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Complete Monotone Quasiconcave Duality[1]

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Abstract

We introduce a notion of complete monotone quasiconcave duality and we show that it holds for important classes of quasiconcave functions.

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1 Introduction

Since the seminal studies of de Finetti [8] and Fenchel [10], Quasiconvex Analysis has been the subject of very active research.[1] Throughout its history, this field has been deeply influenced by Economic Theory. For example, the original work of de Finetti was motivated by problems in Paretian ordinal utility and, more recently, research in quasiconvex duality has been partly motivated by the dual description of preferences and technologies in Microeconomics (see, e.g., Diewert [9]).

This paper is in keeping with this tradition. In fact, our purpose here is to study a notion of duality that was originally motivated by a problem from Economic Theory. Specifically, in [3] we introduce a general class of uncertainty averse preferences that generalize the variational preferences of [18]. Uncertainty averse preferences $\succsim$ are complete, monotone, and convex binary relations defined on the classic space $B_0(\Omega, \Sigma, C)$ of decision theory, where Ω is a state space, Σ is an event algebra, and C is a convex set of consequences. Elements $f \in B_0(\Omega, \Sigma, C)$ are simple Σ-measurable functions $f : \Omega \to C$, interpreted as the acts available to a decision maker.[2]

Under suitable behavioral conditions, [3] shows that a preference $\succsim$ on $B_0(S, \Sigma, X)$ is uncertainty averse if and only if there is a lower semicontinuous quasiconvex function $G : \mathbb{R}\times\Delta \to (-\infty, \infty]$, increasing in the first argument, such that the functional $I : B_0(\Omega, \Sigma, C) \to \mathbb{R}$ given by

$$I(u(f)) = \min_{p\in\Delta} G\left(\int u(f)\, dp, p\right), \tag{1}$$

represents $\succsim$, where Δ is the set of the probabilities on (Ω, Σ) and $u : C \to \mathbb{R}$ is an affine function with $u(C) = \mathbb{R}$.

As [3] shows, the quasiconvex function G can be interpreted as the decision maker's index of uncertainty aversion. For this decision theoretic interpretation of G to be meaningful, it is crucial that, given I and u, the quasiconvex G that satisfies (1) be unique. In fact, this is what allows to behaviorally pin down G from the decision maker's preferences, and makes comparative statics meaningful.

Mathematically, the functionals I are monotone and quasiconcave over $B_0(\Omega, \Sigma, \mathbb{R})$, and the relation (1) can be viewed as their dual description. In particular, the uniqueness of G requires the existence of a one-to-one relation between the functionals $I : B_0(\Omega, \Sigma, \mathbb{R}) \to \mathbb{R}$ and the quasiconvex functions $G : \mathbb{R}\times\Delta \to (-\infty, \infty]$. This led us to study in depth monotone quasiconcave functionals and their dual properties.

Specifically, in this paper we study monotone quasiconcave functionals $g : X \to [-\infty, \infty]$ defined over normed Riesz spaces with unit.[3] We associate to any quasiconcave function $g : X \to [-\infty, \infty]$, an auxiliary function

$$G_\xi(t) = \sup_{x\in X} \{g(x) : \langle \xi, x\rangle \le t\},$$

where $t \in \mathbb{R}$ and ξ is an element of the topological dual X^*. Because of the positive 0-homogeneity of the map $(t, \xi) \mapsto G_\xi(t)$, it is enough to consider its restriction on $\mathbb{R} \times S^*$, where S^* is the unit sphere of X^*. When g is monotone, we can actually consider the map $(t, \xi) \mapsto G_\xi(t)$ on $\mathbb{R} \times \Delta$, where $\Delta \subset S^*$ is the set of the positive functionals with unit norm.

[1] We refer the reader to Penot [25] for a recent survey. See also Crouzeix [5], [6], and [7], Martinez-Legaz [20] and [21], and Penot and Volle [24].

[2] This is the Anscombe and Aumann [2] version of the classic Savage [30] set up. See [11] and [17].

[3] To be precise we consider the class of M spaces (see Subsection 2.1). These spaces are essentially function spaces equipped with their supnorm, $B_0(\Omega, \Sigma, \mathbb{R})$ is an example.

It is well known that under mild assumptions quasiconcave functionals $g : X \to [-\infty, \infty]$ can be recovered from their dual functions $G_\xi(t)$ through the relation

$$g(x) = \inf_{\xi \in S^*} G_\xi(\langle \xi, x \rangle). \tag{2}$$

When g is monotone, the sphere S^* can be replaced by Δ (see Theorem 1).[4] The duality relation between g and the map $(t, \xi) \mapsto G_\xi(t)$ provided by (2) is the mathematical underpinning of the decision theoretic representation (1).

However, as we already pointed out, a key issue in (1) is the uniqueness of G. This is not ensured by the duality relation (2), which is indeed an incomplete duality. In fact, given a quasiconcave function $g : X \to [-\infty, \infty]$ there may be many $G : \mathbb{R} \times S^* \to [-\infty, \infty]$ such that $g(x) = \inf_{\xi \in S^*} G(\langle \xi, x \rangle, \xi)$. This "inverse problem," of fundamental decision theoretic importance, is what characterizes our notion of duality. Notice that the classic Fenchel conjugation is complete: the Fenchel duality map $f \mapsto f^*$ is one-to-one on the space of upper semicontinuous concave functions (see, e.g., [29, Theorem 5]). In our quasiconcave setting the problem turns out to be more difficult. To be more precise, we now introduce formally our duality notion for the monotone case, which is the one we consider in the paper and the most relevant for economic applications (see the discussion at the end of the section).

Assume that X is a normed Riesz spaces with unit and denote by $\mathcal{M}_{qc}(X)$ the set of all quasiconcave monotone functions $g : X \to [-\infty, \infty]$. Moreover, denote by $\mathcal{M}(\mathbb{R} \times \Delta)$ the space of functions $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$ such that:

(A.1) $G(\cdot, \xi)$ is increasing for each $\xi \in \Delta$,

(A.2) $\lim_{t \to +\infty} G(t, \xi) = \lim_{t \to +\infty} G(t, \xi')$ for all $\xi, \xi' \in \Delta$.

Consider the operator $\mathbf{T} : \mathcal{M}_{qc}(X) \to \mathcal{M}(\mathbb{R} \times \Delta)$ given by

$$(\mathbf{T}g)(t, \xi) = G_\xi(t) = \sup_{x \in X} \{g(x) : \langle \xi, x \rangle \leq t\}, \quad \forall g \in \mathcal{M}_{qc}(X),$$

and the operator $\mathbf{Q} : \mathcal{M}(\mathbb{R} \times \Delta) \to \mathcal{M}_{qc}(X)$ given by

$$(\mathbf{Q}G)(x) = \inf_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi), \quad \forall G \in \mathcal{M}(\mathbb{R} \times \Delta). \tag{3}$$

We can now define our notion of (complete) quasiconcave monotone duality.

Definition 1 *Two subsets $A \subset \mathcal{M}_{qc}(X)$ and $B \subset \mathcal{M}(\mathbb{R} \times \Delta)$ form a (complete monotone) quasiconcave duality pair, written $\langle A, B \rangle_{qc}$, if $\mathbf{T}$ is injective on A, $\mathbf{T}(A) = B$, and $\mathbf{T}^{-1} = \mathbf{Q}$ on B.*

In other words, we have $\langle A, B \rangle_{qc}$ when for every $g \in A$ the *only* $G \in B$ such that

$$g(x) = \inf_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi), \qquad \forall x \in X, \tag{4}$$

is given by

$$G(t, \xi) = \sup\{g(x) : \langle \xi, x \rangle \leq t\}, \qquad \forall (t, \xi) \in \mathbb{R} \times \Delta, \tag{5}$$

and, conversely, for every $G \in B$ there is (a unique) $g \in A$ such that (5) holds. Such g is given by (4), that is, the duality is complete.

[4] A first formulation of (2) goes back to de Finetti [8]. Since then, it has been extensively studied (see the references in Section 2.3). As emphasized by [24], there is a close relation between Fenchel conjugation of the Convex Analysis and (2), in which quasiaffine functions $x \to G_\xi(\langle \xi, x \rangle)$ play the role of affine functions.

The purpose of this paper is to identify significant dual pairs $\langle A, B\rangle_{qc}$. This is mainly done in Section 3. Our main results, Theorems 2, 3, and 6, are based on minimax arguments (in particular, Theorems 3 and 6 use Sion [31] and Tuy [32]) and show that our complete duality holds, respectively, for the very important classes of monotone evenly quasiconcave functions, of monotone lower semicontinuous quasiconcave functions, and of monotone uniformly continuous quasiconcave functions. These results are then specialized in Sections 6 and 7, where Theorems 9 and 10 establish duality results for monotone and quasiconcave functionals that are, respectively, translation invariant and positively homogeneous. Together these results show the wide applicability of our notion of duality. They are summarized in Section 8, which also provides a glossary of our main notation.

It is important to observe that our notion of complete duality is different from the one usually studied in Microeconomics (see, e.g., [14], [7] and [9]), which associates to a utility function u defined on a cone K a normalized indirect utility v defined by

$$v(\xi) = \sup_{x \in K} \{u(x) : \langle \xi, x\rangle \leq 1\}, \qquad \forall \xi \in K^*, \tag{6}$$

where K^* is the positive dual cone of K. Here ξ is a (linear) price functional and 1 is normalized income. The dual relation is given by

$$u(x) = \inf_{\xi \in K^*} \{v(\xi) : \langle \xi, x\rangle \leq 1\}, \qquad \forall x \in K. \tag{7}$$

This duality thus associates increasing quasiconcave functions $u : K \to [-\infty, \infty]$ to decreasing quasiconvex functions $v : K^* \to [-\infty, \infty]$. For this duality, Martinez-Legaz [21] established a uniqueness result through evenly quasiconcave/quasiconvex pairs (u, v), with an additional mild assumption on the behavior of the functions on the boundaries of K and K^*.

We believe that also our duality is natural for consumer theory purposes. In fact, our indirect utility function $v : \mathbb{R} \times \Delta \to [-\infty, \infty]$ is given by

$$v(t, \xi) = \sup_{x \in X} \{u(x) : \langle \xi, x\rangle \leq t\},$$

which, relative to (6), allows to better keep track of income effects, captured by changes in t. Though a thorough investigation of our duality – and of its relations with the duality (7) – is beyond the scope of this paper, in Section 4 we characterize the set of all possible indirect utility functions $v : \mathbb{R} \times \Delta \to [-\infty, \infty]$. In other words, we are able to identify all real valued functions on $\mathbb{R} \times \Delta$ that may arise from a standard utility maximization problem. Since in Economics it is often convenient to begin the analysis by specifying a given indirect utility function, this characterization is important as it shows what are the functions $v : \mathbb{R} \times \Delta \to [-\infty, \infty]$ that is legitimate to assume to be an indirect utility function.

We close by observing that our duality theory relies on two important assumptions: the space X is a normed Riesz space with interior unit, and the quasiconcave functional $g : X \to [-\infty, \infty]$ is monotone. In particular, monotonicity makes it possible to consider the function G_ξ restricted on the simplex Δ, which in turn is convex when the supnorm is considered. In any case, both assumptions are satisfied in most economic applications, which are the original motivation of this paper. These assumptions are also common in Finance: for example, we are using the results of this paper to study risk measures when the riskless asset is illiquid or the interest rate is stochastic (see [4] for details).

2 Preliminaries

2.1 Set Up

Throughout the paper, X is a normed vector space and X^* is its topological dual. Elements of X^* are usually denoted by ξ, while $\langle \xi, x\rangle$, with $x \in X$, denotes the bilinear pairing. We denote by $S^* = \{\xi \in X^* : \|\xi\| = 1\}$ the unit sphere of X^*.

If $(X, \geq)$ is an ordered normed space, we denote by X_+ its positive cone $\{x \in X : x \geq 0\}$ and by X_+^* the set of all positive functionals in X^*. We also set $\Delta = \left\{\xi \in X_+^* : \|\xi\| = 1\right\}$. In the sequel, X^* and any of its subsets will be always equipped with the weak* topology, particularly the simplex Δ.[5]

We will often assume that X is an M-space with unit. Recall that an M-space is a normed Riesz space for which $\|x \vee y\| = \|x\| \vee \|y\|$ holds for all $x, y \in X_+$. It is well known that any normed Riesz space with order unit e can be turned into an M-space,[6] provided e is interior to the positive cone X_+. The supnorm $\|x\|_e = \inf \{\lambda \in \mathbb{R} : |x| \leq \lambda e\}$ generated by e is actually an equivalent M-norm ([1, ch. 9]).

Throughout the paper all the M-spaces that we consider will have a unit element e. For example, given an algebra Σ of subsets of a space Ω, the space $B_0(\Omega, \Sigma, \mathbb{R})$ with unit 1_Ω and its supnorm closure are examples of M-spaces that play an important role in decision theory; two other important classes of M-spaces are the spaces of real-valued continuous functions defined on compact Hausdorff topological spaces and the L^∞ spaces on finite measure spaces.

If X is an M-space, its closed unit ball is $[-e, e] = \{x \in X : -e \leq x \leq e\}$. Hence, $\|\xi\| = \langle \xi, e\rangle$ for all $\xi \in X_+^*$, and so $\Delta = \left\{\xi \in X_+^* : \langle \xi, e\rangle = 1\right\}$, which is therefore a convex and weak* compact set.

A subset C of X is *evenly convex* if it is the intersection of a family of open half spaces.[7] Evenly convex sets are convex, and intersections of any family of evenly convex sets are evenly convex. The next lemma is well known.

Lemma 1 *A set C is evenly convex if and only if for each $\bar{x} \notin C$ there is $\bar{\xi} \in X^* \setminus \{0\}$ such that $\langle \bar{\xi}, \bar{x}\rangle < \langle \bar{\xi}, x\rangle$ for all $x \in C$.*

By standard separation results, both open convex sets and closed convex sets are then evenly convex.

A function $g : X \to [-\infty, \infty]$ is:

(i) *lower semicontinuous* if the sets $\{g \leq \alpha\}$ are closed for all $\alpha \in \mathbb{R}$;

(ii) *upper semicontinuous* if the sets $\{g \geq \alpha\}$ are closed for all $\alpha \in \mathbb{R}$;

(iii) *positively homogeneous* if $g(\lambda x) = \lambda g(x)$ for all $\lambda > 0$ and $x \in X$;

(iv) *quasiconcave* if the sets $\{g \geq \alpha\}$ are convex for all $\alpha \in \mathbb{R}$;

(v) *evenly quasiconcave* if the sets $\{g \geq \alpha\}$ are evenly convex for all $\alpha \in \mathbb{R}$;

(vi) *strictly evenly quasiconcave* if the sets $\{g > \alpha\}$ are evenly convex for all $\alpha \in [-\infty, \infty)$.

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[5] See, e.g., [1] and [15] for all notions on ordered vector spaces that will be used.

[6] A positive element e is a *unit* if for all $x \in X$ there is some $\alpha \geq 0$ such that $|x| \leq \alpha e$.

[7] With the convention that such intersection is X if the family is empty. The notion of even convexity and its basic properties are due to Fenchel [10]. Evenly quasiconvex functions were independently introduced by Martinez-Legaz [20] and Passy and Prisman [23].

Quasiconvex notions are similarly defined. In particular, a function that is both quasiconcave and quasiconvex is called *quasiaffine*. Moreover, a function $g : X \to [-\infty, \infty]$ is (extended-valued) continuous if and only if it is both lower and upper semicontinuous; i.e., $\lim_{x\to x_0} g(x) = g(x_0) \in [-\infty, \infty]$ for all $x_0 \in X$.

Clearly, an evenly quasiconcave function $g : X \to [-\infty, \infty]$ is quasiconcave. Moreover, it is easy to check that:

(i) g is evenly quasiconcave if it is strictly evenly quasiconcave;

(ii) g is evenly quasiconcave if it is upper semicontinuous and quasiconcave;

(iii) g is strictly evenly quasiconcave if it is lower semicontinuous and quasiconcave.[8]

Observe that when g is positively homogeneous, then $g(0) = \lambda g(0)$ for all $\lambda > 0$, so that either $g(0) = \pm\infty$ or $g(0) = 0$. In particular, $g(0) = 0$ if g is real valued.

Suppose X is an ordered space. A function $g : X \to [-\infty, \infty]$ is *monotone* (or *increasing*) if $x \geq y$ implies $g(x) \geq g(y)$. If X has an order unit e, then $g : X \to [-\infty, \infty]$ is:

(i) *normalized* if $g(\lambda e) = \lambda$ for all $\lambda \in \mathbb{R}$;

(ii) *translation invariant* if $g(x + \lambda e) = g(x) + \lambda$ for all $\lambda \in \mathbb{R}$ and $x \in X$.

If g is translation invariant and real valued, then $g - g(0)$ is normalized. Moreover, it is easy to see ([19]) that a real valued g is translation invariant if and only if $g(x + \lambda e) = g(x) + \lambda$ for all $\lambda \geq 0$ and $x \in X$.

Lemma 2 *Let X be an M-space. If $g : X \to [-\infty, \infty]$ is a normalized and monotone function, then it is real valued, with*

$$\min_{\xi\in\Delta} \langle \xi, x\rangle \leq g(x) \leq \max_{\xi\in\Delta} \langle \xi, x\rangle, \qquad \forall x \in X. \tag{8}$$

In particular, $|g(x)| \leq \|x\|$ for all $x \in X$.

Proof. Let $x \in X$, it can be shown that $\min_{\xi\in\Delta} \langle \xi, x\rangle = \sup\{\lambda : x \geq \lambda e\}$ and $\max_{\xi\in\Delta} \langle \xi, x\rangle = \inf\{\lambda : x \leq \lambda e\}$. Hence, since

$$e \sup\{\lambda : x \geq \lambda e\} \leq x \leq e \inf\{\lambda : x \leq \lambda e\},$$

we have that $g(x) \in [\min_{\xi\in\Delta} \langle \xi, x\rangle, \max_{\xi\in\Delta} \langle \xi, x\rangle]$. Therefore,

$$|g(x)| \leq \max\left\{\max_{\xi\in\Delta} |\langle \xi, -x\rangle|, \max_{\xi\in\Delta} |\langle \xi, x\rangle|\right\} \leq \max\{\|-x\|, \|x\|\} = \|x\|,$$

as desired. ■

This lemma fails without monotonicity. For instance, on $\mathbb{R}^2$ with the usual unit $e = (1, 1)$, the function

$$g(x) = \begin{cases} \lambda & \text{if } x = \lambda e \text{ for some } \lambda \in \mathbb{R} \\ -\infty & \text{else} \end{cases}$$

is normalized, but clearly neither monotone nor finite.

[8] In fact, if g is quasiconcave and lower semicontinuous, then all its upper level sets $\{g > \alpha\}_{\alpha\in\mathbb{R}}$ are open convex by definition. Moreover, $\{g > -\infty\}$ is open, being the union of open sets, and is convex by the quasiconcavity of g.

Example 1 Mean functionals are a fundamental example of normalized, monotone, and quasiconcave functionals. For, given the function space $L^\infty(\mu)$, where μ is a probability measure on some measurable space (Ω, Σ), consider the mean functional $I : L^\infty(\mu) \to \mathbb{R}$ defined by

$$I(f) = \phi^{-1}\left(\int_\Omega \phi(f(\omega))\, d\mu(\omega)\right),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous. Clearly, I is monotone and normalized. If, in addition, ϕ is concave, then I is quasiconcave as well. ▲

2.2 Two Key Auxiliary Functions

Given $\xi \in X^*$, to each function $g : X \to [-\infty, \infty]$ we can associate two auxiliary scalar functions, defined for all $t \in \mathbb{R}$ by:

$$g_\xi(t) = \sup_{x\in X}\{g(x) : \langle \xi, x\rangle = t\} \quad \text{and} \quad G_\xi(t) = \sup_{x\in X}\{g(x) : \langle \xi, x\rangle \le t\}.$$

These two functions, which will play a key role in what follows, can take values on $[-\infty, \infty]$. The function G_ξ is increasing and dominates g_ξ for each $\xi \in X^*$. In fact, $G_\xi(t) = \sup_{k\le t} g_\xi(k)$. Moreover:

(i) $g_{\lambda\xi}(\lambda t) = g_\xi(t)$ for all $0 \ne \lambda \in \mathbb{R}$;

(ii) $G_{\lambda\xi}(\lambda t) = G_\xi(t)$ for all $\lambda > 0$.

Denote by g_ξ^+ and G_ξ^+ the lower semicontinuous envelopes of g_ξ and G_ξ, respectively. Clearly, $g_\xi^+ \ge g_\xi$ and $G_\xi^+ \ge G_\xi$. In particular, $G_\xi^+(t) = \inf\{G_\xi(t') : t' > t\}$ since G_ξ is increasing.

The next lemmas give some basic properties of the mapping $(t, \xi) \mapsto G_\xi(t)$.

