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Abstract

This paper considers a family of admission mechanisms, with multiple applications and application costs. Multiple applications impose serious coordination problems to colleges, but application costs restore stability. Without application costs and under incomplete information unstable allocations emerge.
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1 Introduction

This paper analyzes a class of admission mechanisms: each student sends costly applications to some colleges, then each college selects the applicants to accept, finally each accepted student selects the college to join among the ones that accepted her. This procedure resembles many real world mechanisms, for instance the admissions procedures to Graduate Schools and decentralized job markets.

The mechanism extends the Students-Propose-and-Colleges-Choose Mechanism presented in Alcalde and Romero-Medina (2000), where application costs were zero and each student was allowed to a unique application. Such a mechanism implements the stable correspondence in Subgame Perfect Equilibrium (SPE). But in the real world applicants are rarely restricted to a unique application and often application fees or information gathering costs are present.

With multiple applications each college selects a group of applicants and, at the same time, proposes them like in the Colleges-Propose-and-Students-Choose-Mechanism in Alcalde and Romero-Medina (2000). A problem of coordination among colleges emerges. Regardless of the restriction on applications, the set of equilibrium outcomes contains the stable set (Proposition 2) and with positive application costs, they coincide (Theorem 1). If application costs are zero or if information is incomplete (Examples 1 and 2, respectively) the coordination problems among colleges induce unstable allocations. Future research should clarify if the introduction of additional stages in which new applications and offers are done might help to eliminate unstable allocations (see also Sotomayor (2003) and Alcalde and Romero-Medina (2005)) and favor information release.

The paper is organized as follows. Section 2 introduces the model and Section 3 contains the main results.

2 The Model

A bilateral matching market is represented by a triplet (C, S, P): $C = \{c_1, ..., c_k\}$ is the set of colleges, $S = \{s_1, ..., s_t\}$ is the set of students, $C \cap S = \varnothing$, $P = (P_{c_1}, ..., P_{c_k}, P_{s_1}, ...P_{s_t})$ is the vector of agents' preferences. Let $c \in C$. P_c represents college c's preferences, a strict order on 2^S. Let $S' \subset S$. The **choice set** of c from S' is $Ch_c(S', P_c) = \arg\max_{P_c} \{S'' : S'' \subset S'\}$ is the favorite group of students for college c among the ones belonging to S'. Any student s such that $\varnothing P_c s$ is **unacceptable to** c. Otherwise s is **acceptable to** c. The set of c's acceptable students is denoted by $A(c, P_c)$, c's **quota** is $q_c = \max \{\sharp S' : Ch_c(S', P_c) \neq \varnothing\}$. Let $s \in S$. P_s denotes student s's preferences, a strict order on $C \cup \{s\}$. Any c such that sP_sc is **unacceptable to** s. Otherwise c is **acceptable to** s. Weak preferences are denoted by R. For each $s \in S$, u_s denotes a function, $u_s : C \cup \{s\} \to \mathbf{R}$ representing P_s.

Definition 1 *A **matching** on (C, S) is a function $\mu : C \cup S \to 2^S \cup C$, such that, for every $(c, s) \in C \times S$:*
(i) $\mu(c) \in 2^S$.

(ii) $\mu(s) \in C \cup \{s\}$.
(iii) $\mu(s) = c \Leftrightarrow s \in \mu(c)$.

Definition 2 *μ is* ***individually rational for*** *$s \in S$ if $\mu(s) R_s s$.*
μ is ***individually rational for*** *$c \in C$ if $r R_c \varnothing$, for all $r \in \mu(c)$.*

Definition 3 *μ is* ***blocked by a pair*** *$(c, s) \in C \times S$ if:*
(i) $c P_s \mu(s)$.
(ii) $s \in Ch_c(\mu(c) \cup \{s\})$.

A matching μ is unstable if there are a college c and a student s who are not paired together but: (i) s would prefer to join c rather than her mate under μ, (ii) c would accept s among its students if it was given to choose its students among the ones in $\mu(c) \cup \{s\}$.

