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Mas-Colell Bargaining Set of Large Games[1]

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Abstract

We study the equivalence between the MB-set and the core in the general context of games with a measurable space of players. In the first part of the paper, we study the problem without imposing any restriction on the class of games we consider. In the second part, we apply our findings to specific classes of games for which we provide new equivalence results. These include non-continuous convex games, exact non-atomic market games and non-atomic non-exact games. We also introduce, and characterize, a new class of games, which we call *thin* games. For these, we show not only that the MB-set is equal to the core, but also that it is the unique stable set in the sense of von Neumann and Morgenstern. Finally, we study the relation between thin games, market games and convex games.

JEL Classification: C 71

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1 Introduction

The Mas-Colell Bargaining set as a solution for cooperative games was introduced by A. Mas-Colell in [26], where he showed that for atomless economies his bargaining set consists of the Walrasian equilibria. Shitoviz [30] later extended this result to more general economies. The idea of Mas-Colell Bargaining set is in the same spirit as that underlying various other Bargaining sets that, starting with Davis and Maschler's [8], have been proposed in the literature (see [10], [33] and [34]). Roughly speaking, it goes as follows. An allocation ξ of the resources is proposed. A subset A of the players objects to ξ if they can allocate among themselves the resources available to them in such a way that every member of A is better off with respect to the allocation ξ. A counter-objection to this objection is another objection to ξ, which guarantees that a certain subset of players $D \subset A$ is better off with respect to the previous objection. An objection is called justified if there is no counter-objection to it, and a Bargaining set is the set of all allocations for which there is no justified objection. The rationale is clear: if a subset A of players can make a justified objection to ξ, then ξ will never take place because players in A have both the incentives and the power to make sure that that would not happen. The actual definition of Mas-Colell Bargaining set (MB-set, henceforth) has later been refined by Einy et al. in [13]. In that paper, they showed that for continuous convex games the MB-set not only coincides with the core, but it is also stable in the sense of von Neumann and Morgenstern. Later, Marinacci and Montrucchio [23, Corollary 2] (in a paper which is mainly devoted to the stability of the core) showed that the equivalence between the MB-set and the core also holds for exact non-atomic market games with finite-dimensional cores.

Here, we study the equivalence between the MB-set and the core in the general context of games with a measurable space of players. The paper is divided into two parts. In Part I, we study the problem of equivalence without imposing any restriction on the class of games we consider. In Part II, we apply our findings to specific classes of games for which we provide new equivalence results. For each part, we provide a separate roadmap so to allow the reader to select the path better suited to his/her own interests.

Notation. The notation employed throughout the paper is standard. For a measurable space (Ω, Σ), $B(\Sigma)$ denotes the Banach space of bounded Σ-measurable functions equipped with the sup-norm. The subset of $B(\Sigma)$ consisting of all $f \in B(\Sigma)$ such that $0 \leq f \leq 1$ is called the *space of ideal coalitions*, and denoted by $B_1(\Sigma)$. The indicator function of a set E, will be usually denoted by χ_E.

The dual of $B(\Sigma)$, that is the space of bounded charges endowed with the variation norm, is denoted by $ba(\Sigma)$. If $m \in ba(\Sigma)$ and $A \in \Sigma$, by m^A we denote the charge in $ba(\Sigma)$ defined by $m^A(C) = m(A \cap C)$ for all $C \in \Sigma$. The subsets of $ba(\Sigma)$ consisting of σ-additive measures and of non-atomic measures are denoted by $ca(\Sigma)$ and $na(\Sigma)$, respectively. The notation $ca_\alpha(\Sigma)$ ($na_\alpha(\Sigma)$) indicates all measures μ in $ca(\Sigma)$ ($na(\Sigma)$) for which $\mu(\Omega) = \alpha$. If $\lambda \in ca(\Sigma)$, $ca(\lambda)$ denotes the set of all measures in $ca(\Sigma)$ which are absolutely continuous with respect to λ. We shall use the short-hand notation ba and ca whenever no confusion may arise. Finally, if X is a Banach space and X^* is its dual, we often denote the duality pairing by $\langle x, x^* \rangle$, $x \in X$ and $x^* \in X^*$. All unexplained notation and terminology is also fully standard, and can be found nearly in any paper on the subject (see, for instance, [24] for a rather comprehensive reference).

PART I: THEORY

After stating the main definition (Section 2), we give a characterization of justified objections in Section 3. This readily leads to establishing necessary and sufficient conditions for the equivalence between the MB-set and the core (Theorem 1, Section 4). Throughout this part, we stress the fundamental role played by the maximal excess game introduced by Solymosi in [32] and further studied by Holzman in [18]. In Section 5, we provide some criteria for checking whether or not a given game satisfies the conditions of Theorem 1. In Section 6, we introduce a new condition, called Property (H), which implies that for any allocation ξ not in the core, one can find a justified objection against it by using elements of the core only. Transparently, Property (H) guarantees the equivalence between the MB-set and the core. At an intuitive level, Property (H) is a demand that the core be "large" in that it has to contain sufficiently many justified objections. This intuition is confirmed in Proposition 9 (Section 7), where we show that Property (H) is equivalent to two other conditions: one is, indeed, a "core-largeness" condition previously introduced by Marinacci and Montrucchio in [23] and the other is the existence of a maximum for the maximal excess game (condition (SM) of Theorem 1). Section 7's main result is Theorem 4, which states that the last two conditions are, in turn, equivalent to the existence of a certain saddle point. As a sample of the applicability of our theory, we conclude the section (and with it PART I of the paper) by giving a proof of the equivalence result mentioned by [5] for unitary glove market games.

2 Mas-Colell bargaining set

A transferable utility (TU) game is a triple (ν, Ω, Σ), where Ω is the set of players, Σ is a σ-algebra of coalitions of Ω and $\nu : \Sigma \to \mathbb{R}$, with $\nu(\varnothing) = 0$. In this paper, we consider only games that are bounded, that is such that $\sup_{E\in\Sigma} |\nu(E)| < \infty$. Also, in order to avoid tedious complications, we will always assume that the σ-algebra Σ contains all the singletons of Ω. This ensures that for any non-empty $A \in \Sigma$ there is a positive measure $m \in ca(\Sigma)$ for which $m(A) > 0$. The *core* of a game ν is

$$core(\nu) = \{m \in ba : m(\Omega) = \nu(\Omega) \text{ and } m(E) \geq \nu(E) \text{ for all } E \in \Sigma\}.$$

The core is a weak*-compact subset of ba. Games having nonempty cores are called *balanced*. Given a game (ν, Ω, Σ), the set $I^*(\nu) = \{\xi \in ba : \xi(\Omega) \leq \nu(\Omega)\}$ is the set of all *preimputations*.

We are now ready to give the definition of the Mas-Colell Bargaining set. We adopt the formulation of Einy et al. [13]. Given a preimputation $\xi \in I^*(\nu)$, an *objection* to ξ is a pair (A, η), $A \in \Sigma$ and $\eta \in ba$, which satisfies $\eta(A) \leq \nu(A)$, $\xi(A) < \eta(A)$ and $\xi(B) \leq \eta(B)$ for all $B \subseteq A$. A *counter objection* to the objection (A, η), is a pair (C, ζ) for which:

(i) $C \in \Sigma$, $\zeta \in ba$ and $\zeta(C) \leq \nu(C)$,

(ii) If $B \subseteq A \cap C$ then $\zeta(B) \geq \eta(B)$,

(iii) If $D \subseteq C \setminus A$ then $\zeta(D) \geq \xi(D)$,

(iv) $\zeta(C) > \eta(A \cap C) + \xi(C \setminus A)$.

An objection to ξ is called *justified* if there is no counter objection to it.

Definition 1 *The Mas-Colell bargaining set, denoted by $MB(\nu)$, is the collection of all preimputations against which there is no justified objection.*

All measures (preimputations, objections and counter-objections) which appear in the definition are elements of ba. One could give a more restrictive definition of MB-set by requiring that either objections or counter-objections be elements of ca. This gives rise to four different definitions of MB-set. To keep track of these possibilities, we will be using the notation $MB(\nu) - (\cdot, \cdot)$, where $(\cdot, \cdot)$ denotes, respectively, the type of objections and counter-objections that are allowed. For instance, we shall write $MB(\nu) - (ca, ba)$ to mean that objections are in ca, while counter objections are in ba. $MB(\nu) - (ca, ba)$ is the largest bargaining set, $MB(\nu) - (ba, ca)$ is the smallest and the other two are sandwiched between them. The notation $MB(\nu)$ without any qualification stands for $MB(\nu) - (ba, ba)$. We conclude this section by observing the well-known fact that elements of $core(\nu)$ are immune to any objection.

Proposition 1 *A member $\xi \in I^*(\nu)$ is immune to any objection if and only if $\xi \in core(\nu)$. Thus, $core(\nu) \subseteq MB(\nu) - (ba, ca)$.*

Proof. It suffices to prove that each $\xi \in I^*(\nu) \setminus core(\nu)$ has at least one objection. By definition $\xi(U) < \nu(U)$ for some coalition U. Let $m \in ca^+$ be such that $m(U) > 0$. Then, for $\alpha > 0$ and sufficiently small, we have $\xi(U) + \alpha m(U) \leq \nu(U)$ and one easily sees that $(U, \xi + \alpha m)$ is an objection against ξ. ■

3 Maximal excess games

If a preimputation $\xi \notin core(\nu)$, then (from Proposition 1) there always exists an objection (A, η) to ξ. In order to understand the structure of the MB-set, it is clear from its very definition that we need to know when it is that (A, η) is a justified objection. This is accomplished by Proposition 2 below, which also brings to light the fundamental role played by the game

$$\nu_\xi(A) = \sup\{\nu(E) - \xi(E) : E \subseteq A,\ E \in \Sigma\},$$

in the study of the MB-set. The game ν_ξ – called the *maximal excess game* associated with ν and ξ – is clearly monotone and bounded provided ν is bounded. Moreover, $\nu_\xi \equiv 0$ if and only if $\xi \in core(\nu)$.

Proposition 2 *Let $\xi \notin core(\nu)$ be a preimputation. The following conditions are equivalent:*

(i) (A, η) is a justified objection at ξ;

(ii) (A, η) is an objection at ξ, and for all $C \in \Sigma$

$$\nu(C) \leq \eta(A \cap C) + \xi(C \setminus A); \tag{1}$$

(iii) for the pair (A, η),

$$\sup_{E \in \Sigma} \nu(E) - \xi(E) = \nu_\xi(\Omega) = \nu(A) - \xi(A) \tag{2}$$

holds and $\eta^A - \xi^A \in core(\nu_\xi)$;

Proof. (i) $\Longrightarrow$ (ii) Suppose that

$$\nu(C) > \eta(A \cap C) + \xi(C \setminus A) \tag{3}$$

holds for some C. Pick $m \in ca^+$ so that $m(C) > 0$. We are going to show that for $\varepsilon > 0$ small enough and for $\zeta = \eta^A + \xi^{C \setminus A} + \varepsilon m$, the pair (C, ζ) is a counter objection at (A, η), thus contradicting (i). In fact, if $D \subseteq A \cap C$, then $\zeta(D) = \eta(D) + \varepsilon m(D) \geq \eta(D)$. If $D \subseteq C \setminus A$, then $\zeta(D) = \xi(D) + \varepsilon m(D) \geq \xi(D)$. Moreover,

$$\begin{aligned} \zeta(C) &= \eta(A \cap C) + \xi(C \setminus A) + \varepsilon m(C) \\ &> \eta(A \cap C) + \xi(C \setminus A) \end{aligned}$$

and, by (3), $\zeta(C) \leq \nu(C)$ for ε small enough. That is, (C, ζ) is a counter objection at (A, η).

(ii) $\Longrightarrow$ (iii) From (1), by subtracting $\xi(C)$ on both sides, we obtain

$$\nu(C) - \xi(C) \leq \eta(A \cap C) - \xi(A \cap C) = \left(\eta^A - \xi^A\right)(C)$$

for all C. Since $\eta(C) \geq \xi(C)$ for all $C \subseteq A$, we have $\eta^A - \xi^A \geq 0$. Hence, $\nu_\xi(E) \leq \left(\eta^A - \xi^A\right)(E)$ for all $E \in \Sigma$. Moreover,

$$\left(\eta^A - \xi^A\right)(\Omega) = \eta(A) - \xi(A) \leq \nu(A) - \xi(A) \leq \nu_\xi(\Omega).$$

Hence, $\left(\eta^A - \xi^A\right)(\Omega) = \nu_\xi(\Omega) \Longrightarrow \eta^A - \xi^A \in core(\nu_\xi)$. Also, $\eta(A) - \xi(A) = \nu_\xi(\Omega)$ and (2) is true.

(iii) $\Longrightarrow$ (i). Assume that (iii) holds for a pair (A, η). We first show that (A, η) is an objection at ξ. Since $\xi \notin core(\nu)$, $\nu_\xi(\Omega) > 0$. Hence, $\left(\eta^A - \xi^A\right)(\Omega) > 0 \Longrightarrow \eta(A) > \xi(A)$. Similarly, $\eta(B) \geq \xi(B)$ for all $B \subseteq A$. Moreover,

$$\left(\eta^A - \xi^A\right)(\Omega) = \eta(A) - \xi(A) = \nu_\xi(\Omega) = \nu(A) - \xi(A).$$

Therefore, $\eta(A) = \nu(A)$ and (A, η) is an objection at ξ. Finally, for all $C \in \Sigma$,

$$\begin{aligned} \left(\eta^A - \xi^A\right)(C) &\geq \nu_\xi(C) \geq \nu(C) - \xi(C) \Longleftrightarrow \\ \nu(C) &\leq \left(\eta^A - \xi^A\right)(C) + \xi(C) = \eta(A \cap C) + \xi(C \setminus A). \end{aligned}$$

It is clear that (A,η) is immune to counter objections. For, if (C,ζ) were a counter objection, then we would have $\zeta(C) \leq \nu(C) \leq \eta(A \cap C) + \xi(C \setminus A)$, which violates point (iv) in the definition of bargaining set. ■

The next proposition is useful in that it gives a method for constructing a justified objection (if any) to a given preimputation ξ.

Proposition 3 *Any of the conditions in Proposition 2 is implied by*
(iv) for the pair (A,η), (2) holds and $\eta = \xi + u$ with $u \in core(\nu_\xi)$.