Lemma 3 *For any function $g : X \to [-\infty, \infty]$,*

$$\lim_{t\to+\infty} G_\xi(t) = \sup_{\zeta\in X^*}\sup_{t\in\mathbb{R}} G_\zeta(t) = \sup_{x\in X} g(x) \in [-\infty, \infty], \qquad \forall \xi \in X^*. \tag{9}$$

More generally,

$$\lim_{t\to+\infty}\left(\inf_{\xi\in C} G_\xi(t)\right) = \sup_{x\in X} g(x), \tag{10}$$

holds for any nonempty relatively compact set C of X^*.

Proof. By definition, $G_\zeta(t) \le \sup_{x\in X} g(x)$ for all $t \in \mathbb{R}$ and all $\zeta \in X^*$, so that $\sup_{\zeta\in X^*}\sup_{t\in\mathbb{R}} G_\zeta(t) \le \sup_{x\in X} g(x)$. Similarly, $g(x) \le G_\xi(\langle \xi, x\rangle)$ for all $x \in X$ and all $\xi \in X^*$.

There exists a sequence $\{x_n\}_n$ such that $g(x_n) \uparrow \sup_{x\in X} g(x)$. Since $t \mapsto G_\xi(t)$ is increasing, we have $g(x_n) \le G_\xi(\langle \xi, x_n\rangle) \le \lim_{t\to+\infty} G_\xi(t)$ for all n. Hence,

$$\sup_{x\in X} g(x) = \lim_n g(x_n) \le \lim_{t\to+\infty} G_\xi(t) \le \sup_{\zeta\in X^*}\sup_{t\in\mathbb{R}} G_\zeta(t) \le \sup_{x\in X} g(x),$$

which proves (9).

As to (10), it is clear that the mapping such that $t \mapsto \inf_{\xi\in C} G_\xi(t)$ is increasing. Therefore,

$$\lim_{t\to+\infty}\left(\inf_{\xi\in C} G_\xi(t)\right) = \sup_{t\in\mathbb{R}}\inf_{\xi\in C} G_\xi(t) \le \sup_{x\in X} g(x).$$

Suppose, by contradiction that $\sup_{t\in\mathbb{R}} \inf_{\xi\in C} G_\xi(t) = \alpha < \sup_{x\in X} g(x)$. There is then a point $\overline{x}$ such that $g(\overline{x}) > \alpha$. Let $\overline{t} = \sup_{\xi\in C} \langle \xi, \overline{x}\rangle$. As C is relatively compact, $\overline{t} < \infty$. Hence

$$\alpha = \sup_{t\in\mathbb{R}} \inf_{\xi\in C} G_\xi(t) \geq \inf_{\xi\in C} G_\xi\left(\overline{t}\right) \geq g(\overline{x}) > \alpha,$$

that leads to a contradiction. ■

A straightforward adaptation of a well known result from Microeconomics shows that the mapping $(t,\xi) \mapsto G_\xi(t)$ is quasiconvex over $\mathbb{R}\times X^*$. However, more is true. Set $\mathbb{R}^\diamond = \mathbb{R}\backslash\{0\}$, say that a subset C of $\mathbb{R}\times\Delta$ is $\Diamond$-*evenly convex* if for each $\left(\overline{t},\overline{\xi}\right) \notin C$ there exists $(s,x)\in\mathbb{R}^\diamond\times X$, such that $\overline{t}s + \left\langle\overline{\xi}, x\right\rangle < ts + \langle\xi, x\rangle$ for all $(t,\xi)\in C$. Similarly, a function defined on $\mathbb{R}\times\Delta$ is $\Diamond$-*evenly quasiconvex* if all its lower contour sets are $\Diamond$-evenly convex.[9]

Lemma 4 *For any function $g : X \to [-\infty,\infty]$, the mapping $(t,\xi)\mapsto G_\xi(t)$ is quasiconvex over $\mathbb{R}\times X^*$ and $\Diamond$-evenly quasiconvex on $\mathbb{R}\times\Delta$.*

Proof. Let $(t_1,\xi_1),(t_2,\xi_2)\in\mathbb{R}\times X^*$. Consider $\lambda\in(0,1)$ and the point (t',ξ'), with $\xi' = \lambda\xi_1 + (1-\lambda)\xi_2$ and $t' = \lambda t_1 + (1-\lambda)t_2$. We have

$$\{x\in X : \langle\xi', x\rangle \leq t'\} \subset \{x\in X : \langle\xi_1, x\rangle \leq t_1\} \cup \{x\in X : \langle\xi_2, x\rangle \leq t_2\},$$

which implies $G_{\xi'}(t') \leq \max\{G_{\xi_1}(t_1), G_{\xi_2}(t_2)\}$, and the mapping is quasiconvex.

Set $L_\alpha = \{(t,\xi)\in\mathbb{R}\times\Delta : G_\xi(t)\leq\alpha\}$, with $\alpha\in\mathbb{R}$. If $\left(\overline{t},\overline{\xi}\right)\in\mathbb{R}\times\Delta$ and $\left(\overline{t},\overline{\xi}\right)\notin L_\alpha$, then $G_{\overline{\xi}}\left(\overline{t}\right) > \alpha$. This implies that there exists a point $\overline{x}$ such that $\left\langle\overline{\xi},\overline{x}\right\rangle \leq \overline{t}$ and $g(\overline{x}) > \alpha$. But $G_\xi(t)\leq\alpha$ for all $(t,\xi)\in L_\alpha$, which implies that $\langle\xi,\overline{x}\rangle > t$ for all $(t,\xi)\in L_\alpha$. This, in turn, implies that

$$\langle\xi,\overline{x}\rangle - t > 0 \geq \left\langle\overline{\xi},\overline{x}\right\rangle - \overline{t}, \qquad \forall (t,\xi)\in L_\alpha.$$

■

Clearly, the map $t\mapsto G_\xi(t)$ is both evenly quasiconvex and evenly quasiconcave since G_ξ is monotone. Namely, $t\mapsto G_\xi(t)$ is evenly quasiaffine.

Lemma 5 *Given $g : X\to[-\infty,\infty]$, if g is lower semicontinuous, then the map $(t,\xi)\mapsto G_\xi(t)$ is lower semicontinuous on $\mathbb{R}\times(X^*\setminus\{0\})$.*

Proof. Let $\alpha\in\mathbb{R}$ and $\left(\overline{t},\overline{\xi}\right)\in\mathbb{R}\times X^*\backslash\{0\}$ be such that $G_{\overline{\xi}}\left(\overline{t}\right) > \alpha$ and $\left\|\overline{\xi}\right\|\neq 0$. There is $x_0\in X$ such that $\left\langle\overline{\xi}, x_0\right\rangle \leq \overline{t}$ and $g(x_0) > \alpha$. As

$$\left\|\overline{\xi}\right\| = \sup\left\{\left\langle\overline{\xi}, x\right\rangle : \|x\| = 1\right\},$$

there is $u\in X$, with $\|u\| = 1$, such that $\left\langle\overline{\xi}, u\right\rangle \geq 2^{-1}\left\|\overline{\xi}\right\| > 0$. The sequence $x_n = x_0 - n^{-1}u \to x_0$. Hence, there is $\bar{n}\in\mathbb{N}$ such that $g(x_{\bar{n}}) > \alpha$. Moreover, $\left\langle\overline{\xi}, x_{\bar{n}}\right\rangle = \left\langle\overline{\xi}, x_0\right\rangle - \overline{n}^{-1}\left\langle\overline{\xi}, u\right\rangle \leq \overline{t} - \delta$ for $\delta = 2^{-1}\bar{n}^{-1}\left\|\overline{\xi}\right\| > 0$.

The set $U = \left\{\xi\in X^*\backslash\{0\} : \langle\xi, x_{\bar{n}}\rangle < \left\langle\overline{\xi}, x_{\bar{n}}\right\rangle + \delta/2\right\}$ is open in the topology induced by the weak* topology, and for all $(t,\xi)\in\left(\overline{t}-\delta/2,\infty\right)\times U$ we have

$$\langle\xi, x_{\bar{n}}\rangle < \left\langle\overline{\xi}, x_{\bar{n}}\right\rangle + \delta/2 \leq \overline{t} - \delta + \delta/2 = \overline{t} - \delta/2 < t.$$

[9]We require that s is nonzero, which is stronger than requiring that s or x are nonzero, as in Lemma 1. As a result, $\Diamond$-even convexity is slightly more than an extension to products of topological vector spaces of the notion of even convexity introduced in Subsection 2.1 for normed vector spaces. Clearly, $\emptyset$ and $\mathbb{R}\times\Delta$ are $\Diamond$-evenly convex. Moreover, $\Diamond$-evenly convex sets are evenly convex, and $\Diamond$-evenly quasiconvex functions are evenly quasiconvex.

Hence, $G_\xi(t) \geq g(x_{\bar{n}}) > \alpha$, and the map $(t,\xi) \mapsto G_\xi(t)$ is lower semicontinuous. ■

Remark. Since the increasing map $t \mapsto G_\xi(t)$ is lower semicontinuous, it is also left continuous.

Lemma 6 *Given any $h : X \to [-\infty,\infty]$, let $\varphi : \overline{h(X)} \to [-\infty,\infty]$ be extended-valued continuous and monotone, and set $g = \varphi \circ h$. Then, $G_\xi(t) = \varphi(H_\xi(t))$ and $g_\xi(t) = \varphi(h_\xi(t))$ for all $(t,\xi) \in \mathbb{R} \times \Delta$.*

Proof. It is enough to prove that $\sup_{x\in C} g(x) = \varphi(\sup_{x\in C} h(x))$ for a nonempty subset C of X. Set $H = \sup_{x\in C} h(x)$. Since $C \neq \emptyset$, there exists a sequence $\{x_n\}_n$ in C such that $h(x_n) \uparrow H$. Therefore, $H \in \overline{h(C)} \subset \overline{h(X)}$. Monotonicity of φ implies $\varphi(H) \geq \sup_{x\in C} g(x)$. Suppose per contra $m = \sup_{x\in C} g(x)$ and $\varphi(H) > m \geq g(x)$ for all $x \in C$. Continuity of φ implies

$$\varphi(H) = \lim_{n\to\infty} \varphi(h(x_n)) = \lim_{n\to\infty} g(x_n) \leq m < \varphi(H),$$

a contradiction. ■

The positive cone X_+ in an ordered space X is said to be *quasi-reproducing* if $X = \overline{X_+ - X_+}$. By the Riesz decomposition, positive cones in normed Riesz spaces are "reproducing," namely, $X = X_+ - X_+$. For later reference, next we give two elementary properties.

Lemma 7 *X_+ is quasi-reproducing if and only if for any $\xi \in X_+^* \setminus \{0\}$ there is $z \in X_+$ such that $\langle \xi, z\rangle > 0$.*

Proof. Assume $X = \overline{X_+ - X_+}$ and suppose, by contradiction, that $\langle \xi, x\rangle = 0$ for some $\xi \in X_+^* \setminus \{0\}$ and all $x \in X_+$. It follows that $\langle \xi, u\rangle = 0$ for all $u \in \overline{X_+ - X_+}$. Hence, the closed vector space $\overline{X_+ - X_+}$ would be included into the hyperplane $\langle \xi, x\rangle = 0$, a contradiction with $\xi \neq 0$.

Conversely, assume that for any $\xi \in X_+^* \setminus \{0\}$ there is $z \in X_+$ such that $\langle \xi, z\rangle > 0$, and suppose, by contradiction that $\overline{X_+ - X_+} \neq X$. Then, the closed subspace $\overline{X_+ - X_+}$ would be contained into an hyperplane. Hence, $\langle \xi, u\rangle = 0$ for all $u \in \overline{X_+ - X_+}$ and for some $\xi \in X^* \setminus \{0\}$. In particular, we would have $\langle \xi, x\rangle = 0$ for all $x \in X_+$, which implies $\xi \in X_+^* \setminus \{0\}$ and leads to a contradiction. ■

Lemma 8 *If X_+ is quasi-reproducing and $g : X \to [-\infty,\infty]$ is monotone, then $G_\xi = g_\xi$ for all $\xi \in X_+^* \setminus \{0\}$.*

Proof. By definition, we have $g_\xi(t) \leq G_\xi(t)$. Suppose, by contradiction that $g_\xi(t) < G_\xi(t)$ for some $\xi \in X_+^* \setminus \{0\}$ and $t \in \mathbb{R}$. This implies the existence of a point $x \in X$ for which $g_\xi(t) < g(x) \leq G_\xi(t)$ and $\langle \xi, x\rangle < t$. By Lemma 7, we have $\langle \xi, x + \alpha z\rangle = t$, for some $z \in X_+$ and $\alpha > 0$. Hence, $g(x) \leq g(x + \alpha z) \leq g_\xi(t)$ that leads to a contradiction. ■

2.3 A Representation Result

Evenly quasiconcave functions g can be recovered from the scalar functions $g_\xi(t)$ and $G_\xi(t)$. Though formula (11) is essentially well known, this result is the starting point of our analysis and for this reason we now present it in detail. An early version of this result for the function g_ξ can be found in de Finetti [8, p. 178], in his seminal paper on quasiconcavity (the function $g_\xi(t)$ is de Finetti's "profile" function). Other relevant references are Greenberg and Pierskalla [12] and Crouzeix [6]. A general formulation can be found in Penot and Volle [24, Theorem 2.6].

Theorem 1 *A function $g : X \to [-\infty,\infty]$, where X is a normed space, is evenly quasiconcave if and only if*

$$g(x) = \inf_{\xi\in S^*} G_\xi(\langle \xi, x\rangle) = \inf_{\xi\in S^*} g_\xi(\langle \xi, x\rangle), \qquad \forall x \in X. \tag{11}$$

Moreover:

(i) If X is ordered, then g is monotone if and only if in (11) we can replace S^* with Δ.

(ii) g is strictly evenly quasiconcave if and only if the infima in (11) are attained at all $x \in X$.

(iii) g is upper semicontinuous if and only if in (11) we can replace G_ξ and g_ξ with G_ξ^+ and g_ξ^+, respectively.

Remark. Unlike formula (11), points (i)-(iii) are novel. In particular, since quasiconcave lower semicontinuous functions are strictly evenly quasiconcave, point (ii) implies that for them the infima in (11) are attained at all $x \in X$.

Proof. "Only if." Suppose g is evenly quasiconcave. The result is trivially true if $g \equiv -\infty$. Assume $g \not\equiv -\infty$. We have

$$g(x) \leq g_\xi(\langle \xi, x\rangle) \leq G_\xi(\langle \xi, x\rangle), \qquad \forall x \in X, \forall \xi \in X^* \setminus \{0\}, \tag{12}$$

and so

$$g(x) \leq \inf_{\xi \in X^* \setminus \{0\}} g_\xi(\langle \xi, x\rangle) \leq \inf_{\xi \in X^* \setminus \{0\}} G_\xi(\langle \xi, x\rangle), \qquad \forall x \in X. \tag{13}$$

Pick $\bar{x} \in X$. If $\bar{x}$ is a global maximum for g on X, equality holds in (12), and so in (13). Assume that $\bar{x} \in X$ is not a global maximum. Note that by (13) it suffices to prove the statements only for the functions G_ξ.

Case 1: Suppose $g(\bar{x}) \in \mathbb{R}$. Since $\bar{x}$ is not a global maximum, there is $\bar{\varepsilon} > 0$ such that $\{g \geq g(\bar{x}) + \varepsilon\} \neq \emptyset$ for all $\varepsilon \in (0, \bar{\varepsilon}]$. For all such ε, $\bar{x} \notin \{g \geq g(\bar{x}) + \varepsilon\}$. Since this upper set is evenly convex, there is $\bar{\xi} \in X^* \setminus \{0\}$ such that $\langle \bar{\xi}, \bar{x}\rangle < \langle \bar{\xi}, x\rangle$ for all $x \in \{g \geq g(\bar{x}) + \varepsilon\}$. That is, $\{g \geq g(\bar{x}) + \varepsilon\} \subset \{\bar{\xi} > \langle \bar{\xi}, \bar{x}\rangle\}$. Namely, $\{\bar{\xi} \leq \langle \bar{\xi}, \bar{x}\rangle\} \subset \{g < g(\bar{x}) + \varepsilon\}$. Thus, $G_{\bar{\xi}}(\langle \bar{\xi}, \bar{x}\rangle) \leq g(\bar{x}) + \varepsilon$ and

$$g(\bar{x}) \leq \inf_{\xi \in X^* \setminus \{0\}} G_\xi(\langle \xi, \bar{x}\rangle) \leq G_{\bar{\xi}}(\langle \bar{\xi}, \bar{x}\rangle) \leq g(\bar{x}) + \varepsilon$$

for all $\varepsilon \in (0, \bar{\varepsilon}]$. This implies equality in (13).

Case 2: Suppose $g(\bar{x}) \notin \mathbb{R}$. We can suppose $g(\bar{x}) = -\infty$, because $g(\bar{x}) = \infty$ implies that $\bar{x}$ is a global maximum. Since $g \not\equiv -\infty$, there is $\bar{l} > 0$ large enough so that $\{g \geq -l\} \neq \emptyset$ for all $l \geq \bar{l}$. For all such l, $\bar{x} \notin \{g \geq -l\}$. Since this set is evenly convex, there is $\bar{\xi} \in X^* \setminus \{0\}$ such that $\langle \bar{\xi}, \bar{x}\rangle < \langle \bar{\xi}, x\rangle$ for all $x \in \{g \geq -l\}$. That is, $\{g \geq -l\} \subset \{\bar{\xi} > \langle \bar{\xi}, \bar{x}\rangle\}$ and $\{\bar{\xi} \leq \langle \bar{\xi}, \bar{x}\rangle\} \subset \{g < -l\}$. Thus $G_{\bar{\xi}}(\langle \bar{\xi}, \bar{x}\rangle) \leq -l$ and

$$g(\bar{x}) \leq \inf_{\xi \in X^* \setminus \{0\}} G_\xi(\langle \xi, \bar{x}\rangle) \leq G_{\bar{\xi}}(\langle \bar{\xi}, \bar{x}\rangle) \leq -l$$

for all $l \geq \bar{l}$. This implies $\inf_{\xi \in X^* \setminus \{0\}} G_\xi(\langle \xi, \bar{x}\rangle) = -\infty$, and so equality holds in (13).

To complete the proof of (11), observe that, for all $\xi \in X^* \setminus \{0\}$,

$$G_\xi(\langle \xi, \bar{x}\rangle) = G_{\|\xi\|^{-1}\xi}\left(\left\langle \|\xi\|^{-1}\xi, \bar{x}\right\rangle\right) \quad \text{and} \quad g_\xi(\langle \xi, \bar{x}\rangle) = g_{\|\xi\|^{-1}\xi}\left(\left\langle \|\xi\|^{-1}\xi, \bar{x}\right\rangle\right).$$

"If." Suppose (11) holds, i.e., $g(x) = \inf_{\xi \in S^*} G_\xi(\langle \xi, x\rangle)$ for all $x \in X$. We prove that the set $\{g \geq \alpha\}$ is evenly convex by using Lemma 1. If $\{g \geq \alpha\} = X$, there is nothing to prove. Otherwise, let $\bar{x} \notin \{g \geq \alpha\}$, i.e., $g(\bar{x}) < \alpha$. Then, there exists $\bar{\xi} \in S^*$ for which $G_{\bar{\xi}}(\langle \bar{\xi}, \bar{x}\rangle) < \alpha$. Let $y \in \{g \geq \alpha\}$. Suppose, by contradiction that $\langle \bar{\xi}, y\rangle \leq \langle \bar{\xi}, \bar{x}\rangle$. Then, $g(y) \leq G_{\bar{\xi}}(\langle \bar{\xi}, y\rangle) \leq G_{\bar{\xi}}(\langle \bar{\xi}, \bar{x}\rangle) < \alpha$, a contradiction. Therefore, $\langle \bar{\xi}, \bar{x}\rangle < \langle \bar{\xi}, y\rangle$ for all $y \in \{g \geq \alpha\}$ and $\{g \geq \alpha\}$ is evenly convex.

(i) Suppose that X is an ordered vector space. If g is monotone, proceed as above, and notice that the separating $\bar{\xi}$ is positive. In fact, fix $z \in X_+$ and take $y \in \{g \geq g(\bar{x}) + \varepsilon\}$ (resp. $y \in \{g \geq -l\}$). Notice that $y + nz \in \{g \geq g(\bar{x}) + \varepsilon\}$ (resp. $y + nz \in \{g \geq -l\}$) for all $n \in \mathbb{N}$, and so $\langle \bar{\xi}, \bar{x} \rangle < \langle \bar{\xi}, y \rangle + n \langle \bar{\xi}, z \rangle$. Then, $\langle \bar{\xi}, z \rangle > n^{-1} \left(\langle \bar{\xi}, \bar{x} \rangle - \langle \bar{\xi}, y \rangle \right)$ for all $n \in \mathbb{N}$, which implies $\langle \bar{\xi}, z \rangle \geq 0$, as desired.

