run strategies over $1/N$. In Figure 1 we have plotted recursive, real time one-step ahead predicted returns from the VAR(1)-ALL model. The figure suggest that towards the end of 2004 and throughout mid-2007, eREITs just recovered from a pre-2000 downturn in predicted mean returns, whilst displaying relatively constant correlations with other asset classes and volatility. Thus, it is not surprising the in ex-post terms, the overweighting on eREITs implied by most (all) optimizing strategies relative to $1/N$ may have handsomely paid off over the longer sample investigated in this paper.

On the one hand, the presence of instability cautions us before rushing to a conclusion that naive diversification strategies have the potential to reduce households ex-post realized welfare. On the other hand, we must also point out that our results on the outperformance of optimizing strategies relative to $1/N$ may actually represent a lower bound. First and foremost, competing models exist to capture predictability (like the non-linear models in Detemple, Garcia, and Rindisbacher, 2003) and/or competing methods to map predictability in portfolio decisions (see Brandt, 1999, or Ait-Sahalia and Brandt, 2001). Second, a few recent papers propose methods to enhance the out-of-sample performance of market timing strategies (e.g., DeMiguel et al., 2009b). Third, even within the linear framework, the real estate finance literature indicates which specific state variables may predict real estate returns: given the importance of eREITs for our results, it seems that even the out-of-sample performance of VARs can only improve, relative to what we have reported here.

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Appendix: Computing Optimal Portfolios under Predictable Returns

Classical Portfolios

Under equation (5), the (conditional) distribution of cumulative future real returns (i.e. the first four elements in $z_{t,T} \equiv \sum_{k=1}^{T} z_{t+k}$) is multivariate normal with mean and covariance matrix given by the appropriately selected elements of:

$$\begin{aligned} E_{t-1}[\mathbf{z}_{t,T}] &= T\boldsymbol{\mu} + (T-1)\boldsymbol{\Phi\mu} + (T-2)\boldsymbol{\Phi}^2\boldsymbol{\mu} + ... + \boldsymbol{\Phi}^{T-1}\boldsymbol{\mu} + (\boldsymbol{\Phi} + \boldsymbol{\Phi}^2 + ... + \boldsymbol{\Phi}^T)\mathbf{z}_{t-1} \\ Var_{t-1}[\mathbf{z}_{t,T}] &= \boldsymbol{\Sigma} + (\mathbf{I}+\boldsymbol{\Phi})\boldsymbol{\Sigma}(\mathbf{I}+\boldsymbol{\Phi})' + (\mathbf{I}+\boldsymbol{\Phi}+\boldsymbol{\Phi}^2)\boldsymbol{\Sigma}(\mathbf{I}+\boldsymbol{\Phi}+\boldsymbol{\Phi}^2)' + \\ &\quad ... + (\mathbf{I}+\boldsymbol{\Phi}+...+\boldsymbol{\Phi}^{T-1})\boldsymbol{\Sigma}(\mathbf{I}+\boldsymbol{\Phi}+...+\boldsymbol{\Phi}^{T-1})', \end{aligned} \tag{12}$$

where $\mathbf{I}$ is the identity matrix of dimension $N + M$ and $\boldsymbol{\Phi}^k \equiv \prod_{i=1}^{k} \boldsymbol{\Phi}$. Since the parametric form of the predictive distribution of $\mathbf{z}_{t,T}$ is known, it is possible to approach directly the problem in (3), or equivalently

$$\max_{\boldsymbol{\omega}_t} \int \frac{W_{t+T}^{1-\gamma}}{1-\gamma} \phi\left(E_t[\mathbf{z}_{t,T}], Var_t[\mathbf{z}_{t,T}]\right) \cdot d\mathbf{z}_{t,T} \tag{13}$$

where $(\phi\left(E_t[\mathbf{z}_{t,T}], Var_t[\mathbf{z}_{t,T}]\right)$ is a multivariate normal with mean $E_t[\mathbf{z}_{t,T}]$ and covariance matrix $Var_t[\mathbf{z}_{t,T}])$, by simulation methods. Indeed, it is possible to solve this problem by employing simulation methods similar to Kandel and Stambaugh (1996), Barberis (2000), and Guidolin and Timmermann (2008):

$$\max_{\boldsymbol{\omega}_t} \frac{1}{S} \sum_{i=1}^{S} \left[\frac{\{\omega_t^s \exp(R_{t,T}^{s,i}) + \omega_t^b \exp(R_{t,T}^{b,i}) + \omega_t^r \exp(R_{t,T}^{r,i}) + (1-\omega_t^s-\omega_t^b-\omega_t^r)\exp(R_{t,T}^{f,i})\}^{1-\gamma}}{1-\gamma} \right], \tag{14}$$

where $\{R_{t,T}^{s,i},\, R_{t,T}^{b,i},\, R_{t,T}^{r,i},\, R_{t,T}^{f,i}\}_{i=1}^{N}$ are obtained simulating from the process in (1) S times. To obtain sufficiently precise results, we have employed $N = 100,000$ Monte Carlo trials in order to minimize any residual random errors in optimal weights induced by simulations.

Bayesian Portfolios

In this case, Monte Carlo methods require drawing a large number of times from $p(\mathbf{z}_{t,T})$ and then 'extracting' cumulative returns from the resulting vector. Given the problem

$$\max_{\boldsymbol{\omega}_t} \int \frac{W_{t+T}^{1-\gamma}}{1-\gamma} p(\mathbf{z}_{t,T}|\ddot{\mathbf{Z}}_t, \boldsymbol{\theta}) p(\boldsymbol{\theta}|\ddot{\mathbf{Z}}_t) \cdot d\mathbf{z}_{t,T},$$

this task is simplified by the fact that predictive draws can be obtained by drawing from the posterior distribution of the parameters and then, for each set of parameters drawn, by sampling one point from the distribution of returns conditional on past data and the parameters. At this point, equation (5) can be re-stated as:

$$\begin{bmatrix} \mathbf{z}_2' \\ \mathbf{z}_3' \\ \vdots \\ \mathbf{z}_t' \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{z}_1' \\ 1 & \mathbf{z}_2' \\ \vdots & \vdots \\ 1 & \mathbf{z}_{t-1}' \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}' \\ \boldsymbol{\Phi}' \end{bmatrix} + \begin{bmatrix} \boldsymbol{\epsilon}_2' \\ \boldsymbol{\epsilon}_3' \\ \vdots \\ \boldsymbol{\epsilon}_t' \end{bmatrix},$$

or simply $\mathbf{Z} = \mathbf{XC} + \mathbf{E}$, where $\mathbf{Z}$ is a $(t-1, N+M)$ matrix with the observed vectors as rows, $\mathbf{X}$ is a $(t-1, N+M+1)$ matrix of regressors, and $\mathbf{E}$ a $(t-1, N+M)$ matrix of error terms, respectively. All the coefficients are instead collected in the $(N+M+1, N+M)$ matrix $\mathbf{C}$. If we consider the following standard uninformative diffuse prior:[30]

$$p(\mathbf{C}, \boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-\frac{N+2}{2}},$$

then the posterior distribution for the coefficients in θ, $p(\mathbf{C}, \boldsymbol{\Sigma}^{-1}|\ddot{\mathbf{Z}}_t)$ can be characterized as:

$$\begin{aligned} \boldsymbol{\Sigma}^{-1}|\ddot{\mathbf{Z}}_t &\sim \text{Wishart}(t-N-2, \hat{\mathbf{S}}^{-1}) \\ vec(\mathbf{C})|\boldsymbol{\Sigma}^{-1}, \ddot{\mathbf{Z}}_t &\sim N\left(vec(\hat{\mathbf{C}}), \boldsymbol{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1}\right) \end{aligned}$$

where $\hat{\mathbf{S}} = (\mathbf{Z} - \mathbf{X}\hat{\mathbf{C}})'(\mathbf{Z} - \mathbf{X}\hat{\mathbf{C}})$ and $\hat{\mathbf{C}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}$, i.e. the classical OLS estimators for the coefficients and covariance matrix of the residuals.