Proof. We prove that (iv) $\Longrightarrow$ (iii). Let $\nu_\xi(\Omega) = \nu(A) - \xi(A)$ and $\eta = u + \xi$ for some $u \in core(\nu_\xi)$. We have $u(\Omega) = \nu_\xi(\Omega) = \nu(A) - \xi(A)$ and $u(A) \geq \nu_\xi(A) \geq \nu(A) - \xi(A)$. As $u \geq 0$, $u = u^A$. Consequently, $\eta^A - \xi^A = u^A = u \in core(\nu_\xi)$ and (iii) holds. ■

By virtue of Proposition 3, we can construct a justified objection (if any) to a preimputation ξ as follows: first, take any coalition A for which $\nu(A) - \xi(A) = \nu_\xi(\Omega)$; then, pick any element $u \in core(\nu_\xi)$ (if nonempty). Then, $(A, \xi + u)$ is a justified objection to ξ.

3.1 Related literature

Maximal excess games were introduced by Solymosi [32] in his study of the Davis-Maschler bargaining set. Later, they have been used by Holzman [18] in his work on the comparability between the Davis-Maschler bargaining set and the MB-set. Holzman made the important observation that the characterization of the MB-set is intimately related to the balancedness of games ν_ξ, whenever ξ runs in the space of preimputations. Both Solymosi's and Holzman's analysis are restricted to games with finitely many players.

Proposition 2 is a straightforward extension of Observation 2.2 of Holzman [18], who also remarked the role played by condition (2). Condition (2), however, plays here a much more significant role than it does in finite games where it is automatically satisfied. Condition (1) has also been stated by [14, Lemma 5.1] for games with a countable set of players.

4 MB-set and the core

In this section as well as in Section 6, we study conditions guaranteeing the equivalence between the core and the MB-set. Necessary and sufficient conditions for this equivalence are easily obtained from Proposition 2 (see also [18, Cor. 2.3]). We have

Theorem 1 *$MB(\nu) = core(\nu)$ if and only if for all $\xi \in I^*(\nu)$, the following two conditions are both satisfied*
(SM) $\sup_{E\in\Sigma} \nu(E) - \xi(E)$ is a maximum; and
(C) $core(\nu_\xi) \neq \varnothing$.

Proof. Suppose that for some ξ either (SM) or (C) is violated. Clearly, $\xi \notin core(\nu)$. By Proposition 2 there is no justified objection at ξ. Hence $\xi \in MB(\nu)$, and $MB(\nu) \neq core(\nu)$.

Conversely, assume that both (SM) and (C) hold, and suppose that there exists $\xi \in MB(\nu) \setminus core(\nu)$. Let $A \in \Sigma$ be such that $\nu_\xi(\Omega) = \nu(A) - \xi(A)$, and let $u \in core(\nu_\xi)$.

Then, by (iv) Proposition 3, the pair $(A, u + \xi)$ is a justified objection to ξ, thus contradicting $\xi \in MB(\nu)$. ■

Thus, Theorem 1 states that two conditions of a rather different nature must be satisfied for the equality $MB(\nu) = core(\nu)$ to hold. Of course, this raises the question of how likely it is for an arbitrary game to satisfy both (SM) and (C). The next example describes a rather typical phenomenon.

Example 1 *Let $\left(\nu, \mathbb{N}, 2^{\mathbb{N}}\right)$ be a σ-additive and positive game on the natural numbers $\mathbb{N}$ (i.e., $\nu \in ca^{+}$). Let $\nu(\{i\}) > 0$ for all $i \in \mathbb{N}$. Clearly, $core(\nu) = \{\nu\}$. If $\pi \in ba^{+}$ is any purely additive measure such that $\pi(\mathbb{N}) \leq \nu(\mathbb{N})$ then $\pi \in MB(\nu) - (ba, ba)$. To see this, it suffices by Theorem 1 to check that (2) fails. We have*

$$\sup_{E \in \Sigma} \nu(E) - \pi(E) \leq \sup_{E \in \Sigma} \nu(E) = \nu(\mathbb{N}). \tag{4}$$

Clearly $\pi(F) = 0$, if $F \subset \mathbb{N}$ is finite. Hence,

$$\sup_{F \text{ is finite}} \nu(F) - \pi(F) = \nu(\mathbb{N}).$$

It follows that $\nu_{\pi}(\mathbb{N}) = \nu(\mathbb{N})$. This value, however, cannot be reached. For if it could, then we would have $\nu(A) - \pi(A) = \nu(\mathbb{N})$ for some coalition A, which implies $-\pi(A) = \nu(A^{c}) \Leftrightarrow \nu(A^{c}) = \pi(A) = 0$. As the support of ν is $\mathbb{N}$, it would follow that $A = \mathbb{N}$ and $\pi(\mathbb{N}) = 0$, a contradiction.

Example 1 makes it clear that there is little hope to establish core-equivalence results at a significant level of generality if we do not restrict the space of preimputations to be in ca. Because of this, for the remainder of the paper we are going to restrict our analysis to the σ-bargaining set, $MB^{\sigma}(\nu) = MB(\nu) \cap ca$. Consequently, the following reformulation of Theorem 1 is handy. The notation $core^{\sigma}(\cdot)$ stands for $core(\cdot) \cap ca$.

Theorem 2 *$MB^{\sigma}(\nu) - (ba, ba) = core^{\sigma}(\nu)$ holds, if and only if for all $\xi \in I^{*}(\nu) \cap ca$,*
(SM) $\sup_{E \in \Sigma} \nu(E) - \xi(E)$ is a maximum;
(C) $core(\nu_{\xi}) \neq \varnothing$.
Similarly, $MB^{\sigma}(\nu) - (ca, ba) = core^{\sigma}(\nu)$, if and only if (SM) holds and
(C^{σ}) $core^{\sigma}(\nu_{\xi}) \neq \varnothing$.

The easy proof is omitted. With regard to the second part of the statement, it should be noticed that while condition $core^{\sigma}(\nu_{\xi}) \neq \varnothing$ is clearly more demanding than $core(\nu_{\xi}) \neq \varnothing$, it is also true that $MB^{\sigma}(\nu) - (ba, ba) \subseteq MB^{\sigma}(\nu) - (ca, ba)$.

We close this section by recording one more consequence of Proposition 2. It asserts that, when we restrict to σ-additive preimputations, the four bargaining sets discussed in Section 2 are indeed fewer.

Proposition 4 *We have $MB^{\sigma}(\nu) - (ca, ba) = MB^{\sigma}(\nu) - (ca, ca)$.*

Proof. We have already observed that $MB^{\sigma}(\nu) - (ca, ca) \subseteq MB^{\sigma}(\nu) - (ca, ba)$. Let $\xi \notin MB^{\sigma}(\nu) - (ca, ca)$ and $\xi \in ca$. By definition, there is an objection (A, η) at ξ with $\eta \in ca$ which is immune to counter objections from ca. By mimicking the proof of Proposition 2, one sees that necessarily (1) holds. In turn, this implies that there are no counter objections in ba. That is, $\xi \notin MB^{\sigma}(\nu) - (ca, ba)$. ■

5 Conditions (SM) and (C)

It is often difficult to check whether or not a given game satisfies conditions (SM) and (C) for all $\xi \in ca$. Scope of this section is to provide some criteria for accomplishing this task. The next proposition shows that, when a game is λ-continuous with respect to $\lambda \in ca^{+}$, it suffices to check that (SM) and (C) hold only for preimputations in $ca(\lambda)$. We recall that a game is λ-continuous with respect to $\lambda \in ca^{+}$ if $\lambda(N) = 0$ implies that N is ν-null coalition. Namely, $\nu(E \cup N) = \nu(E)$ for all $E \in \Sigma$.

Proposition 5 *Let ν be a λ-continuous game with $\lambda \in ca^{+}$. Conditions (SM) and (C) hold provided they do for all preimputations $\xi \in ca(\lambda)$.*

Proof. Let $\xi \in ca$. By the Lebesgue Decomposition Theorem it holds $\xi = \xi^{a} + \xi^{s}$, with $\xi^{a} \perp \xi^{s}$ and $\xi^{a} \in ca(\lambda)$. Hence, there exists a coalition N such that $\lambda(N) = 0$ and $|\xi^{s}|(\Omega \setminus N) = 0$. By the Jordan Decomposition Theorem, there exists a decomposition $\xi^{s} = \xi^{s}_{+} - \xi^{s}_{-}$ and a decomposition $N = N^{+} \cup N^{-}$ such that ξ^{s} is positive on N^{+} and negative on N^{-}.

Fix a coalition A. For $E \subseteq A$, consider the coalition $\widetilde{E} = (E \setminus N) \cup (N^{-} \cap A) \subseteq A$ and notice that any coalition $E_1 \subseteq A$ admits the representation $E_1 = \widetilde{E}$ for some $E \subseteq A$ (it suffices to set $E = (E_1 \setminus N) \cup (N^{+} \cap A)$). Clearly

$$\nu\left(\widetilde{E}\right) - \xi\left(\widetilde{E}\right) = \nu(E) - \xi^{a}(E) + \xi^{s}_{-}(A).$$

Thus, we get

$$\sup_{E \subseteq A} \nu(E) - \xi(E) = \xi^{s}_{-}(A) + \sup_{E \subseteq A} \nu(E) - \xi^{a}(E). \tag{5}$$

Namely, $\nu_{\xi}(A) = \nu_{\xi^{a}}(A) + \xi^{s}_{-}(A)$ for all A. This implies $core(\nu_{\xi}) = core\left(\nu_{\xi^{a}}\right) + \xi^{s}_{-}$ which proves the first part of our claim. By setting $A = \Omega$ in (5), we obtain

$$\sup_{E \in \Sigma} \nu(E) - \xi(E) = \xi^{s}_{-}(\Omega) + \sup_{E \in \Sigma} \nu(E) - \xi^{a}(E). \tag{6}$$

We now prove that if $\sup_{E \in \Sigma} \nu(E) - \xi^{a}(E)$ is a maximum then $\sup_{E \in \Sigma} \nu(E) - \xi(E)$ is a maximum as well. Suppose that $\sup_{E \in \Sigma} \nu(E) - \xi^{a}(E) = \nu(A) - \xi^{a}(A)$, and consider the coalition $A_1 = (A \setminus N) \cup N^{-}$. Then

$$\nu(A_1) - \xi(A_1) = \nu(A) - \xi^{a}(A) + \xi^{s}_{-}(\Omega)$$

and the second part of the claim is true by (6). ■

The next theorem concerns condition (C) only. This is the only one that matters when the game is finite. In that setting, Solymosi [32] has identified several types of games having a "closure property", that is such that, for any preimputation, the maximal excess game and the original game are of same type. Convex games (see Proposition 15, below; see also Maschler, Peleg and Shapley [25]), monotonic veto-controlled games and assignment games (Granot and Granot [17]) all fall into this category. In such cases, balancedness of ν_ξ is easy to deduce. Here we give a result of different nature, which is based on the properties of the game's extension to the space of the ideal coalitions $B_1(\Sigma)$.

Theorem 3 *Assume that the game ν has an extension $\widehat{\nu} : B_1(\Sigma) \to \mathbb{R}$ satisfying:*
i) $\widehat{\nu}(\chi_A) \geq \nu(A)$ for all $A \in \Sigma$;
ii) $\widehat{\nu}$ is bounded, positively homogeneous and concave;
iii) $\sup_{f \in B_1(\Sigma)} [\widehat{\nu}(f) - \langle \xi, f \rangle] = \nu_\xi(\Omega)$ holds for any $\xi \in ba$.
Then $core(\nu_\xi) \neq \varnothing$.

Proof. Define the functional $\widehat{\nu_\xi} : B_1(\Sigma) \to \mathbb{R}$ by

$$\widehat{\nu_\xi}(f) = \sup_{0 \leq \varphi \leq f} \widehat{\nu}(\varphi) - \langle \xi, \varphi \rangle,$$

for all $f \in B_1(\Sigma)$. It is easy to check that $\widehat{\nu_\xi}$ is concave and linearly homogeneous. In particular, $\widehat{\nu_\xi}(\alpha\chi_\Omega) = \alpha\widehat{\nu_\xi}(\chi_\Omega)$ for $\alpha \in [0,1]$. By (i), we have $\widehat{\nu_\xi}(\chi_A) \geq \nu_\xi(A)$. Further, (iii) implies $\widehat{\nu_\xi}(\chi_\Omega) = \nu_\xi(\Omega)$.

The game $\widehat{\nu_\xi}$ is bounded by condition (ii). Therefore $\widehat{\nu_\xi}$ is sup-norm continuous and superdifferentiable over the interior of B_1. Denoting by ∂ the superdifferential operator, we have that $\partial\widehat{\nu_\xi}\left(2^{-1}\chi_\Omega\right) \neq \varnothing$. Let $\overline{m} \in \partial\widehat{\nu_\xi}\left(2^{-1}\chi_\Omega\right)$ with $\overline{m} \in ba$. It follows

$$\widehat{\nu_\xi}(f) \leq \widehat{\nu_\xi}\left(2^{-1}\chi_\Omega\right) + \left\langle \overline{m}, f - 2^{-1}\chi_\Omega \right\rangle \tag{7}$$

for all $f \in B_1(\Sigma)$. Setting $f = 0$, we get $m(\Omega) \leq \nu_\xi(\Omega)$. While, by setting $f = \chi_\Omega$, we obtain $m(\Omega) \geq \nu_\xi(\Omega)$. Hence, (7) becomes $\widehat{\nu_\xi}(f) \leq \langle \overline{m}, f \rangle$. In particular, $\nu_\xi(A) \leq \widehat{\nu_\xi}(\chi_A) \leq \overline{m}(A)$. Consequently, $\overline{m} \in core(\nu_\xi) \neq \varnothing$. ■

From the second part of Theorem 2, it is also useful to have a criterion for checking when $core^\sigma(\nu_\xi) \neq \varnothing$. The next proposition gives an easy one, which we will be using in several occasions. We recall that a game is inner (outer) continuous if $\nu(A_n) \to \nu(A)$ for all the sequences $A_n \uparrow A$ ($A_n \downarrow A$) and for all $A \in \Sigma$. In particular, a game is continuous if it is both inner and outer continuous.