Conversely, suppose $g(x) = \inf_{\xi \in \Delta} G_\xi(\langle \xi, x \rangle)$ for all $x \in X$. Then $x \geq y$ implies $\langle \xi, x \rangle \geq \langle \xi, y \rangle$ for all $\xi \in \Delta$, and so $G_\xi(\langle \xi, x \rangle) \geq G_\xi(\langle \xi, y \rangle)$ for all $\xi \in \Delta$ by monotonicity of G_ξ. Hence, $g(x) = \inf_{\xi \in \Delta} G_\xi(\langle \xi, x \rangle) \geq \inf_{\xi \in \Delta} G_\xi(\langle \xi, y \rangle) = g(y)$, as wanted.

(ii) Suppose g is strictly evenly quasiconcave, in light of (12) it is enough to prove that the first infimum is attained. If $\overline{x} \in X$ is a global maximum, then $G_\xi(\langle \xi, \overline{x} \rangle) = g(\overline{x})$ for each $\xi \in S^*$. Hence the infimum is attained. Assume $\overline{x} \in X$ is not a global maximum. The set $\{g > g(\overline{x})\}$ is nonempty, evenly convex, and $\overline{x} \notin \{g > g(\overline{x})\}$. Consequently, there is a functional $\overline{\xi} \in X^* \setminus \{0\}$ such that $\langle \overline{\xi}, \overline{x} \rangle < \langle \overline{\xi}, x \rangle$ for all $x \in \{g > g(\overline{x})\}$. Hence, $\{\overline{\xi} \leq \langle \overline{\xi}, \overline{x} \rangle\} \subset \{g \leq g(\overline{x})\}$, and so $G_{\overline{\xi}}(\langle \overline{\xi}, \overline{x} \rangle) \leq g(\overline{x})$. Given the positive 0-homogeneity of the map $(t, \xi) \mapsto G_\xi(t)$ and its definition, this implies that $G_{\|\overline{\xi}\|^{-1}\overline{\xi}}\left(\left\langle \|\overline{\xi}\|^{-1} \overline{\xi}, \overline{x} \right\rangle\right) = g(\overline{x})$ where $\|\overline{\xi}\|^{-1} \overline{\xi} \in S^*$.

Conversely, suppose that the first infimum in (11) is attained. Let $\alpha \in [-\infty, \infty)$ and consider the strict upper level $\{g > \alpha\}$. Let $\overline{x} \notin \{g > \alpha\}$, i.e., $g(\overline{x}) \leq \alpha$. Since the infimum is attained, there is some $\overline{\xi} \in S^*$ such that $G_{\overline{\xi}}(\langle \overline{\xi}, \overline{x} \rangle) = g(\overline{x}) \leq \alpha$. This implies that $\langle \overline{\xi}, x \rangle > \langle \overline{\xi}, \overline{x} \rangle$ if $x \in \{g > \alpha\}$. By Lemma 1, we can conclude that the set $\{g > \alpha\}$ is evenly convex.

(iii) From $g_\xi \leq G_\xi$, it follows $g_\xi^+ \leq G_\xi^+$. Therefore, it suffices to prove the statement for the functions G_ξ^+.

Let $\overline{x} \in X$. If $\overline{x}$ is a global maximum for g on X, then, by (12) and the definition of lower semicontinuous envelope,

$$g(\overline{x}) \leq G_\xi(\langle \xi, \overline{x} \rangle) \leq G_\xi^+(\langle \xi, \overline{x} \rangle) \leq G_\xi(\langle \xi, \overline{x} \rangle + 1) \leq g(\overline{x}), \qquad \forall \xi \in S^*,$$

and $g(\overline{x}) = \inf_{\xi \in S^*} G_\xi^+(\langle \xi, \overline{x} \rangle)$.

If $\overline{x}$ is not a global maximum for g on X, then, $g(\overline{x}) \in [-\infty, \infty)$ and there exists a sequence $\{\lambda_n\}_n \subset \mathbb{R}$ such that $\lambda_n \downarrow g(\overline{x})$ and $\overline{x} \notin \{g \geq \lambda_n\} \neq \emptyset$ for each $n \in \mathbb{N}$. Since $\{g \geq \lambda_n\}$ are nonempty, closed, and convex, by a strong separation theorem there is a sequence $\{\xi_n\}_n \subset X^* \setminus \{0\}$ such that $\langle \xi_n, \overline{x} \rangle + \varepsilon_n < \langle \xi_n, x \rangle$ for all $x \in \{g \geq \lambda_n\}$, where $\varepsilon_n > 0$. Hence, $\{g \geq \lambda_n\} \subset \{\xi_n > \langle \xi_n, \overline{x} \rangle + \varepsilon_n\}$ for all $n \in \mathbb{N}$. That is, $\{\xi_n \leq \langle \xi_n, \overline{x} \rangle + \varepsilon_n\} \subset \{g < \lambda_n\}$. This implies $G_{\xi_n}(\langle \xi_n, \overline{x} \rangle + \varepsilon_n) \leq \lambda_n$. Therefore,

$$\begin{aligned} g(\overline{x}) &\leq G_{\|\xi_n\|^{-1}\xi_n}\left(\left\langle \|\xi_n\|^{-1} \xi_n, \overline{x} \right\rangle\right) \leq G^+_{\|\xi_n\|^{-1}\xi_n}\left(\left\langle \|\xi_n\|^{-1} \xi_n, \overline{x} \right\rangle\right) \\ &\leq G_{\|\xi_n\|^{-1}\xi_n}\left(\left\langle \|\xi_n\|^{-1} \xi_n, \overline{x} \right\rangle + \|\xi_n\|^{-1} \varepsilon_n\right) \leq \lambda_n, \end{aligned}$$

which yields the result. Conversely, if in (11) we can replace G_ξ with G_ξ^+, we have that g is the lower envelope of a family of quasiconcave and upper semicontinuous functions on X, i.e. $x \mapsto G_\xi^+(\langle \xi, x \rangle)$ for $\xi \in S^*$. Therefore, g is quasiconcave and upper semicontinuous. ■

3 Main Duality Results

Theorem 1 associates to an evenly quasiconcave function g on X a quasiconvex function on $\mathbb{R} \times S^*$ (or on $\mathbb{R} \times \Delta$) that satisfies (11). This duality is, however, incomplete. In fact, there is no uniqueness:

to an evenly quasiconcave function g is, in principle, possible to associate multiple functions with the properties of $G_\xi(t)$. As a result, the duality is only one directional: to a function g we can associate a function like $G_\xi(t)$, but not vice versa.

As discussed in the Introduction, the complete duality notion we introduced in Definition 1 addresses this problem and in this section we identify some important dual pairs $\langle A, B\rangle_{qc}$. In particular, in this section we establish complete quasiconcave monotone duality for three very important classes of monotone quasiconcave functions: (i) evenly quasiconcave functions (Theorem 2), (ii) lower semicontinuous functions (Theorem 3), and (iii) uniformly continuous functions (Theorem 5). As a by-product of the lower semicontinuous duality, in Theorem 4 we establish a duality for continuous functions.

We begin by reporting the main properties of the operators $\mathbf{T}$ and $\mathbf{Q}$ discussed in the Introduction. Recall that $\mathbf{T} : \mathcal{M}_{qc}(X) \to \mathcal{M}(\mathbb{R}\times\Delta)$ is given for all g by

$$(\mathbf{T}g)(t,\xi) = G_\xi(t), \qquad \forall (t,\xi) \in \mathbb{R}\times\Delta, \tag{14}$$

and $\mathbf{Q} : \mathcal{M}(\mathbb{R}\times\Delta) \to \mathcal{M}_{qc}(X)$ is given for all G by

$$(\mathbf{Q}G)(x) = \inf_{\xi\in\Delta} G(\langle \xi, x\rangle, \xi), \qquad \forall x \in X. \tag{15}$$

It is convenient to consider the natural extension of $\mathbf{T}$ to the set $[-\infty,\infty]^X$ of all functions $g : X \to [-\infty,\infty]$ given by (14). Moreover, we denote by $\mathcal{M}_{eqc}(X)$ the set of all $g \in \mathcal{M}_{qc}(X)$ that are evenly quasiconcave, and by $\mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$ the set of all functions $G \in \mathcal{M}(\mathbb{R}\times\Delta)$ such that:

(A.3) $(t,\xi) \mapsto G(t,\xi)$ is $\Diamond$-evenly quasiconvex on $\mathbb{R}\times\Delta$.

Proposition 1 *Let X be an ordered normed vector space. $\mathbf{T} : [-\infty,\infty]^X \to \mathcal{M}(\mathbb{R}\times\Delta)$ and $\mathbf{Q} : \mathcal{M}(\mathbb{R}\times\Delta) \to \mathcal{M}_{qc}(X)$ be defined as above. Then:*

(i) $\mathbf{T}, \mathbf{Q}$ *are well defined,* $\mathbf{T}\left([-\infty,\infty]^X\right) \subset \mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$ *and* $\mathbf{Q}(\mathcal{M}(\mathbb{R}\times\Delta)) \subset \mathcal{M}_{eqc}(X)$;

(ii) $\mathbf{T}, \mathbf{Q}$ *are monotone;*

(iii) $\mathbf{QT} \geq \mathbf{I}$ *and* $\mathbf{TQ} \leq \mathbf{I}$;

(iv) $\mathbf{Q}$ *is the hypo-epi-inverse of* $\mathbf{T}$*, i.e.,*[10]

$$G \geq \mathbf{T}g \Longleftrightarrow \mathbf{Q}G \geq g, \qquad \forall g \in \mathcal{M}_{qc}(X), \forall G \in \mathcal{M}(\mathbb{R}\times\Delta);$$

(v) $\mathbf{QT}g = g$ *if and only if* $g \in \mathcal{M}_{eqc}(X)$;

(vi) for all $g \in [-\infty,\infty]^X$, $\mathbf{QT}g$ *is the least monotone and evenly quasiconcave function greater than* g;

(vii) $\mathbf{T}$ *is injective on* $\mathcal{M}_{eqc}(X)$ *and* $\mathbf{Q}$ *is its left inverse.*

Proof. (i) First we show that $\mathbf{T}$ is well defined and $\mathbf{T}\left([-\infty,\infty]^X\right) \subset \mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$. Let $g \in [-\infty,\infty]^X$. We already observed that $\mathbf{T}g : (t,\xi) \mapsto G_\xi(t)$ is increasing in the first component, thus A.1 holds. Lemma 3 guarantees that $\mathbf{T}g$ satisfies A.2. Thus $\mathbf{T}g$ belongs to $\mathcal{M}(\mathbb{R}\times\Delta)$ and $\mathbf{T}$ is well defined. Lemma 4 guarantees that $\mathbf{T}g$ satisfies A.3, that is $\mathbf{T}g \in \mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$.

[10] On hypo-epi inversion we refer to [24].

To show that $\mathbf{Q}$ is well defined and $\mathbf{Q}(\mathcal{M}(\mathbb{R}\times\Delta)) \subset \mathcal{M}_{eqc}(X)$, it is sufficient to observe that, for all $G \in \mathcal{M}(\mathbb{R}\times\Delta)$ and $\xi \in \Delta$, the functions $G(\langle \xi, \cdot \rangle, \xi)$ is monotone and evenly quasiconcave. Therefore, the lower envelope $\mathbf{Q}G(\cdot) = \inf_{\xi\in\Delta} G(\langle \xi, \cdot \rangle, \xi)$ is monotone and evenly quasiconcave too.

(ii) It is easily checked that $\mathbf{T}$ and $\mathbf{Q}$ are monotone.

(iii) $\mathbf{QT} \geq \mathbf{I}$ is equivalent to the fact that $\inf_{\xi\in\Delta} G_\xi(\langle \xi, x\rangle) \geq g(x)$ for all $x \in X$. Next we prove the relation $\mathbf{TQ} \leq \mathbf{I}$. Let $G \in \mathcal{M}(\mathbb{R}\times\Delta)$ and let $g = \mathbf{Q}G$, that is $g(x) = \inf_{\xi\in\Delta} G(\langle \xi, x\rangle, \xi)$ for all $x \in X$. We want to show $\mathbf{T}g \leq G$. By the minimax inequality, for each $(t, \xi) \in \mathbb{R}\times\Delta$, we have,

$$G_{\bar{\xi}}(\bar{t}) = \sup_{\langle \bar{\xi}, x\rangle \leq \bar{t}} g(x) = \sup_{\langle \bar{\xi}, x\rangle \leq \bar{t}} \inf_{\xi\in\Delta} G(\langle \xi, x\rangle, \xi) \leq \inf_{\xi\in\Delta} \sup_{\langle \bar{\xi}, x\rangle \leq \bar{t}} G(\langle \xi, x\rangle, \xi) = G(\bar{t}, \bar{\xi}).$$

The last equality is proved in Lemma 9 below.

(iv) From $G \geq \mathbf{T}g$, (ii), and (iii), it follows that $\mathbf{Q}G \geq \mathbf{QT}g \geq g$. Conversely, $\mathbf{Q}G \geq g$, by (ii), implies $\mathbf{TQ}G \geq \mathbf{T}g$ and (iii) delivers $G \geq \mathbf{TQ}G \geq \mathbf{T}g$.

(v) The equivalence $\mathbf{QT}g = g$ if and only if $g \in \mathcal{M}_{eqc}(X)$ is (i) of Theorem 1.

(vi) Recall that $\mathbf{Q}G$ is monotone and evenly quasiconcave, for all $G \in \mathcal{M}(\mathbb{R}\times\Delta)$. Hence, $\mathbf{QT}g = \mathbf{Q}(\mathbf{T}g) \geq g$ is monotone and evenly quasiconcave for all $g \in [-\infty, \infty]^X$. Moreover, if $g' \geq g$ is monotone and evenly quasiconcave, then $g' = \mathbf{QT}g' \geq \mathbf{QT}g$, as wanted.

(vii) Point (v) implies that $\mathbf{QT} = \mathbf{I}$ on $\mathcal{M}_{eqc}(X)$, hence $\mathbf{T}$ is injective on $\mathcal{M}_{eqc}(X)$ and $\mathbf{Q}$ is its left inverse.[11] ■

The next simple corollary, and especially the equivalence between (i) and (iii), will be very useful to prove duality results.

Corollary 1 *The following statements are equivalent for $A \subset \mathcal{M}_{qc}(X)$ and $B \subset \mathcal{M}(\mathbb{R}\times\Delta)$:*

(i) A and B form a complete monotone quasiconcave duality pair;

(ii) $A \subset \mathcal{M}_{eqc}(X)$ and $B = \mathbf{T}(A)$;

(iii) $A \subset \mathcal{M}_{eqc}(X)$, $\mathbf{T}(A) \subset B$, $\mathbf{Q}(B) \subset A$, and $\mathbf{TQ} = \mathbf{I}$ on B.

Proof. (i) implies (ii). By definition, $\mathbf{T}$ is injective on A, $\mathbf{T}(A) = B$, and $\mathbf{T}^{-1} = \mathbf{Q}$ on B. Therefore, $\mathbf{QT}g = g$ for all $g \in A$ and point (v) of Proposition 1 guarantees that $A \subset \mathcal{M}_{eqc}(X)$.

(ii) implies (iii). We only have to show that $\mathbf{Q}(B) \subseteq A$, $\mathbf{TQ} = \mathbf{I}$ on B. Since $A \subset \mathcal{M}_{eqc}(X)$, point (v) of Proposition 1 guarantees that $\mathbf{T}$ is injective on A and $\mathbf{Q} : \mathbf{T}(A) \to A$ is its inverse. Then $B = \mathbf{T}(A)$ implies $\mathbf{Q}(B) = \mathbf{QT}(A) = \mathbf{I}(A) = A$. Moreover, for all $G \in B = \mathbf{T}(A)$, $G = \mathbf{T}\left(\mathbf{T}^{-1}G\right) = \mathbf{TQ}G$, that is $\mathbf{TQ} = \mathbf{I}$ on B.

(iii) implies (i). Since $A \subset \mathcal{M}_{eqc}(X)$, point (v) of Proposition 1 guarantees that $\mathbf{T}$ is injective on A and $\mathbf{Q} : \mathbf{T}(A) \to A$ is its inverse. Then $\mathbf{T}(A) \subset B$, $\mathbf{Q}(B) \subset A$, $\mathbf{TQ} = \mathbf{I}$ on B, imply $B = \mathbf{T}_{|A}\mathbf{Q}_{|B}(B) \subset \mathbf{T}_{|A}(A) = \mathbf{T}(A)$. Finally, $B = \mathbf{T}(A)$ and we already observed that $\mathbf{Q}$ is the inverse of $\mathbf{T}$ on $\mathbf{T}(A)$. ■

By point (ii) of the corollary, it holds $\langle \mathcal{M}_{eqc}(X), \mathbf{T}(\mathcal{M}_{eqc}(X))\rangle_{qc}$. The set $\mathbf{T}(\mathcal{M}_{eqc}(X))$ will be characterized momentarily in Theorem 2.

[11] Recall that a left inverse of a function $F : L \to M$ is a function $F' : F(L) \to L$ such that $F' \circ F = I_L$. A function admits left inverse if and only if it is injective, and in this case the only left inverse is $F^{-1} : F(L) \to L$.

3.1 Even Quasiconcave Duality

Theorem 2 *Let X be an M-space. Then,* $\left\langle \mathcal{M}_{eqc}(X), \mathcal{M}_{qcx}^{\diamondsuit}(\mathbb{R}\times\Delta)\right\rangle_{qc}$.

In other words, the map $\mathbf{T} : \mathcal{M}_{eqc}(X) \to \mathcal{M}_{qcx}^{\diamondsuit}(\mathbb{R}\times\Delta)$ given by

$$(\mathbf{T}g)(t,\xi) = G_{\xi}(t), \qquad \forall (t,\xi) \in \mathbb{R}\times\Delta,$$

is one-to-one and onto. Its inverse $\mathbf{T}^{-1} : \mathcal{M}_{qcx}^{\diamondsuit}(\mathbb{R}\times\Delta) \to \mathcal{M}_{eqc}(X)$ is

$$\left(\mathbf{T}^{-1}G\right)(x) = \inf_{\xi\in\Delta} G\left(\langle \xi, x\rangle, \xi\right), \qquad \forall x \in X.$$

This is our basic duality result, which establishes a complete duality for the class $\mathcal{M}_{eqc}(X)$ of the evenly quasiconcave functions $g : X \to [-\infty,\infty]$, a key class in Quasiconvex Analysis in view of Theorem 1.

By point (i) of Proposition 1 and point (iii) of Corollary 1, to prove Theorem 2 it is sufficient to show that $\mathbf{TQ} = \mathbf{I}$ on $\mathcal{M}_{qcx}^{\diamondsuit}(\mathbb{R}\times\Delta)$. This is done in Lemma 10, which, in turn builds on the following:

Lemma 9 *Let X be an ordered normed vector space. If $G \in \mathcal{M}(\mathbb{R}\times\Delta)$ and $\left(\bar{t},\bar{\xi}\right) \in \mathbb{R}\times\Delta$, then*

$$G\left(\bar{t},\bar{\xi}\right) = \min_{\xi\in\Delta}\left(\sup_{x\in\left\{\bar{\xi}\le\bar{t}\right\}} G\left(\langle \xi, x\rangle, \xi\right)\right).$$

Proof. Consider the program

$$\rho\left(\xi,\bar{\xi},\bar{t}\right) = \sup_{x\in\left\{\bar{\xi}\le\bar{t}\right\}} G\left(\langle \xi, x\rangle, \xi\right),$$

with $\xi \in \Delta$. It is sufficient to show that $\rho\left(\xi,\bar{\xi},\bar{t}\right) \ge \rho\left(\bar{\xi},\bar{\xi},\bar{t}\right) = G\left(\bar{t},\bar{\xi}\right)$ for all $\xi \in \Delta$. For the second equality just notice that, by monotonicity of G in the first component, $\rho\left(\bar{\xi},\bar{\xi},\bar{t}\right) = \sup_{x\in\left\{\bar{\xi}\le\bar{t}\right\}} G\left(\left\langle\bar{\xi}, x\right\rangle, \bar{\xi}\right) \le G\left(\bar{t},\bar{\xi}\right)$. Furthermore, since $\bar{\xi} \ne 0$, there exists $\bar{x} \in X$ such that $\left\langle \bar{\xi}, \bar{x}\right\rangle = \bar{t}$ implying that the sup is attained. Consider two cases.