We adopt the following simulation method. First, we draw S independent variates from $p(\mathbf{C}, \boldsymbol{\Sigma}^{-1}|\ddot{\mathbf{Z}}_t)$. This is done by first sampling from a marginal Wishart for $\boldsymbol{\Sigma}^{-1}$ and then (after calculating $\boldsymbol{\Sigma}$) from the conditional $N\left(vec(\hat{\mathbf{C}}), \boldsymbol{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1}\right)$, where $\hat{\mathbf{C}}$ is easily calculated. Second, for each set $(\mathbf{C}, \boldsymbol{\Sigma})$ obtained, the algorithm samples cumulated returns from a multivariate normal with mean vector and covariance matrix given by (12). In particular, since applying Monte Carlo methods implies a double simulation scheme (i.e., one pass to characterize the predictive density of returns, and a second pass to solve the portfolio choice problem), S is set to a relatively large value of 300,000 independent trials that are intended to approximate the joint predictive density of real returns and predictors.

[30] Uninformative priors may be a suboptimal choice, even in in-sample exercises. Hoovernaars et al. (2007) develop the concept of robust portfolio, i.e., the portfolio of an investor with a prior that has minimal welfare costs when evaluated under a wide range of alternative priors. We do not purse this extension in the our paper.

Table 1

Summary Statistics for Portfolio Returns and Predictors

The table reports standard summary statistics for monthly portfolio returns on stocks, long-term bonds, eREITs, and 3-month T-bills and for 4 predictor variables (CPI inflation, the dividend yield, the term spread and the default spread). The dividend yield, term, and default spreads are expressed in annualized terms. The sample period is 1972:01 – 2007:12. JB stands for the Jarque-Bera normality test. The null hypothesis of a zero median is tested using the nonparametric Wilcoxon signed rank test. In the case of kurtosis, the null hypothesis is that the kurtosis coefficient equals the Gaussian benchmark level of 3.

	Mean	Median	Std. Dev.	Uncond. Sharpe Ratio	Minimum	Maximum	Skewness	Kurtosis	J-B test
Real Stock Returns	0.349	0.745**	4.493	0.043	-26.39	14.96	-0.843**	6.483**	269.6**
Long-term Govt. Bonds Real Returns	0.117	0.147	2.205	-0.018	-7.631	8.911	0.011	4.028*	19.05**
Equity REIT Real Returns	0.489*	0.749**	4.057	0.082	-17.39	12.31	-0.633**	5.163**	113.0**
1-month T-bill Returns	0.534**	0.486**	0.287	__	0.072	2.142	1.529**	7.303**	501.7**
CPI Inflation rate	0.378**	0.320**	0.354	__	-0.806	1.790	0.532**	3.969*	37.28**
Dividend Yield (annual MA)	2.851**	2.640**	1.010	__	1.250	6.060	0.707**	3.045	36.02**
Default Spread (Baa-Aaa, annualized)	2.018**	1.907**	0.561	__	0.923	3.780	0.684**	3.130	34.03**
Riskless Term Spread (annualized)	1.523**	1.689**	2.194	__	-15.16	5.88	-1.818**	11.828**	1640**

* significance at 5%, ** significance at 1%.

Table 2

MLE Estimates of a VAR(1) Including All Predictors with Equity REITs

The table reports the MLE estimation outputs for the Gaussian VAR(1) model:

$$z_t = \mu + \Phi z_{t-1} + \boldsymbol{\varepsilon}_t$$

where z_t includes continuously compounded monthly real asset returns and the dividend yield, the term spread, the default spread, the real short rate and inflation. $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Sigma)$. t statistics are reported in parenthesis under the corresponding point estimates. Bold coefficients imply a p-value of 10% or lower.

	Stock Returns$_t$	Bond Returns$_t$	E-REITs$_t$	Dividend Yield $_t$	Term Spread $_t$	Default Spread $_t$	Real Short Rate $_t$	Inflation $_t$
				μ′				
	0.006	-0.008	0.001	0.000	**0.001**	0.000	**-0.004**	**0.003**
	(-0.492)	(-1.314)	(-0.130)	(-0.660)	(-3.283)	(-0.196)	(-4.503)	(-3.505)
				Φ				
Stocks $_{t-1}$	-0.022	**-0.059**	0.074	0.001	0.001	0.000	-0.005	0.004
	(-0.382)	(-2.073)	(-1.432)	(-0.664)	(-0.560)	(-1.151)	(-1.181)	(-1.168)
Bonds $_{t-1}$	0.028	0.027	**0.264**	0.000	**0.007**	**-0.001**	**0.019**	**-0.026**
	(-0.261)	(-0.512)	(-2.731)	(-0.118)	(-1.865)	(-1.950)	(-2.637)	(-3.947)
E-REITs $_{t-1}$	0.102	-0.015	0.000	-0.003	**0.008**	**-0.001**	**-0.009**	0.002
	(-1.596)	(-0.478)	(-0.007)	(-1.419)	(-3.588)	(-2.461)	(-2.131)	(-0.599)
Dividend Yield $_{t-1}$	**0.801**	0.213	**0.699**	**0.977**	**-0.017**	**-0.002**	**-0.052**	**0.070**
	(-2.842)	(-1.550)	(-2.780)	(-99.93)	(-1.834)	(-2.025)	(-2.705)	(-4.065)
Term Spread $_{t-1}$	**-5.366**	-1.214	**-3.064**	**0.233**	**0.563**	**0.018**	**0.750**	**-0.339**
	(-2.845)	(-1.316)	(-1.819)	(-3.559)	(-9.112)	(-2.988)	(-5.876)	(-2.937)
Def Spread $_{t-1}$	3.239	**6.872**	4.784	**-0.358**	-0.174	**0.976**	**1.061**	**-0.954**
	(-0.664)	(-2.880)	(-1.097)	(-2.115)	(-1.091)	(-61.23)	(-3.215)	(-3.191)
Real Short rate $_{t-1}$	**-3.127**	-0.687	**-2.940**	0.073	0.037	**0.013**	**1.022**	-0.085
	(-2.066)	(-0.928)	(-2.175)	(-1.391)	(-0.750)	(-2.616)	(-9.984)	(-0.914)
Inflation $_{t-1}$	**-5.437**	**-1.621**	**-4.619**	**0.156**	-0.011	**0.018**	**0.649**	**0.341**
	(-3.196)	(-1.949)	(-3.041)	(-2.648)	(-0.199)	(-3.163)	(-5.642)	(-3.278)
				Correlation matrix				
Stocks $_t$	1	0.13	0.547	-0.887	0.099	-0.218	0.109	-0.176
Bonds $_t$		1	0.113	-0.195	-0.361	0.35	0.208	-0.09
E-REITs $_t$			1	-0.542	0.09	-0.206	0.059	-0.112
Dividend Yield $_t$				1	-0.113	0.186	-0.117	0.191
Term Spread $_t$					1	-0.288	-0.445	-0.006
Def Spread $_t$						1	0.073	0.005
Real Short rate $_t$							1	-0.886
Inflation $_t$								1

Table 3

Conditional Moments of Predicted Returns

The table reports the annualized conditional mean and variances (upper panel) as well as their conditional correlations (bottom panel) of real returns under 1- and 24-month head predictive densities implied by a classical VAR(1) model that includes all predictors.