Proposition 6 *$core^\sigma(\nu_\xi) = core(\nu_\xi)$ for all $\xi \in ca$, provided ν is inner continuous.*

Proof. As ν_ξ is monotone, a sufficient condition for all charges in its core to be σ-additive is that ν_ξ is continuous at Ω (see [28] and [24]). Namely $\nu_\xi(A_n) \to \nu_\xi(\Omega)$, as $A_n \uparrow \Omega$. Let $A_n \uparrow \Omega$ and E be any coalition. Clearly, $A_n \cap E \uparrow E$. We have $\nu_\xi(A_n) \geq \nu(A_n \cap E) - \xi(A_n \cap E)$. By taking limits, $\lim_n \nu_\xi(A_n) \geq \nu(E) - \xi(E)$, which implies $\lim_n \nu_\xi(A_n) \geq \nu_\xi(\Omega)$ thus proving the claim. ■

6 A sufficient condition for MB = Core

In this section, we introduce a new condition which guarantees that the equality MB-set = core holds. The reason for doing so is twofold. First, our condition clarifies a good deal of what is behind equivalence results previously established (see [13] and [23]). In fact, our condition suggests that equivalence results should be linked, in most cases, to a property of the core of being large in the (loose) sense of containing sufficiently many justified objections. We will make this intuition precise in the next section. Second, most of new equivalence results that we prove in the second part will be derived from this condition (precisely, from Corollary 1 below).

We recall that, given a (signed) charge μ, a *Hahn decomposition* for μ is a partition $\{A, A^c\}$ of Ω ($A \in \Sigma$) such that $\mu(B) \geq 0$ for every $B \subseteq A$ and $\mu(B) \leq 0$ for every $B \subseteq A^c$. A charge need not admit a Hahn decomposition (H-decomposition, for short), while a measure always does. Clearly, if $\{A, A^c\}$ is a Hahn decomposition for μ, we have $\mu(A) = \sup_{E\in\Sigma} \mu(E)$.

Definition 2 *We say that a game ν has Property (H) if for any $\xi \notin core(\nu)$, there exists $\eta \in core(\nu)$ such that:*
(i) $\{A, A^c\}$ is a Hahn decomposition for the measure $\eta - \xi$;
(ii) $\nu(A) = \eta(A)$.

Proposition 7 *Suppose that ν has Property (H). Then, for any $\xi \notin core(\nu)$ there exists a justified objection, (A, η) to ξ with $\eta \in core(\nu)$.*

Proof. Let $\xi \notin core(\nu)$. By assumption, there exists $\eta \in core(\nu)$ satisfying (i) and (ii) of the Definition 2 above. Consider the pair (A, η). We are going to show that (A, η) satisfies condition (iii) in Proposition 2. We have,

$$\begin{aligned}
\sup_{E\in\Sigma} \nu(E) - \xi(E) &\leq \sup_{E\in\Sigma} \eta(E) - \xi(E) = \eta(A) - \xi(A) \\
&= \nu(A) - \xi(A)
\end{aligned}$$

and hence (2) holds. Moreover, $\nu_\xi(\Omega) = \eta(A) - \xi(A) = \eta^A(\Omega) - \xi^A(\Omega)$. Finally, if C is any coalition, then

$$\begin{aligned}
\nu_\xi(C) &= \sup_{B\subseteq C} \nu(B) - \xi(B) \leq \sup_{B\subseteq C} \eta(B) - \xi(B) \\
&= \eta(C\cap A) - \xi(C\cap A) \leq \eta^A(C) - \xi^A(C).
\end{aligned}$$

Hence, $\eta^A - \xi^A \in core(\nu_\xi)$, and this completes the proof. ■

Proposition 7 readily leads to the following Corollary, which we specialize to the countably additive case.

Corollary 1 *Suppose that for all $\xi \in I^*(\nu) \cap ca$, ν has Property (H) and that, in addition, the measure η in Definition 2 is in $core^\sigma(\nu)$. Then, $MB^\sigma(\nu) - (ca, ba) = core^\sigma(\nu)$.*

We remark that this result remains true for any formulation of the bargaining set where objections are restricted to lie in a fixed subspace containing $core^\sigma(\nu)$.

7 Largeness and saddle property

At an intuitive level, Property (H) is a demand that the core of the game be in some sense "large". Following this intuition, it is then natural to ask whether or not there is a link between Property (H) and other "core-largeness" conditions that have been studied in the literature. A classic one is as follows.

$[L_0]$ A balanced game ν satisfies largeness condition (L_0) *if* $\xi \geq \nu$, $\xi \in ca$, *implies that there is* $\eta \in core\,(\nu)$ *such that* $\xi \geq \eta \geq \nu$.

This was introduced by Sharkey [29]. Marinacci and Montrucchio [23] strengthened this to the following one, which is easily seen to imply (L_0).

$[L_1]$ A balanced game ν satisfies largeness condition (L_1) if $\xi + k \geq \nu$, $\xi \in ca$ *and* k *is a non-negative scalar, implies that there is* $\eta \in core\,(\nu)$ *such that* $\xi + k \geq \eta \geq \nu$. The game ν satisfies (L_1^{σ}) if η can be selected in $core^{\sigma}\,(\nu)$.

Largeness condition (L_1) is linked to the maximal excess game by means of the following proposition, whose proof we omit because elementary.

Proposition 8 *A game* ν *satisfies largeness condition* (L_1) *if and only if for all* $\xi \in ca$ *there is* $\eta \in core\,(\nu)$ *such that* $\xi + \nu_{\xi}\,(\Omega) \geq \eta$.

The next proposition shows that Property (H) is intimately related to Condition (L_1).

Proposition 9 *A game* ν *has properties* (L_1) *and* (SM) *if and only if it has Property* (H).

Proof. Assume that ν has (L_1) and (SM). Then, by (SM) there exists $A \in \Sigma$ such that $\nu(A) - \xi(A) = \sup_{E\in\Sigma}[\nu(E) - \xi(E)]$. By Property (L_1), $\forall \xi \in ca$ there exists $\eta \in core(\nu)$ such that $\xi + \nu_{\xi}(\Omega) \geq \eta$. Hence, $\nu(A) - \xi(A) \geq \sup_{E\in\Sigma}[\eta(E) - \xi(E)]$.

Let $\{C, C^c\}$ be an H-decomposition for $\eta - \xi$. By the preceding

$$\nu(A) - \xi(A) \geq \eta(C) - \xi(C) \geq \eta(A) - \xi(A) \geq \nu(A) - \xi(A),$$

where the last inequality follows from $\eta \in core(\nu)$. That is,

$$\nu(A) - \xi(A) = \eta(C) - \xi(C).$$

Now, suppose that $A \neq C$ modulo $(\eta - \xi)$-measure 0 sets ($\mathrm{mod}(\eta - \xi) - 0$, for short). Then,

$$\nu(A) - \xi(A) = \eta(C) - \xi(C) > \eta(A) - \xi(A) \geq \nu(A) - \xi(A),$$

a contradiction. Thus $A = C$ $\mathrm{mod}(\eta - \xi) - 0$, $\{A, A^c\}$ is an H-decomposition for $\eta - \xi$ and $\eta(A) = \nu(A)$.

For the converse, just observe that Property (H) immediately implies both (SM) and the existence of $\eta \in core(\nu)$ such that $\xi + \nu_{\xi}(\Omega) \geq \eta$. ■

As a consequence of Proposition 9, we obtain the following easy corollary.

Corollary 2 $MB^{\sigma}\,(\nu) = core^{\sigma}\,(\nu)$ *for any balanced game satisfying* (L_1) *and condition* (SM). *Furthermore,* $MB^{\sigma}\,(\nu) - (ca, ba) = core^{\sigma}\,(\nu)$, *whenever* ν *satisfies* (L_1^{σ}).

In connection with Theorem 1, the result of Proposition 9 naturally raises the question of when a game satisfies condition (L_1). This is answered by Proposition 10, below. In order to state and prove it, we need to introduce another auxiliary game.

Definition 3 *Given a balanced game ν and $\xi \in ba$, the dual of the maximal excess game ν_ξ is the game $\widetilde{\nu}_\xi : \Sigma \longrightarrow \mathbb{R}$ defined by*

$$\widetilde{\nu}_\xi (A) = \inf_{\mu \in core(\nu)} (\mu - \xi)^+ (A) . \tag{8}$$

We refer to discussion following the proof of Proposition 10 for an explanation of why $\widetilde{\nu}_\xi$ is the dual game of ν_ξ. In the meantime, we observe that, generally speaking, while the set $M = \{(\mu - \xi)^+ : \mu \in core(\nu)\}$ is always relatively weak*-compact and convex, it may not be weak*-closed. Thus, the inf in (8) cannot be replaced by a minimum. If, however, $core(\nu) \subset ca$, then it follows from Mazur's theorem that $core(\nu)$ is weakly compact, M is weakly closed and the inf in (8) is actually a minimum. We also observe that $\widetilde{\nu}_\xi$ is always a totally balanced game. The study of the relation between $\widetilde{\nu}_\xi$ and ν_ξ leads to a full characterization of condition (L_1).

Proposition 10 *For any balanced game ν, we have $\nu_\xi \leq \widetilde{\nu}_\xi$. Moreover, the game ν satisfies largeness condition (L_1) if and only if $\nu_\xi (\Omega) = \widetilde{\nu}_\xi (\Omega)$ and, in addition, the inf is a minimum, that is:*

$$\nu_\xi (\Omega) = \min_{\mu \in core(\nu)} (\mu - \xi)^+ (\Omega) . \tag{9}$$

In this case, $core(\nu_\xi) \neq \varnothing$ for all $\xi \in ca$. Furthermore, $core^\sigma (\nu_\xi) \neq \varnothing$ for all $\xi \in ca$, provided ν satisfies (L_1^σ).

Proof. Let ν be any balanced game. If $\mu \in core(\nu)$ and $A \in \Sigma$,

$$\nu_\xi (A) = \sup_{B \subseteq A} [\nu (B) - \xi (B)] \leq \sup_{B \subseteq A} [\mu (B) - \xi (B)] = (\mu - \xi)^+ (A) .$$

Hence, $\nu_\xi (A) \leq \widetilde{\nu}_\xi (A)$.

Let ν satisfy (L_1). Then, there is $\mu \in core(\nu)$ such that $\nu_\xi (\Omega) + \xi \geq \mu$. Namely, $\nu_\xi (\Omega) \geq \mu - \xi$ or, equivalently, $\nu_\xi (\Omega) \geq \mu (A) - \xi (A)$ for all $A \in \Sigma$. Consequently,

$$\nu_\xi (\Omega) \geq \sup_{E \in \Sigma} [\mu (E) - \xi (E)] = (\mu - \xi)^+ (\Omega) .$$

Since $\nu_\xi \leq \widetilde{\nu}_\xi$, we get (9).

Conversely, assume that (9) holds. We have $\nu_\xi (\Omega) = (\mu - \xi)^+ (\Omega)$ for some $\mu \in core(\nu)$. Hence,

$$(\mu - \xi)^+ (\Omega) = \sup_{E \in \Sigma} [\mu (E) - \xi (E)] = \nu_\xi (\Omega) ,$$

which amounts to saying $\mu - \xi \leq \nu_\xi (\Omega)$. Equivalently, $\mu \leq \nu_\xi (\Omega) + \xi$ which proves that ν satisfies (L_1). To conclude, if $\overline{\mu}$ is a minimum point in (9), we have $(\overline{\mu} - \xi)^+ (\Omega) = \nu_\xi (\Omega)$. Moreover, for any coalition A,

$$\nu_\xi (A) \leq \widetilde{\nu}_\xi (A) \leq (\overline{\mu} - \xi)^+ (A) .$$

Hence $(\overline{\mu}-\xi)^{+} \in core\,(\nu_\xi)$ and $core\,(\nu_\xi) \neq \varnothing$. If ν satisfies L_1^σ, it is easy to see that $\overline{\mu} \in ca$ and therefore $core^\sigma\,(\nu_\xi) \neq \varnothing$, and the claims are proved. ■

The maximal excess game ν_ξ is transparently associated to the extremum problem

$$\sup_{E\in\Sigma}\left\{\inf_{\eta\in core(\nu)}[\eta(E)-\xi(E)]\right\} \tag{A}$$

Its dual problem is

$$\inf_{\eta\in core(\nu)}\left\{\sup_{E\in\Sigma}[\eta(E)-\xi(E)]\right\} \tag{B}$$

which is what allows us to think of $\widetilde{\nu}_\xi$ as of the dual of ν_ξ. A quick glance at problems (A) and (B) also tells that Property (H) is equivalent to the existence of a saddle point. Thus, the following theorem comes with no surprise. Recall that a game ν is exact if $core(\nu) \neq \varnothing$ and

$$\nu\,(A) = \min_{\eta\in core(\nu)} \eta\,(A)$$

for all $A \in \Sigma$. For $\xi \in ca(\Sigma)$, define the Lagrangian function $L_\xi : \Sigma \times core\,(\nu) \to \mathbb{R}$, by

$$L_\xi\,(A,\eta) = (\eta-\xi)\,(A)\,.$$

Theorem 4 *Let ν be an exact game. Then,*

(1) ν satisfies (L_1) if and only if

$$\sup_{A\in\Sigma}\min_{\eta\in core(\nu)} L_\xi\,(A,\eta) = \min_{\eta\in core(\nu)}\max_{A\in\Sigma} L_\xi\,(A,\eta) \tag{10}$$

for all $\xi \in I^*\,(\nu)$.

(2) ν satisfies (L_1) and (SM) if and only if $L_\xi\,(A,\eta)$ has a saddle point for all $\xi \in I^*\,(\nu)$. That is,

$$\max_{A\in\Sigma}\min_{\eta\in core(\nu)} L_\xi\,(A,\eta) = \min_{\eta\in core(\nu)}\max_{A\in\Sigma} L_\xi\,(A,\eta)\,. \tag{11}$$

Every saddle point $\left(\overline{A},\overline{\eta}\right)$ of L_ξ is a justified objection to ξ.

Proof. Note that

$$\min_{\eta\in core(\nu)} L_\xi\,(A,\eta) = \nu\,(A) - \xi\,(A)\,,$$

if ν is exact. While,

$$\max_{A\in\Sigma} L_\xi\,(A,\eta) = (\eta-\xi)^{+}\,(\Omega)\,.$$

Consequently, relation (10) is nothing but (9), and the claim follows from Proposition 10.