(i) Suppose $\xi \in span\left(\bar{\xi}\right)$. Then $\xi = \alpha\bar{\xi}$ and $1 = \|\xi\| = \left\|\alpha\bar{\xi}\right\| = |\alpha|\left\|\bar{\xi}\right\| = |\alpha|$. If $\xi = \bar{\xi}$, then $\rho\left(\xi,\bar{\xi},\bar{t}\right) = \rho\left(\bar{\xi},\bar{\xi},\bar{t}\right) = G\left(\bar{t},\bar{\xi}\right)$. Else $\xi = -\bar{\xi}$ and

$$\rho\left(\xi,\bar{\xi},\bar{t}\right) = \sup_{x\in\left\{\bar{\xi}\le\bar{t}\right\}} G\left(\left\langle -\bar{\xi}, x\right\rangle, -\bar{\xi}\right) = \sup_{x\in\left\{-\bar{\xi}\ge-\bar{t}\right\}} G\left(\left\langle -\bar{\xi}, x\right\rangle, -\bar{\xi}\right)$$

but, since $\bar{\xi} \ne 0$, for all $t \ge -\bar{t}$ there is $x_t \in X$ such that $\left\langle -\bar{\xi}, x_t\right\rangle = t \ge -\bar{t}$, thus

$$\sup_{x\in\left\{-\bar{\xi}\ge-\bar{t}\right\}} G\left(\left\langle -\bar{\xi}, x\right\rangle, -\bar{\xi}\right) \ge \lim_{t\to+\infty} G\left(t,-\bar{\xi}\right) = \lim_{t\to+\infty} G\left(t,\bar{\xi}\right) \ge G\left(\bar{t},\bar{\xi}\right).$$

(ii) Suppose $\xi \notin span\left(\bar{\xi}\right)$. By the Fundamental Theorem of Duality (see, e.g., [1, Theorem 5.91]), $\ker\left(\bar{\xi}\right) \not\subseteq \ker(\xi)$. That is, there is $y \in X$ such that $\left\langle \bar{\xi}, y\right\rangle = 0$ and $\langle \xi, y\rangle \ne 0$. Choose $\bar{x} \in X$ such that $\left\langle \bar{\xi}, \bar{x}\right\rangle = \bar{t}$, then the straight line $\bar{x} + \alpha y$ is thus included into the hyperplane $\left\{\bar{\xi} = \bar{t}\right\}$. Hence,

$$\rho\left(\xi,\bar{\xi},\bar{t}\right) \ge \lim_{t\to+\infty} G\left(t,\xi\right) = \lim_{t\to+\infty} G\left(t,\bar{\xi}\right) \ge G\left(\bar{t},\bar{\xi}\right).$$

In sum, $\rho\left(\xi,\bar{\xi},\bar{t}\right) \ge G\left(\bar{t},\bar{\xi}\right)$ for all $\xi \in \Delta$ and $\rho\left(\bar{\xi},\bar{\xi},\bar{t}\right) = G\left(\bar{t},\bar{\xi}\right)$. ■

Lemma 10 *Let X be an M-space. We have* $\mathbf{TQ}G = G$ *for all* $G \in \mathcal{M}^{\diamondsuit}_{qcx}(\mathbb{R}\times\Delta)$. *That is, we have*

$$\sup_{x\in X}\{g(x) : \langle \xi, x\rangle \leq t\} = G(t,\xi), \qquad \forall (t,\xi)\in\mathbb{R}\times\Delta, \tag{16}$$

where $g(x) = \inf_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi)$ *for all* $x \in X$.

Proof. Fix $(\bar{t},\bar{\xi}) \in \mathbb{R}\times\Delta$. By definition, $\sup_{x\in X}\left\{g(x) : \langle\bar{\xi},x\rangle \leq \bar{t}\right\} = \sup_{x\in\{\bar{\xi}\leq\bar{t}\}}\inf_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi)$ and, by Lemma 9, $\inf_{\xi\in\Delta}\sup_{x\in\{\bar{\xi}\leq\bar{t}\}} G(\langle\xi,x\rangle,\xi) = G(\bar{t},\bar{\xi})$. Since it is well known that

$$\sup_{x\in\{\bar{\xi}\leq\bar{t}\}}\inf_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi) \leq \inf_{\xi\in\Delta}\sup_{x\in\{\bar{\xi}\leq\bar{t}\}} G(\langle\xi,x\rangle,\xi) = G(\bar{t},\bar{\xi}), \tag{17}$$

it remains to prove the converse inequality. If $G(\bar{t},\bar{\xi}) = \inf_{(t,\xi)\in\mathbb{R}\times\Delta} G(t,\xi)$, the equality in (17) is easily checked. Otherwise, set $\alpha = G(\bar{t},\bar{\xi})$. We have $\alpha > -\infty$. Moreover, for each scalar $\beta < \alpha$, $L_\beta = \{(t,\xi)\in\mathbb{R}\times\Delta : G(t,\xi)\leq\beta\}$ is $\diamondsuit$-evenly convex and $(\bar{t},\bar{\xi}) \notin L_\beta$. If β is large enough, L_β is neither empty nor $\mathbb{R}\times\Delta$. Therefore, there is $\overline{x}\in X$ and $s \neq 0$ such that,

$$\langle\xi,\overline{x}\rangle + st > \langle\bar{\xi},\overline{x}\rangle + s\bar{t}, \qquad \forall (t,\xi)\in L_\beta. \tag{18}$$

Since G is increasing in the first component, it is easy to see that $s < 0$.[12] Set $\lambda = \bar{t} - \left\langle\bar{\xi},\frac{\overline{x}}{|s|}\right\rangle$ and $\hat{x} = \frac{\overline{x}}{|s|} + \lambda e$, then $\langle\bar{\xi},\hat{x}\rangle = \bar{t}$ and for all $(t,\xi)\in L_\beta$

$$\begin{aligned}
\langle\xi,\overline{x}\rangle + st > \langle\bar{\xi},\overline{x}\rangle + s\bar{t} &\Longrightarrow \left\langle\xi,\frac{\overline{x}}{|s|}+\lambda e\right\rangle - t > \left\langle\bar{\xi},\frac{\overline{x}}{|s|}+\lambda e\right\rangle - \bar{t}\\
&\Longrightarrow \langle\xi,\hat{x}\rangle - t > \langle\bar{\xi},\hat{x}\rangle - \bar{t}\\
&\Longrightarrow \langle\xi,\hat{x}\rangle - t > 0.
\end{aligned}$$

Therefore, if $\langle\xi,\hat{x}\rangle - t \leq 0$, then $(t,\xi)\notin L_\beta$.

If for each $\xi\in\Delta$ we pick $t_\xi = \langle\xi,\hat{x}\rangle$, then $\langle\xi,\hat{x}\rangle - t_\xi = 0$. Therefore, $(t_\xi,\xi) = (\langle\xi,\hat{x}\rangle,\xi)\notin L_\beta$ for each $\xi\in\Delta$. This implies $G(\langle\xi,\hat{x}\rangle,\xi) > \beta$ for each $\xi\in\Delta$. Since $\hat{x}\in\{\bar{\xi}\leq\bar{t}\}$, we have that

$$\alpha \geq \sup_{x\in\{\bar{\xi}\leq\bar{t}\}}\inf_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi) \geq \inf_{\xi\in\Delta} G(\langle\xi,\hat{x}\rangle,\xi) \geq \beta.$$

This is true for each β in a left neighborhood of α, thus $\sup_{x\in\{\bar{\xi}\leq\bar{t}\}}\inf_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi) = \alpha$, as desired. ■

3.2 Lower Semicontinuous Duality

Let $\mathcal{M}_{lsc}(X)$ be the set of all $g\in\mathcal{M}_{qc}(X)$ that are lower semicontinuous with values in $[-\infty,\infty]$. Note that if $g\in\mathcal{M}_{lsc}(X)$, g is strictly evenly quasiconcave and therefore $\mathcal{M}_{lsc}(X)\subset\mathcal{M}_{eqc}(X)$.

Denote by $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)\subset\mathcal{M}(\mathbb{R}\times\Delta)$ the class of functions $G:\mathbb{R}\times\Delta\to[-\infty,\infty]$ such that:

(A.4) $(t,\xi)\mapsto G(t,\xi)$ is lower semicontinuous and quasiconvex on $\mathbb{R}\times\Delta$.

Theorem 3 *Let X be an M-space. Then,* $\langle\mathcal{M}_{lsc}(X),\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)\rangle_{qc}$. *In particular, the* inf *in (3) is achieved.*

[12] If per contra $s > 0$, take $(t',\xi')\in L_\beta$, then by monotonicity $(t'-n,\xi')\in L_\beta$ for all $n\in\mathbb{N}$. therefore $\langle\xi',\overline{x}\rangle + st' - sn > \langle\bar{\xi},\overline{x}\rangle + s\bar{t}$ for all $n\in\mathbb{N}$, which is absurd.

That is, the map $\mathbf{T} : \mathcal{M}_{lsc}(X) \to \mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ given by

$$(\mathbf{T}g)(t,\xi) = G_\xi(t), \qquad \forall (t,\xi) \in \mathbb{R}\times\Delta,$$

is one-to-one and onto. Its inverse $\mathbf{T}^{-1} : \mathcal{L}_{qcx}(\mathbb{R}\times\Delta) \to \mathcal{M}_{lsc}(X)$ is

$$\left(\mathbf{T}^{-1}G\right)(x) = \min_{\xi\in\Delta} G(\langle \xi, x\rangle, \xi), \qquad \forall x \in X.$$

By Theorem 3, there is a complete duality for the class $\mathcal{M}_{lsc}(X)$ of the quasiconcave and lower semicontinuous monotone functions $g : X \to [-\infty,\infty]$. The proof of this result follows from the next few lemmas. Given $G : \mathbb{R}\times\Delta \to [-\infty,\infty]$, define $\Gamma : X\times\Delta \to [-\infty,\infty]$ by $\Gamma(x,\xi) = G(\langle \xi, x\rangle, \xi)$.

It is noteworthy that Theorem 3 has the non obvious implication that $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta) \subset \mathcal{M}^{\diamondsuit}_{qcx}(\mathbb{R}\times\Delta)$, namely, that quasiconvex and lower semicontinuous functions of $\mathcal{M}(\mathbb{R}\times\Delta)$ are necessarily $\diamondsuit$-quasiconvex.

Lemma 11 *Suppose that $G : \mathbb{R}\times\Delta \to [-\infty,\infty]$ is increasing in the first argument, lower semicontinuous and quasiconvex on $\mathbb{R}\times\Delta$. Then,*

(i) Γ is lower semicontinuous on $X\times\Delta$, quasiconvex on Δ, and quasiaffine on X.

(ii) If G is real valued, given any closed and convex subset Z of X, we have

$$\inf_{\xi\in\Delta}\sup_{x\in Z}\Gamma(x,\xi) = \sup_{x\in Z}\inf_{\xi\in\Delta}\Gamma(x,\xi). \tag{19}$$

Proof. (i) Consider a net $\{(x_\alpha,\xi_\alpha)\}_{\alpha\in A} \subset X\times\Delta$ such that $(x_\alpha,\xi_\alpha) \to (x,\xi)$ in the product topology. This is equivalent to $x_\alpha \to x$ and $\xi_\alpha \to \xi$. It follows that $\langle \xi_\alpha, x_\alpha\rangle \to \langle \xi, x\rangle$. For,

$$\begin{aligned}|\langle \xi_\alpha, x_\alpha\rangle - \langle \xi, x\rangle| &\le |\langle \xi_\alpha, x_\alpha\rangle - \langle \xi_\alpha, x\rangle| + |\langle \xi_\alpha, x\rangle - \langle \xi, x\rangle| = |\langle \xi_\alpha, x_\alpha - x\rangle| + |\langle \xi_\alpha, x\rangle - \langle \xi, x\rangle| \\ &\le \|\xi_\alpha\|\,\|x_\alpha - x\| + |\langle \xi_\alpha, x\rangle - \langle \xi, x\rangle| = \|x_\alpha - x\| + |\langle \xi_\alpha, x\rangle - \langle \xi, x\rangle| \to 0.\end{aligned}$$

Since G is lower semicontinuous, it then follows that

$$\liminf_\alpha \Gamma(x_\alpha,\xi_\alpha) = \liminf_\alpha G(\langle \xi_\alpha, x_\alpha\rangle, \xi_\alpha) \ge G(\langle \xi, x\rangle, \xi) = \Gamma(x,\xi),$$

and so Γ is lower semicontinuous.

Clearly, Γ is quasiconvex on both Δ and X, separately. Let us show that Γ is quasiconcave on X. Fix $\xi \in \Delta$. Let $x_1, x_2 \in \{x \in X : \Gamma(x,\xi) \ge \alpha\}$ where $\alpha \in \mathbb{R}$. Wlog, suppose $\langle \xi, x_1\rangle \ge \langle \xi, x_2\rangle$. Therefore, $\langle \xi, \lambda x_1 + (1-\lambda)x_2\rangle \ge \langle \xi, x_2\rangle$ for each $\lambda \in (0,1)$. Since G is monotone in the first argument, $\Gamma(\lambda x_1 + (1-\lambda)x_2, \xi) \ge \Gamma(x_2,\xi) \ge \alpha$, as desired. This completes the proof of (i).

(ii) Γ is real valued given that G is real valued. In view of the properties of Γ established in point (i), the minimax equality (19) follows from the Minimax Theorem of [32, Corollary 2] (see also [16, Theorem 1.3], and [13, Theorem 3.1]). ■

Lemma 12 *A function $g : X \to [-\infty,\infty]$ is quasiconcave, monotone, and lower semicontinuous if and only if there exists a $G : \mathbb{R}\times\Delta \to [-\infty,\infty]$ increasing in the first argument, lower semicontinuous, and quasiconvex on $\mathbb{R}\times\Delta$, such that*

$$g(x) = \min_{\xi\in\Delta} G(\langle \xi, x\rangle, \xi), \qquad \forall x \in X. \tag{20}$$

Proof. We first prove necessity. Consider the mapping such that $(t,\xi) \mapsto G_\xi(t)$ for each $(t,\xi) \in \mathbb{R}\times\Delta$. By definition, it is increasing in the first component. Since g is lower semicontinuous, by Lemma 5 $G_\xi(t)$ is lower semicontinuous on $\mathbb{R}\times\Delta$. Moreover, by Lemma 4, it is quasiconvex on $\mathbb{R}\times\Delta$. Finally, (ii) of Theorem 1 implies that it satisfies (20).

Conversely, suppose that $G : \mathbb{R}\times\Delta \to [-\infty,\infty]$ is increasing in the first argument and lower semicontinuous on $\mathbb{R}\times\Delta$, and that (20) holds. Clearly, g is monotone. Moreover, by Lemma 11, $G(\langle\xi,\cdot\rangle,\xi) : X \to [-\infty,\infty]$ is quasiconcave for each $\xi \in \Delta$, and so is g (since it is the infimum of quasiconcave functions).

It remains to prove lower semicontinuity. First, given (20), we can write

$$g(x) = \min_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi) = \min_{\xi\in\Delta} \Gamma(x,\xi), \qquad \forall x \in X.$$

Next, consider $\{x_n\}_n \subset X$ such that $x_n \to x$. Then, there exists a subsequence $\{x_{n_k}\}_k$ such that $\lim_k g(x_{n_k}) = \liminf_n g(x_n)$. Furthermore, by (20), for each k there exists $\xi_{n_k} \in \Delta$ such that $g(x_{n_k}) = \Gamma(x_{n_k},\xi_{n_k})$.

Since Δ is compact, there exists a subnet $\left\{\xi_{n_{k_\alpha}}\right\}_{\alpha\in A}$ such that $\xi_{n_{k_\alpha}} \to \bar{\xi} \in \Delta$. Given Lemma 11, we have that Γ is lower semicontinuous on $X\times\Delta$. Hence,

$$\begin{aligned}\liminf_n g(x_n) &= \lim_k g(x_{n_k}) = \lim_\alpha g\left(x_{n_{k_\alpha}}\right) = \lim_\alpha \Gamma\left(x_{n_{k_\alpha}},\xi_{n_{k_\alpha}}\right)\\ &\geq \Gamma\left(x,\bar{\xi}\right) \geq \min_{\xi\in\Delta}\Gamma(x,\xi) = g(x).\end{aligned}$$

This proves that g is lower semicontinuous. ■

Lemma 13 *Let X be an M-space. We have $\mathbf{TQ}G = G$ for all $G \in \mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$. That is, we have*

$$\sup_{x\in X}\{g(x) : \langle\xi,x\rangle \leq t\} = G(t,\xi), \qquad \forall (t,\xi) \in \mathbb{R}\times\Delta,$$

where $g(x) = \min_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi)$ for all $x \in X$.

Proof. Let us first consider the case in which G is real valued; i.e., $G : \mathbb{R}\times\Delta \to \mathbb{R}$. Fix $\left(\bar{t},\bar{\xi}\right) \in \mathbb{R}\times\Delta$. Define $Z = \left\{x \in X : \left\langle\bar{\xi},x\right\rangle \leq \bar{t}\right\}$. By (19),

$$G_{\bar{\xi}}\left(\bar{t}\right) = \sup_{x\in Z} g(x) = \sup_{x\in Z}\inf_{\xi\in\Delta}\Gamma(x,\xi) = \inf_{\xi\in\Delta}\sup_{x\in Z}\Gamma(x,\xi) = \inf_{\xi\in\Delta}\sup_{x\in Z} G(\langle\xi,x\rangle,\xi). \tag{21}$$

By Lemma 9, $G\left(\bar{t},\bar{\xi}\right) = \inf_{\xi\in\Delta}\sup_{x\in Z} G(\langle\xi,x\rangle,\xi)$. Then, by (21), it follows that $G_\xi(t) = G(t,\xi)$ for all $(t,\xi) \in \mathbb{R}\times\Delta$. This completes the proof for G real valued. Consider now $G : \mathbb{R}\times\Delta \to [-\infty,\infty]$. Let $\varphi : [-\infty,\infty] \to \mathbb{R}$ be a strictly increasing, extended-valued continuous, and bounded function.[13] The function $\widehat{G} = \varphi\circ G : \mathbb{R}\times\Delta \to \mathbb{R}$ is real valued and belongs to $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$. Moreover, the function $\widehat{g} = \varphi\circ g : X \to \mathbb{R}$ is such that $\widehat{g}(x) = \min_{\xi\in\Delta}\widehat{G}(\langle\xi,x\rangle,\xi)$ for all $x \in X$.

By the first part of the proof, $\widehat{G}_\xi(t) = \widehat{G}(t,\xi)$ for all $(t,\xi) \in \mathbb{R}\times\Delta$. Hence, by Lemma 6

$$\varphi(G_\xi(t)) = \widehat{G}_\xi(t) = \widehat{G}(t,\xi) = \varphi(G(t,\xi)), \qquad \forall (t,\xi) \in \mathbb{R}\times\Delta,$$

and so $G_\xi(t) = G(t,\xi)$ for all $(t,\xi) \in \mathbb{R}\times\Delta$. ■

[13] E.g.,

$$\varphi(t) = \begin{cases} -\frac{\pi}{2} & \text{if } t = -\infty \\ \arctan t & \text{if } t \in \mathbb{R} \\ \frac{\pi}{2} & \text{if } t = \infty \end{cases}$$

Proof of Theorem 3. Consider $\mathbf{T} : \mathcal{M}_{lsc}(X) \to \mathcal{M}(\mathbb{R} \times \Delta)$. By Lemmas 3, 4, and 5, we have that $T(\mathcal{M}_{lsc}(X)) \subset \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$. Hence, $\mathbf{T} : \mathcal{M}_{lsc}(X) \to \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$. By Proposition 1, $\mathbf{T}$ is injective on $\mathcal{M}_{lsc}(X) \subset \mathcal{M}_{eqc}(X)$. Let $G \in \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$. By Lemma 12, $\mathbf{Q}G \in \mathcal{M}_{lsc}(X)$. By Lemma 13, $\mathbf{TQ}G = G$ therefore T is surjective. That is, $\mathbf{T}^{-1} = \mathbf{Q}$ on $\mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$. ■

Denote by $\mathcal{C}(\mathbb{R} \times \Delta) \subset \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ the class of functions $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$ such that:

(A.5) $G(\cdot, \xi)$ is extended-valued continuous on $\mathbb{R}$ for each $\xi \in \Delta$.[14]

The next corollary is an interesting consequence of Lemmas 12 and 13.

Corollary 2 *Let X be an M-space. Then, $\mathcal{C}(\mathbb{R} \times \Delta) \subset \mathbf{T}(\mathcal{M}_{lsc}(X))$, namely, $\langle \mathbf{Q}\mathcal{C}(\mathbb{R} \times \Delta), \mathcal{C}(\mathbb{R} \times \Delta)\rangle_{qc}$. In particular,*

$$G(t, \xi) = \sup_{x \in X} \{g(x) : \langle \xi, x\rangle \leq t\} = G_\xi(t).$$

where $g : X \to [-\infty, \infty]$ is the monotone, continuous, and quasiconcave function defined by

$$g(x) = \min_{\xi \in \Delta} G(\langle \xi, x\rangle, \xi), \qquad \forall x \in X.$$

with $G \in \mathcal{C}(\mathbb{R} \times \Delta)$.