	Annualized Returns			
	Stock	**Bond**	**e-REITs**	**3-m T-bills**
	Horizon: 1 month			
Mean	0.037	0.011	0.047	0.020
Standard Error	(0.001)	(0.000)	(0.001)	(0.000)
Standard Deviation	0.084	0.024	0.084	0.012
	Horizon: 24 month			
Mean	0.038	0.013	0.05	0.021
Standard Error	(0.001)	(0.000)	(0.001)	(0.000)
Standard Deviation	0.084	0.036	0.096	0.012

	Correlations					
	Stock Bond	**Stock e-REITs**	**Stock Real Tbill**	**Bond e-REITs**	**Bond Real Tbill**	**e-REITs Real Tbill**
	Horizon: 1 month					
Correlation	0.196	0.590	0.174	0.135	0.253	0.090
Standard Error	(0.004)	(0.003)	(0.002)	(0.001)	(0.002)	(0.002)
	Horizon: 24 month					
Correlation	0.415	0.572	0.290	0.409	0.045	0.067
Standard Error	(0.010)	(0.003)	(0.006)	(0.003)	(0.004)	(0.004)

Table 4

Recursive Sample Averages for Classical Portfolios Weights: VAR(1) ALL vs. Benchmarks

The table reports mean portfolio weights from the recursive solution of classical optimal portfolio problems under alternative specifications of the investment horizon and of the constant coefficient of relative risk aversion (γ). The sample period over which the recursive calculation has been performed is 1994:12 – 2007:12. Besides the predictability weights implied by the VAR(1) model that includes all predictors, the table reports sample mean portfolio weights for two key benchmarks: the Gaussian IID case in which there is no predictability (but the portfolio problem has to be solved numerically) and the equally-weighted 1/N case (where the weights are simply imputed as part of the portfolio rule). Two asset menus are also considered: with and without e-REITs.

	Gaussian IID Predictability						
	Stocks RE	Stocks No RE	Bonds RE	Bonds No RE	e-REITs	3-month T-bills RE	3-month T-bills No RE
	Horizon: 1 month						
$\gamma = 2$	0.175	0.963	0.000	0.037	0.825	0.000	0.000
$\gamma = 5$	0.103	0.857	0.166	0.093	0.608	0.123	0.050
$\gamma = 10$	0.050	0.810	0.069	0.042	0.421	0.459	0.148
	Horizon: 3 months						
$\gamma = 2$	0.175	0.963	0.000	0.037	0.825	0.000	0.000
$\gamma = 5$	0.103	0.857	0.166	0.093	0.608	0.123	0.050
$\gamma = 10$	0.050	0.810	0.069	0.042	0.421	0.459	0.148
	Horizon: 6 months						
$\gamma = 2$	0.175	0.963	0.000	0.037	0.825	0.000	0.000
$\gamma = 5$	0.103	0.857	0.166	0.093	0.608	0.123	0.050
$\gamma = 10$	0.050	0.810	0.069	0.042	0.421	0.459	0.148
	Horizon: 12 months						
$\gamma = 2$	0.175	0.963	0.000	0.037	0.825	0.000	0.000
$\gamma = 5$	0.103	0.857	0.166	0.093	0.608	0.123	0.050
$\gamma = 10$	0.050	0.810	0.069	0.042	0.421	0.459	0.148
	Horizon: 24 months						
$\gamma = 2$	0.175	0.963	0.000	0.037	0.825	0.000	0.000
$\gamma = 5$	0.103	0.857	0.166	0.093	0.608	0.123	0.050
$\gamma = 10$	0.050	0.810	0.069	0.042	0.421	0.459	0.148
	Horizon: 60 months						
$\gamma = 2$	0.175	0.963	0.000	0.037	0.825	0.000	0.000
$\gamma = 5$	0.103	0.857	0.166	0.093	0.608	0.123	0.050
$\gamma = 10$	0.050	0.810	0.069	0.042	0.421	0.459	0.148

	Equally weighted (1/N)						
	Stocks RE	Stocks No RE	Bonds RE	Bonds No RE	e-REITs	3-month T-bills RE	3-month T-bills No RE
	Horizon: 1 month						
$\gamma = 2$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 5$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 10$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
	Horizon: 3 months						
$\gamma = 2$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 5$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 10$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
	Horizon: 6 months						
$\gamma = 2$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 5$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 10$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
	Horizon: 12 months						
$\gamma = 2$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 5$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 10$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
	Horizon: 24 months						
$\gamma = 2$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 5$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 10$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
	Horizon: 60 months						
$\gamma = 2$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 5$	0.250	0.333	0.250	0.333	0.250	0.250	0.334
$\gamma = 10$	0.250	0.333	0.250	0.333	0.250	0.250	0.334

	VAR(1) -- ALL						
	Stocks RE	Stocks No RE	Bonds RE	Bonds No RE	e-REITs	3-month T-bills RE	3-month T-bills No RE
	Horizon: 1 month						
$\gamma = 2$	0.166	0.677	0.000	0.225	0.834	0.000	0.098
$\gamma = 5$	0.110	0.378	0.273	0.459	0.541	0.076	0.163
$\gamma = 10$	0.053	0.186	0.134	0.214	0.277	0.536	0.600
	Horizon: 3 months						
$\gamma = 2$	0.164	0.750	0.000	0.148	0.836	0.000	0.102
$\gamma = 5$	0.124	0.402	0.196	0.309	0.542	0.138	0.289
$\gamma = 10$	0.053	0.195	0.077	0.127	0.276	0.594	0.678
	Horizon: 6 months						
$\gamma = 2$	0.171	0.621	0.000	0.091	0.829	0.000	0.288
$\gamma = 5$	0.131	0.418	0.168	0.241	0.550	0.151	0.341
$\gamma = 10$	0.054	0.192	0.060	0.229	0.277	0.609	0.579
	Horizon: 12 months						
$\gamma = 2$	0.158	0.679	0.000	0.060	0.842	0.000	0.261
$\gamma = 5$	0.138	0.428	0.078	0.171	0.570	0.214	0.401
$\gamma = 10$	0.050	0.194	0.030	0.203	0.286	0.634	0.603
	Horizon: 24 months						
$\gamma = 2$	0.159	0.694	0.000	0.043	0.841	0.000	0.263
$\gamma = 5$	0.159	0.431	0.023	0.076	0.586	0.232	0.493
$\gamma = 10$	0.057	0.319	0.013	0.041	0.302	0.628	0.640
	Horizon: 60 months						
$\gamma = 2$	0.159	0.720	0.000	0.032	0.841	0.000	0.248
$\gamma = 5$	0.247	0.553	0.001	0.110	0.620	0.132	0.337
$\gamma = 10$	0.118	0.273	0.006	0.073	0.330	0.546	0.654

Table 5

Recursive Sample Averages for Bayesian Portfolios Weights: VAR(1) ALL and VAR(1)-DY vs. No Predictability

The table reports mean portfolio weights from the recursive solution of Bayesian optimal portfolio problems under alternative specifications of the investment horizon and of the constant coefficient of relative risk aversion (γ). The sample period over which the recursive calculation has been performed is 1994:12 – 2007:12. Besides the predictability weights implied by the VAR(1)-ALL and VAR(1)-DY models that include all predictors and the dividend yield, respectively, the table reports sample mean portfolio weights for the Gaussian IID case in which there is no predictability (but the portfolio problem has to be solved numerically). Bayesian weights are computed under a joint predictive density for real asset returns that is computed from standard uninformative priors on the unknown model coefficients. Two asset menus are also considered: with and without e-REITs.