The first statement is obvious, for the minimax condition (11) is equivalent to the fact that the sup of $\sup_{A\in\Sigma}[\nu\,(A)-\xi\,(A)]$ is attained.

Let $\left(\overline{A},\overline{\eta}\right)$ be a saddle point. Namely,

$$L_\xi\,(A,\overline{\eta}) \leq L_\xi\left(\overline{A},\overline{\eta}\right) \leq L_\xi\left(\overline{A},\eta\right)$$

for all $A \in \Sigma$ and $\eta \in core(\nu)$. The first inequality means $(\overline{\eta} - \xi)\left(\overline{A}\right) \geq (\overline{\eta} - \xi)(A)$. Hence, $\left(\overline{A}, \overline{A}^c\right)$ is a Hahn decomposition of $\overline{\eta} - \xi$. The second inequality amount saying that $\nu\left(\overline{A}\right) = \overline{\eta}\left(\overline{A}\right)$. By Proposition (7) $\left(\overline{A}, \overline{\eta}\right)$ is a justified objection to ξ. ■

For the remainder of the paper, we shall say that a game ν has Property (SP) if $L_\xi(A, \eta)$ has a saddle point for all $\xi \in I^*(\nu)$. Summing up, for exact games

$$(H) \Longleftrightarrow (L_1) \ \& \ (SM) \Longleftrightarrow (SP) \Longrightarrow [MB^\sigma(\nu) = core^\sigma(\nu)].$$

More can be said about exact games satisfying (SP), which are, in addition, continuous.

Corollary 3 *If ν is a continuous exact games with property (SP), then*

$$MB^\sigma(\nu) = core(\nu) = V \cap ca$$

for all von Neumann-Morgenstern stable sets V.

This follows at once from [23, Proposition 3] by observing that Property (H) implies the conditions stated therein for the stability of the core. Notice also that, if we are willing to restrict to countably additive imputations, then Corollary 3 says that the MB-set (hence, the core) is the unique stable set.

We saw in Proposition 7 of Section 6 that Property (H) implies that, for any preimputation not in the core, there exists an objection (A, η) against it with $\eta \in core(\nu)$. It thus makes sense to ask if a converse to Proposition 7 holds. That is, if a game ν is such that any preimputation can be objected against by means of elements of the core, does ν necessarily satisfy condition (L_1)?[1] The next proposition shows that the answer is affirmative if either ν is inner continuous and satisfies condition (L_0) or if ν satisfies a mild strengthening of condition (L_0) (see the remark following Proposition 11)

Proposition 11 *Let ν be inner continuous and satisfy (L_0). Then*

$$core(\nu_\xi) \subseteq core(\widetilde{\nu}_\xi) \tag{12}$$

for all $\xi \in ca$. Furthermore, if $core(\nu_\xi) \neq \varnothing$ then $core(\nu_\xi) = core(\widetilde{\nu}_\xi)$. Consequently, if this is true for all $\xi \in ca$, then ν satisfies (L_1).

Proof. If $core(\nu_\xi) = \varnothing$, (12) is trivial. Suppose $m \in core(\nu_\xi)$. As ν in inner continuous, $core(\nu_\xi) = core^\sigma(\nu_\xi)$ by Proposition 6. Therefore, $m \in ca$. Since $m(A) \geq \nu_\xi(A) \geq \nu(A) - \xi(A)$ for all A, it holds $m + \xi \geq \nu$. By (L_0), $m + \xi \geq \mu$ for some $\mu \in core(\nu)$. Namely, $m \geq \mu - \xi$. As $m \geq 0$, $m \geq (\mu - \xi)^+ \geq \widetilde{\nu}_\xi$. On the other hand, we have $m(\Omega) = \nu_\xi(\Omega) \leq \widetilde{\nu}_\xi(\Omega)$. Consequently, m lies in the core of the game $\widetilde{\nu}_\xi$, and the first claim is proved. Assume now that $core(\nu_\xi) \neq \varnothing$, i.e., $m(\Omega) = \nu_\xi(\Omega) = \widetilde{\nu}_\xi(\Omega)$ for some $m \in ca$. For any $\varepsilon > 0$, there is some $\mu_\varepsilon \in core(\nu)$ such that $(\mu_\varepsilon - \xi)^+(\Omega) \leq \nu_\xi(\Omega) + \varepsilon$. That is, $(\mu_\varepsilon - \xi)(A) \leq \nu_\xi(\Omega) + \varepsilon$ for all A. Let $\mu^* \in core(\nu)$ be a cluster point of the net $\{\mu_\varepsilon\}$. Clearly, $(\mu^* - \xi)(A) \leq \nu_\xi(\Omega)$ which implies $\xi + \nu_\xi(\Omega) \geq \mu^*$. Hence ν satisfies (L_1). By Proposition 10, $core(\nu_\xi) = core(\widetilde{\nu}_\xi)$. ■

For future reference, the following is worth recording.

[1] Under our hypothesis, (SM) is automatically satisfied.

FACT 1 In Proposition 11, the condition of inner continuity of the game ν can be dispensed with if we assume the stronger largeness condition (L_0^*): $\xi \geq \nu \Longrightarrow \xi \geq \mu \geq \nu$ for all $\xi \in ba$.

An interesting by-product of Proposition 11 is the following Corollary. The interest stems from the fact that, generally speaking, (L_0) does not imply (L_1), not even for finite exact games.

Corollary 4 *If ν in inner continuous and satisfies (L_0), then $MB^{\sigma}(\nu) - (ca, ba) = core(\nu)$ if and only if ν satisfies (L_1) and (SM).*

Proof. If $MB^{\sigma}(\nu) - (ca, ba) = core(\nu)$, by Theorem 2 Property (SM) holds and $core^{\sigma}(\nu_{\xi}) = core(\nu_{\xi}) \neq \varnothing$. From Proposition 11, it follows that ν satisfies (L_1). Conversely, assume that (SM) and (L_1) hold. By Proposition 10 $core^{\sigma}(\nu_{\xi}) \neq \varnothing$. Now, Theorem 2 implies the desired result. ■

7.1 Glove-market games

As a sample of the applicability of the theory we have developed in this section, we provide a new proof of a result from [5]. In the game we are going to consider, the space $\Omega = \{1, 2, ..., n\}$ is the *traders space* and the space $J = \{1, 2, ..., m\}$ is the *commodity space*. Every trader holds a unit of some commodity in J. Thus, Ω can be partitioned as $\Omega = \cup_{r \in J} A_r$, where $i \in A_r$ if trader i holds good $r \in J$. We also assume

$$|A_1| = |A_2| = ... = |A_m| = s.$$

That is, there are exactly s traders holding the same commodity r for each $r \in J$. The *unitary glove-market game* is the exact game defined by

$$\nu(A) = \min_{r \in J} |A \cap A_r|. \tag{13}$$

The interpretation is that there is demand only for equal quantities of each commodity, each valued at unit price.

Game (13) belongs to the class of glove-market games, which is of fundamental importance in the theory of games as Kalai and Zemel [19] have shown the equivalence between the class of glove-market games and that of totally balanced games. Production games with Leontief technologies are also equivalent to glove market games. Glove-market games have been studied in a recent paper by Apartsin and Holzman ([5]), especially for what concerns the relation between the core and the Maschler bargaining set. In [5], Apartsin and Holzman also stated, without an explicit proof, the equivalence between the Mas-Colell bargaining set and the core of the game (13). Here we give a proof such equivalence as an application of our theory. In the course of the proof, we will make use of the following elementary fact.

FACT 2 Let $x, y \in \mathbb{R}_+^m$ with $\sum_{r=1}^m x_r \leq 1 \leq \sum_{r=1}^m y_r$. There exists $\overline{\alpha} \in \Delta$ such that $x_r \leq \overline{\alpha}_r \leq y_r$ for every r (Δ denotes the unit simplex in $\mathbb{R}^m$).

Proposition 12 *Market games (13) satisfy* (L_1). *Consequently,* (SP) *holds and* $MB(\nu) = core(\nu)$.

Proof. It is convenient to re-index individual i as $i \equiv (r,k)$, where $r \in J$ and $k \in \{1,2,...,s\}$. Moreover, given a preimputation $\xi = (\xi_i)_{i\in\Omega}$, we enumerate the second index k of (r,k) so that

$$\xi^+_{(r,1)} \leq \xi^+_{(r,2)} \leq ..... \leq \xi^+_{(r,s)} \tag{14}$$

for all $r \in J$. If we sum over $r \in J$, there is an integer h such that

$$\sum_{r\in J} \xi^+_{(r,1)} \leq .... \leq \sum_{r\in J} \xi^+_{(r,h)} \leq 1 \leq \sum_{r\in J} \xi^+_{(r,h+1)} \leq .... \leq \sum_{r\in J} \xi^+_{(r,s)}. \tag{15}$$

Note that since we allow for the two extreme cases $1 \leq \sum_{r\in J} \xi^+_{(r,1)}$ and $\sum_{r\in J} \xi^+_{(r,s)} \leq 1$, the integer h may run over $\{0,1,...,s\}$.

Let Δ denote the unit simplex in $\mathbb{R}^m$. The core of unitary glove market game is (see [5])

$$core(\nu) = \left\{ \sum_{r\in J} \alpha_r |A_r \cap \cdot| : \alpha \in \Delta \right\}.$$

Hence, it is easy to see that for the dual game we have

$$\widetilde{\nu}_\xi(\Omega) = \min_{\alpha\in\Delta} \sum_{r\in J}\sum_{i\in A_r} (\alpha_r - \xi_i)^+ = \min_{\alpha\in\Delta} \sum_{r\in J}\sum_{k=1}^{s} \left(\alpha_r - \xi_{(r,k)}\right)^+.$$

Setting $A_r^+ = \{i \in A_r : \alpha_r - \xi_i \geq 0\}$, we also have

$$\widetilde{\nu}_\xi(\Omega) = \min_{\alpha\in\Delta} \left[\sum_{r\in J} \alpha_r |A_r^+| - \sum_{r\in J}\sum_{i\in A_r^+} \xi_i \right].$$

Fix ξ. In view of (14), (15) and FACT 2, there exists an element $\overline{\alpha} \in \Delta$ such that

$$\xi^+_{(r,h)} \leq \overline{\alpha}_r \leq \xi^+_{(r,h+1)} \tag{16}$$

for all $r \in J$. Hence, for that $\overline{\alpha} \in \Delta$, we get $|A_r^+| = h$, unless $\xi_{(r,h)}, \xi_{(r,h+1)} < 0$, where $|A_r^+| > h$. In this case, (16) implies $\overline{\alpha}_r = 0$. It will be apparent, however, that this does not affect the next step. In fact,

$$\sum_{r\in J} \overline{\alpha}_r |A_r^+| - \sum_{r\in J}\sum_{i\in A_r^+} \xi_i = h - \sum_{r\in J}\sum_{i\in A_r^+} \xi_i.$$

Moreover, for the coalition $\overline{A} = \cup_{r\in J} A_r^+$

$$\nu\left(\overline{A}\right) - \xi\left(\overline{A}\right) = h - \sum_{r\in J}\sum_{i\in A_r^+} \xi_i.$$

Consequently, $\widetilde{\nu}_\xi(\Omega) = \nu\left(\overline{A}\right) - \xi\left(\overline{A}\right) = \nu_\xi(\Omega)$, and unitary glove-market games satisfy (L_1). ■

PART II: APPLICATIONS

In this part, we apply the theory we have developed so far in order to obtain new equivalence theorems for various classes of games such as convex games, exact non-atomic market games, non-exact non-atomic market games as well as a new class of games, which we introduce here and call *thin games*. This part unfolds as follows. In Section 8, we define thin games and provide the equivalence result for these games. We, then, devote two more sections to thin games: since this is a new class, it makes sense to inquire into their relation with other well-known classes of games. In Section 9, we study the relation between thin games and exact non-atomic market games. We show that these two classes share several properties (Theorems 6, 7 and 8) and, in fact, their intersection includes a wide class of games, which we *Schur games* (Definition 6). Schur games contain all exact non-atomic market games whose core is finite dimensional, but there is plenty of Schur games with infinite dimensional cores (Example 4 and Theorem 9). In combination with Theorem 5, these results improve upon [23] by showing that the equivalence between the MB-set and the core obtains for a class of exact non-atomic market games with infinite-dimensional cores. In Section 10, we study the relation between thin games and non-atomic convex games. We show that the overlap between the two classes is trivial. Precisely, a non-atomic convex games is thin if and only if it is a measure (Theorem 11). In Section 11, we move to non-atomic market games which are not necessarily exact (in fact, we require that they would only be totally balanced). We show that the MB-set coincides with the core whenever the core is finite dimensional, thus improving upon a result of [23]. We conclude the paper by applying our theory to convex games. These have been the object of several papers. The main result in the area is due to Einy et al. [13], who showed that for bounded, continuous convex games the MB-set coincides with the core. We are going to improve upon their result by weakening the continuity assumption (Theorem 14, Section 12).

8 Thin Games

Thin games are going to be defined as a subclass of continuous exact non-atomic games. Thus, Schmeidler's theorem [28] applies, and the core of a thin game will always be a subset of $ca(\lambda)$ for some non-atomic measure λ. Hence, the core of a thin game is isometrically isomorphic to a weakly compact subset of $\mathcal{L}^1(\Omega, \Sigma, \lambda)$. The crucial concept is stated in Definition 4, below. For $M \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ and $S \in \Sigma$, the subset $M(S)^\perp \subset \mathcal{L}^\infty(\Omega, \Sigma, \lambda)$ is given by

$$M(S)^\perp = \{\varphi \in \mathcal{L}^\infty(\Omega, \Sigma, \lambda) : \langle f, \varphi \rangle = 0 \text{ for all } f \in M \text{ and } \varphi\chi_{S^c} = 0\}.$$

Definition 4 (Kingman and Robertson [20]) *A set $M \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ is thin, if and only if $M(S)^\perp \neq \{0\}$ for all S such that $\lambda(S) > 0$.*[2]

[2] Thin sets play a key role in the study of infinite dimensional vector measures [20] (see also Section IX of [12]).

It is easy to see that non-atomicity of λ is essential for the definition to have a bite. It is also easy to check that a set M is thin if and only if $\overline{linM}$ (the closed linear span of M) is thin, and that every finite dimensional subspace of $\mathcal{L}^1(\Omega, \Sigma, \lambda)$ is thin. Thin sets, however, need not be finite dimensional (see next section). In what follows, we will call a subset of $na_\alpha(\Sigma, \lambda)$ thin, if the corresponding set of densities is thin in the sense of Kingman and Robertson.