3.2.1 Continuous Duality

The lower semicontinuous duality established in Theorem 3 implies a duality for continuous functions. This duality is based on the lower envelope G^+ of G in its first argument: given a function $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$, the *lower envelope* $G^+ : \mathbb{R} \times \Delta \to [-\infty, \infty]$ in its first argument is given by

$$G^+(t, \xi) = \inf\{G(t', \xi) : t' > t\}, \qquad \forall (t, \xi) \in \mathbb{R} \times \Delta.$$

Lemma 14 *If G belongs to $\mathcal{M}(\mathbb{R} \times \Delta)$ and is quasiconvex, then $G^+ \in \mathcal{M}(\mathbb{R} \times \Delta)$ is quasiconvex on $\mathbb{R} \times \Delta$ and upper semicontinuous in the first argument.*

Proof. The envelope G^+ is easily seen to be monotone and upper semicontinuous in the first component. It is quasiconvex on $\mathbb{R} \times \Delta$. Consider two points $(t_1, \xi_1), (t_2, \xi_2) \in \mathbb{R} \times \Delta$ and $\lambda \in (0, 1)$, and define $(t_\lambda, \xi_\lambda) = \lambda(t_1, \xi_1) + (1 - \lambda)(t_2, \xi_2)$. Then, since G is quasiconvex, for each $n \geq 1$,

$$G\left(t_\lambda + \frac{1}{n}, \xi_\lambda\right) \leq \max\left\{G\left(t_1 + \frac{1}{n}, \xi_1\right), G\left(t_2 + \frac{1}{n}, \xi_2\right)\right\}.$$

Since G is monotone, we then have $G^+(t_\lambda, \xi_\lambda) \leq \max\{G^+(t_1, \xi_1), G^+(t_2, \xi_2)\}$, and so G^+ is quasiconvex. Moreover, for fixed $\xi \in \Delta$, we have

$$G(t, \xi) \leq G^+(t, \xi) \leq G(t + \varepsilon, \xi), \quad \forall t \in \mathbb{R}, \forall \varepsilon \in (0, \infty).$$

Then, G being increasing in its first component,

$$\lim_{t \to \infty} G(t, \xi) = \lim_{t \to \infty} G^+(t, \xi), \quad \forall \xi \in \Delta. \tag{22}$$

Since $G \in \mathcal{M}(\mathbb{R} \times \Delta)$, (22) implies $\lim_{t \to \infty} G^+(t, \xi') = \lim_{t \to \infty} G^+(t, \xi)$ for all $\xi, \xi' \in \Delta$. We conclude that $G^+ \in \mathcal{M}(\mathbb{R} \times \Delta)$. ■

[14] That is, $\lim_{t \to t_0} G(t, \xi) = G(t_0, \xi) \in (-\infty, \infty]$ for all $\xi \in \Gamma$.

By Lemma 14, the lower envelopes G^+ belong to the domain of $\mathbf{Q}$, and so we can write $\mathbf{Q}G^+$. Using this observation, denote by $\mathcal{CO}(\mathbb{R}\times\Delta)\subset\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ the class of functions $G:\mathbb{R}\times\Delta\to[-\infty,\infty]$ such that $\mathbf{Q}G^+=\mathbf{Q}G$.

Moreover, let $\mathcal{M}_c(X)\subset\mathcal{M}_{lsc}(X)$ be the set of all continuous $g:X\to[-\infty,\infty]$ that belong to $\mathcal{M}_{qc}(X)$. We can now state the announced duality.

Theorem 4 *Let X be an M-space. Then, $\langle\mathcal{M}_c(X),\mathcal{CO}(\mathbb{R}\times\Delta)\rangle_{qc}$. In particular, the* inf *in (3) is achieved.*

The proof of this theorem is based on the following lemma.

Lemma 15 *A function $g:X\to[-\infty,\infty]$ is quasiconcave, monotone, and continuous if and only if there exists $G\in\mathcal{CO}(\mathbb{R}\times\Delta)$ such that*

$$g(x)=\min_{\xi\in\Delta}G(\langle\xi,x\rangle,\xi),\qquad\forall x\in X.\tag{23}$$

Proof. We first prove necessity. By Theorem 3, since $g\in\mathcal{M}_c(X)\subset\mathcal{M}_{lsc}(X)$, the mapping $(t,\xi)\mapsto G_\xi(t)$ belongs to $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$. By Theorem 1-(iii) and Proposition 14, we have that $\mathbf{Q}G^+=\mathbf{Q}G$. Conversely, suppose that $G\in\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$. Then, by Theorem 3 and (23), $\mathbf{Q}G=g\in\mathcal{M}_{lsc}(X)$.

It remains to prove upper semicontinuity. First, given (23) and $G\in\mathcal{CO}(\mathbb{R}\times\Delta)$, we can write

$$g(x)=\inf_{\xi\in\Delta}G^+(\langle\xi,x\rangle,\xi),\qquad\forall x\in X.$$

Since, for each $\xi\in\Delta$, the function $x\mapsto G^+(\langle\xi,x\rangle,\xi)$ is upper semicontinuous, being a composition of an upper semicontinuous function with a continuous one, so is the function g. ■

Proof of Theorem 4. Consider $\mathbf{T}:\mathcal{M}_c(X)\to\mathcal{CO}(\mathbb{R}\times\Delta)$. By Lemma 15, $\mathbf{T}$ is well defined. By Theorem 3 and Lemma 15, $\mathbf{T}$ is bijective and $\mathbf{T}^{-1}=\mathbf{Q}$. ■

We close with a characterization of the class of functions $\mathcal{C}(\mathbb{R}\times\Delta)$.

Proposition 2 *$\mathcal{C}(\mathbb{R}\times\Delta)\subset\mathcal{CO}(\mathbb{R}\times\Delta)$. Moreover, $G\in\mathcal{C}(\mathbb{R}\times\Delta)$ if and only if $G\in\mathcal{CO}(\mathbb{R}\times\Delta)$ and $G^+\in\mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$; in this case $G=G^+$.*

Proof. Suppose that $G\in\mathcal{C}(\mathbb{R}\times\Delta)$. By definition, $G\in\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ and $G(\cdot,\xi)$ for each $\xi\in\Delta$ is extended-valued continuous and monotone. Therefore, $G^+=G$ and this prove the first part of the statement.

Suppose that $G\in\mathcal{CO}(\mathbb{R}\times\Delta)$ and $G^+\in\mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$. Since $G\in\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ then, by Theorem 2, $G\in\mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$. Given that $G^+\in\mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)$ and that $\mathbf{Q}G^+=\mathbf{Q}G$, by Theorem 2 $G^+=G$. Therefore, $G(\cdot,\xi)=G^+(\cdot,\xi)$ for each $\xi\in\Delta$, implying that $G(\cdot,\xi)$ is extended-valued continuous for each $\xi\in\Delta$. ■

We illustrate these results with an example.

Example 2 Let X be an M-space and $\varphi:\mathbb{R}\to\mathbb{R}$ an increasing function such that for some $t_0\in\mathbb{R}$ the function φ is convex on $(-\infty,t_0)$, $\lim_{t\to t_0^-}\varphi(t)=\infty$, and $\varphi(t)=\infty$ for all $t\geq t_0$. Consider two convex and lower semicontinuous mappings $c_1,c_2:\Delta\to(-\infty,\infty]$. Set

$$G(t,\xi)=\varphi(t+c_1(\xi))+c_2(\xi),\qquad\forall(t,\xi)\in\mathbb{R}\times\Delta.$$

It is easy to see that the conditions of Corollary 2 are satisfied. The function $G(t,\xi)$ is actually lower semicontinuous and convex, and satisfies A.5. Hence, $G \in \mathcal{C}(\mathbb{R} \times \Delta)$. The quasiconcave, and monotone function

$$g(x) = \inf_{\xi \in \Delta} \left(\varphi\left(\langle \xi, x\rangle + c_1(\xi)\right) + c_2(\xi)\right) = \inf_{\xi \in \Delta} \left\{\varphi\left(\langle \xi, x\rangle + c_1(\xi)\right) + c_2(\xi) : \langle \xi, x\rangle + c_1(\xi) < t_0\right\}$$

is such that $G_\xi(t) = G(t,\xi)$. Observe that g is in general not concave. Moreover, we have $G = G^+$. Consequently $G \in CO(\mathbb{R} \times \Delta)$, and g is continuous by Theorem 4. ▲

3.3 Uniform Continuous Duality

We turn now to real valued and uniformly continuous quasiconcave functions $g : X \to \mathbb{R}$. We show that a neat complete duality holds for them as well. We begin by showing what form Theorem 1 takes for uniformly continuous functionals. Here, $\mathrm{dom}G_\xi = \{t \in \mathbb{R} : G_\xi(t) < \infty\}$.

Theorem 5 *A function $g : X \to \mathbb{R}$ is uniformly continuous and quasiconcave if and only if*

$$g(x) = \min_{\xi \in S^*} g_\xi\left(\langle \xi, y\rangle\right) = \min_{\xi \in S^*} G_\xi\left(\langle \xi, x\rangle\right), \qquad \forall x \in X, \tag{24}$$

where $\mathrm{dom}G_\xi \in \{\emptyset, \mathbb{R}\}$ for all $\xi \in S^*$, and $\{G_\xi\}_{\xi \in S^* : \mathrm{dom}G_\xi = \mathbb{R}}$ is a nonempty family of real valued uniformly equicontinuous functions.[15]

If, in addition, X is ordered, then g is monotone if and only if in (24) we can replace S^* with Δ.

Proof. Suppose g is real valued, quasiconcave, and uniformly continuous. By (ii) of Theorem 1, we have the representation (24). As g is uniformly continuous, for all $\varepsilon > 0$, there is some $\delta > 0$ such that $\|x - x'\| \le \delta \Longrightarrow |g(x) - g(x')| \le \varepsilon$. In particular, if $u \in X$ and $\|u\| = 1$, then

$$g(x + \delta u) \le g(x) + \varepsilon \text{ and } g(x - \delta u) \ge g(x) - \varepsilon, \tag{25}$$

hold for all $x \in X$ and $\|u\| = 1$. Fix now $\xi \in S^*$. As

$$\|\xi\| = \sup_{\|x\|=1} \langle \xi, x\rangle = 1,$$

there exists an element $u \in X$, with $\|u\| = 1$ and $\langle \xi, u\rangle \ge 1/2$.

Given an $\varepsilon > 0$, let δ be such that (25) is satisfied. Let $t \in \mathrm{dom}G_\xi$ and $t' \in \mathbb{R}$ with $|t - t'| \le \delta/2$. Consider two cases:

Case 1: $t' \le t$. Then,

$$\begin{aligned} G_\xi(t) - \varepsilon &= \sup\left\{g(x) - \varepsilon : \langle \xi, x\rangle \le t\right\} \le \sup\left\{g(x - \delta u) : \langle \xi, x\rangle \le t\right\} \\ &= \sup\left\{g(x) : \langle \xi, x + \delta u\rangle \le t\right\} = \sup\left\{g(x) : \langle \xi, x\rangle \le t - \delta\langle \xi, u\rangle\right\} \\ &= G_\xi\left(t - \delta\langle \xi, u\rangle\right) \le G_\xi(t - \delta/2) \le G_\xi(t') \le G_\xi(t). \end{aligned}$$

Therefore, $|G_\xi(t) - G_\xi(t')| \le \varepsilon$, under our assumption.

Case 2: $t' \ge t$. Then,

$$\begin{aligned} G_\xi(t) &\le G_\xi(t') \le G_\xi(t + \delta/2) \le G_\xi\left(t + \delta\langle \xi, u\rangle\right) \\ &= \sup\left\{g(x) : \langle \xi, x\rangle \le t + \delta\langle \xi, u\rangle\right\} = \sup\left\{g(x) : \langle \xi, x - \delta u\rangle \le t\right\} \\ &= \sup\left\{g(x + \delta u) : \langle \xi, x\rangle \le t\right\} \le \varepsilon + \sup\left\{g(x) : \langle \xi, x\rangle \le t\right\} = \varepsilon + G_\xi(t), \end{aligned}$$

[15] That is, for every $\varepsilon > 0$ there is $\delta > 0$ such that $|t - t'| \le \delta$ implies $\left|G_\xi(t) - G_\xi(t')\right| \le \varepsilon$, for all $t, t' \in \mathbb{R}$ and all $\xi \in S^*$ such that $\mathrm{dom}\left(G_\xi\right) = \mathbb{R}$.

and once again we get $|G_\xi(t) - G_\xi(t')| \le \varepsilon$.

In sum, we established that:

- Fix $\varepsilon > 0$ and $\|\xi\| = 1$ with $\mathrm{dom}G_\xi \neq \emptyset$. If $t \in \mathrm{dom}G_\xi$, then $[t-\delta/2, t+\delta/2] \subset \mathrm{dom}G_\xi$; that is, $\mathrm{dom}G_\xi = \mathbb{R}$. Notice that, for a generic $x \in X$, we have that $G(\langle \xi_x, x\rangle, \xi_x) = g(x) \in \mathbb{R}$ for some $\xi_x \in S^*$. Hence, $\mathrm{dom}G_{\xi_x} = \mathbb{R}$.
- Since g is real valued, by definition, $G_\xi > -\infty$ for all $\xi \in S^*$.
- For all $\varepsilon > 0$ there is $\delta > 0$ such that $|G_\xi(t) - G_\xi(t')| \le \varepsilon$ for all $\|\xi\| = 1$ with $\mathrm{dom}G_\xi = \mathbb{R}$ and all $t, t' \in \mathbb{R}$ with $|t - t'| \le \delta/2$.

As to the converse, assume that (24) holds and $\{G_\xi\}_{\xi \in S^* : \mathrm{dom}G_\xi = \mathbb{R}}$ is a real valued, nonempty family of equicontinuous functions. By definition, given $\varepsilon > 0$, there is $\delta > 0$ such that $|G_\xi(t) - G_\xi(t')| \le \varepsilon$ for all $\xi \in S^*$ with $\mathrm{dom}G_\xi = \mathbb{R}$ and all $t, t' \in \mathbb{R}$ with $|t - t'| \le \delta$.

Take $x, y \in X$ such that $\|x - y\| \le \delta$. There is $\xi_x \in S^*$ such that $g(x) = G_{\xi_x}(\langle \xi_x, x\rangle)$. Since $\{G_\xi\}_{\xi \in S^* : \mathrm{dom}G_\xi = \mathbb{R}}$ is a nonempty family of real valued functions, this implies that $g(x) \in \mathbb{R}$, then $\mathrm{dom}G_{\xi_x} = \mathbb{R}$. Moreover, if $\|x - y\| \le \delta$, then $|\langle \xi_x, x\rangle - \langle \xi_x, y\rangle| \le \|\xi_x\| \|x - y\| \le \delta$. By uniform equicontinuity $|G_{\xi_x}(\langle \xi_x, x\rangle) - G_{\xi_x}(\langle \xi_x, y\rangle)| \le \varepsilon$, and so

$$\begin{aligned} g(x) &= \min_{\xi \in S^*} G_\xi(\langle \xi, x\rangle) = G_{\xi_x}(\langle \xi_x, x\rangle) \ge G_{\xi_x}(\langle \xi_x, y\rangle) - \varepsilon \\ &\ge \min_{\xi \in S^*} G_\xi(\langle \xi, y\rangle) - \varepsilon = g(y) - \varepsilon. \end{aligned}$$

Exchanging the two points x and y, we get $|g(x) - g(y)| \le \varepsilon$, and so g is uniformly continuous. ■

Set $\mathrm{dom}G(\cdot, \xi) = \{t \in \mathbb{R} : G(t, \xi) < \infty\}$. Denote by $\mathcal{E}(\mathbb{R} \times \Delta) \subset \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ the set of functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ that have the following additional properties:

(A.6) $\mathrm{dom}G(\cdot, \xi) \in \{\emptyset, \mathbb{R}\}$ for all $\xi \in \Delta$, and there exists at least one $\bar{\xi}$ such that $\mathrm{dom}G\left(\cdot, \bar{\xi}\right) = \mathbb{R}$.

(A.7) $G(\cdot, \xi)$ are uniformly equicontinuous on $\mathbb{R}$ for all $\xi \in \Delta$ such that $\mathrm{dom}G(\cdot, \xi) = \mathbb{R}$.[16]

Finally, let $\mathcal{M}_{uc}(X) \subset \mathcal{M}_c(X)$ be the set of all functions $g : X \to \mathbb{R}$ that are monotone, quasiconcave, and uniformly continuous.

Theorem 6 *Let X be an M-space. Then, $\langle \mathcal{M}_{uc}(X), \mathcal{E}(\mathbb{R} \times \Delta)\rangle_{qc}$. In particular, the* inf *in (3) is achieved.*

By Theorem 6, we thus have a complete duality also for the important class $\mathcal{M}_{uc}(X)$ of the quasiconcave and uniformly continuous monotone functions $g : X \to \mathbb{R}$.

Observe how the additional continuity property that characterizes the functions g in $\mathcal{M}_{uc}(X)$ among those in $\mathcal{M}_{lsc}(X)$ is reflected in the duality by the additional continuity property that the functions G have in $\mathcal{E}(\mathbb{R} \times \Delta)$ among those in $\mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$. The duality $\langle \mathcal{M}_{uc}(X), \mathcal{E}(\mathbb{R} \times \Delta)\rangle_{qc}$ can thus be viewed as a "continuous" specification of the duality $\langle \mathcal{M}_{lsc}(X), \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)\rangle_{qc}$.

The proof of Theorem 6 is based on the following lemma.

[16] That is, for every $\varepsilon > 0$ there is $\delta > 0$ such that $|t - t'| \le \delta$ implies $|G(t, \xi) - G(t', \xi)| \le \varepsilon$, for all $t, t' \in \mathbb{R}$ and all $\xi \in \Delta$ such that $\mathrm{dom}G(\cdot, \xi) = \mathbb{R}$.

Lemma 16 *A function $g : X \to [-\infty, \infty]$ is real valued, quasiconcave, monotone, and uniformly continuous if and only if there exists a $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ in $\mathcal{E}(\mathbb{R} \times \Delta)$ such that*

$$g(x) = \min_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi), \qquad \forall x \in X. \tag{26}$$

Proof. We first prove necessity. By the proof of Lemma 12, the mapping $(t, \xi) \mapsto G_\xi(t)$ belongs to $\mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ and satisfies (26). Since g is real valued, such mapping takes values on $(-\infty, \infty]$ and, by Theorem 5, the mapping satisfies A.6 and A.7.

Conversely, for sufficiency, suppose that $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is in $\mathcal{E}(\mathbb{R} \times \Delta)$ and that (26) holds. Since G satisfies (26) and A.6, for each $x \in X$ there exists $\xi_x \in \Delta$ such that

$$\infty > G\left(\left\langle \bar{\xi}, x \right\rangle, \bar{\xi}\right) \geq g(x) = G(\langle \xi_x, x \rangle, \xi_x) > -\infty.$$

This implies that g is real valued. Clearly, g is monotone. Moreover, by Lemma 11, $G(\langle \xi, \cdot \rangle, \xi) : X \to (-\infty, \infty]$ is quasiconcave for each $\xi \in \Delta$, and so is g being the infimum of quasiconcave functions.

It remains to prove uniform continuity. By definition, given $\varepsilon > 0$, there is $\delta > 0$ such that $|G(t, \xi) - G(t', \xi)| \leq \varepsilon$ for all $\xi \in \Delta$ with $\mathrm{dom} G(\cdot, \xi) = \mathbb{R}$ and all $t, t' \in \mathbb{R}$ with $|t - t'| \leq \delta$.

Take $x, y \in X$ such that $\|x - y\| \leq \delta$. There is $\xi_x \in \Delta$ such that $g(x) = G(\langle \xi_x, x \rangle, \xi_x)$. Since $g(x) \in \mathbb{R}$, then $\mathrm{dom} G(\cdot, \xi_x) = \mathbb{R}$. Moreover, if $\|x - y\| \leq \delta$, then $|\langle \xi_x, x \rangle - \langle \xi_x, y \rangle| \leq \|\xi_x\| \|x - y\| \leq \delta$. By uniform equicontinuity, $|G(\xi_x, \langle \xi_x, x \rangle) - G(\xi_x, \langle \xi_x, y \rangle)| \leq \varepsilon$, and so

$$\begin{aligned} g(x) &= \min_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi) = G(\langle \xi_x, x \rangle, \xi_x) \geq G(\langle \xi_x, y \rangle, \xi_x) - \varepsilon \\ &\geq \min_{\xi \in \Delta} G(\langle \xi, y \rangle, \xi) - \varepsilon = g(y) - \varepsilon. \end{aligned}$$

Exchanging the two points x and y, we get $|g(x) - g(y)| \leq \varepsilon$, and so g is uniformly continuous. ■

Proof of Theorem 6. Consider $\mathbf{T} : \mathcal{M}_{uc}(X) \to \mathcal{M}(\mathbb{R} \times \Delta)$. By Lemmas 3, 4, and 5 and by Theorem 5, we have that $\mathbf{T}(\mathcal{M}_{uc}(X)) \subset \mathcal{E}(\mathbb{R} \times \Delta)$. Hence, $\mathbf{T} : \mathcal{M}_{uc}(X) \to \mathcal{E}(\mathbb{R} \times \Delta)$. By Proposition 1, $\mathbf{T}$ is injective on $\mathcal{M}_{uc}(X) \subset \mathcal{M}_{eqc}(X)$. Let $G \in \mathcal{E}(\mathbb{R} \times \Delta) \subset \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$. By Lemma 16, $\mathbf{Q}G \in \mathcal{M}_{uc}(X) \subset \mathcal{M}_{lsc}(X)$. By Lemma 13, $\mathbf{TQ}G = G$ therefore T is surjective. That is, $\mathbf{T}^{-1} = \mathbf{Q}$ on $\mathcal{E}(\mathbb{R} \times \Delta)$. ■

4 Characterization of Indirect Utilities

The results on even quasiconcave duality provide a characterization of indirect utilities for preferences defined on M-spaces. In fact, interpret $[-\infty, \infty]^X$ as the set of all utility functions (thus allowing for non-monotone utility functions), and Δ as the set of all normalized prices. Under this interpretation, $\mathbf{T}\left([-\infty, \infty]^X\right)$ is the set of all indirect utilities.