	Gaussian IID Predictability							VAR(1) -- ALL							VAR(1) -- DY						
	RE	No RE	RE	No RE	e-REITs	RE	No RE	RE	No RE	RE	No RE	e-REITs	RE	No RE	RE	No RE	RE	No RE	e-REITs	RE	No RE
	Stocks		Bonds			3-month T-bills		Stocks		Bonds			3-month T-bills		Stocks		Bonds			3-month T-bills	
	Horizon: 1 month							Horizon: 1 month							Horizon: 1 month						
$\gamma = 2$	0.236	0.674	0.010	0.324	0.754	0.000	0.002	0.169	0.553	0.007	0.182	0.823	0.001	0.265	0.164	0.752	0.004	0.247	0.832	0.000	0.001
$\gamma = 5$	0.111	0.238	0.246	0.279	0.390	0.253	0.483	0.097	0.338	0.264	0.279	0.464	0.175	0.383	0.089	0.329	0.247	0.267	0.464	0.200	0.404
$\gamma = 10$	0.053	0.139	0.104	0.121	0.194	0.649	0.740	0.045	0.163	0.118	0.248	0.234	0.603	0.589	0.044	0.164	0.113	0.119	0.233	0.610	0.717
	Horizon: 3 months							Horizon: 3 months							Horizon: 3 months						
$\gamma = 2$	0.231	0.666	0.006	0.333	0.763	0.000	0.001	0.168	0.644	0.010	0.260	0.822	0.000	0.096	0.153	0.726	0.016	0.273	0.831	0.000	0.001
$\gamma = 5$	0.110	0.279	0.246	0.278	0.393	0.251	0.443	0.088	0.326	0.172	0.242	0.418	0.322	0.432	0.073	0.315	0.142	0.208	0.402	0.383	0.477
$\gamma = 10$	0.053	0.137	0.104	0.120	0.196	0.647	0.743	0.034	0.152	0.059	0.094	0.211	0.696	0.754	0.033	0.156	0.048	0.071	0.206	0.713	0.773
	Horizon: 6 months							Horizon: 6 months							Horizon: 6 months						
$\gamma = 2$	0.235	0.669	0.006	0.330	0.759	0.000	0.001	0.177	0.645	0.004	0.227	0.819	0.000	0.128	0.152	0.731	0.012	0.268	0.836	0.000	0.001
$\gamma = 5$	0.110	0.281	0.243	0.278	0.389	0.258	0.441	0.096	0.326	0.113	0.195	0.416	0.375	0.479	0.070	0.313	0.102	0.171	0.400	0.428	0.516
$\gamma = 10$	0.054	0.138	0.102	0.120	0.194	0.650	0.742	0.036	0.148	0.033	0.073	0.210	0.721	0.779	0.030	0.154	0.031	0.046	0.206	0.733	0.800
	Horizon: 12 months							Horizon: 12 months							Horizon: 12 months						
$\gamma = 2$	0.234	0.674	0.007	0.326	0.759	0.000	0.000	0.188	0.639	0.001	0.186	0.811	0.000	0.175	0.156	0.741	0.007	0.258	0.836	0.001	0.001
$\gamma = 5$	0.111	0.283	0.236	0.275	0.388	0.265	0.442	0.106	0.326	0.054	0.140	0.414	0.426	0.534	0.074	0.316	0.081	0.149	0.403	0.442	0.535
$\gamma = 10$	0.053	0.139	0.100	0.118	0.193	0.654	0.743	0.034	0.136	0.020	0.059	0.208	0.738	0.805	0.029	0.155	0.025	0.035	0.208	0.738	0.810
	Horizon: 24 months							Horizon: 24 months							Horizon: 24 months						
$\gamma = 2$	0.232	0.683	0.008	0.316	0.760	0.000	0.001	0.201	0.700	0.000	0.142	0.799	0.000	0.158	0.170	0.756	0.005	0.244	0.825	0.000	0.000
$\gamma = 5$	0.113	0.287	0.231	0.266	0.383	0.273	0.447	0.102	0.309	0.025	0.100	0.401	0.472	0.591	0.089	0.325	0.070	0.136	0.402	0.439	0.539
$\gamma = 10$	0.054	0.140	0.097	0.114	0.191	0.658	0.746	0.028	0.124	0.014	0.058	0.207	0.751	0.818	0.036	0.159	0.021	0.029	0.208	0.735	0.812
	Horizon: 60 months							Horizon: 60 months							Horizon: 60 months						
$\gamma = 2$	0.221	0.699	0.014	0.300	0.765	0.000	0.001	0.261	0.712	0.000	0.131	0.739	0.000	0.157	0.213	0.757	0.005	0.242	0.782	0.000	0.001
$\gamma = 5$	0.122	0.297	0.217	0.247	0.382	0.279	0.456	0.125	0.291	0.013	0.094	0.351	0.511	0.615	0.126	0.334	0.056	0.115	0.376	0.442	0.551
$\gamma = 10$	0.058	0.144	0.091	0.105	0.188	0.663	0.751	0.035	0.110	0.035	0.094	0.186	0.744	0.796	0.055	0.164	0.017	0.023	0.193	0.735	0.813

Table 6

Ex-post Performance of Classical Portfolios

The table reports the mean ex-post performance of classical optimal portfolios recursively computed over the sample 1994:12-2007:12, along with those of the 1/N and the Gaussian IID case (columns 1 and 2). Performance measures are computed for an investor with constant relative risk aversion respectively equal to 2, 5, 10 and different investment horizons (from 1 to 60 months). Five different forecasting models are considered: the four models including one predictor only (in columns 4 to 7) and the model including all predictors(ALL, column 3).Two asset menus are also considered: with and without E-REITs. Performance measures that are boldfaced are "best" for a given asset menu, investment horizon, and risk aversion coefficient, where "best" means maximum in the case of mean returns, Sharpe ratios, and certainty equivalent (CER), and minimum in the case of volatility.