Definition 5 *A continuous non-atomic exact game ν is thin if its core is (isometrically isomorphic to) a thin subset of $\mathcal{L}^1(\Omega, \Sigma, \lambda)$.*

As anticipated, for thin games a core-equivalence theorem holds.

Theorem 5 *Let ν be a thin game. Then, $MB^\sigma(\nu) = core(\nu)$. In addition, for any von Neumann - Morgenstern stable set V, we have $V \cap ca = MB^\sigma(\nu)$.*

Proof. In order to prove the equality $\mathrm{MB}^\sigma = \mathrm{core}$, we are going to show that a continuous exact game with thin core satisfies conditions (L_1) and (SM). To this end, it will suffice to prove (by Proposition 9 and Theorem 4) that the Lagrangian $L_\xi : \Sigma \times core(\nu) \longrightarrow \mathbb{R}$, $L_\xi(A, \eta) = (\eta - \xi)(A)$, has a saddle point $\left(\overline{A}, \overline{\eta}\right)$ for all $\xi \in ca(\lambda)$. Let $[\mathcal{L}^\infty(\lambda)]_1 = \{\varphi \in \mathcal{L}^\infty(\lambda) : 0 \leq \varphi \leq 1\}$, and consider the function $\widetilde{L}_\xi : [\mathcal{L}^\infty(\lambda)]_1 \times core(\nu) \to \mathbb{R}$, given by $\widetilde{L}_\xi(\varphi, \eta) = (\eta - \xi)(\varphi)$. By Sion's minimax theorem [31, Corollary 3.3], it has a saddle point $(\overline{\varphi}, \overline{\eta})$. That is,

$$\widetilde{L}_\xi(\varphi, \overline{\eta}) \leq \widetilde{L}_\xi(\overline{\varphi}, \overline{\eta}) \leq \widetilde{L}_\xi(\overline{\varphi}, \eta)$$

for all $(\varphi, \eta) \in [\mathcal{L}^\infty(\lambda)]_1 \times core(\nu)$. Namely,

$$\begin{aligned} (\overline{\eta} - \xi)(\varphi) &\leq (\overline{\eta} - \xi)(\overline{\varphi}) \\ \overline{\eta}(\overline{\varphi}) &\leq \eta(\overline{\varphi}). \end{aligned}$$

Consider now Kingman-Robertson's map $u : [\mathcal{L}^\infty(\lambda)]_1 \to \mathbb{R}^{I+1}$ defined as

$$u(\varphi) = \left(\langle \xi, \varphi \rangle, (\langle \eta, \varphi \rangle)_{\eta \in core(\nu)}\right).$$

As $core(\nu) \cup \{\xi\}$ is thin, [20, Theorem 1] ensures that

$$u[\mathcal{L}^\infty(\lambda)]_1 = u\{\chi_E : E \in \Omega\}.$$

Consequently, there exists a coalition $\overline{A}$ such that $\eta\left(\overline{A}\right) = \eta(\overline{\varphi})$ for all $\eta \in core(\nu)$ and $\xi\left(\overline{A}\right) = \xi(\overline{\varphi})$. Hence,

$$\begin{aligned} (\overline{\eta} - \xi)(A) &\leq (\overline{\eta} - \xi)\left(\overline{A}\right) \\ \overline{\eta}\left(\overline{A}\right) &\leq \eta\left(\overline{A}\right). \end{aligned}$$

That is, $\left(\overline{A}, \overline{\eta}\right)$ is a saddle point of $L_\xi(A, \eta)$, and ν satisfies the conditions (L_1) and (SM). Now, the statements follow from Theorem 4 and Corollary 3. ■

9 Thin games vs market games

Recall that a game ν is a market game if it is super-additive and it admits a positively homogeneous na-continuous extension to the set of ideal coalitions (see [27] and [4]). Non-atomic exact market games are of special importance in Economics because of Aumann and Shapley's [7] celebrated result that perfectly competitive pure-exchange economies are market games.

We begin this section by observing that thin games and non-atomic exact market games have several properties in common. One is the m-closure property (originally introduced in [22]): games in both classes can be defined as lower envelopes of certain sets of measures and the game's core coincides with the set defining it. This is shown in the next two theorems. Theorem 6 was proved in [4, Theorem 3]. The notation $\overline{(\cdot)}$ stands for norm-closure, $\overline{(\cdot)}^w$ for weak-closure and, finally, $\overline{(\cdot)}^*$ stands for weak*-closure.

Theorem 6 ([4]) *ν is an exact non-atomic market game if and only if ν is the lower envelope of some norm-relatively compact subset K of $na_\alpha(\Sigma)$. Moreover,*

$$\overline{co}(K) = \overline{co}^*(K) = core(\nu).$$

Theorem 7 *ν is a thin game if and only if ν is the lower envelope of some weak-compact thin subset K of $na_\alpha(\Sigma)$. Moreover,*

$$\overline{co}(K) = \overline{co}^*(K) = core(\nu).$$

Proof. One direction is the definition of thin games. In the other, let ν be the lower envelope of some weak-compact thin subset K of $na_\alpha(\Sigma)$. That is, $\nu = \nu_K$, where $\nu_K(A) = \min_{\eta \in K} \eta(A)$. Obviously,

$$\overline{co}(K) \subseteq \overline{co}^*(K) \subseteq core(\nu) \subset na_\alpha(\Sigma).$$

Now, if there exists $\mu \in core(\nu) \backslash \overline{co}(K)$, then by the separation theorem in [3, Theorem 1] there exists an $A \in \Sigma$ for which

$$\nu(A) \leq \mu(A) < \inf_{\eta \in K} \langle \eta, A \rangle = \nu_K(A)$$

which contradicts ν being the lower envelope of K. ■

Notice that Theorems 6 and 7 also tell us that any exact non-atomic market game with finite dimensional core is thin. Thus, the two classes of games certainly have a non-trivial intersection.

Another property common to both thin games and exact non-atomic market games is that they are both naturally associated with certain compact operators. To see this, let us begin by observing that any continuous exact game is naturally associated with a linear operator $\mathcal{L}^\infty(\lambda) \to l^\infty(\mathcal{I})$. For if ν is continuous and exact, then $core(\nu)$ is isometrically isomorphic (by Schmeidler's theorem) to a subset $\mathcal{F} = \{f_i\}_{i \in I} \subset \mathcal{L}^1(\lambda)$, and we can define the operator $T_\nu : \mathcal{L}^\infty(\lambda) \to l^\infty(\mathcal{I})$ by

$$T_\nu \varphi = (\int \varphi f_i d\lambda)_{i \in \mathcal{I}}.$$

T_ν is the integral of the vector measure $\boldsymbol{\mu} : \Sigma \longrightarrow l^\infty(\mathcal{I})$ defined by

$$A \longmapsto (\int \chi_A f_i d\mu)_{i\in\mathcal{I}}.$$

We observe that (a) $\mathcal{F}$ weakly compact $\Longrightarrow \boldsymbol{\mu}$ bounded and countably additive; and (b) the Bartle-Dunford-Schwartz theorem implies that T_ν is always weak* to weak continuous (see [12, Corollary 7, p. 14]).

Theorem 8 *T_ν is a compact operator when ν is either an exact non-atomic market game or ν is a thin game.*

Proof. (1) If ν is an exact non-atomic market game, then $core(\nu) \sim \mathcal{F} = \{f_i\}_{i\in I} \subset \mathcal{L}^1(\lambda)$ and is compact in the $\mathcal{L}^1$-norm by Theorem 6. Let $R \equiv T_\nu \mid_{B_{\mathcal{L}^\infty(\lambda)}}$, that is R is the restriction of T_ν to the unit ball in $B_{\mathcal{L}^\infty(\lambda)}$. We are going to show that R is weak* to norm continuous, which immediately implies compactness of T_ν. To this end, consider the family of linear functionals $\mathcal{F} = \left\{\int \cdot f_i d\lambda \mid i \in I\right\}$ on $\mathcal{L}^\infty(\lambda)$. By considering the restrictions of the functionals to $B_{\mathcal{L}^\infty(\lambda)}$, we can view $\mathcal{F}$ as a subset of $\mathcal{C}(B_{\mathcal{L}^\infty(\lambda)})$. Since $\mathcal{F}$ is norm-compact, it is equicontinuous by Arzelà-Ascoli's theorem. Hence, for any $\varphi^* \in B_{\mathcal{L}^\infty(\lambda)}$ and $\forall\varepsilon > 0$, there exists a weak* neighborhood $U(\varphi^*)$ such that

$$\varphi \in U(\varphi^*) \Longrightarrow \|R\varphi - R\varphi^*\|_\infty = \sup_{i\in I}\left|\int \varphi f_i d\lambda - \int \varphi^* f_i d\lambda\right| < \varepsilon,$$

which proves that T_ν is bounded-weak* to norm continuous.

(2) Let $core(\nu) \sim \mathcal{F} = \{f_i\}_{i\in I} \subset \mathcal{L}^1(\lambda)$. If ν is thin, then the norm-closure of the range of the vector measure $\boldsymbol{\mu}$ is convex [20]. By letting K_0 denote the set of indicator functions in $\mathcal{L}^\infty(\lambda)$, and by $K = [\mathcal{L}^\infty(\lambda)]_1$, we have

$$T_\nu(K) = \overline{T_\nu(K_0)}^w = \overline{\boldsymbol{\mu}(K_0)}^w = \overline{\boldsymbol{\mu}(K_0)}.$$

The first equality follows from the fact – noted above – that T_ν is always weak* to weak continuous, the second is a definition and the third follows from $\overline{\boldsymbol{\mu}(K_0)}^w = \overline{\left[\overline{\boldsymbol{\mu}(K_0)}\right]}^w = \overline{\boldsymbol{\mu}(K_0)}$. In fact, $\overline{\boldsymbol{\mu}(K_0)}$ is convex, and we have $\overline{\left[\overline{\boldsymbol{\mu}(K_0)}\right]}^w = \overline{\left[\overline{\boldsymbol{\mu}(K_0)}\right]} = \overline{\boldsymbol{\mu}(K_0)}$. Now, $T_\nu(K)$ norm-compact $\Longrightarrow T_\nu(B_{\mathcal{L}^\infty(\lambda)}) = T_\nu(K - K)$ – the image of the ball – is norm-compact. Hence, T_ν is a compact operator. ■

Summing up, thin games and exact non-atomic market games are both associated to compact operators, they both have the m-closure property and their intersection contains all market games with finite dimensional cores. These results raise three obvious questions: (i) Are there market games with infinite dimensional cores which are thin?; (ii) Are there thin games which are not market games?; and, finally, (iii) Are there exact non-atomic market games which are not thin? The next three subsections answer all these questions in the affirmative.

9.1 Schur games

A Banach space X has the Schur property if every weakly convergent sequence in X converges in norm. The space l^1 is an example of a space with the Schur property.

Definition 6 *Let ν be a continuous exact game, Let core $(\nu) \sim \mathcal{F} \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$. The game ν is called a Schur game if the linear space $\overline{lin\mathcal{F}}$ has the Schur property.*

Clearly, by Theorem 6, a Schur game is a market game. Under a mild assumption it is thin as well.

Proposition 13 *Assume that (Ω, Σ) is standard Borel and that λ nonatomic. If the closed subspace $V \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ is Schur, then V is thin.*

Proof. Fix $S \in \Sigma$ with $\lambda(S) > 0$. Let us first prove that if $V \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ is Schur, then the subspace of $\mathcal{L}^1(S, \Sigma_S, \lambda_S)$

$$V(S) = \left\{ f \in \mathcal{L}^1(S, \Sigma_S, \lambda_S) : f = \varphi_{|S} \text{ with } \varphi \in V \right\}$$

is Schur. Let $f_n \to 0$ weakly, with $f_n \in V(S)$. Consider the sequence $\widetilde{f_n} \in V$ where $\widetilde{f_n}(\omega) = f_n(\omega)$, if $\omega \in S$, and $\widetilde{f_n}(\omega) = 0$, if $\omega \in S^c$. For any $\varphi \in \mathcal{L}^\infty(\Omega, \Sigma, \lambda)$, we have $\left\langle \widetilde{f_n}, \varphi \right\rangle = \left\langle f_n, \varphi_{/S} \right\rangle$. Hence, $\left\langle \widetilde{f_n}, \varphi \right\rangle \to 0$. As $\widetilde{f_n} \to 0$ weakly, then $\left\| \widetilde{f_n} \right\| \to 0$. Namely, $\|f_n\| \to 0$ in $\mathcal{L}^1(S, \Sigma_S, \lambda_S)$ and consequently $V(S)$ has the Schur property.

On the other hand, under our assumptions, $\mathcal{L}^1(S, \Sigma_S, \lambda_S)$ is isomorphic to $\mathcal{L}^1(0,1)$ which does not have the Schur property. Hence, $V(S) \neq \mathcal{L}^1(S, \Sigma_S, \lambda_S)$ for all $\lambda(S) > 0$. This amounts to saying that V is thin. ■

Hence, under a standardness assumption, we have the following inclusions:

Finite Dim. games $\subset$ *Schur games* $\subset$ *Thin games* $\cap$ *Market games.*

Notice also that, since $\mathcal{L}^1(\Omega, \Sigma, \lambda)$ contains an isomorphic copy of l^1, there is plenty of infinite dimensional thin subspaces in $\mathcal{L}^1(\Omega, \Sigma, \lambda)$. Two concrete examples are given below. By Theorem 7, the corresponding games are thin games with infinite dimensional cores.

Example 2 *Let $\{A_i\}_{i=1}^\infty$ be a countably and measurable partition of Ω, with $\lambda(A_i) > 0$. Let Σ_1 be the σ-algebra generated by $\{A_i\}_{i=1}^\infty$. Clearly, the vector space $\mathcal{L}^1(\Omega, \Sigma_1, \lambda) \simeq \ell_1 \subset \mathcal{L}^1(\Omega, \Sigma_1, \lambda)$ is Schur.*

Example 3 *Consider a disjoint normalized sequence $f_n \in \mathcal{L}^1(\Omega, \Sigma, \lambda)$, that is, $|f_n| \wedge |f_m| = 0$ if $n \neq m$ and $\|f_n\| = 1$. The space $\overline{lin\{f_n\}_{n=1}^\infty}$ is isomorphic (but not lattice isomorphic) to l^1, and therefore is Schur.*

9.2 Thin games which are not market games

To ask whether or not there are thin games which are not market games is, by virtue of Theorems 6 and 7, the same as asking whether or not there are weakly compact cores that are thin but are not norm-compact. The example below shows that such cores do exist by exhibiting a thin subspace of $\mathcal{L}^1(\lambda)$ which does not have the Schur property. This example is especially insightful, and leads to the far more general result of Theorem 9, which is of independent interest.