Lemma 17 *Let X be an M-space. Given any $g : X \to [-\infty, \infty]$, there exists a unique $h \in \mathcal{M}_{eqc}(X)$ such that*

$$\sup_{x \in X} \{g(x) : \langle \xi, x \rangle \leq t\} = \sup_{x \in X} \{h(x) : \langle \xi, x \rangle \leq t\}, \qquad \forall (t, \xi) \in \mathbb{R} \times \Delta.$$

Specifically, h is the least monotone evenly quasiconcave function greater than g, and

$$\arg\max_{x \in X} \{g(x) : \langle \xi, x \rangle \leq t\} \subseteq \arg\max_{x \in X} \{h(x) : \langle \xi, x \rangle \leq t\} \qquad \forall (t, \xi) \in \mathbb{R} \times \Delta.$$

Proof. By point Proposition 1-(i), $\mathbf{T}\left([-\infty,\infty]^X\right) \subset \mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta)$. By Theorem 2,

$$\mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta) = \mathbf{T}(\mathcal{M}_{eqc}(X)) \subset \mathbf{T}\left([-\infty,\infty]^X\right).$$

Thus, $\mathbf{T}(\mathcal{M}_{eqc}(X)) = \mathbf{T}\left([-\infty,\infty]^X\right)$.

As $\mathbf{T}$ is injective over $\mathcal{M}_{eqc}(X)$, for any function g there a unique function $h \in \mathcal{M}_{eqc}(X)$ such that $\mathbf{T}g = \mathbf{T}h$. This implies $\mathbf{QT}g = \mathbf{QT}h = h$. By (iii) of Proposition 1, $h \geq g$. On the other hand, if $h_1 \geq g$, with $h \in \mathcal{M}_{eqc}(X)$, we have $h_1 = \mathbf{QT}h_1 \geq \mathbf{QT}g = h$. Let $\overline{x} \in \arg\max_{x\in X}\{g(x) : \langle \xi, x\rangle \leq t\}$. As $\langle \xi, \overline{x}\rangle \leq t$, we have

$$\sup_{x\in X}\{h(x) : \langle \xi, x\rangle \leq t\} \geq h(\overline{x}) \geq g(\overline{x}) = \sup_{x\in X}\{g(x) : \langle \xi, x\rangle \leq t\}.$$

Hence, $h(\overline{x}) = \sup_{x\in X}\{h(x) : \langle \xi, x\rangle \leq t\}$, as desired. ∎

This lemma shows that in the standard utility maximization problems of Microeconomics, it is without loss of generality (in terms of optimal values) to consider functions in $\mathcal{M}_{eqc}(X)$, that is, evenly quasiconcave and monotone utility functions. Therefore, the set of all possible indirect utility functions is given by $\mathbf{T}(\mathcal{M}_{eqc}(X)) = \mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta)$. In other words, all functions in $\mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta)$, and only them, can be viewed as arising from a maximization problem. As observed in the Introduction, this is important in applications. A slightly stronger result is actually true:

Theorem 7 *Let X be an M-space. Then,*

$$\mathbf{T}\left([-\infty,\infty]^X\right) = \mathbf{T}(\mathcal{M}_{eqc}(X)) = \mathcal{M}^{\lozenge}_{qbx}(\mathbb{R}\times\Delta), \tag{27}$$

and $\mathbf{Q}_{|\mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta)} = \max \mathbf{T}^{-1}$.

Proof. By the proof of Lemma 17, we have that $\mathbf{T}(\mathcal{M}_{eqc}(X)) = \mathbf{T}\left([-\infty,\infty]^X\right)$. By Theorem 2, it follows that $\mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta) = \mathbf{T}(\mathcal{M}_{eqc}(X))$. Thus, (27) holds.

It remains to prove that $\mathbf{Q}(G) = \max \mathbf{T}^{-1}(G)$ for each $G \in \mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta)$. Notice that for each indirect utility G and each utility h that induces G (i.e., such that $\mathbf{T}h = G$), Proposition 1-(iii) implies $\mathbf{Q}G = \mathbf{QT}h \geq h$, while Theorem 2 guarantees $\mathbf{T}(\mathbf{Q}G) = G$. That is, $\mathbf{Q}G \in \mathbf{T}^{-1}(G)$ and it is the greatest utility that induces G. ∎

Though we leave for future research a thorough study of $\mathcal{M}^{\lozenge}_{qcx}(\mathbb{R}\times\Delta)$ as a set of indirect utilities, Lemma 17 and Theorem 7 show that our notion of complete duality is relevant for this important topic in Microeconomics.

5 Concavity and Fenchel Duality

In this section we consider our duality results for concave functions $g : X \to [-\infty,\infty)$. Besides its intrinsic interest, this allows us to investigate the relationships between our duality and the classic Fenchel conjugation $g^*(\xi) = \inf_{x\in X}\{\langle \xi, x\rangle - g(x)\}$. In this section, for each function $g : X \to [-\infty,\infty)$ we set $\mathrm{dom} g = \{x \in X : g(x) > -\infty\}$ and $\mathrm{dom} g_\xi = \{t \in \mathbb{R} : g_\xi(t) > -\infty\}$.

5.1 Preliminary Lemmas

The results of this section rest on the following two lemmas. The first one is true under no assumptions on the functions.

Lemma 18 *Given any $g : X \to [-\infty, \infty)$, we have*

$$(g_\xi)^* (\lambda) = g^* (\lambda\xi), \qquad \forall \xi \in X^*, \ \forall \lambda \in \mathbb{R}. \tag{28}$$

Proof. First observe that the linear map $x \mapsto \langle \xi, x \rangle$ from X to $\mathbb{R}$ has full rank if $\xi \neq 0$. Therefore,

$$\inf_{x \in X} \varphi(x) = \inf_{t \in \mathbb{R}} \inf_{\langle \xi, x \rangle = t} \varphi(x) \tag{29}$$

for any function $\varphi : X \to [-\infty, \infty]$ and any $0 \neq \xi \in X^*$. Hence, for all $\xi \neq 0$ and for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} g^* (\lambda\xi) &= \inf_{x \in X} \{\lambda \langle \xi, x \rangle - g(x)\} = \inf_{t \in \mathbb{R}} \inf_{\langle \xi, x \rangle = t} \{\lambda \langle \xi, x \rangle - g(x)\} = \inf_{t \in \mathbb{R}} \inf_{\langle \xi, x \rangle = t} \{\lambda t - g(x)\} \\ &= \inf_{t \in \mathbb{R}} \left\{ \lambda t - \sup_{\langle \xi, x \rangle = t} g(x) \right\} = \inf_{t \in \mathbb{R}} \{\lambda t - g_\xi(t)\} = (g_\xi)^* (\lambda). \end{aligned}$$

If $\xi = 0$ we have that $g^* (\lambda\xi) = g^* (0) = -\sup_{x \in X} g(x)$ for all $\lambda \in \mathbb{R}$. By definition of g_ξ, we have that $g_0(t) = -\infty$ for each $t \in \mathbb{R} \setminus \{0\}$ while $g_0(0) = \sup_{x \in X} g(x)$. Hence, for all $\lambda \in \mathbb{R}$,

$$(g_0)^* (\lambda) = \inf_{t \in \mathbb{R}} \{\lambda t - g_0(t)\} = -g_0(0) = -\sup_{x \in X} g(x) = g^* (\lambda\xi).$$

Therefore, (28) holds for all $\xi \in X^*$ and for all $\lambda \in \mathbb{R}$. ∎

The case with $\lambda = 1$ of Lemma 18 is especially interesting and is reported in the next corollary, for later reference.

Corollary 3 *Let $g : X \to [-\infty, \infty)$, we have*

$$g^* (\xi) = \inf_{t \in \mathbb{R}} \{t - g_\xi(t)\}, \qquad \forall \xi \in X^*, \tag{30}$$

and, when g is monotone and X_+ is quasi-reproducing,

$$g^* (\xi) = \inf_{t \in \mathbb{R}} \{t - G_\xi(t)\}, \qquad \forall \xi \in X_+^*. \tag{31}$$

In the concave case we have the following relation between G_ξ and the Fenchel conjugate g^*.

Lemma 19 *Let $g : X \to \mathbb{R}$ be concave. Then,*

$$G_\xi(t) = \min_{\lambda \geq 0} [\lambda t - g^* (\lambda\xi)], \qquad \forall t \in \mathbb{R}, \forall \xi \in X^* \setminus \{0\}.$$

Proof. Fix $(t, \xi) \in \mathbb{R} \times X^* \setminus \{0\}$. We can write $G_\xi(t) = \sup_{x \in X} [g(x) - \delta_\xi(x)]$, where δ_ξ is the convex indicator function:

$$\delta_\xi(x) = \begin{cases} 0 & \text{if } \langle \xi, x \rangle \leq t, \\ \infty & \text{if } \langle \xi, x \rangle > t. \end{cases}$$

By the Fenchel-Rockafellar Duality Theorem ([27]),

$$\sup_{x \in X} [g(x) - k(x)] = \min_{\xi \in X^*} [k_*(\xi) - g^* (\xi)],$$

when $g : X \to [-\infty, \infty)$ is a proper concave function, $k : X \to (-\infty, \infty]$ is a proper convex function, k is finite and continuous at some point of $\operatorname{dom} g$, and k_* is the (convex) Fenchel conjugate of k. If we set $k(x) = \delta_\xi(x)$ for each $x \in X$, being g real valued, we have that the assumptions hold for the fixed ξ and t. It remains to calculate $k_*(\xi)$. We have $k_*(\bar{\xi}) = \sup_{x \in X} [\langle \bar{\xi}, x \rangle : \langle \xi, x \rangle \leq t]$. It is easy to check that

$$k_*(\bar{\xi}) = \begin{cases} \lambda t & \text{if } \bar{\xi} = \lambda\xi, \ \lambda \geq 0, \\ \infty & \text{else.} \end{cases}$$

Therefore, $G_\xi(t) = \sup_{x \in X} [g(x) - \delta_\xi(x)] = \min_{\lambda \geq 0} [\lambda t - g^* (\lambda\xi)]$. ∎

5.2 Characterizations of Concavity

We now give some characterizations of concavity that follow from Theorem 1. The equivalence of (i) and (ii) was actually the motivation of de Finetti's early version of Theorem 1. Indeed, the fact that concavity of g implies concavity of both functions g_ξ and G_ξ is well known. We report here the proof for completeness.

Proposition 3 *Given an evenly quasiconcave function $g : X \to [-\infty, \infty)$, consider the following properties:*

(i) g is concave;

(ii) g_ξ is concave for each $\xi \in S^*$;

(iii) G_ξ is concave for each $\xi \in S^*$;

(iv) $g_\xi^+ (t) = \inf_{\lambda\in\mathbb{R}} \{\lambda t - g^* (\lambda\xi)\}$ for each $t \in \overline{\{g_\xi > -\infty\}}$ and $\xi \in S^*$.

Then,

$$(i) \iff (ii) \iff (iii) \implies (iv)$$

Moreover, all properties are equivalent provided g is upper semicontinuous.

Proof. First we will prove that (i), (ii) and (iii) are equivalent.

(i) implies (ii) and (iii). Pick $\xi \in S^*$, then for all $t, r \in \mathbb{R}$ and $\lambda \in (0, 1)$,

$$\begin{aligned}
g_\xi (\lambda t + (1-\lambda) r) &= \sup_{x\in X} \{g (x) : \langle \xi, x\rangle = \lambda t + (1-\lambda) r\} \\
&\geq \sup_{y,z\in X} \{g (\lambda y + (1-\lambda) z) : \langle \xi, y\rangle = t,\ \langle \xi, z\rangle = r\} \\
&\geq \sup_{y,z\in X} \{\lambda g (y) + (1-\lambda) g (z) : \langle \xi, y\rangle = t,\ \langle \xi, z\rangle = r\} \\
&= \lambda \sup_{y\in X} \{g (y) : \langle \xi, y\rangle = t\} + (1-\lambda) \sup_{z\in X} \{g (z) : \langle \xi, z\rangle = r\} \\
&= \lambda g_\xi (t) + (1-\lambda) g_\xi (r).
\end{aligned}$$

The proof for G_ξ follows just as easily.

(ii) or (iii) imply (i). By hypothesis, it follows that for each $\xi \in S^*$ the function such that $x \mapsto g_\xi (\langle \xi, x\rangle, \xi)$ or such that $x \mapsto G_\xi (\langle \xi, x\rangle, \xi)$ is concave, being the composite of an affine function with a concave function. By Theorem 1, it follows that the lower envelope g is concave as well.

(ii) implies (iv). By Lemma 18, $(g_\xi)^* (\lambda) = g^* (\lambda\xi)$. Hence, as g_ξ is concave,

$$cl (g_\xi) (t) = (g_\xi)^{**} (t) = \inf_{\lambda\in\mathbb{R}} \{\lambda t - g^* (\lambda\xi)\}.$$

Where $cl (g_\xi)$ is the "closure" of g_ξ. We have that $cl (g_\xi) (t) = g_\xi^+ (t)$ for all $t \in \mathbb{R}$ if $g_\xi < \infty$ and $\mathrm{dom} g_\xi \neq \emptyset$ or if $t \in \overline{\mathrm{dom} g_\xi}$. Otherwise, $cl (g_\xi) = \infty$ (see [28, pag 307] and [29]).

We have thus the desired implication (ii) $\implies$ (iv), as well as the first part of the statement.

(iv) implies (i). Suppose g is upper semicontinuous. We always have $(g_\xi)^{**} \geq g_\xi^+$ and $(g_\xi)^{**}$ is concave for all $\xi \in S^*$. By Theorem 1, it follows that

$$g (x) = \inf_{\xi\in S^*} g_\xi (\langle \xi, x\rangle) = \inf_{\xi\in S^*} g_\xi^+ (\langle \xi, x\rangle) \leq \inf_{\xi\in S^*} (g_\xi)^{**} (\langle \xi, x\rangle)$$

for all $x \in X$. Suppose $\overline{x} \in \text{dom}g$. From $g(\overline{x}) > -\infty$ it follows $g_\xi(\langle \xi, \overline{x}\rangle) > -\infty$ for all $\xi \in S^*$. Hence, $\langle \xi, \overline{x}\rangle \in \text{dom}g_\xi$ for all $\xi \in S^*$. Consequently, $g_\xi^+(\langle \xi, \overline{x}\rangle) = (g_\xi)^{**}(\langle \xi, \overline{x}\rangle)$ for all $\xi \in S^*$ and

$$g(x) = \inf_{\xi \in S^*} (g_\xi)^{**}(\langle \xi, x\rangle) \tag{32}$$

for all $x \in \text{dom}g$. (32) implies that g is concave on $\text{dom}g$, since it is the lower envelope of concave functions on a convex set. It follows that g is concave on X. ■

For real valued functions, Proposition 3 takes the following form.

Corollary 4 *Given an evenly quasiconcave function $g : X \to \mathbb{R}$, the following are equivalent facts:*

(i) g is concave;

(ii) g_ξ is concave for each $\xi \in S^*$;

(iii) G_ξ is concave for each $\xi \in S^*$;

(iv) $g_\xi(t) = \inf_{\lambda \in \mathbb{R}} \{\lambda t - g^*(\lambda\xi)\}$ for each $(t, \xi) \in \mathbb{R} \times S^*$.

In particular, $\text{dom}g_\xi = \mathbb{R}$ and either $g_\xi < \infty$ or $g_\xi = \infty$.

Proof. From Proposition 3 we have that (i), (ii) and (iii) are equivalent.

(i) implies (iv). Suppose g is concave. Since g is real valued, by definition, $g_\xi(t) > -\infty$ for all $(t, \xi) \in \mathbb{R} \times S^*$. Therefore, it follows that $\text{dom}\, g_\xi = \mathbb{R}$. Then, by concavity of g_ξ, it follows that $g_\xi < \infty$ or $g_\xi = \infty$. In both cases, g_ξ is an upper semicontinuous function, for all $\xi \in S^*$. Hence, $g_\xi = g_\xi^+$ for all $\xi \in S^*$. By Proposition 3, we have that

$$g_\xi(t) = \inf_{\lambda \in \mathbb{R}} \{\lambda t - g^*(\lambda\xi)\}, \quad \forall t \in \mathbb{R}.$$

(iv) implies (i). If $g_\xi(t) = \inf_{\lambda \in \mathbb{R}} \{\lambda t - g^*(\lambda\xi)\}$ for each $(t, \xi) \in \mathbb{R} \times S^*$, then g_ξ is concave for all $\xi \in S^*$. It follows that (ii) is satisfied and g is concave. ■

Similarly, if X an ordered space, for monotone functions Proposition 3 takes the following form.

Corollary 5 *Given an evenly quasiconcave and monotone function $g : X \to [-\infty, \infty)$, consider the following properties:*

(i) g is concave;

(ii) g_ξ is concave for each $\xi \in \Delta$;

(iii) G_ξ is concave for each $\xi \in \Delta$;

(iv) If X is quasi-reproducing, $g_\xi^+(t) = \inf_{\lambda \in \mathbb{R}_+} \{\lambda t - g^*(\lambda\xi)\}$ for each $t \in \overline{\{g_\xi > -\infty\}}$ and $\xi \in \Delta$.

Then,

$$(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii) \Longrightarrow (iv)$$

and all properties are equivalent provided g is upper semicontinuous and X is quasi-reproducing.

Proof. (i), (ii) and (iii) are equivalent. The proof is similar to the one of Proposition 3.

(i) implies (iv). If $g = -\infty$ then $g_\xi = -\infty$ for all $\xi \in \Delta$ and there is nothing to prove. Otherwise, there exists $\bar{x} \in X$ such that $g(\bar{x}) > -\infty$. By Proposition 3, we have that $g_\xi^+(t) = \inf_{\lambda\in\mathbb{R}} \{\lambda t - g^*(\lambda\xi)\}$ for each $t \in \overline{\{g_\xi > -\infty\}}$. Pick $\xi \notin X_+^*$ then there exists an element $\bar{z} \in X^+$ such that $\langle \xi, \bar{z}\rangle < 0$. Since g is monotone we have that,

$$g^*(\xi) = \inf_{x\in X} \{\langle \xi, x\rangle - g(x)\} \leq \inf_{\lambda\in\mathbb{R}_+} \{\langle \xi, \bar{x} + \lambda\bar{z}\rangle - g(\bar{x} + \lambda\bar{z})\} \leq \inf_{\lambda\in\mathbb{R}_+} \{\langle \xi, \bar{x} + \lambda\bar{z}\rangle - g(\bar{x})\} = -\infty. \tag{33}$$

If $\xi \in \Delta$ for each $\lambda < 0$ we have that $\lambda\xi \notin X_+^*$, since X is quasi-reproducing, then,

$$g_\xi^+(t) = \inf_{\lambda\in\mathbb{R}} \{\lambda t - g^*(\lambda\xi)\} = \inf_{\lambda\in\mathbb{R}_+} \{\lambda t - g^*(\lambda\xi)\}.$$

(iv) implies (i). Suppose g is upper semicontinuous then the proof is basically the same of Proposition 3. ■

The case in which g is real valued and monotone is particularly simple and interesting.

Proposition 4 *An evenly quasiconcave and monotone function $g : X \to \mathbb{R}$ is concave if and only if*

$$g_\xi(t) = \inf_{\lambda\in\mathbb{R}_+} \{\lambda t - g^*(\lambda\xi)\}, \qquad \forall (t,\xi) \in \mathbb{R} \times \left(X^* \backslash -X_+^*\right). \tag{34}$$

In this case, we have:

(i) $G_\xi(t) = g_\xi(t)$ *for all* $t \in \mathbb{R}$ *and all* $\xi \in X^* \backslash -X_+^*$,

(ii) $G_\xi(t) = \sup_{x\in X} g(x)$ *for all* $t \in \mathbb{R}$ *and all* $\xi \notin X_+^*$,

(iii) $(G_\xi)^*(\lambda) = g^*(\lambda\xi)$ *for all* $\xi \in X^* \backslash -X_+^*$,

(iv) $g_\xi(t) = G_\xi(t) \wedge G_{-\xi}(-t)$ *for all* $t \in \mathbb{R}$ *and* $\xi \in X^* \backslash \{0\}$.