Definition 4 *μ is* ***stable in market*** *(C, S, P) if it is individually rational for all $x \in C \cup S$ and if no pair blocks it. Otherwise μ is* ***unstable***.

The set of matchings that are stable in market (C, S, P) is the **stable set, denoted by** $\square(C, S, P)$.
Two properties on all colleges' preferences are imposed: substitutability and separability. Substitutability assures the non-emptiness of the stable set. A college's preferences are substitutable if it wants to enroll a student even when other students become unavailable.

Definition 5 *Let $c \in C$. P_c are said to be* ***substitutable*** *if, for each $A \in 2^S$ and for all $s, s' \in S$, $s \neq s'$:*

$$s \in Ch_c(A, P_c) \Rightarrow s \in Ch_c(A \square \{s'\}, P_c).$$

Preferences are quota q separable when adding additional acceptable students makes any given set of students of less than q elements a better one.

Definition 6 *Let $c \in C$ and let $q > 0$ be a natural number. P_c are* ***quota q-separable*** *if for all $S' \subset S$*

$$\sharp S' < q, \;\; s \notin S', \; s \in A(c, P_c) \Longleftrightarrow (S' \cup \{s\}) \, P_c S'.$$

$$\sharp S' > q \Rightarrow \varnothing P_c S'.$$

This assumption, weaker than responsiveness, implies that the set of unmatched students is the same in all stable matchings (Martinez and al (2000)), a property used in the proof of Lemma 1.

The paper analyzes implementation in SPE. Let Φ be a class of matching markets and let F be a correspondence on the set of matchings on (C, S). A mechanism **implements** F **in SPE** if the set of SPE outcomes coincides with the allocations prescribed by F, for each $(C, S, P) \in \Phi$.

2.1 The Admission Mechanism

For all s_i, let n_i representing the maximum number of colleges student s_i is allowed to apply to. Let $\delta \geq 0$ be the cost that each student pays to apply to each college. Application costs are assumed to be small: if $u_s(c) > u_s(s)$ then $u_s(c) \square \delta > u_s(s)$[1].

The **Sequential Admission Mechanism (SAM) with restriction** $n = (n_1, ..., n_t)$ is described by the following procedure

> **Stage 1: Application.** s_i sends applications to at most n_i colleges. Let $C_1(s_i)$, $\sharp C_1(s_i) \leq n_i$ be the set of colleges s_i applies to. Let $S_1(c) = \bigcup_{c \in C_1(s)} \{s\}$ the set of students who applied to $c \in C$.
>
> **Stage 2: Acceptation.** c accepts a subset of students, $S_2(c) \subset S_1(c)$. For each student s let $C_2(s) = \bigcup_{s \in S_2(c)} \{s\}$ be the set of colleges that accepted s.
>
> **Stage 3: Matching.** Student s decides which college to join among the ones in $C_2(s)$.

Let μ be the matching resulting from such procedure. The payoff of student s, is $u_s(\mu(s)) \square \delta \sharp C_1(s)$.

Let Z_2 be the set of subgames starting at the second stage. Each $z_2 \in Z_2$ is characterized by the family of sets of students who applied to each college $\{S_1(c, z_2)\}_{c \in C}$, or equivalently by the family of sets of colleges each student applied to, $\{C_1(s, z_2)\}_{s \in S}$.
$P(z_2)$ denotes the following profile of preferences:

(i) for each $c \in C$: $P_c(z_2) = P_c$.

(ii) for each $s \in S$: if $c \notin C_1(s, z_2)$ or if sP_sc then $sP_s(z_2)c$. If $c, c' \in C_1(s, z_2)$ then $cP_s(z_2)c'$ iff cP_sc'.

$P(z_2)$ coincide with P but for one aspect: each student ranks as unacceptable all the colleges she did not apply to.
Let $n = (n_1, ..., n_t)$ be a restriction on the number of applications, n is assumed to be public knowledge.

3 The Results

The first result characterizes the outcomes of the second stage subgames.