Example 4 *Let $([0,1],\mathcal{B},\lambda)$ be the standard Lebesgue space. Let $\{f_n\} \in \mathcal{L}^1([0,1],\mathcal{B},\lambda)$ be a sequence which converges to 0 weakly, but not in the $\mathcal{L}^1$-norm. A classical example is the sequence $f_n(t) = \sin(2^n \pi t)$. Define the sequence $\widetilde{f}_n \in \mathcal{L}^1\left([0,1]^2, \mathcal{B}\otimes\mathcal{B}, \lambda\otimes\lambda\right)$, by $\widetilde{f}_n(t,s) \equiv f_n(t)$.*

The sequence $\widetilde{f}_n \to 0$ weakly in $\mathcal{L}^1\left([0,1]^2, \mathcal{B}\otimes\mathcal{B}, \lambda\otimes\lambda\right)$ but not strongly. Moreover, the set $\left\{\widetilde{f}_n\right\}_n$ is thin.

Proof. The first property is a straightforward exercise. Let us prove that $\left\{\widetilde{f}_n\right\}_n$ is thin. Let $E \subseteq [0,1]^2$ be any measurable set with positive measure. Denote by $\chi_E(t,s)$ its indicator function. Set

$$\alpha(t) = \frac{\int_0^1 s\chi_E(t,s)\,ds}{\int_0^1 \chi_E(t,s)\,ds}. \tag{17}$$

By (17) it follows that

$$\int_0^1 (\alpha(t) - s)\,\chi_E(t,s)\,ds = 0, \tag{18}$$

holds for all $t \in [0,1]$. Define the function $\varphi \in \mathcal{L}^\infty\left([0,1]^2, \mathcal{B}\otimes\mathcal{B}, \lambda\otimes\lambda\right)$ as

$$\varphi(t,s) = (\alpha(t) - s)\,\chi_E(t,s).$$

By construction $\varphi(t,s) = 0$ outside E. It is also easy to see that $\|\varphi\|_\infty \neq 0$. To conclude, we have

$$\int_{[0,1]^2} f_n(t)\,(\alpha(t) - s)\,\chi_E(t,s)\,dtds = 0$$

for all n, because (18) holds. Hence, the set $\left\{\widetilde{f}_n\right\}_n$ is thin. ■

The next Theorem provides a complete characterization of thin subspaces of $\mathcal{L}^1$ in the case in which they are sublattices.

Theorem 9 *Let $V \subset \mathcal{L}^1(\Omega,\Sigma,\lambda)$ be a closed subspace, and denote by Σ_+ the class of set A for which $\lambda(A) > 0$. If V is thin, then:*

(i) for all $A \in \Sigma_+$ there is some $E \subseteq A$ such that $\chi_E \notin V$.

If V is a sublattice containing the constant function χ_Ω, then the condition (i) is sufficient for V to be thin. In this case, $V \simeq \mathcal{L}^1(\Omega,\Sigma_1,\lambda)$ where Σ_1 is a σ-subalgebra of Σ.

Proof. Necessity of (i). Suppose (i) does not hold. Then, there exists some $S \in \Sigma_+$ such that $\chi_E \in V$ for all $E \subseteq S$. That is, $V(S) = \mathcal{L}^1(S, \Sigma_S, \lambda_S)$ with $\lambda(S) > 0$. Hence, V is not thin as $V^\perp(S) = \{0\}$.

We now prove that (i) suffices, if V is a sublattice containing the constant functions. In such a case, it is well-known that the collection Σ_1 of the sets $A \in \Sigma$ such that $\chi_A \in V$ forms a σ-subalgebra of Σ. This implies that V can be identified with the sublattice $\mathcal{L}^1(\Omega, \Sigma_1, \lambda)$ of $\mathcal{L}^1(\Omega, \Sigma, \lambda)$ (for details, see for instance [2, Theorem 12.11]). We must prove that (i) implies $\mathcal{L}^1(\Omega, \Sigma_1, \lambda)$ is thin.

Fix $A \in \Sigma_+$ and consider the set $E \notin \Sigma_1$ of condition (i). Clearly, $\lambda(E) > 0$. Set

$$\varphi = \chi_E - \frac{\mathbb{E}[\chi_E \mid \Sigma_1]}{\mathbb{E}[\chi_A \mid \Sigma_1]}\chi_A, \tag{19}$$

with the convention $0/0 = 0$.

From $0 \leq \chi_E \leq \chi_A$, it follows $0 \leq \mathbb{E}[\chi_E \mid \Sigma_1] \leq \mathbb{E}[\chi_A \mid \Sigma_1]$. Hence, $\mathbb{E}[\chi_A \mid \Sigma_1](\omega) = 0 \implies \mathbb{E}[\chi_E \mid \Sigma_1](\omega) = 0$. Consequently, φ is well defined, it vanishes outside A and $\varphi \in \mathcal{L}^\infty(\Omega, \Sigma, \lambda)$.

Let us prove that $\int f\varphi d\lambda = 0$, for all $f \in \mathcal{L}^1(\Omega, \Sigma_1, \lambda)$. From (19),

$$\mathbb{E}[\varphi \mid \Sigma_1] = \mathbb{E}[\chi_E \mid \Sigma_1] - \frac{\mathbb{E}[\chi_E \mid \Sigma_1]}{\mathbb{E}[\chi_A \mid \Sigma_1]}\mathbb{E}[\chi_A \mid \Sigma_1] = 0,$$

whenever $\mathbb{E}[\chi_A \mid \Sigma_1](\omega) \neq 0$. Define

$$N = \{\omega \in \Omega = \mathbb{E}[\chi_A \mid \Sigma_1](\omega) \neq 0\}.$$

For any $f \in V = \mathcal{L}^1(\Omega, \Sigma_1, \lambda)$, we have

$$\begin{aligned}
\int_\Omega f\varphi d\lambda &= \int_\Omega \mathbb{E}[f\varphi \mid \Sigma_1] d\lambda = \int_\Omega f\mathbb{E}[\varphi \mid \Sigma_1] d\lambda \\
&= \int_N f\mathbb{E}[\varphi \mid \Sigma_1] d\lambda + \int_{N^c} f\mathbb{E}[\varphi \mid \Sigma_1] d\lambda \\
&= \int_{N^c} f\mathbb{E}[\chi_E \mid \Sigma_1] d\lambda = 0.
\end{aligned}$$

To conclude the proof, it remains to show that $\|\varphi\|_\infty > 0$. Assume, by contradiction, that $\varphi = 0$ a.s. Then

$$\chi_E = \frac{\mathbb{E}[\chi_E \mid \Sigma_1]}{\mathbb{E}[\chi_A \mid \Sigma_1]}\chi_A.$$

Hence, $E = E \cap A = \{\mathbb{E}[\chi_E \mid \Sigma_1] \leq 0\}$, that implies $E \in \Sigma_1$, a contradiction. Summing up, V is thin. ■

Theorem 9 provides us with a method to construct a large amount of thin subspaces without the Schur property: Let $(\Omega, \Sigma, \lambda)$ be the product of countably many copies of the probability space $[0,1]$; if we consider the canonical filtration $(\Sigma_n)_{n=1}^\infty$ in $(\Omega, \Sigma, \lambda)$, then the subspaces $\mathcal{L}^1(\Omega, \Sigma_n, \lambda)$ trivially satisfy condition (i) of Theorem 9. Note that since $\mathcal{L}^1(\Omega, \Sigma_n, \lambda) \simeq \mathcal{L}^1[0,1]$, these subspaces are not Schur.

Having shown that thin games need not be Schur, we conclude this subsection by giving a necessary and sufficient condition for a thin game to be a market game. It is stated in terms of the properties of the operator T_ν encountered above.

Theorem 10 *A thin game ν is a Schur game if and only if the associated operator T_ν is bounded-weak* to norm continuous.*[3]

Proof. If ν is a Schur game, then $core(\nu) \sim \mathcal{F} = \{f_i\}_{i\in I} \subset \mathcal{L}^1(\lambda)$ and is compact in the $\mathcal{L}^1$-norm by Theorem 6. By the proof of Theorem 8 part (a), T_ν is bounded-weak* to norm continuous.

In the converse direction, assume that T_ν is bounded-weak* to norm continuous. We are going to show that this implies that the support functional of $\mathcal{F}$, $\sigma_{\mathcal{F}} : \mathcal{L}^\infty(\lambda) \to \mathbb{R}$,

$$\sigma_{\mathcal{F}}(\varphi) = \sup_{f\in\mathcal{F}} \int \varphi f_i d\lambda,$$

is continuous for the bounded weak*-topology. In fact, let $\varphi^* \in \mathcal{L}^\infty(\lambda)$. There exists $\alpha \in \mathbb{R}$ such that $\varphi^* \in \alpha B_{\mathcal{L}^\infty(\lambda)}$. Since $T_\nu \mid_{\alpha B_{\mathcal{L}^\infty(\lambda)}}$ is weak*-to-norm continuous, $\forall \varepsilon > 0$ here exists a weak* neighborhood $U(\varphi^*)$ such that

$$\varphi \in U(\varphi^*) \cap \alpha B_{\mathcal{L}^\infty(\lambda)} \Longrightarrow \sup_{i\in I} \left| \int \varphi f_i d\lambda - \int \varphi^* f_i d\lambda \right| < \varepsilon.$$

Combining this with the elementary inequality: If $w, z \in \mathbb{R}^I$ are bounded, then

$$\left| \sup_I w - \sup_I z \right| \leq \sup_I |w - z|$$

we get (by observing that for $\varphi \in B_{\mathcal{L}^\infty(\mu)}$ and $f_i \in \mathcal{L}^1(\lambda)$, the mapping $w = (\int \varphi f_i d\mu)_{i\in I}$ is a bounded element of $\mathbb{R}^I$)

$$\varphi \in U(\varphi^*) \cap \alpha B_{\mathcal{L}^\infty(\lambda)} \Longrightarrow |\sigma_{\mathcal{F}}(\varphi) - \sigma_{\mathcal{F}}(\varphi^*)| \leq \sup_{i\in I} \left| \int \varphi f_i d\lambda - \int \varphi^* f_i d\lambda \right| < \varepsilon.$$

Now, $\sigma_{\mathcal{F}}$ continuous for the bounded weak*-topology implies, by Lemma 3 in [4], that $\mathcal{F}$ is compact in the norm topology. That is, ν is Schur. ■

9.3 Exact non-atomic market games which are not thin

Above, we saw that there is always plenty of thin games which are not market games. The next example shows that there are exact non-atomic market games which are not thin. The idea is to construct a norm-compact subset $C \subset \mathcal{L}^1[0,1]$, containing a Schauder basis for $\mathcal{L}^1[0,1]$. Thus, C spans $\mathcal{L}^1[0,1]$ (and, thus, it is not thin) but, by Theorem 6, the set C defines (and is the core of) an exact non-atomic market game.

[3] See [9] for the definition and properties of the bounded weak* topology.

Example 5 *Let $\mathcal{L}^1[0,1]$ and P_n be the uniform partition of the interval $[0,1]$. Namely,*

$$P_n = \left\{ \left[sn^{-1}, (s+1)\, n^{-1} \right) \right\}_{s=0}^{n-1}$$

for all $n \geq 1$. Let $\mathcal{B}_n = \sigma(P_n)$ denote the sigma algebra generated by the partition P_n. If $f_0 \in \mathcal{L}^1[0,1]$ is a fixed element, then the sequence $f_n = \mathbb{E}[f_0 \mid \mathcal{B}_n]$ converges in $\mathcal{L}^1$ to f_0. Consequently, the set $\{f_n\}_{n=1}^{\infty}$ is relatively norm compact. Thus, by Theorem 6, the game

$$\nu(A) = \min_{f \in \overline{co}\{f_n\}_{n=1}^{\infty}} \int_A f d\lambda$$

is an exact market game.

Assume that $f_n = \mathbb{E}[f_0 \mid \mathcal{B}_n] \neq f_{n+1} = \mathbb{E}[f_0 \mid \mathcal{B}_{n+1}]$ for all $n \geq 1$. For instance, this property holds if f_0 is injective. In this case, we can employ the family of bounded projectors $S_n : \mathcal{L}^1[0,1] \to \mathcal{L}^1[0,1]$, $S_n(f) = \mathbb{E}[f \mid \mathcal{B}_n]$, to construct a Schauder basis for $\mathcal{L}^1[0,1]$ (see for instance Proposition 1.1.7 of [1]). In fact, the sequence $\{\varphi_n\}_{n=1}^{\infty}$, with $\varphi_n = f_n - f_{n-1}$ and $\varphi_1 = f_1$ is one such (conditional) Schauder basis for $\mathcal{L}^1[0,1]$, and, clearly,

$$\overline{lin\,[core(\nu)]} = \overline{lin\,\{\varphi_n\}_{n=1}^{\infty}} = \mathcal{L}^1[0,1].$$

10 Thin games vs Convex games

We have seen that thin games and market games have a non-trivial intersection, which includes the Schur games. The situation is dramatically different when it comes to the relation between thin games and non-atomic convex games: their intersection consists of measures only (Theorem 11, below). This result complements [23, Proposition 4] which shows that convex games have only a trivial intersection with market games.

Theorem 11 *Let ν be a nonatomic convex game. Then, its core is thin if and only if ν is a singleton.*

Proof. *Step 1.* Let A be any linear coalition, that is a coalition such that $\nu(A) + \nu(A^c) = \nu(\Omega)$. Then $\eta(A) = \nu(A)$ for all $\eta \in core(\nu)$. Let us prove that

$$\nu(A \cup E) + \nu(A \cap E) = \nu(A) + \nu(E) \tag{20}$$

holds for all E.