		(1)		(2)		(3)		(4)		(5)		(6)		(7)	
		1/N		**Gaussian IID**		**VAR(1) -- DY**		**VAR(1) -- TERM Spread**		**VAR(1) -- DEF Spread**		**VAR(1) -- CPI Inflation**		**VAR(1) -- ALL**	
		RE	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**
		Investment Horizon: 1 month													
γ=2	**Annual Mean Returns**	6.07	4.97	7.93	**7.60**	8.49	6.76	8.49	6.47	8.44	6.54	**8.50**	6.43	8.40	4.81
	Annualized Volatility	**5.92**	**5.01**	13.97	14.48	13.92	13.07	13.98	12.88	13.96	13.11	13.91	12.79	13.96	11.05
	Annualized Sharpe Ratio	**0.33**	0.17	0.27	**0.24**	0.31	0.20	0.31	0.19	0.31	0.18	0.31	0.18	0.31	0.06
	Annualized CER	5.72	4.72	5.94	**5.45**	6.51	5.00	6.49	4.76	6.45	4.77	**6.53**	4.75	6.41	3.55
γ=5	**Annual Mean Returns**	6.07	4.97	**6.45**	**7.32**	6.20	4.99	5.97	4.74	6.14	6.49	6.05	4.76	6.18	4.27
	Annualized Volatility	**5.92**	**5.01**	10.87	14.15	8.94	6.12	8.50	5.89	9.20	6.49	8.41	5.75	9.23	6.33
	Annualized Sharpe Ratio	**0.33**	0.17	0.21	**0.22**	0.23	0.14	0.21	0.10	0.22	0.09	0.23	0.11	0.22	0.02
	Annualized CER	**5.18**	**4.35**	3.34	1.95	4.13	4.03	4.09	3.86	3.93	3.67	4.21	3.92	3.95	3.26
γ=10	**Annual Mean Returns**	**6.07**	4.97	5.42	**7.43**	5.19	4.61	5.06	4.46	5.18	4.52	5.09	4.51	5.16	4.22
	Annualized Volatility	5.92	5.01	8.79	14.09	4.51	3.05	**4.23**	2.89	4.68	3.24	4.24	**2.83**	4.67	3.10
	Annualized Sharpe Ratio	**0.33**	0.17	0.14	**0.23**	0.23	0.15	0.22	0.11	0.22	0.11	0.22	0.18	0.22	0.02
	Annualized CER	**4.26**	3.70	1.12	-4.17	4.14	**4.14**	4.13	4.04	4.04	3.98	4.17	4.10	4.02	3.73
		Investment Horizon: 3 months													
γ=2	**Annual Mean Returns**	6.16	4.96	8.71	**7.76**	9.28	6.94	9.28	6.61	9.20	6.68	**9.29**	6.53	9.16	7.04
	Annualized Volatility	**6.01**	**4.93**	13.97	14.48	14.03	13.44	14.08	13.14	14.08	13.49	14.03	13.03	14.17	11.31
	Annualized Sharpe Ratio	0.35	0.16	0.32	0.24	0.37	0.21	0.36	0.19	0.36	0.19	**0.37**	0.18	0.35	**0.26**
	Annualized CER	5.91	4.72	6.72	5.50	7.31	5.12	7.29	4.87	7.22	4.85	**7.32**	4.82	7.15	**5.73**
γ=5	**Annual Mean Returns**	6.26	4.96	**7.11**	**7.38**	6.73	5.04	6.39	4.77	6.62	4.85	6.47	4.83	6.64	4.57
	Annualized Volatility	**6.01**	**4.93**	10.83	14.72	8.86	6.17	8.34	5.84	9.21	6.41	8.34	5.68	9.38	6.13
	Annualized Sharpe Ratio	**0.35**	0.16	0.27	**0.22**	0.29	0.15	0.27	0.11	0.27	0.11	0.29	0.12	0.27	0.07
	Annualized CER	**5.37**	**4.36**	4.09	1.65	4.73	4.10	4.62	3.92	4.47	3.81	4.70	4.03	4.41	3.64
γ=10	**Annual Mean Returns**	**6.26**	4.96	6.05	**7.45**	5.52	4.68	5.34	4.47	5.47	4.52	5.38	4.54	5.39	4.37
	Annualized Volatility	6.01	4.93	8.87	14.65	4.38	3.14	**4.06**	2.93	4.57	3.26	4.12	**2.89**	4.59	3.01
	Annualized Sharpe Ratio	**0.35**	0.16	0.21	**0.23**	0.31	0.17	0.29	0.11	0.29	0.12	0.30	0.14	0.27	0.07
	Annualized CER	4.46	3.78	1.55	-4.87	**4.54**	**4.19**	4.50	4.04	4.41	3.99	4.51	4.13	4.32	3.92
		Investment Horizon: 6 months													
γ=2	**Annual Mean Returns**	6.24	4.81	9.06	**7.56**	9.67	6.81	9.65	6.40	9.58	6.48	**9.67**	6.31	9.49	5.39
	Annualized Volatility	**5.70**	**4.73**	14.33	14.61	14.22	13.15	14.27	12.74	14.29	13.15	14.24	12.64	14.39	12.84
	Annualized Sharpe Ratio	0.37	0.14	0.34	**0.23**	**0.39**	0.20	0.39	0.18	0.38	0.18	0.39	0.17	0.37	0.10
	Annualized CER	5.93	4.59	7.06	**5.35**	**7.70**	5.05	7.66	4.74	7.59	4.71	7.69	4.68	7.46	3.66
γ=5	**Annual Mean Returns**	6.24	4.81	**7.34**	**7.16**	6.93	5.05	6.57	4.68	6.80	4.76	6.70	4.73	6.93	4.82
	Annualized Volatility	**5.70**	**4.73**	10.63	14.56	8.78	6.01	8.09	5.60	9.15	6.17	8.07	5.45	9.34	6.25
	Annualized Sharpe Ratio	**0.37**	0.14	0.30	**0.21**	0.32	0.15	0.30	0.10	0.29	0.10	0.32	0.11	0.30	0.11
	Annualized CER	**5.46**	**4.27**	4.47	1.27	4.99	4.13	4.92	3.89	4.69	3.78	5.06	3.98	4.75	3.83
γ=10	**Annual Mean Returns**	6.24	4.81	**6.25**	**7.26**	5.67	4.68	5.40	4.45	5.53	4.47	5.51	4.52	5.57	4.35
	Annualized Volatility	5.70	4.73	8.57	14.54	4.32	3.19	**3.90**	2.94	4.47	3.22	3.94	**2.90**	4.41	3.03
	Annualized Sharpe Ratio	**0.37**	0.14	0.25	**0.21**	0.35	0.17	0.32	0.10	0.31	0.10	0.35	0.13	0.32	0.07
	Annualized CER	4.68	3.72	1.85	-6.00	4.72	**4.17**	4.63	4.02	4.52	3.95	**4.72**	4.11	4.59	3.90

Table 6 (continued)