Lemma 1 *For all $z_2 \in Z_2$, z_2 implements $\square(C, S, P(z_2))$ in SPE.*

Proof. Let μ be a SPE outcome of z_2. It is easily seen that μ is individually rational for colleges and for students. Let (c, s) be a college-student pair. If (c, s)

[1] If costs the same are higher the results of the paper apply to the market where, $\forall\ s \in S$, every college with $u_s(c) \square \delta < u_s(s)$ is eliminated from s's list of acceptable colleges.

blocks μ let c accepting $Ch_c(\mu(c) \cup \{s\})$. Such deviation would be profitable to c: at the last stage s accepts the best offer she holds at each SPE.
Let $\mu \in \square(C, S, P(z_2))$, then μ is a SPE outcome of z_2. Consider the following strategy for college c: accept only the applicants in $\mu(c)$. Let students selecting their best available college at the third stage. The stability of μ implies that no college can profitably deviate so μ is a SPE outcome of z_2. ■

The result implies that the Colleges-Propose-And-Students-Choose-Mechanism implements the stable set in SPE (Theorem 4.1 in Alcalde and Romero-Medina (2000)).

The colleges become somehow "irrelevant" in the game. Indeed, to analyze the SAM it is sufficient to analyze the associated **Reduced Admission Mechanism (RAM) with restriction** n denoted by H^n. Here, only students play and the outcomes are determined according their optimal stable allocations. Let S be the set of the players. Each student s_i's message space is $M_s^{n_i} = \{C' \subset C : \sharp C' \leq n_i\}$ and the outcome function, h is defined as follows. Let $m = (m_1, ..., m_t) \in M^n = \prod_{i=1}^{t} M_{s_i}^{n_i}$. Let $z_2 = z_2(m)$ be the second stage subgame of the SAM induced by each student s_i applying to the colleges in m_i. Finally, set $h(m) = \mu_{z_2}^S$, where $\mu_{z_2}^S$ is the students optimal stable matching of $(C, S, P(z_2))$. The payoff for player s_i is $u_{s_i}(h(m)(s_i)) \square \delta \sharp m_i$.

Proposition 1 *(i) If μ^* is the outcome matching of the SAM with restriction n and $z_2^* \in Z_2$ is the second stage subgame on the equilibrium path, then μ^* is the students' optimal stable matching of $(C, S, P(z_2^*))$.*
(ii) μ^* is a SPE outcome matching of the SAM with restriction $n = (n_1, ..., n_t)$ if and only if it is a NE outcome of H^n

Proof. (i) Let μ^S be the students' optimal stable matching of $(C, S, P(z_2^*))$. From Lemma 1 all students prefer μ^S to μ^*. If $\mu^S(s) P_s \mu^*(s)$ for some s, consider the following deviation: s applies only to $\mu^S(s)$. Let z_2 be the second stage subgame induced by such deviation. $\mu^S \in \square(C, S, P(z_2))$, $\mu^S(s)$ is the unique s's stable partner. From Lemma 1 the outcome of this deviation belongs to $\square(C, S, P(z_2))$. Since colleges' preferences are substitutable and quota q-separable, s is never unmatched in $\square(C, S, P(z_2))$: s would reduce application costs then the deviation would be profitable to her.
(ii) It is sufficient to prove that $(C_1^*(s_1), ..., C_1^*(s_t))$ are first stage strategies forming part of a SPE of the sequential game if and only if they constitute a NE of the RAM and that both equilibria yield the same outcome matching. Let μ^* be a SPE outcome and let $(C_1^*(s_1), ..., C_1^*(s_t))$ be a first stage SPE strategy leading to μ^*. From (i) it follows that it must be a NE strategy for the RAM. Now, let $(C_1^*(s_1), ..., C_1^*(s_t))$ be a NE of the RAM and let μ^* be its outcome. Consider the following profile of strategies for the players of the sequential mechanism. At the first stage each student s applies to $C_1^*(s)$. At the second stage, for each subgame z_2, each college c accepts only the students in $\mu_{z_2}^S(c)$, the students' optimal stable matching of $(C, S, P(z_2))$. At the third stage students conform to SPE strategy. It is easily verified that such strategy profile constitute a SPE of the SAM yielding μ^* as outcome. ■

First, a weak implementation result is proven.