As $A \cap E \subseteq E \subseteq A \cup E$, it follows from A being linear that there is some $\eta \in core(\nu)$ for which $\eta(A \cap E) = \nu(A \cap E)$, $\eta(E) = \nu(E)$ and $\eta(A \cup E) = \nu(A \cup E)$. Hence,

$$\begin{aligned} \nu(A \cup E) + \nu(A \cap E) &= \eta(A \cup E) + \eta(A \cap E) \\ &= \eta(A) + \eta(E) = \nu(A) + \nu(E). \end{aligned}$$

In particular, by (20) we have

$$\nu(A \cup E) = \nu(A) + \nu(E)$$

for all linear set A and all E such that $A \cap E = \varnothing$. It follows that

$$\nu(F \cup E) = \nu(F) + \nu(E) \tag{21}$$

holds for all F and E such that F is contained in some linear set A, i.e., $F \subseteq A$ and $E \cap A = \varnothing$. For

$$\nu(E) \leq \nu(F \cup E) - \nu(F) \leq \nu(A \cup E) - \nu(A) = \nu(E).$$

Step 2. We construct a chain of linear coalitions $\{A_\alpha\}_{\alpha \in [0,1]}$ with the following properties.

$$\begin{aligned} A_\alpha &\subseteq A_\beta \text{ if } \alpha \leq \beta \\ A_0 &= \varnothing,\ A_1 = \Omega \\ \lambda(A_\alpha) &= \alpha \\ \nu(A_\alpha) + \nu(A_\alpha^c) &= \nu(\Omega) \text{ for all } \alpha \in [0,1]. \end{aligned}$$

Here λ is the positive control measure which is supposed to be normalized to 1.

The construction is similar to [7, Lemma 5.4]. Under our assumption, $\mathit{core}(\nu) \sim \{f_i\}_{i \in \mathcal{I}} \subset \mathcal{L}^1(\lambda)$. Moreover, $\{f_i\}_{i \in \mathcal{I}}$ thin implies that $\{f_i\}_{i \in \mathcal{I}} \cup \{1\}$ is thin as well, and the linear operator $u : \mathcal{L}^\infty(\lambda) \to \mathbb{R}^{\mathcal{I}+1}$, $u(\varphi) = \left(\int \varphi d\lambda, (\int \varphi f_i d\lambda)_{i \in \mathcal{I}}\right)$ has the property (see [20]) that $u(K) = u(K_0) = \boldsymbol{\mu}(\Sigma)$, where $\boldsymbol{\mu}$ is the vector measure associated to u as in Section 9. Thus, $\boldsymbol{\mu}(\Sigma)$ is compact and convex.

Set $A_0 = \varnothing$ and $A_1 = \Omega$. As $u(\chi_\Omega) = (1, \nu(\Omega))$, there is a set $A_{1/2}$ such that $u\left(\frac{1}{2}\chi_\Omega\right) = u\left(\chi_{A_{1/2}}\right)$. Clearly, $\nu\left(A_{1/2}\right) = \frac{1}{2}\nu(\Omega)$, $\lambda\left(A_{1/2}\right) = 1/2$ and $A_{1/2}$ is a linear set. Applying this procedure to the sets $A_1 \setminus A_{1/2}$ and $A_{1/2} \setminus A_0$ and so on, we get a chain A_α with the desired properties, for each dyadic rational α. Now if β is an arbitrary number of $[0,1]$, define $A_\beta = \cup_{\alpha < \beta} A_\alpha$ the union being extended over all the α that are dyadic rationals.

Step 3. Consider now two disjoint sets $E \cap F = \varnothing$. By using the family $\{A_\alpha\}_{\alpha \in [0,1]}$ constructed in Step 2, we have also $(E \cap A_\alpha) \cap F = \varnothing$ for all α. By (21),

$$\nu[(E \cap A_\alpha) \cup F] = \nu(E \cap A_\alpha) + \nu(F).$$

Clearly, $(E \cap A_\alpha) \cup F \subseteq E \cup F$, and $\lambda([E \cup F] \setminus [E \cap A_\alpha \cup F]) = \lambda(E \cap A_\alpha^c) \leq \lambda(A_\alpha^c) = 1 - \alpha$. Analogously, $\lambda(E \setminus (E \cap A_\alpha)) \leq 1 - \alpha$. Therefore, we get as $\alpha \to 1$,

$$\nu(E \cup F) = \nu(E) + \nu(F).$$

This proves that ν is necessarily additive. ■

11 Non-exact nonatomic market games

It is shown in [23] that the core coincides with the MB-set for *exact* non-atomic market games, provided that their cores are finite dimensional. Here, we show that this result obtains even when the exactness assumption is dropped.

Recall that a games is *dna*-continuous if it admits a *dna*-continuous extension, ν^*, to $B_1(\Sigma)$. Mertens's [27] *dna*-extension generalizes the classical *na*-extension of Aumann and Shapley [7]. Following [15], we say that a game ν is *dna-uniformly continuous* if for every $\varepsilon > 0$ there exists a vector measure $\boldsymbol{\mu} = (\mu_i)_{i=1}^n$ with $\mu_i \in na$, such that $\mu(A) = \mu(B) \Longrightarrow |\nu(A) - \nu(B)| < \varepsilon$. Clearly *dna*-uniformly continuous games are *dna*-continuous. The simplest examples of *dna*-uniformly continuous games are the measure games $\nu = f(\boldsymbol{\mu})$, where f is a real-valued function defined on the range $R(\boldsymbol{\mu})$ of the non-atomic vector measure μ.

A coalition $N \in \Sigma$ is called *na-null*, if $\mu(N) = 0$ for all $\mu \in na^+$. This amounts to saying that $\chi_N \in \overline{\{0\}}$, the *dna*-closure of $\{0\}$. We first record some useful facts.

Proposition 14 *i) If ν is dna-uniformly continuous, then ν is λ-continuous for some $\lambda \in na^1$ and core$(\nu) \subset na$;*

ii) if N is na-null then N is ν-null, provided ν is dna-continuous;

iii) if ν is dna-continuous and ν is continuous at $\varnothing$ and Ω, then core$(\nu) \subset na$, provided (Ω, Σ) is the standard Borel space.[4]

Proof. i) This is proved in [15, Prop. 2.1].

ii) Let N be *na*-null. Clearly, any $N_1 \subseteq N$ is *na*-null as well. We have

$$\nu(A \cup N) = \nu(A \cup (N \setminus A)) = \nu^*\left(\chi_A + \chi_{N \setminus A}\right) = \nu^*(\chi_A) = \nu(A)$$

for every $A \in \Sigma$. Thus, N is ν-null.

iii) As ν is continuous at $\varnothing$ and Ω, then *core*$(\nu) \subset ca$ (see [24]). Let $m \in$ *core*(ν) and N be any *na*-null coalition. By (ii), N is ν-null. Let us show that N is m-null. Take any $M \subseteq N$, we have $m(M) \geq \nu(M) = 0$ and $m(M^c) = \nu(\Omega) - m(M) \geq \nu(M^c)$. Namely, $m(M) \leq \nu(\Omega) - \nu(M^c) = 0$. Hence, $m(M) = 0 \Longrightarrow |m|(N) = 0$. To conclude, $|m|(N) = 0$ holds for all N *na*-null. In particular, all the singleton $N = \{\omega\}$ are *na*-null. Hence, $|m|\{\omega\} = 0$. In Borel spaces this implies that m is non-atomic. ■

The next result, which is a consequence of Theorem 3, ensures the non-emptiness of the core of the maximal excess games in this setting. Notice that the games covered by this Theorem are essentially *dna*-generalization of market games

Theorem 12 *Let ν have dna-continuous extension which is concave and linearly homogeneous, then core$(\nu_\xi) \neq \varnothing$ for every $\xi \in na$. In addition, if ν is λ-continuous for some $\lambda \in na^1$, then core$(\nu_\xi) \neq \varnothing$ for every $\xi \in ca$.*

Proof. Given $\xi \in na$, use Theorem 3 for the *dna*-extension $\nu^* : B_1(\Sigma) \to \mathbb{R}$. As ν is bounded, by a density argument, we infer that the function $\nu^*(f)$ is bounded on $B_1(\Sigma)$. It suffices to prove (iii) of Theorem 3. On the other hand, the function $f \to \nu^*(f) - \langle \xi, f \rangle$ is *dna*-continuous, if $\xi \in na$. As the indicator functions are *dna*-dense in $B_1(\Sigma)$, condition (iii)

[4] Notice that if ν is *na*-continuous, then *core*$(\nu) \subset na$, under no additional assumption. In addition, if a standardness assumption is made, then ν is necessarily λ-continuous for some $\lambda \in na$.

follows. Consequently, $core(\nu_\xi) \neq \varnothing$ for all $\xi \in na$. The last claim is a direct consequence of Proposition 5. ■

The result of Theorem 12 alone does not suffice to ensure equivalence theorems for non-atomic games, since the additional condition (SM) might still fail. The next theorem specifies a class of non-atomic games for which (SM) holds.

Theorem 13 *Let ν be dna-continuous and λ-continuous for some $\lambda \in na^1$. Then, $MB^\sigma(\nu)-(ca, ba) = core(\nu)$ if and only if (SM) holds for all $\xi \in na \cap I^*(\nu)$. In particular, this is the case for the measure games $\nu = f(\boldsymbol{\mu})$, where $\boldsymbol{\mu} = (\mu_i)_{i=1}^n$ is a non-atomic signed vector measure and $f : R(\boldsymbol{\mu}) \to \mathbb{R}$ is continuous, concave and linearly homogeneous on the range $R(\boldsymbol{\mu})$.*

Proof. The first statement is a consequence of Theorems 12, 2 and Proposition 5. Let us prove the last claim. The function $\nu^*(\cdot) = f(\boldsymbol{\mu}(\cdot))$ is the *dna*-continuous and concave extension of ν. It is bounded as f is continuous on the compact and convex set $R(\boldsymbol{\mu})$. The game ν is clearly *dna*-uniformly continuous. Part (i) of Proposition 14 implies that $core(\nu) \subset na$. By Theorem 12, $core(\nu_\xi) \neq \varnothing$ for every $\xi \in na$. In fact, by (i) of Proposition 14, ν is λ-continuous for some $\lambda \in na^+$ and, hence, $core(\nu_\xi) \neq \varnothing$ for every $\xi \in ca$.

To end the proof, we must check that (SM) holds. Assume first that $\xi \in span(\mu_1, \mu_2, ..., \mu_n)$, namely, $\xi = a \cdot \boldsymbol{\mu}$, with $a \in \mathbb{R}^n$. We have

$$\sup_{E\in\Sigma} \nu(E) - \xi(E) = \sup_{E\in\Sigma} [f(\boldsymbol{\mu}(E)) - a \cdot \boldsymbol{\mu}(E)] = \sup_{x\in R(\boldsymbol{\mu})} [f(x) - a \cdot x].$$

Clearly, the sup is attained at some point $x_0 \in R(\boldsymbol{\mu})$. By Lyapunov Theorem, there is some A such $\boldsymbol{\mu}(A) = x_0$. If $\xi \notin span(\mu_1, \mu_2, ..., \mu_n)$, then $\xi \in span(\mu_1, \mu_2, ..., \mu_n, \xi)$, and we obtain the same result by considering the new measure game

$$f_0(\mu_1, \mu_2, ..., \mu_n, \xi) = f(\mu_1, \mu_2, ..., \mu_n).$$

An appeal to Theorem 2 concludes the proof. ■

Theorem 13 is not striking, as the measure games it applies to overlap with the atomless market games treated by Mas-Colell. Thus, the present result could be derived from Mas-Colell's theorem. A sharp comparison between Theorem 13 and Mas-Colell's one is, however, difficult since the underlying assumptions are not identical (see also [5, Remark page 199]).

12 Convex games

It is known from Einy et al. [13] that $core(\nu) = MB^\sigma(\nu)$ holds for bounded and continuous convex games. We are going to show that this property may hold true under weaker assumptions than continuity. To this end, a useful observation is that only Condition (SM) matters here, as Condition (C) of Theorem 1 is automatically satisfied independently of any continuity assumption. The following result is due to [25]. We include a proof as we will be using it later on.

Proposition 15 ([25]) *Let ν be convex. Then, the game ν_ξ is convex for all $\xi \in ba$. Consequently, core $(\nu_\xi) \neq \varnothing$ if ν is convex and bounded.*

Proof. Given two coalitions A and B, and $\varepsilon > 0$, there are $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $\nu(A_1) - \xi(A_1) \geq \nu_\xi(A) - \varepsilon/2$ and $\nu(B_1) - \xi(B_1) \geq \nu_\xi(B) - \varepsilon/2$. Hence,

$$\begin{aligned} \nu_\xi(A \cup B) + \nu_\xi(A \cap B) &\geq \nu(A_1 \cup B_1) + \nu(A_1 \cap B_1) - \xi(A_1 \cup B_1) - \xi(A_1 \cap B_1) \\ &\geq \nu(A_1) + \nu(B_1) - \xi(A_1) - \xi(B_1) \\ &\geq \nu_\xi(A) + \nu_\xi(B) - \varepsilon. \end{aligned}$$

Hence, ν_ξ is convex. As ν_ξ is convex and bounded, we infer that *core* $(\nu_\xi) \neq \varnothing$ (see [21] and [24]). ■

Another useful insight comes from the fact, shown in Proposition 16 below, that bounded convex games satisfy largeness condition (L_1). Once again, this is independent of any continuity assumption. Proposition 16 is straightforward consequence of the following Lemma for which we only sketch a proof, since it is more or less known. We could not find, however, a precise reference, at least for the general setting we are dealing with.

Lemma 1 *Every bounded convex game ν satisfies largeness condition (L_0^*): for all $\xi \in ba$, $\xi \geq \nu \Longrightarrow \exists \mu \in core(\nu)$ such that $\xi \geq \mu \geq \nu$.*

Proof. [Proof (Sketch)] Observe that $\xi \geq \nu$ implies $\langle \xi, \varphi \rangle \geq \nu_c(\varphi)$ for all $\varphi \in B^+(\Sigma)$, where $\nu_c(\varphi)$ is the Choquet integral $\nu_c(\varphi) = \int \varphi d\nu$. The functional ν_c is concave over $B^+(\Sigma)$, whenever ν is bounded and convex. Hence $\xi \in \partial\nu_c(0)$. By a standard argument of convex analysis, we have that $\partial\nu_c(0) = core(\nu) + ba^+$. Hence, the desired result. ■

Proposition 16 *Any bounded convex game satisfies (L_1). It satisfies (L_1^σ), provided it is inner continuous. Bounded convex games have the saddle property (SP) if and only if they satisfy (SM). In particular, (SP) holds for continuous convex games.*

Proof. Lemma 1, Proposition 11, FACT 1 and Proposition 15 imply condition L_1 (or L_1^σ) for convex games. As bounded convex games are exact, Theorem 4 guarantees that Property (SP) holds provided that Property (SM) holds.