Ex-post Performance of Classical Portfolios

		(1)		(2)		(3)		(4)		(5)		(6)		(7)	
		1/N		Gaussian IID		VAR(1) -- DY		VAR(1) -- TERM Spread		VAR(1) -- DEF Spread		VAR(1) -- CPI Inflation		VAR(1) -- ALL	
		RE	No RE	RE	No RE	RE	No RE	RE	No RE	RE	No RE	RE	No RE	RE	No RE
		Investment Horizon: 12 months													
γ=2	Annual Mean Returns	6.38	4.66	10.09	**7.57**	10.75	6.90	10.73	6.39	10.66	6.54	**10.79**	6.30	10.67	7.01
	Annualized Volatility	**5.98**	**3.99**	15.18	16.39	15.10	14.97	15.21	14.28	15.25	14.84	15.14	14.14	15.23	14.48
	Annualized Sharpe Ratio	0.37	0.10	0.32	**0.21**	0.44	0.18	0.43	0.16	0.43	0.16	**0.44**	0.15	0.43	0.20
	Annualized CER	6.05	4.40	7.99	4.77	8.68	4.54	8.62	4.24	8.53	4.21	**8.70**	4.20	8.55	**4.79**
γ=5	Annual Mean Returns	6.38	4.66	**8.21**	**7.12**	7.76	5.06	7.18	4.62	7.52	4.75	7.35	4.67	7.98	5.18
	Annualized Volatility	**5.98**	**3.99**	10.76	16.40	9.26	7.13	8.11	6.40	9.38	6.98	8.09	6.21	9.88	6.00
	Annualized Sharpe Ratio	0.37	0.10	0.38	**0.18**	0.39	0.13	0.37	0.07	0.36	0.09	**0.40**	0.08	0.39	0.17
	Annualized CER	5.57	3.99	5.55	-0.50	5.74	3.76	5.63	3.56	5.45	3.44	**5.80**	3.67	5.71	**4.28**
γ=10	Annual Mean Returns	6.38	4.66	**7.03**	**7.21**	6.03	4.71	5.69	4.42	5.87	4.42	5.80	4.47	6.03	4.41
	Annualized Volatility	5.98	3.99	8.10	16.38	4.40	3.90	**3.74**	3.46	4.43	3.67	3.85	**3.45**	4.22	3.79
	Annualized Sharpe Ratio	0.37	0.10	0.36	**0.19**	0.43	0.14	0.41	0.08	0.39	0.07	**0.43**	0.09	0.42	0.07
	Annualized CER	4.80	3.30	4.00	-8.52	**5.10**	**3.96**	5.02	3.83	4.92	3.72	5.09	3.87	5.06	3.69
		Investment Horizon: 24 months													
γ=2	Annual Mean Returns	6.46	4.56	10.86	**7.46**	11.59	6.78	11.55	6.30	11.47	6.46	**11.63**	6.21	11.46	6.22
	Annualized Volatility	**6.67**	**6.13**	17.91	18.74	17.91	17.79	18.03	16.80	18.14	17.54	17.95	16.59	17.97	17.55
	Annualized Sharpe Ratio	0.35	0.07	0.38	**0.18**	0.42	0.15	0.41	0.13	0.40	0.13	**0.42**	0.13	0.41	0.12
	Annualized CER	6.07	**4.20**	8.29	3.71	9.05	3.46	8.97	3.32	8.85	3.18	**9.07**	3.30	8.90	3.04
γ=5	Annual Mean Returns	6.46	4.55	8.74	**7.09**	8.12	4.93	7.51	4.50	8.02	4.64	5.95	4.39	**8.89**	4.70
	Annualized Volatility	**6.67**	6.13	12.42	7.09	10.44	8.59	8.97	7.70	10.71	8.41	9.03	7.48	11.92	**5.81**
	Annualized Sharpe Ratio	0.35	0.07	0.37	**0.16**	0.38	0.09	0.38	0.05	0.36	0.06	0.39	0.05	**0.40**	0.10
	Annualized CER	5.48	3.64	5.76	-3.07	5.84	3.07	5.84	2.99	5.66	2.76	5.99	3.14	**6.06**	**3.96**
γ=10	Annual Mean Returns	6.46	4.55	**7.33**	**7.21**	6.15	4.61	5.80	4.33	6.06	4.34	5.95	4.38	6.40	4.56
	Annualized Volatility	6.67	6.13	8.54	18.89	4.68	4.92	**3.87**	4.41	4.75	4.63	4.01	**4.39**	5.17	4.85
	Annualized Sharpe Ratio	0.35	0.07	0.37	**0.16**	0.43	0.09	0.43	0.04	0.40	0.04	**0.45**	0.05	0.44	0.09
	Annualized CER	4.55	2.67	**5.31**	-10.54	5.22	**3.49**	5.17	3.43	5.08	3.29	5.25	3.48	5.28	3.44
		Investment Horizon: 60 months													
γ=2	Annual Mean Returns	5.78	3.76	9.61	**4.25**	10.72	3.74	10.60	3.45	10.50	3.62	**10.82**	3.40	10.80	3.26
	Annualized Volatility	**6.12**	**5.86**	22.14	20.30	22.06	20.34	22.63	18.39	23.11	20.27	22.69	17.98	22.01	19.48
	Annualized Sharpe Ratio	0.27	-0.07	0.25	**0.01**	0.30	-0.02	0.29	-0.04	0.28	-0.03	0.29	-0.04	**0.30**	-0.05
	Annualized CER	5.48	**3.49**	6.64	1.59	7.84	1.09	7.56	1.22	7.30	1.00	7.77	1.25	**7.94**	0.96
γ=5	Annual Mean Returns	5.78	3.76	8.01	4.01	8.08	4.30	6.88	3.44	7.64	3.53	7.14	3.58	**9.01**	**5.66**
	Annualized Volatility	**6.12**	**5.86**	15.79	19.96	12.42	11.51	9.69	8.13	11.93	7.89	9.86	7.92	14.41	10.04
	Annualized Sharpe Ratio	0.27	-0.07	0.25	-0.01	0.32	0.01	0.28	-0.09	0.29	-0.08	0.30	-0.07	**0.34**	**0.15**
	Annualized CER	5.01	3.15	4.68	-0.76	5.26	2.47	5.25	2.36	5.37	2.47	5.48	2.56	**5.51**	**4.22**
γ=10	Annual Mean Returns	5.78	3.76	**6.58**	4.09	6.28	4.33	5.49	3.89	5.82	3.87	5.69	3.93	6.65	**4.64**
	Annualized Volatility	6.19	5.86	8.50	20.00	4.92	7.01	**3.44**	5.29	4.21	**4.57**	3.55	5.22	5.79	6.87
	Annualized Sharpe Ratio	0.27	-0.07	0.29	0.00	**0.43**	0.04	0.39	-0.05	0.40	-0.06	0.43	-0.04	0.43	**0.07**
	Annualized CER	4.22	2.71	4.89	-2.46	5.23	3.16	5.01	3.04	5.15	3.18	5.18	3.10	**5.26**	**3.41**

Table 7

Ex-post Performance of Bayesian Portfolios

The table reports the mean ex-post performance of Bayesian optimal portfolios recursively computed over the sample 1994:12-2007:12, along with those of the 1/N and the Gaussian IID case (columns 1 and 2). Performance measures are computed for an investor with constant relative risk aversion respectively equal to 2, 5, 10 and different investment horizons (from 1 to 60 months). Five different forecasting models are considered: the four models including one predictor only (in columns 4 to 7) and the model including all predictors(ALL, column 3).Two asset menus are also considered: with and without E-REITs. Performance measures that are boldfaced are "best" for a given asset menu, investment horizon, and risk aversion coefficient, where "best" means maximum in the case of mean returns, Sharpe ratios, and certainty equivalent (CER), and minimum in the case of volatility.