Lemma 2 *Let $\delta \geq 0$, and let μ be a stable matching. Then there exists a SPE of the SAM yielding μ as outcome.*

Proof. Consider $(\mu(s_1), ..., \mu(s_t))$ as strategy profile for the RAM. From the stability of μ it follows that such strategies are a Nash Equilibrium of the RAM. The claim follows from Proposition 1. ■

If $n = (1, ..., 1)$, Lemma 2 implies that the SAM implements the stable set in SPE, even when $\delta = 0$, recovering the main result of Alcalde and Romero-Medina (2000). When costs are positive the mechanism implements the stable set in SPE independently on the restrictions on the number of applications.

Theorem 1 *Let $\delta > 0$. The SAM implements the stable set in SPE.*

Proof. At equilibrium each student applies to at most one college. Let $m^* = (C_1^*(s_1), ..., C_1^*(s_t))$ be a NE strategy of the RAM and let μ^* be the outcome matching of m^*. Let z_2^* be the subgame induced by such strategy as first stage strategy of the SAM, then $\mu^* = \mu_{z_2}^S$. If $\mu^*(s) = s$ then $C_1^*(s) = \varnothing$ otherwise s could save application costs by not applying to any college. If $\mu^*(s) = c \in C$ and $\sharp C_1^*(s) > 1$, let z_2' be the subgame obtained by the following deviation $C_1(s) = \{c\}$. Then $\mu^* = \mu_{z_2}^S = \mu_{z_2'}^S$. Such deviation is profitable to s because she is enrolled by the same college and saves strictly positive costs, a contradiction. The other part of the claim follows from Lemma 2. ■

Proposition 1 results helpful to prove that the SAM implements unstable allocations if costs are zero[2].

Example 1 *Let $\delta = 0$, $t = k = 3$ $n_i \geq (2, 1, 2)$.*
Let $C = \{c_1, c_2, c_3\}$ and let $S = \{s_1, s_2, s_3\}$ and set

$P_{c_1} = s_1, s_2, s_3.$ $\quad$ $P_{s_1} = c_2, c_1, c_3.$
$P_{c_2} = s_3, s_1, s_2.$ $\quad$ $P_{s_2} = c_1, c_2, c_3.$
$P_{c_3} = s_1, s_2, s_3.$ $\quad$ $P_{s_3} = c_1, c_2, c_3.$

Then $q_1 = q_2 = q_3 = 1$, □$(C, S, P) = \{\mu\}$, where :

$$\mu = \begin{matrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{matrix}$$

Let $n \geq (2, 1, 2)$. μ is a NE outcome of the RAM. Consider the following strategy profile of the RAM:
$C(s_1) = \{c_1, c_2\}$, $C(s_2) = \{c_3\}$, $C(s_3) = \{c_1, c_2\}$.
It results in ν

$$\nu = \begin{matrix} c_1 & c_2 & c_3 \\ s_3 & s_1 & s_2 \end{matrix}$$

ν is blocked by (c_1, s_2). At ν, s_1 and s_3 are matched with their first choices, so they cannot profitably deviate. Proposition 1, (i) implies that by including c_3 in

[2] A longer direct proof is available upon request.

her application s_2 ends matched to c_3 like in equilibrium, otherwise ends single, so the proposed strategies are a NE of the RAM. Then the SAM implements unstable allocation, too.

Incomplete information undermines the result, too.

Example 2 *Let students' preferences be public known and coinciding with the ones defined in the proof of Proposition 1. Let colleges preferences be the following with probability 1/2: $\overline{P_{c_1}} = s_1$, $\overline{P_{c_2}} = s_3$, $\overline{P_{c_3}} = s_2$. Assume that they are like in the proof of Proposition 1 with probability 1/2. If δ is small enough there exists a sequential equilibrium of the SAM with restriction in which the students apply like in Proposition 1, and in which each college makes offer to $\mu^S_{z_2}$. The outcome is then unstable with probability 1/2.*

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