Proof. We first prove (i). Pick $t \in \mathbb{R}$ and $\xi \notin -X_+^*$. By Lemma 19, $G_\xi(t) = \min_{\lambda\geq 0} [\lambda t - g^*(\lambda\xi)]$. Since g is monotone, by (33), $g^*(\xi) = -\infty$ for all $\xi \notin X_+^*$. If $\xi \notin -X_+^*$, then $\lambda\xi \notin X_+^*$ for all $\lambda \in (-\infty, 0)$ and $g^*(\lambda\xi) = -\infty$. It follows that

$$G_\xi(t) = \min_{\lambda\in\mathbb{R}} \{\lambda t - g^*(\lambda\xi)\}. \tag{35}$$

By Corollary 4 and 0-homogeneity of the mapping $(t,\xi) \mapsto g_\xi(t)$, $g_\xi(t) = \inf_{\lambda\in\mathbb{R}} \{\lambda t - g^*(\lambda\xi)\}$ for all $(t,\xi) \in \mathbb{R} \times (X^* \backslash \{0\})$, and so $g_\xi(t) = G_\xi(t)$ for all $t \in \mathbb{R}$ and $\xi \notin -X_+^*$.

Having established (i), suppose now that g is concave. Then, by (i) and Lemma 19

$$g_\xi(t) = G_\xi(t) = \min_{\lambda\geq 0} \{\lambda t - g^*(\lambda\xi)\}, \qquad \forall (t,\xi) \in \mathbb{R} \times \left(X^* \backslash -X_+^*\right).$$

Vice versa, if g_ξ satisfies (34), then g_ξ is concave for all $\xi \in \Delta$, and so g is concave as well

(ii) If $\xi \notin X_+^*$, by (33), $g^*(\lambda\xi) = -\infty$ for all $\lambda > 0$. Therefore, by Lemma 19, it follows that

$$G_\xi(t) = \min_{\lambda\geq 0} \{\lambda t - g^*(\lambda\xi)\} = -g^*(0) = \sup_{x\in X} g(x).$$

(iii) If $\xi \in X^* \backslash -X_+^*$ denote by φ_ξ the linear mapping $\lambda \mapsto \lambda\xi$ from $\mathbb{R}$ into X^*. We have $g^*(\lambda\xi) = g^*\varphi_\xi(\lambda) = \left(\varphi_\xi^* g\right)^*(\xi)$, where $\varphi_\xi^* : X \to \mathbb{R}$ is the transpose map. In view of (35), we can write $G_\xi = \left(\varphi_\xi^* g\right)^{**}$. Hence, $(G_\xi)^* = \left(\varphi_\xi^* g\right)^{***} = \left(\varphi_\xi^* g\right)^* = g^*\varphi_\xi$. That is, $(G_\xi)^*(\lambda) = g^*(\lambda\xi)$.

(iv) Lemma 19, Corollary 4, and 0-homogeneity of the mapping $(t,\xi)\mapsto g_\xi(t)$ imply

$$g_\xi(t)=\inf_{\lambda\in\mathbb{R}}\{\lambda t-g^*(\lambda\xi)\}=\inf_{\lambda\geq 0}\{\lambda t-g^*(\lambda\xi)\}\wedge\inf_{\lambda\leq 0}\{\lambda t-g^*(\lambda\xi)\}=G_\xi(t)\wedge G_{-\xi}(-t),$$

as desired. ■

We close this section by establishing a few further properties of real valued concave functions. The first property shows that even quasiconcavity is not a novel notion for real valued concave functions. Property (iii) below is well known, but it is here reported in order to show its close connection with Theorem 1.

Proposition 5 *Given a concave function $g:X\to\mathbb{R}$, then,*

(i) g is evenly quasiconcave if and only if is upper semicontinuous;

(ii) g is strictly evenly quasiconcave if and only if is upper semicontinuous and superdifferentiable at every point;

(iii) g is continuous and everywhere superdifferentiable if and only if is lower semicontinuous;

(iv) if X is a Banach space, g is evenly quasiconcave if and only if is continuous.

Proof. (i) If g is upper semicontinuous then each upper contour set of g is closed hence evenly convex, implying that g is evenly quasiconcave. Vice versa, if g is evenly quasiconcave, as g is real-valued and concave, by Corollary 4, we have that

$$g_\xi(t)=\inf_{\lambda\in\mathbb{R}}\{\lambda t-g^*(\lambda\xi)\},\qquad\forall(t,\xi)\in\mathbb{R}\times S^*.$$

It then follows that any function g_ξ is upper semicontinuous for all $\xi\in S^*$ and so is the function such that $x\mapsto g_\xi(\langle\xi,x\rangle,\xi)$ for all $\xi\in S^*$. Since g is evenly quasiconcave, Theorem 1 implies that $g(x)=\inf_{\xi\in S^*}g_\xi(\langle\xi,x\rangle)$ for all $x\in X$. It follows that g is upper semicontinuous, being the lower envelope of upper semicontinuous functions.

(ii) If g is strictly evenly quasiconcave, by Theorem 1-(ii),

$$g(x)=\min_{\xi\in S^*}g_\xi(\langle\xi,x\rangle),\qquad\forall x\in X.$$

Fix $\bar{x}\in X$. It follows that there exists $\xi_{\bar{x}}\in S^*$ such that $g_{\xi_{\bar{x}}}(\langle\xi_{\bar{x}},\bar{x}\rangle)=g(\bar{x})\in\mathbb{R}$. Since g is concave, by Corollary 4, $g_{\xi_{\bar{x}}}$ is concave and real valued on $\mathbb{R}$. Therefore, $g_{\xi_{\bar{x}}}$ is continuous and superdifferentiable. Hence, there exists $l\in\mathbb{R}$ such that for each $s\in\mathbb{R}$

$$g_{\xi_{\bar{x}}}(s)\leq g_{\xi_{\bar{x}}}(\langle\xi_{\bar{x}},\bar{x}\rangle)+l(s-\langle\xi_{\bar{x}},\bar{x}\rangle).$$

Therefore, if $y\in X$ it follows that,

$$g(y)\leq g_{\xi_{\bar{x}}}(\langle\xi_{\bar{x}},y\rangle)\leq g_{\xi_{\bar{x}}}(\langle\xi_{\bar{x}},\bar{x}\rangle)+\langle l\xi_{\bar{x}},y-\bar{x}\rangle.$$

This, in turn, means $l\xi_{\bar{x}}\in\partial g(\bar{x})$.

Conversely, suppose g is upper semicontinuous and superdifferentiable at every point in X. Then, g is evenly quasiconcave and, by Theorem 1, $g(x)=\inf_{\xi\in S^*}G_\xi(\langle\xi,x\rangle)$. Fix $\overline{x}\in X$ and let $\overline{\xi}\in\partial g(\overline{x})$. This implies that

$$g(x)\leq g(\overline{x})+\langle\overline{\xi},x-\overline{x}\rangle.$$

Hence, if $\langle \overline{\xi}, x\rangle \leq \langle \overline{\xi}, \overline{x}\rangle$ then $g(x) \leq g(\overline{x})$. Consequently, if $\overline{\xi} = 0$ then $\overline{x}$ is a global maximizer and it easily follows that $g(\overline{x}) = \min_{\xi \in S^*} G_\xi(\langle \xi, \overline{x}\rangle)$. Otherwise, $g(\overline{x}) = G_{\|\overline{\xi}\|^{-1}\overline{\xi}}\left(\left\langle \|\overline{\xi}\|^{-1}\overline{\xi}, \overline{x}\right\rangle\right)$, and again we obtain that $g(\overline{x}) = \min_{\xi \in S^*} G_\xi(\langle \xi, \overline{x}\rangle)$. It follows that $g(x) = \min_{\xi \in S^*} G_\xi(\langle \xi, x\rangle)$ for all $x \in X$, Theorem 1-(ii) implies that g is strictly evenly quasiconcave.

(iii) If g is lower semicontinuous, then each strict upper contour set is open, and so evenly convex. It turns out that g is strictly evenly quasiconcave. By (ii), g is upper semicontinuous and superdifferentiable. This yields the desired result. Vice versa, if g is continuous and superdifferentiable at each point, g is clearly lower semicontinuous.

(iv) If g is continuous, by (i), it follows that g is evenly quasiconcave. Conversely, by (i), if g is evenly quasiconcave then it is upper semicontinuous. As well known, real-valued and upper semicontinuous concave functions are continuous, provided X is complete (see [26, Proposition 3.3]). ■

5.3 Duality

We now establish duality results, our main object of interest, for the concave case. We need some notation. Given a normed ordered vector space X, let $\mathcal{M}_{conc}(X) \subset \mathcal{M}_{qc}(X)$ be the collection of all functions $g : X \to \mathbb{R}$ that are monotone, upper semicontinuous, and concave. Moreover, given $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ define the functional $\gamma : X^* \to [-\infty, \infty)$ as

$$\gamma(\xi) = \begin{cases} G^*\left(\|\xi\|, \frac{\xi}{\|\xi\|}\right) & \text{if } \xi \in X_+^* \setminus \{0\}, \\ \limsup_{\substack{\|\zeta\| \to 0 \\ \zeta \neq 0}} \gamma(\zeta) & \text{if } \xi = 0, \\ -\infty & \text{if } \xi \notin X_+^*, \end{cases} \tag{36}$$

where $G^*(\cdot, \xi)$ is the Fenchel conjugate of $G(\cdot, \xi)$.

Finally, let $Co(\mathbb{R} \times \Delta) \subset \mathcal{M}(\mathbb{R} \times \Delta)$ be the collection of all functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ that satisfy the following two properties:

(C.1) $G(\cdot, \xi)$ is concave and closed for all $\xi \in \Delta$.

(C.2) $\gamma : X^* \to [-\infty, \infty)$ is proper, upper semicontinuous, cofinite, and concave.[17]

Theorem 8 *Let X be a normed ordered vector space. Then, $\langle \mathcal{M}_{conc}(X), Co(\mathbb{R} \times \Delta)\rangle_{qc}$. In particular, the* inf *in (3) is achieved.*

Proof. We first prove that the mapping $\mathbf{T}$ is well defined. Fix $g \in \mathcal{M}_{conc}(X)$. Since g is real valued the mapping such that $(t, \xi) \mapsto G_\xi(t)$, takes values strictly greater than $-\infty$. By Proposition 1, $\mathbf{T}g \in \mathcal{M}(\mathbb{R} \times \Delta)$. By Corollary 4, we have that G_ξ satisfies (C.1) for each $\xi \in \Delta$. Consider the Fenchel conjugate g^* of g. If $\xi \in X_+^* \setminus \{0\}$, by Proposition 4,

$$g^*(\xi) = g^*\left(\|\xi\| \frac{\xi}{\|\xi\|}\right) = G^*_{\frac{\xi}{\|\xi\|}}(\|\xi\|).$$

If $\xi \notin X_+^*$, $g^*(\xi) = -\infty$ since g is monotone. Therefore, $g^*(\xi) = \gamma(\xi)$, for each $\xi \in X^* \setminus \{0\}$ and $g^*(0) = \gamma(0) = \limsup_{\substack{\|\zeta\| \to 0 \\ \zeta \neq 0}} \gamma(\zeta)$. Since g is real valued and belongs to $\mathcal{M}_{conc}(X)$ and g^* is upper

[17] The function γ is cofinite if, for all $x \in X$, there exists $\alpha \in \mathbb{R}$ such that $\gamma(\xi) \leq \langle \xi, x\rangle + \alpha$ for all ξ. This implies that, if $dom\gamma \neq \emptyset$, its Fenchel conjugate is real valued.

semicontinuous and concave, it turns out that γ is proper, cofinite, upper semicontinuous, and concave. Condition (C.2) is thus satisfied, and we conclude that $\mathbf{T}g \in Co(\mathbb{R} \times \Delta)$. We can then conclude that the mapping T is well defined.

Since $\mathcal{M}_{conc}(X) \subset \mathcal{M}_{eqc}(X)$, by Proposition 1, $\mathbf{T}$ is injective.

Finally, we show that $\mathbf{T}$ is onto. Pick $G \in Co(\mathbb{R} \times \Delta)$, we will show that there is $g \in \mathcal{M}_{conc}(X)$ so that $\mathbf{T}g = G(t, \xi)$. Fix $G \in Co(\mathbb{R} \times \Delta)$ and let $\gamma : X^* \to [-\infty, \infty)$ be the associated functional (36). Consider $g : X \to [-\infty, \infty]$ such that $g = \gamma^*$. By C.2, g is real valued, monotone, upper semicontinuous, and concave. If $\xi \in \Delta$, since $g^* = \gamma$ and by Lemma 19, C.2 and C.1, we have that

$$G_\xi(t) = \inf_{\lambda \geq 0} \{\lambda t - g^*(\lambda \xi)\} = \inf_{\lambda \geq 0} \{\lambda t - \gamma(\lambda \xi)\} = \inf_{\lambda \geq 0} \{\lambda t - G^*(\lambda, \xi)\} = cl\ G(t, \xi) = G(t, \xi).$$

We can conclude that $\mathbf{T}$ is well defined and bijective. By Proposition 1, $\mathbf{Q}$ is the left inverse of $\mathbf{T}$ over $\mathcal{M}_{eqc}(X)$, therefore it is the inverse of $\mathbf{T}$ on $\mathcal{M}_{conc}(X)$. ∎

Remarks. (i) Theorem 8 holds even if X is not an M-space. (ii) In general, the space $Co(\mathbb{R} \times \Delta)$ is not included into $\mathcal{C}(\mathbb{R} \times \Delta)$. For, though C.1 implies that all functions belonging to $Co(\mathbb{R} \times \Delta)$ satisfy (A.1), (A.2), and (A.5), condition (A.4) may fail since G might not be lower semicontinuous. However, if the space X is complete, then the functions in $\mathcal{M}_{conc}(X)$ are continuous, and so Lemma 5 implies that the mapping such that $(t, \xi) \mapsto G_\xi(t)$ is lower semicontinuous. As a result, $Co(\mathbb{R} \times \Delta) \subset \mathcal{C}(\mathbb{R} \times \Delta)$ provided X is a Banach ordered space. (iii) Let X be an M-space and let $\mathcal{M}_{coco}(X)$ be the space of monotone, real valued continuous and concave functions, and let $\mathcal{L}_{conc}(\mathbb{R} \times \Delta)$ be the space of functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ such that $G \in \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$, $G(\cdot, \xi)$ is concave for each $\xi \in \Delta$ and $G(\cdot, \xi) < \infty$ for some $\xi \in \Delta$. It is immediate to see that $\mathcal{L}_{conc}(\mathbb{R} \times \Delta)$ is a subset of $\mathcal{C}(\mathbb{R} \times \Delta)$. Moreover, it is easy to see that Theorem 3 implies $\langle \mathcal{M}_{coco}(X), \mathcal{L}_{conc}(\mathbb{R} \times \Delta) \rangle_{qc}$.

6 Translation Invariance

In this section we establish a duality result for real valued, monotone, quasiconcave, and translation invariant functions. Without loss of generality, throughout the section we consider translation invariant functions g that are normalized, that is, given the translation invariance, such that $g(0) = 0$.[18] Throughout this section X will be considered to be an M-space.

6.1 Basic Properties

We begin with a few useful properties.

Lemma 20 *Let $g : X \to \mathbb{R}$ be monotone and evenly quasiconcave. Then, g is normalized if and only if $\inf_{\xi \in \Delta} G_\xi(t) = t$ for all $t \in \mathbb{R}$.*

Proof. By Theorem 1, $g(x) = \inf_{\xi \in \Delta} G_\xi(\langle \xi, x \rangle)$ for all $x \in X$. If g is normalized then,

$$t = g(te) = \inf_{\xi \in \Delta} G_\xi(\langle \xi, te \rangle) = \inf_{\xi \in \Delta} G_\xi(t), \qquad \forall t \in \mathbb{R}.$$

Conversely, if $\inf_{\xi \in \Delta} G_\xi(t) = t$ for all $t \in \mathbb{R}$, then

$$g(te) = \inf_{\xi \in \Delta} G_\xi(\langle \xi, te \rangle) = \inf_{\xi \in \Delta} G_\xi(t) = t, \qquad \forall t \in \mathbb{R}.$$

∎

[18]Indeed, notice that for any given function $g : X \to \mathbb{R}$, g is monotone, quasiconcave, and translation invariant if and only if $g - g(0)$ shares the same properties. Moreover, $g - g(0)$ is normalized.

Lemma 21 *A quasiconcave and translation invariant function $g : X \to [-\infty, \infty)$ is concave. Moreover, g is real valued and Lipschitz continuous if it is monotone.*

Conversely, a monotone, evenly quasiconcave, concave, and normalized function $g : X \to [-\infty, \infty)$ is translation invariant.

Proof. Fix $x_1, x_2 \in X$. If $g(x_1) \wedge g(x_2) = -\infty$, then for each $\lambda \in (0, 1)$

$$g(\lambda x_1 + (1-\lambda) x_2) \geq -\infty = \lambda g(x_1) + (1-\lambda) g(x_2).$$

If $g(x_1) \wedge g(x_2) > -\infty$, there is $\eta \in \mathbb{R}$ such that $g(x_1) = \eta + g(x_2) = g(x_2 + \eta e)$. By quasiconcavity and translation invariance,

$$g(\lambda x_1 + (1-\lambda) x_2) + (1-\lambda)\eta = g(\lambda x_1 + (1-\lambda)(x_2 + \eta e)) \geq g(x_1).$$

This in turn implies

$$g(\lambda x_1 + (1-\lambda) x_2) \geq g(x_1) - (1-\lambda)\eta.$$

On the other hand, $\eta = g(x_1) - g(x_2)$. Inserting this value, we get

$$g(\lambda x_1 + (1-\lambda) x_2) \geq \lambda g(x_1) + (1-\lambda) g(x_2).$$

Suppose g is also monotone. By Lemma 2 it is real-valued. Then, if $x_1, x_2 \in X$ we have that $x_1 \leq x_2 + \|x_1 - x_2\| e$, and so, by monotonicity and translation invariance, $g(x_1) \leq g(x_2) + \|x_1 - x_2\|$. By exchanging the role of x_1 and x_2 the statement follows.

If g is evenly quasiconcave and concave, by Corollary 5, G_ξ is concave for all $\xi \in \Delta$. Since g is monotone and normalized, by Lemma 2, g is real valued. It follows that $G_\xi > -\infty$ for all $\xi \in \Delta$. Moreover, since g is normalized, $G_\xi(t) \geq t$ for all t and all $\xi \in \Delta$. If $G_\xi(t) = \infty$ for some t, by concavity of G_ξ it follows that $G_\xi = \infty$. If $G_\xi(t) < \infty$ for all t, then, define $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(t) = G_\xi(t) - t$. The function ρ is real valued, nonnegative, and concave. This implies that it is constant (see, e.g., [28, Corollary 8.6.2]). Hence, $G_\xi(t) = t + c(\xi)$ for all $(t, \xi) \in \mathbb{R} \times \Delta$ where $c : \Delta \to [0, \infty]$ is quasiconvex. Since g is monotone and evenly quasiconcave, by Theorem 1,

$$g(x + \lambda e) = \inf_{\xi \in \Delta} [\langle \xi, x + \lambda e \rangle + c(\xi)] = \lambda + \inf_{\xi \in \Delta} [\langle \xi, x \rangle + c(\xi)] = \lambda + g(x), \quad \forall \lambda \in \mathbb{R}, \forall x \in X,$$

as desired. ■

Corollary 6 *A quasiconcave, monotone, and normalized function $g : X \to \mathbb{R}$ is concave if and only if is translation invariant.*

Proof. Suppose $g : X \to \mathbb{R}$ is concave. Since X is an M-space and g is monotone and normalized, by Lemma 2, g is continuous at 0. Hence, g is continuous on X ([1, Theorem 5.43]), and so, by Lemma 21, is translation invariant. The converse follows from Lemma 21. ■

Example 3 Consider the mean functional

$$I(f) = \phi^{-1}\left(\int_\Omega \phi(f(\omega))\, d\mu(\omega)\right)$$

of Example 1. By Corollary 6, I is concave if and only if is translation invariant. It can be shown that this is a strong condition that forces ϕ to be either $\phi(t) = at + b$, with $a > 0$, or $\phi(t) = -ae^{-bt} + c$, with $a, b > 0$ (see [3]). ▲

6.2 Duality

Let $\mathcal{M}_{tr}(X) \subset \mathcal{M}_{qc}(X)$ be the collection of quasiconcave, monotone, normalized, and translation invariant functions $g : X \to \mathbb{R}$. By Lemma 21, $\mathcal{M}_{tr}(X) \subset \mathcal{M}_{conc}(X) \cap \mathcal{M}_{uc}(X)$.