		(1)		(2)		(3)		(4)		(5)		(6)		(7)	
		1/N		**Gaussian IID**		**VAR(1) -- DY**		**VAR(1) -- TERM Spread**		**VAR(1) -- DEF Spread**		**VAR(1) -- CPI Inflation**		**VAR(1) -- ALL**	
		RE	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**
		Investment Horizon: 1 month													
γ=2	**Annual Mean Returns**	6.07	4.97	8.19	5.45	8.27	5.91	8.29	5.94	8.27	5.88	8.33	5.94	**8.45**	**6.01**
	Annualized Volatility	**5.92**	**5.01**	12.85	10.31	13.45	11.68	13.44	11.71	13.44	11.67	13.40	11.71	13.55	9.61
	Annualized Sharpe Ratio	**0.33**	0.17	0.32	0.13	0.31	0.15	0.31	0.15	0.31	0.15	0.31	0.15	0.32	**0.19**
	Annualized CER	5.72	4.72	6.51	4.37	6.42	4.50	6.44	4.53	6.42	4.48	6.49	4.53	**6.57**	**5.08**
γ=5	**Annual Mean Returns**	**6.07**	**4.97**	5.38	4.54	6.04	4.47	5.98	4.47	6.02	4.43	6.02	4.47	6.01	4.21
	Annualized Volatility	**5.92**	5.01	6.86	**4.56**	7.66	5.22	7.68	5.25	7.77	5.31	7.67	5.25	7.96	5.32
	Annualized Sharpe Ratio	**0.33**	**0.17**	0.18	0.09	0.25	0.06	0.24	0.06	0.24	0.05	0.24	0.06	0.23	0.01
	Annualized CER	**5.18**	**4.35**	4.16	4.02	4.53	3.78	4.45	3.77	4.45	3.72	4.49	3.77	4.37	3.49
γ=10	**Annual Mean Returns**	**6.07**	**4.97**	4.80	4.36	5.01	4.32	5.07	4.29	5.10	4.30	5.10	4.34	5.08	4.19
	Annualized Volatility	5.92	5.01	**3.37**	**2.24**	3.81	2.62	3.81	2.63	3.87	2.67	3.80	2.63	3.95	2.62
	Annualized Sharpe Ratio	**0.33**	**0.17**	0.19	0.10	0.25	0.06	0.24	0.06	0.25	0.11	0.25	0.07	0.24	0.01
	Annualized CER	4.26	3.70	4.21	**4.11**	**4.35**	3.97	4.32	3.94	4.33	3.94	4.35	3.99	4.28	3.84
		Investment Horizon: 3 months													
γ=2	**Annual Mean Returns**	4.96	6.26	8.86	5.29	8.93	5.84	8.95	5.79	8.96	5.85	8.95	5.80	**9.09**	**7.08**
	Annualized Volatility	**4.93**	**6.01**	13.23	10.13	13.66	11.22	13.62	11.23	13.65	11.34	13.61	11.23	13.72	9.99
	Annualized Sharpe Ratio	0.35	0.16	0.36	0.11	0.35	0.15	0.35	0.15	0.35	0.15	0.35	0.15	**0.36**	**0.29**
	Annualized CER	5.91	4.72	7.10	4.27	7.06	4.58	7.09	4.53	7.09	4.55	7.09	4.53	**7.20**	**6.09**
γ=5	**Annual Mean Returns**	4.96	**6.26**	5.91	4.35	6.04	4.57	5.99	4.53	6.01	4.50	6.02	4.53	**6.10**	4.42
	Annualized Volatility	**4.93**	6.01	6.82	**4.26**	6.61	4.79	6.54	4.72	6.84	4.99	6.57	4.72	7.10	4.94
	Annualized Sharpe Ratio	**0.35**	**0.16**	0.26	0.05	0.29	0.09	0.28	0.08	0.27	0.07	0.29	0.08	0.27	0.06
	Annualized CER	**5.37**	**4.36**	4.75	3.91	4.94	4.00	4.92	3.97	4.83	3.88	4.93	3.98	4.82	3.82
γ=10	**Annual Mean Returns**	4.96	**6.26**	**5.60**	4.30	5.16	4.42	5.16	4.39	5.14	4.36	5.18	4.40	5.14	4.30
	Annualized Volatility	4.93	6.01	3.74	**2.16**	3.29	2.45	4.62	4.10	3.40	2.58	**3.27**	2.43	3.42	2.41
	Annualized Sharpe Ratio	0.35	**0.16**	**0.39**	0.07	0.31	0.11	0.31	0.10	0.29	0.08	0.32	0.10	0.29	0.07
	Annualized CER	4.46	3.78	**4.90**	4.07	4.61	**4.13**	4.62	4.10	4.56	4.03	4.64	4.10	4.55	4.02
		Investment Horizon: 6 months													
γ=2	**Annual Mean Returns**	6.24	4.81	9.34	5.28	9.26	5.65	9.17	5.52	9.26	5.62	9.18	4.70	**9.44**	**6.37**
	Annualized Volatility	**5.70**	**4.73**	13.08	9.78	13.71	10.85	13.63	10.79	13.71	11.08	13.63	10.31	13.90	10.96
	Annualized Sharpe Ratio	0.37	0.14	**0.40**	0.12	0.37	0.14	0.37	0.13	0.37	0.13	0.37	0.05	0.38	**0.20**
	Annualized CER	5.93	4.59	**7.66**	4.33	7.42	4.46	7.35	4.33	7.41	4.37	7.36	3.60	7.55	**5.18**
γ=5	**Annual Mean Returns**	6.24	**4.81**	5.91	4.33	6.17	4.59	6.05	4.56	6.10	4.40	6.10	4.57	**6.28**	4.45
	Annualized Volatility	**5.70**	4.73	6.48	**4.05**	6.33	4.54	6.14	4.46	6.63	4.75	6.14	4.45	6.88	4.81
	Annualized Sharpe Ratio	**0.37**	**0.14**	0.27	0.05	0.32	0.10	0.31	0.09	0.29	0.05	0.32	0.10	0.31	0.06
	Annualized CER	**5.46**	**4.27**	4.88	3.93	5.17	4.08	5.11	4.07	5.01	3.83	5.17	4.08	5.10	3.87
γ=10	**Annual Mean Returns**	**6.24**	**4.81**	5.06	4.27	5.25	4.47	5.19	4.43	5.22	4.39	5.24	4.44	5.23	4.31
	Annualized Volatility	5.70	4.73	3.11	**2.20**	3.14	2.49	3.03	2.43	3.31	2.62	**3.02**	2.43	3.27	2.44
	Annualized Sharpe Ratio	**0.37**	**0.14**	0.29	0.06	0.35	0.13	0.35	0.12	0.33	0.09	0.36	0.12	0.33	0.07
	Annualized CER	4.68	3.72	4.59	4.04	4.75	**4.16**	4.73	4.14	4.68	4.05	**4.78**	4.14	4.69	4.01

Table 7 (continued)