It remains to show that continuous convex games satisfy (SM). This can be done by considering the Choquet extension of game as in Einy et al. [13], and we do not duplicate their proof. The same argument, however, will be also used in the proof of Theorem 14, below. ■

Summing up, by virtue of Propositions 15 and 16, we must focus on those convex games satisfying (SM). The next Proposition illustrates that continuity is not necessary for (SM) to hold. Consider the class of *unanimity games*: given a coalition $C \in \Sigma$, the game ν_C is defined by $\nu_C(A) = 1$, if $C \subseteq A$, and $\nu_C(A) = 0$ otherwise.

Proposition 17 *If ν_C is a unanimity game, then:*
i) ν_C is inner continuous if and only if the coalition C contains finitely many players;
ii) core$^\sigma(\nu_C) = MB^\sigma(\nu_C) - (ba, ba)$ for all C.

Proof. i) Assume that C is infinite. Let $B_1 = \{a_1, a_2, ...\} \subseteq C$ and $B_n = \{a_n, a_{n+1}, ...\}$. We have $B_n \downarrow \varnothing$. Clearly, $\Omega \setminus B_n \uparrow \Omega$ and $\nu(\Omega \setminus B_n) = 0$. Hence, ν is not inner continuous at Ω. Conversely, ν_C is clearly continuous if C is finite.

ii) By Proposition 16, it suffices to check that (SM) holds for all unanimity games. Given $\xi \in ca$, a straightforward though tedious calculation yields

$$\begin{aligned}\nu_\xi(\Omega) &= \sup_{A \in \Sigma} [\nu_C(A) - \xi(A)] \\ &= \max\left\{1 - \xi^+(C) + \xi^-(\Omega); \xi^-(\Omega)\right\}.\end{aligned}$$

Let $\{N^+, N^-\}$ be a partition of Ω for which $\xi^+(N^-) = \xi^-(N^+) = 0$. If $\xi^+(C) \leq 1$, it is easy to check that the maximum is attained at $C \cup N^-$. Likewise, $\xi^+(C) \geq 1$ implies that the maximum is at $N^- \cup (C \setminus N^+)$. ■

We now turn to a general criterion which guarantees the same result as for unanimity games. Note that all unanimity games ν_C are outer continuous. Moreover, $core^\sigma(\nu_C) = \left\{\mu \in ca^1 : \mu(C) = 1\right\}$ is clearly a weak*-dense subset of $core(\nu_C)$.

Theorem 14 *Let ν be a bounded convex game. Under the following three conditions:*
i) $\nu \geq \sigma$ for some $\sigma \in ca$;
ii) ν is λ-continuous for some $\lambda \in ca^1$;
iii) ν is outer continuous;
ν has Property (SP). Consequently,

$$core^\sigma(\nu) = MB^\sigma(\nu) - (ba, ba),$$

holds and $core^\sigma(\nu)$ is a weak-dense subset of $core(\nu)$.*

Proof. By Proposition 16, ν satisfies (L_1). We must prove that (SM) holds. By Corollary 5 in Appendix, $core^\sigma(\nu)$ is a weak* dense subset of $core(\nu)$. From (ii) it follows that $core^\sigma(\nu) \subset ca(\lambda)$. Hence, $core^\sigma(\nu)$ may be identified with a subset of $\mathcal{L}^1(\lambda)$. Since any bounded convex game is of bounded variation, the Choquet integral $\int \varphi d\nu = \int_0^1 \nu(\varphi \geq t)\, dt$ is well-defined for all $\varphi \in B_1(\Sigma)$. By Proposition 5, it suffices to consider elements $\xi \in ca(\lambda)$. We pattern the remainder of the proof after [13]. We have

$$\nu_c(\varphi) = \int \varphi d\nu = \min_{\mu \in core(\nu)} \langle \varphi, \mu \rangle = \inf_{\mu \in core^\sigma(\nu)} \langle \varphi, \mu \rangle$$

for all $\varphi \in [\mathcal{L}^\infty(\lambda)]_1$. Since

$$\nu_c(\varphi) - \langle \xi, \varphi \rangle = \inf_{\mu \in core^\sigma(\nu)} \langle \varphi, \mu - \xi \rangle,$$

$\nu_c(\varphi) - \langle \xi, \varphi \rangle$ is weak* upper semicontinuous on $[\mathcal{L}^\infty(\lambda)]_1$. By Alaoglu's theorem, $[\mathcal{L}^\infty(\lambda)]_1$ is weak* compact. Therefore, $\nu_c(\varphi) - \langle \xi, \varphi \rangle$ attains the maximum value over $[\mathcal{L}^\infty(\lambda)]_1$. Let $\varphi^* \in [\mathcal{L}^\infty(\lambda)]_1$ be a maximum point. Namely,

$$\begin{aligned}k &= \nu_c(\varphi^*) - \langle \xi, \varphi^* \rangle \geq \nu_c(\varphi) - \langle \xi, \varphi \rangle \\ k &= \int_0^1 (\nu - \xi)(\varphi^* \geq t)\, dt \geq \nu_c(\varphi) - \langle \xi, \varphi \rangle,\end{aligned}$$

for all $\varphi \in [\mathcal{L}^{\infty}(\lambda)]_1$. It follows

$$\int_0^1 [k - (\nu - \xi)(\varphi^* \geq t)]\, dt = 0.$$

By setting $\varphi = \chi_{\{f^* \geq t\}}$, we see that $k \geq (\nu - \xi)(\varphi^* \geq t)$. As the function $t \to k - (\nu - \xi)(\varphi^* \geq t)$ is of bounded variation, we have $k - (\nu - \xi)\left(\varphi^* \geq \overline{t}\right) = 0$ for some $\overline{t} \in [0,1]$ (this is true for every $\overline{t}$ at which the previous function is continuous). By letting $A = \left\{\varphi^* \geq \overline{t}\right\}$, we see that $k = \nu(A) - \xi(A) \geq \nu(E) - \xi(E)$ for all $E \in \Sigma$ and, hence, (SM) holds. ■

There are of course convex games for which the conditions of Theorem 14 fail. In such a case, the bargaining set may be rather large. The next example expands Example 2.1 of [13]. Let $\left(\nu, \mathbb{N}, 2^{\mathbb{N}}\right)$ be the convex game defined by $\nu(A) = 1$ if A^c is finite and $\nu(A) = 0$ otherwise. Given a measure $\xi \in ca^+$, let

$$carrier(\xi) = \{i \in \mathbb{N} : \xi(\{i\}) > 0\}.$$

Proposition 18 *If $\xi \in ca^+ \cap I^*(\nu)$, then $\xi \in MB^{\sigma}(\nu) - (ca, ba) \setminus core(\nu)$ if and only if $carrier(\xi)$ is infinite.*

Proof. Observe first that $core^{\sigma}(\nu) = \varnothing$. Otherwise, it would be $\xi(\mathbb{N} \setminus \{i\}) = 1$ for all $i \in \mathbb{N}$, which is a contradiction. Hence, $\xi \notin core(\nu)$. We must prove that condition (SM) fails whenever $carrier(\xi)$ is infinite. As $\nu(A) - \xi(A) = -\xi(A) \leq 0$ whenever A^c is not finite, we obtain

$$\begin{aligned} \sup_{A \in \Sigma} \nu(A) - \xi(A) &= \sup_{A \text{ is cofinite}} 1 - \xi(A) \\ &= 1 - \xi(\mathbb{N}) + \sup_{F \text{ is finite}} \xi(F) = 1. \end{aligned}$$

If the sup were achieved, it would be $\nu(A) - \xi(A) = 1$ for some cofinite element A. That is, $\xi(A) = 0$, which implies that $carrier(\xi)$ is finite, a contradiction. Conversely, if $carrier(\xi) = F_0$ is finite, it is easy to check that the sup is attained at F_0^c. Consequently, $\xi \notin MB^{\sigma}(\nu) - (ca, ba) \setminus core(\nu)$. ■

Appendix: Density of σ-core

In this appendix, we discuss conditions under which the σ-core of an exact game is weak*-dense in the whole core. This issue has been recently studied by Delbaen [11], who related this property to a "Fatou condition". He deals with positive game. Here, we extend his results to games that are not necessarily positive. Let

$$\nu_e(f) = \min_{\mu \in core(\nu)} \langle \mu, f \rangle.$$

Theorem 15 ([11]) *Assume that: (a) ν is exact; (b) $\nu \geq \sigma$ for some $\sigma \in ca$; and (c) ν is λ-continuous for $\lambda \in ca^1$. The following three conditions are equivalent:*

i) $core^\sigma(\nu)$ is a weak-dense subset of $core(\nu)$;*

ii) for all the sequences $0 \leq f_n \leq 1$ in $\mathcal{L}^\infty(\lambda)$,

$$\nu_e\left(\limsup_n f_n\right) \geq \limsup_n \nu_e(f_n) \tag{22}$$

iii) for all the sequences $f_n \downarrow f$ in $\mathcal{L}^\infty(\lambda)$, with $0 \leq f_n \leq 1$:

$$\nu_e\left(\lim_n f_n\right) = \lim_n \nu_e(f_n).$$

Proof. We sketch the proof since it follows Delbaen's [11]. Observe first that there is some $\sigma_1 \in ca(\lambda)$ for which $\nu \geq \sigma_1$. To see this, it suffices to define

$$\sigma_1(A) = \inf_\pi \sum_{A_i \in \pi} \nu(A_i),$$

where the inf is made over all countable measurable partition π of A. It is easy to check that σ_1 is λ-continuous, σ-additive and $\sigma_1 \geq \sigma$. Furthermore, as ν is superadditive, it follows that $\nu \geq \sigma_1 \geq \sigma$.

Clearly, the game $\nu - \sigma_1$ is positive and λ-continuous. Moreover, $core(\nu - \sigma_1) = core(\nu) - \{\sigma_1\}$. Hence, without loss, we can assume $\nu \geq 0$ for the remainder of the proof.

(i) $\Longrightarrow$ (ii). If $core^\sigma(\nu)$ is weak*-dense, then

$$\nu_e(f) = \inf_{\mu \in core^\sigma(\nu)} \langle \mu, f \rangle,$$

with $f \in \mathcal{L}^\infty(\lambda)$. By Fatou's lemma, $\langle \mu, \limsup_n f_n \rangle \geq \limsup_n \langle \mu, f_n \rangle \geq \limsup_n \nu_e(f_n)$, for all $\mu \in core^\sigma(\nu)$. It follows that $\nu_e(\limsup_n f_n) \geq \limsup_n \nu_e(f_n)$.

(ii) $\Longrightarrow$ (i). As (22) holds for all the sequences $0 \leq f_n \leq 1$ and the functional ν_e is translation invariant and positively homogeneous, (22) is true for all the sequences $\|f_n\|_{\mathcal{L}^\infty} \leq 1$. The first step is to prove that the convex cone $K = \{f \in \mathcal{L}^\infty(\lambda) : \nu_e(f) \geq 0\}$ is $\sigma(\mathcal{L}^\infty, \mathcal{L}^\infty)$ closed. By Krein-Smulian's theorem [9], it suffices to prove that $K \cap B_{\mathcal{L}^\infty(\lambda)}$ is weak*-closed. Let $f_\alpha \to f$ in the weak* topology, with $f_\alpha \in K \cap B_{\mathcal{L}^\infty(\lambda)}$. As $\mathcal{L}^\infty(\lambda) \subset \mathcal{L}^1(\lambda)$, $f_\alpha \to f$ weakly in $\mathcal{L}^1(\lambda)$. Hence, $f \in \overline{co(f_\alpha)_\alpha}^w$. This implies that $f \in \overline{co(f_\alpha)_\alpha}$, where the closure is in the strong topology of $\mathcal{L}^1(\lambda)$. Hence there exists a sequence $g_n \to f$ strongly in $\mathcal{L}^1(\lambda)$, with $g_n \in co(f_\alpha)_\alpha \subseteq K \cap B_{\mathcal{L}^\infty(\lambda)}$. By passing to a subsequence g'_n, we have $g'_n \to f$ a.e. Consequently,

$$\nu_e(f) = \nu_e\left(\limsup_n g'_n\right) \geq \limsup_n \nu_e(g'_n) \geq 0$$

and K is weak* closed.

From here, the proof is completed by using the bipolar theorem for the pairing $\langle \mathcal{L}^\infty, \mathcal{L}^1 \rangle$ (see for instance [2, Th. 5.91]) just as in Delbaen's proof.

(iii) $\Longrightarrow$ (ii). Let $\limsup_n f_n = f$. Set $g_n = \sup_{m \geq n} f_m$. Then $g_n \downarrow f$. By assumption, $\nu_e(g_n) \to \nu(f)$. On the other hand, $\nu_e(f_n) \leq \nu_e(g_n)$. So $\limsup_n \nu_e(f_n) \leq \lim_n \nu_e(g_n) = \nu(f)$ which proves the implication. The converse implication, (ii) $\Longrightarrow$ (iii), is obvious. ■

Theorem 15 has a noteworthy specialization for convex games.

Corollary 5 *Assume: ν is convex, $\nu \geq \sigma$ for some $\sigma \in ca$ and ν is λ-continuous for $\lambda \in ca^1$. Then, $core^{\sigma}(\nu)$ is weak*-dense in $core(\nu)$ if and only if ν is outer continuous.*

Proof. In view of (iii) of Theorem 15, the outer continuity is clearly necessary. Let us prove that it is also sufficient. Let $f_n \downarrow f$, with $0 \leq f_n \leq 1$. Clearly, $\{f_n \geq t\} \downarrow \{f \geq t\}$ holds for all t. By the Monotone Convergence Theorem,

$$\lim_n \nu_e(f_n) = \lim_n \int_0^1 \nu(f_n \geq t)\, dt = \int_0^1 \nu(f \geq t)\, dt = \nu_e(f)$$

that provides the desired result. ■

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