Ex-post Performance of Bayesian Portfolios

		(1)		(2)		(3)		(4)		(5)		(6)		(7)	
		1/N		**Gaussian IID**		**VAR(1) -- DY**		**VAR(1) -- TERM Spread**		**VAR(1) -- DEF Spread**		**VAR(1) -- CPI Inflation**		**VAR(1) -- ALL**	
		RE	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**	**RE**	**No RE**
		Investment Horizon: 12 months													
$\gamma=2$	**Annual Mean Returns**	6.38	4.66	10.23	5.10	10.33	**5.68**	10.11	5.40	10.30	5.52	10.06	5.38	**10.40**	3.06
	Annualized Volatility	**5.98**	**3.99**	13.85	10.86	14.43	12.20	14.25	11.84	14.54	12.48	14.29	11.84	14.68	10.36
	Annualized Sharpe Ratio	0.37	0.10	**0.44**	0.09	0.43	**0.13**	0.42	0.11	0.42	0.11	0.41	0.10	0.43	-0.11
	Annualized CER	6.05	**4.40**	**8.49**	3.90	8.43	4.14	8.23	3.94	8.38	3.88	8.20	3.92	8.45	1.94
$\gamma=5$	**Annual Mean Returns**	6.38	**4.66**	6.47	4.20	6.71	4.64	6.48	4.54	6.67	4.49	6.54	4.55	**6.97**	4.53
	Annualized Volatility	5.98	**3.99**	6.44	4.56	6.34	5.25	5.85	5.03	6.69	5.50	**5.82**	5.03	7.06	5.58
	Annualized Sharpe Ratio	0.37	**0.10**	0.36	0.01	0.40	0.09	0.40	0.08	0.38	0.06	**0.41**	0.08	0.40	0.07
	Annualized CER	5.57	**3.99**	5.50	3.69	5.76	3.95	5.67	3.90	5.62	3.71	5.74	3.91	**5.79**	3.73
$\gamma=10$	**Annual Mean Returns**	**6.38**	**4.66**	5.32	4.22	5.52	4.51	5.37	4.40	6.25	4.40	5.38	4.38	5.52	4.34
	Annualized Volatility	5.98	3.99	3.07	**2.57**	3.07	3.05	**2.74**	2.87	3.50	3.18	2.83	2.87	3.19	2.90
	Annualized Sharpe Ratio	0.37	0.10	0.38	0.03	0.45	**0.12**	0.44	0.09	**0.60**	0.08	0.44	0.08	0.43	0.07
	Annualized CER	4.80	3.30	4.87	3.91	5.07	**4.05**	5.01	3.99	**5.67**	3.90	4.99	3.98	5.03	3.93
		Investment Horizon: 24 months													
$\gamma=2$	**Annual Mean Returns**	6.46	4.56	10.76	5.00	11.06	**5.80**	10.66	5.23	10.87	5.25	10.50	5.21	**11.07**	5.72
	Annualized Volatility	**6.67**	**6.13**	15.93	12.00	16.94	14.49	16.69	13.49	17.20	14.32	16.53	13.49	17.07	11.72
	Annualized Sharpe Ratio	0.35	0.07	**0.42**	0.11	0.41	0.11	0.39	0.08	0.39	0.08	0.38	0.08	0.41	**0.13**
	Annualized CER	6.07	4.20	8.72	3.26	**8.78**	3.64	8.43	3.32	8.51	3.05	8.30	3.30	8.75	**4.40**
$\gamma=5$	**Annual Mean Returns**	6.46	4.55	6.62	4.12	7.07	4.68	6.57	4.45	6.95	4.55	6.57	4.46	**7.31**	**4.70**
	Annualized Volatility	6.67	6.13	7.06	**5.52**	7.06	6.61	**6.00**	5.98	7.06	6.48	6.01	5.98	7.49	6.57
	Annualized Sharpe Ratio	0.35	0.07	0.35	0.00	0.41	0.08	0.40	0.05	0.40	0.06	0.40	0.05	**0.42**	**0.08**
	Annualized CER	5.48	**3.64**	5.52	3.39	6.00	3.62	5.80	3.58	5.89	3.49	5.79	3.59	**6.12**	3.63
$\gamma=10$	**Annual Mean Returns**	**6.46**	**4.55**	5.40	4.15	5.64	4.49	5.36	4.30	5.51	4.30	5.43	4.31	5.63	4.36
	Annualized Volatility	6.67	6.13	3.10	**3.36**	3.25	4.01	**2.68**	3.63	3.21	3.74	2.75	3.62	3.28	3.53
	Annualized Sharpe Ratio	0.35	**0.07**	0.40	0.00	0.46	0.09	0.45	0.04	0.43	0.04	**0.47**	0.05	0.45	0.06
	Annualized CER	4.55	2.67	4.97	3.64	**5.18**	3.76	5.05	3.70	5.06	3.64	5.10	3.71	5.15	**3.79**
		Investment Horizon: 60 months													
$\gamma=2$	**Annual Mean Returns**	5.78	3.76	9.32	2.68	9.79	3.64	8.86	2.86	9.30	3.26	8.90	2.29	**9.79**	**6.14**
	Annualized Volatility	**6.12**	**5.86**	18.59	13.00	18.94	15.38	18.97	12.91	19.68	13.01	18.32	12.89	18.31	12.63
	Annualized Sharpe Ratio	0.27	-0.07	0.28	0.00	0.30	-0.03	0.25	-0.10	0.26	-0.07	0.26	-0.10	**0.31**	**0.16**
	Annualized CER	5.48	3.49	7.11	1.36	7.52	2.03	6.59	1.64	6.86	2.03	6.76	1.64	**7.63**	**5.12**
$\gamma=5$	**Annual Mean Returns**	5.78	3.76	6.09	3.44	6.75	4.24	5.92	3.79	6.26	3.81	6.01	3.79	**7.01**	**4.41**
	Annualized Volatility	6.12	5.86	6.86	5.98	6.53	7.54	5.30	6.22	6.07	**5.59**	**5.13**	6.22	6.62	6.61
	Annualized Sharpe Ratio	0.27	-0.07	0.28	0.00	0.40	0.01	0.33	-0.06	0.35	-0.06	0.36	-0.06	**0.43**	**0.04**
	Annualized CER	5.01	3.15	5.21	2.83	5.91	3.34	5.38	3.13	5.59	3.25	5.51	3.13	**6.17**	**3.71**
$\gamma=10$	**Annual Mean Returns**	**5.78**	3.76	5.15	3.86	5.51	**4.32**	5.06	4.01	5.24	4.06	5.18	4.00	5.53	4.28
	Annualized Volatility	6.12	5.86	2.62	4.19	2.53	5.07	1.94	4.31	**1.66**	3.49	1.92	4.32	2.25	**3.68**
	Annualized Sharpe Ratio	0.27	-0.07	0.38	0.00	0.54	0.03	0.47	-0.03	**0.66**	-0.02	0.54	-0.03	0.61	**0.03**
	Annualized CER	4.22	2.71	4.86	3.29	5.24	3.54	4.91	3.39	5.12	3.63	5.02	3.39	**5.32**	**3.81**

Table 8
Differences in CER and Sharpe Ratio Between Best-Performing VAR and 1/N

The table reports the differences (Δ) in CER (upper panel) and Sharpe ratios (bottom panel) between the best predictive/optimizing VAR model and the *1/N* for γ = 5 and asset menus that includes e-REITs, across different time horizons. Negative values mean that 1/N outperforms the best optimizing model. The identity of the best optimizing VAR model is reported in parenthesis.

	T=1		T=3		T=6		T=12		T=24		T=60	
	Δ CER											
Classical	(CPI)	-0.929	(DY)	-0.640	(CPI)	-0.393	(CPI)	0.230	(ALL)	0.580	(ALL)	0.449
Bayesian	(DY)	-0.656	(DY)	-0.432	(DY)	-0.285	(ALL)	0.226	(ALL)	0.642	(ALL)	1.161
	T=1		**T=3**		**T=6**		**T=12**		**T=24**		**T=60**	
	Δ Sharpe Ratio											
Classical	(DY)	-0.095	(DY)	-0.061	(DY,CPI)	-0.050	(CPI)	0.021	(ALL)	0.050	(ALL)	0.070
Bayesian	(DY)	-0.078	(DY)	-0.066	(DY)	-0.066	(CPI)	0.037	(ALL)	0.074	(ALL)	0.166

Figure 1

Conditional Expected Returns

Figure 1 plots one-month ahead predicted means, variances and correlations for annualized returns on stocks, bonds and eREITs These real time forecasts are obtained from the full VAR(1)-ALL estimated using classical methods.