Let $\mathcal{E}_{as}(\mathbb{R} \times \Delta)$ be the collection of functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ that are additively separable, that is, such that

$$G(t, \xi) = t + c(\xi),$$

where $c : \Delta \to [0, \infty]$ is lower semicontinuous, convex, and $\min_{\xi \in \Delta} c(\xi) = 0$.

The next result shows that $\mathcal{M}_{tr}(X)$ is in duality with the additively separable functions $\mathcal{E}_{as}(\mathbb{R} \times \Delta)$.

Theorem 9 *Let X be an M-space. Then, $\langle \mathcal{M}_{tr}(X), \mathcal{E}_{as}(\mathbb{R} \times \Delta) \rangle_{qc}$. In particular, the* inf *in (3) is achieved.*

An immediate consequence of this theorem is that a function $g : X \to \mathbb{R}$ belongs to $\mathcal{M}_{tr}(X)$ if and only if

$$g(x) = \min_{\xi \in \Delta} \{\langle \xi, x \rangle + c(\xi)\},$$

where $c : \Delta \to [0, \infty]$ is lower semicontinuous, convex, and $\min_{\xi \in \Delta} c(\xi) = 0$.

The proof of Theorem 9 will be based on couple of lemmas.

Lemma 22 *A function $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ belongs to $\mathcal{E}_{as}(\mathbb{R} \times \Delta)$ only if it belongs to $\mathcal{E}(\mathbb{R} \times \Delta)$ and $G(\cdot, \xi)$ is translation invariant for each $\xi \in \Delta$.*

Proof. Define $\widetilde{\Delta} = \{\xi \in \Delta : c(\xi) < \infty\}$.

(i) G satisfies A.1. $G(t, \xi) = t + c(\xi)$ for all $t \in \mathbb{R}$ and for all $\xi \in \Delta$. If $\xi \in \widetilde{\Delta}$ then $G(t_1, \xi) \geq G(t_2, \xi)$ if and only if $t_1 \geq t_2$. If $\xi \in \Delta \backslash \widetilde{\Delta}$, $G(\cdot, \xi)$ is constant.

(ii) G satisfies A.2. For each $\xi \in \widetilde{\Delta}$,

$$\lim_{t \to +\infty} G(t, \xi) = \lim_{t \to +\infty} [t + c(\xi)] = \infty,$$

while $G(\cdot, \xi) = \infty$ for each $\xi \in \Delta \backslash \widetilde{\Delta}$.

(iii) G satisfies A.4. By definition, $c : \Delta \to [0, \infty]$ is lower semicontinuous and convex, and the mapping such that $(t, \xi) \mapsto t$ is affine and continuous. Therefore, it follows that G, being the sum of these two functions, is convex and lower semicontinuous.

(iv) G satisfies A.6. If $\xi \in \widetilde{\Delta}$, then for all t in $\mathbb{R}$, we have that $G(t, \xi) = t + c(\xi)$ is a real number, therefore $\operatorname{dom} G(\cdot, \xi) = \mathbb{R}$. Conversely, if $\xi \in \Delta \backslash \widetilde{\Delta}$ then $G(\cdot, \xi) = \infty$ implying that $\operatorname{dom} G(\cdot, \xi) = \emptyset$. Furthermore, since $\min_{\xi \in \Delta} c(\xi) = 0$, if follows that $\operatorname{dom} G(\cdot, \xi) = \mathbb{R}$ for some $\xi \in \widetilde{\Delta}$.

(v) G satisfies A.7. By (iv), $\operatorname{dom} G(\cdot, \xi) = \mathbb{R}$ if and only if $\xi \in \widetilde{\Delta}$. We want to show that for each $\varepsilon > 0$ there is $\delta > 0$ such that $|t - t'| \leq \delta$ implies $|G(t, \xi) - G(t', \xi)| \leq \varepsilon$ for all $t, t' \in \mathbb{R}$ and all $\xi \in \widetilde{\Delta}$. But for any fixed $\varepsilon > 0$ it is enough to pick $\delta = \varepsilon$. Indeed, $|G(t, \xi) - G(t', \xi)| = |t - t'| \leq \varepsilon$ for all $t, t' \in \mathbb{R}$ and all $\xi \in \widetilde{\Delta}$.

By (i)-(v), we conclude that $G \in \mathcal{E}(\mathbb{R} \times \Delta)$.

It remains to prove that $G(\cdot, \xi)$ is translation invariant for each $\xi \in \Delta$. Fix $\xi \in \Delta$ and consider any $t, \lambda \in \mathbb{R}$. We have that $G(t + \lambda \cdot 1, \xi) = t + \lambda \cdot 1 + c(\xi) = t + c(\xi) + \lambda = G(t, \xi) + \lambda$. ■

Lemma 23 *A function $g : X \to [-\infty, \infty]$ is real valued, quasiconcave, monotone, translation invariant if and only if there exists a $G \in \mathcal{E}_{as}(\mathbb{R} \times \Delta)$ such that*

$$g(x) = \min_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi), \qquad \forall x \in X. \tag{37}$$

Proof. "Only if." Define as G the mapping such that $(t, \xi) \mapsto G_\xi(t)$. By Lemma 21, if g is real valued, monotone, quasiconcave and translation invariant then it is uniformly continuous. Since g is quasiconcave, by Theorem 6 $G \in \mathcal{E}(\mathbb{R} \times \Delta)$. Then, by translation invariance of g, we have that for each $t, \lambda \in \mathbb{R}$ and for each $\xi \in \Delta$,

$$\begin{aligned} G_\xi(t) + \lambda &= \sup\{g(x) + \lambda : \langle \xi, x \rangle \le t\} = \sup\{g(x + \lambda e) : \langle \xi, x \rangle \le t\} \\ &= \sup\{g(x) : \langle \xi, x \rangle \le t + \lambda\} = G_\xi(t + \lambda). \end{aligned}$$

Hence, $G_\xi(\lambda) = \lambda + G_\xi(0)$. Define $c : \Delta \to [-\infty, \infty]$ by $c(\xi) = G_\xi(0)$. Then, $G_\xi(t) = t + c(\xi)$ for all $\xi \in \Delta$ and all $t \in \mathbb{R}$. Since $G \in \mathcal{E}(\mathbb{R} \times \Delta)$, it follows that c is lower semicontinuous and quasiconvex. As g is normalized, by Lemma 20, $c(\xi) = G_\xi(0) \ge 0$ for all $\xi \in \Delta$ and $0 = \inf_{\xi \in \Delta} G_\xi(0) = \min_{\xi \in \Delta} c(\xi)$.

It remains to prove the convexity of c. Fix ξ_1 and ξ_2 in Δ and consider $\lambda \in (0, 1)$. If $c(\xi_1) \vee c(\xi_2) = \infty$, then

$$c(\lambda \xi_1 + (1 - \lambda) \xi_2) \le \infty = \lambda c(\xi_1) + (1 - \lambda) c(\xi_2).$$

If $c(\xi_1) \vee c(\xi_2) < \infty$, there exist $t_1, t_2 \in \mathbb{R}$ such that $c(\xi_1) - c(\xi_2) = t_2 - t_1$, i.e., $c(\xi_1) + t_1 = c(\xi_2) + t_2$. As $(t, \xi) \mapsto G_\xi(t) = t + c(\xi)$ is quasiconvex. Then,

$$\lambda t_1 + (1 - \lambda) t_2 + c(\lambda \xi_1 + (1 - \lambda) \xi_2) \le \max\{c(\xi_1) + t_1, c(\xi_2) + t_2\} = c(\xi_2) + t_2.$$

Hence,

$$\begin{aligned} c(\lambda \xi_1 + (1 - \lambda) \xi_2) &\le c(\xi_2) + t_2 - \lambda t_1 - (1 - \lambda) t_2 = c(\xi_2) + \lambda(t_2 - t_1) \\ &= c(\xi_2) + \lambda(c(\xi_1) - c(\xi_2)) = \lambda c(\xi_1) + (1 - \lambda) c(\xi_2), \end{aligned}$$

and so c is convex.

"If." If $G \in \mathcal{E}_{as}(\mathbb{R} \times \Delta)$, then $G \in \mathcal{E}(\mathbb{R} \times \Delta)$ by Lemma 22. By Lemma 16, it follows that g, defined as in (37), is real valued, monotone, and quasiconcave. By Lemma 22, $G(\cdot, \xi)$ is translation invariant for all $\xi \in \Delta$. It follows that for each $x \in X$ and each $\lambda \in \mathbb{R}$,

$$g(x + \lambda e) = \min_{\xi \in \Delta} G(\langle \xi, x + \lambda e \rangle, \xi) = \min_{\xi \in \Delta} G(\langle \xi, x \rangle + \lambda, \xi) = \min_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi) + \lambda = g(x) + \lambda.$$

Hence, g is translation invariant. ■

Proof of Theorem 9. Consider $\mathbf{T} : \mathcal{M}_{tr}(X) \to \mathcal{M}(\mathbb{R} \times \Delta)$. By the proof of Lemma 23, we have that $\mathbf{T}(\mathcal{M}_{tr}(X)) \subset \mathcal{E}_{as}(\mathbb{R} \times \Delta) \subset \mathcal{E}(\mathbb{R} \times \Delta)$. Hence, $\mathbf{T} : \mathcal{M}_{tr}(X) \to \mathcal{E}_{as}(\mathbb{R} \times \Delta)$. By Proposition 1, $\mathbf{T}$ is injective on $\mathcal{M}_{tr}(X) \subset \mathcal{M}_{eqc}(X)$. Let $G \in \mathcal{E}_{as}(\mathbb{R} \times \Delta) \subset \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$. By Lemma 23, $\mathbf{Q}G \in \mathcal{M}_{tr}(X)$. By Lemma 13, $\mathbf{TQ}G = G$ therefore T is surjective. That is, $\mathbf{T}^{-1} = \mathbf{Q}$ on $\mathcal{E}_{as}(\mathbb{R} \times \Delta)$. ■

Remarks. (i) From Theorem 9 and the proof of Lemma 23, it is immediate to see that Lemma 22 turns out to be an "if and only if" statement. (ii) Given that a function $g : X \to \mathbb{R}$ is monotone, quasiconcave and translation invariant if and only if $g - g(0)$ is normalized, Theorem 9 and Lemma 6 provide a more generic dual pair. If we call $\mathcal{M}_{gtr}(X)$ the set of real valued, monotone, quasiconcave and translation invariant functions and we call $\mathcal{E}_{gas}(\mathbb{R}\times\Delta)$ the collection of functions $G:\mathbb{R}\times\Delta\to(-\infty,\infty]$ that are such that

$$G(t,\xi)=t+c(\xi),$$

where $c:\Delta\to(-\infty,\infty]$ is lower semicontinuous, convex, and $\min_{\xi\in\Delta}c(\xi)\in\mathbb{R}$, we have that $\langle\mathcal{M}_{gtr}(X),\mathcal{E}_{gas}(\mathbb{R}\times\Delta)\rangle_{qc}$. It is then clear that the condition $\min_{\xi\in\Delta}c(\xi)=0$ is equivalent to impose the normalization of g. (iii) Observe that, if we define $\mathrm{dom}(g^*)=\{\xi\in X^*:g^*(\xi)>-\infty\}$, in view of Lemma 21 and Theorem 9, Fenchel conjugation $g\mapsto g^*$ establishes a one-to-one correspondence between $\mathcal{M}_{tr}(X)$ and the set $\mathcal{C}^*$ of upper semicontinuous concave functions $g^*:X^*\to[-\infty,\infty)$ with domain included in Δ and with $\max_{\xi\in\Delta}g^*(\xi)=0$. Call such mapping $\mathbf{F}$, it is easy to see that $\mathbf{F}$ is well defined. Define then as π the function from $\mathcal{E}_{as}(\mathbb{R}\times\Delta)$ to $\mathcal{C}^*$ such that $\pi(G)=-c$ where c is extended to X^*, by putting $c=-\infty$ outside Δ. π is clearly well defined and bijective[19]. By Theorem 9, the following diagram

$$\begin{array}{ccc} \mathcal{M}_{tr}(X) & & \\ \mathbf{T}\downarrow & \searrow\mathbf{F} & \\ \mathcal{E}_{as}(\mathbb{R}\times\Delta) & \overset{\pi}{\to} & \mathcal{C}^* \end{array}$$

commutes, where $\mathbf{T}$ is bijective. This proves that the traditional Fenchel conjugation $g\mapsto g^*$ establishes a one-to-one correspondence between $\mathcal{M}_{tr}(X)$ and $\mathcal{C}^*$.

7 Positive Homogeneity

Quasiconcave functions on an M-space X that are positively homogeneous is the last class of functions that we consider.[20] Denote by $\mathcal{M}_{po}(X)\subset\mathcal{M}_{qc}(X)$ the collection of non-degenerate[21] functions $g:X\to(-\infty,\infty]$ that are quasiconcave, monotone, lower semicontinuous, and positively homogeneous.

Similarly, denote by $\mathcal{M}_{upo}(X)\subset\mathcal{M}_{qc}(X)$ the collection of non-degenerate functions $g:X\to\mathbb{R}$ that are quasiconcave, monotone, uniformly continuous, and positively homogeneous. By definition, $\mathcal{M}_{po}(X)\subset\mathcal{M}_{lsc}(X)$ and $\mathcal{M}_{upo}(X)\subset\mathcal{M}_{uc}(X)$.

Let $\mathcal{L}_{ms}(\mathbb{R}\times\Delta)$ be the collection of functions $G:\mathbb{R}\times\Delta\to(-\infty,\infty]$ that are multiplicatively separable, that is, such that

$$G(t,\xi)=\begin{cases}\frac{t}{c_1(\xi)} & \text{if } t\geq 0 \text{ and } \xi\in\widetilde{\Delta}\\ \frac{t}{c_2(\xi)} & \text{if } t\leq 0 \text{ and } \xi\in\widetilde{\Delta}\\ \infty & \text{if } \xi\in\Delta\setminus\widetilde{\Delta}\end{cases}\tag{38}$$

where $\widetilde{\Delta}$ is a closed and convex subset of Δ, and

(i) $c_1:\widetilde{\Delta}\to[0,\infty)$ is concave and upper semicontinuous;

(ii) $c_2:\widetilde{\Delta}\to(0,\infty]$ is convex and lower semicontinuous.

[19] For each g^* in $\mathcal{C}^*$ denote $(g^*)_{|\Delta}$ its restriction to Δ. It follows that $\pi^{-1}:\mathcal{C}^*\to\mathcal{E}_{as}(\mathbb{R}\times\Delta)$ is such that given $g^*\in\mathcal{C}^*$, $\pi^{-1}(g^*)=G$ where

$$G(t,\xi)=t-(g^*)_{|\Delta}(\xi),\ \forall(t,\xi)\in\mathbb{R}\times\Delta.$$

[20] In this section we assume that $0/0=0\cdot\infty=0$.

[21] That is, $g(x)\neq 0$ for at least some $x\in X_+$. If $g:X\to(-\infty,\infty]$ is monotone and positively homogeneous and X admits a order unit e, it is immediate to see that g is non-degenerate if and only if $g(e)>0$.

Let $\mathcal{E}_{ms}(\mathbb{R}\times\Delta)\subset\mathcal{L}_{ms}(\mathbb{R}\times\Delta)$ be the collection of functions $G:\mathbb{R}\times\Delta\to(-\infty,\infty]$ that belong to $\mathcal{E}_{ms}(\mathbb{R}\times\Delta)$ and with $c_1:\widetilde{\Delta}\to[0,\infty)$ such that $\inf_{\xi\in\widetilde{\Delta}}c_1(\xi)>0$, where $\widetilde{\Delta}$ is assumed to be nonempty.

We have $\mathcal{L}_{ms}(\mathbb{R}\times\Delta)\subset\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ and $\mathcal{E}_{ms}(\mathbb{R}\times\Delta)\subset\mathcal{E}(\mathbb{R}\times\Delta)$. That is, $\mathcal{L}_{ms}(\mathbb{R}\times\Delta)$ and $\mathcal{E}_{ms}(\mathbb{R}\times\Delta)$ are, respectively, the multiplicatively separable functions in $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ and $\mathcal{E}(\mathbb{R}\times\Delta)$.

Theorem 10 *Let X be an M-space. Then, $\langle\mathcal{M}_{po}(X),\mathcal{L}_{ms}(\mathbb{R}\times\Delta)\rangle_{qc}$. In particular, the* inf *in (3) is achieved.*

In Theorem 9 we showed that $\mathcal{M}_{tr}(X)$ is in duality with the additively separable functions in $\mathcal{E}(\mathbb{R}\times\Delta)$. Here, Theorem 10 shows that $\mathcal{M}_{po}(X)$ is, instead, in duality with $\mathcal{L}_{ms}(\mathbb{R}\times\Delta)$, the multiplicatively separable functions in $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$. The next corollary completes the picture by showing that $\mathcal{M}_{upo}(X)$ is in duality with $\mathcal{E}_{ms}(\mathbb{R}\times\Delta)$, the multiplicatively separable functions in $\mathcal{E}(\mathbb{R}\times\Delta)$.

Corollary 7 *Let X be an M-space. Then, $\langle\mathcal{M}_{upo}(X),\mathcal{E}_{ms}(\mathbb{R}\times\Delta)\rangle_{qc}$. In particular, the* inf *in (3) is achieved.*

The proof of Theorem 10 is based on few lemmas.

Lemma 24 *Let $(a_i)_{i\in I},(b_i)_{i\in I}\subset\mathbb{R}_+$. The family of functions*

$$f_i(t)=\begin{cases}a_it & \text{if } t\geq 0\\ b_it & \text{if } t\leq 0\end{cases}$$

is uniformly equicontinuous if and only if $\sup_{i\in I}a_i,\sup_{i\in I}b_i<\infty$.

Proof. First observe that a family of monotone functions is uniformly equicontinuous if and only if for every $\varepsilon>0$ there is $\delta>0$ such that

$$f_i(t+\delta)\leq f_i(t)+\varepsilon \tag{39}$$

for all $t\in\mathbb{R}$ and $i\in I$.[22]

In our special case for all $i\in I$, $t\in\mathbb{R}$, and $\delta>0$,

$$\begin{aligned} f_i(t+\delta)-f_i(t) &= \begin{cases}a_i\delta & \text{if } t\geq 0\\ a_it+a_i\delta-b_it & \text{if } -\delta<t<0\\ b_i\delta & \text{if } t+\delta\leq 0 \text{ (i.e. } t\leq-\delta)\end{cases}\\ (a_it<0 \text{ if } -\delta<t<0) &\leq \begin{cases}a_i\delta & \text{if } t\geq 0\\ a_i\delta-b_it & \text{if } -\delta<t<0\\ b_i\delta & \text{if } t\leq-\delta\end{cases}\\ (b_i\delta\geq -b_it\geq 0 \text{ if } -\delta<t<0) &\leq \begin{cases}a_i\delta & \text{if } t\geq 0\\ a_i\delta+b_i\delta & \text{if } -\delta<t<0\\ b_i\delta & \text{if } t\leq-\delta\end{cases}\\ &\leq (a_i+b_i)\,\delta\leq\left(\sup_{i\in I}a_i+\sup_{i\in I}b_i+1\right)\delta. \end{aligned}$$

[22]If for every $\varepsilon>0$ there is $\delta>0$ such that $|t'-t''|\leq\delta$ implies $|f_i(t')-f_i(t'')|\leq\varepsilon$, for all $t',t''\in\mathbb{R}$ and all $i\in I$, then $f_i(t+\delta)-f_i(t)=|f_i(t+\delta)-f_i(t)|\leq\varepsilon$ for all $t\in\mathbb{R}$ and all $i\in I$. Conversely, if condition (39) holds, consider $t',t''\in\mathbb{R}$ with $|t'-t''|\leq\delta$, wlog $t'\geq t''$, then $t'\leq t''+\delta$, $f_i(t''+\delta)\leq f_i(t'')+\varepsilon$, and monotonicity, delivers $f_i(t')\leq f_i(t''+\delta)\leq f_i(t'')+\varepsilon$, whence $|f_i(t')-f_i(t'')|=f_i(t')-f_i(t'')\leq\varepsilon$ for all $i\in I$.

Therefore, if $\sup_{i\in I} a_i, \sup_{i\in I} b_i < \infty$, for all $\varepsilon > 0$ it suffices to take

$$\delta < \frac{\varepsilon}{\left(\sup_{i\in I} a_i + \sup_{i\in I} b_i + 1\right)}$$

to obtain

$$f_i\left(t+\delta\right) - f_i\left(t\right) \leq \left(\sup_{i\in I} a_i + \sup_{i\in I} b_i + 1\right)\delta \leq \varepsilon$$

for all $i \in I$, $t \in \mathbb{R}$, which implies uniform equicontinuity.

If $\sup_{i\in I} a_i = \infty$, then $f_i\left(0+\delta\right) - f_i\left(0\right) = a_i\delta$ for all $\delta > 0$, and so $\sup_{i\in I}\left(f_i\left(0+\delta\right) - f_i\left(0\right)\right) = \infty$, which contradicts condition (39). If $\sup_{i\in I} b_i = \infty$, then for all $\delta > 0$ take $t_\delta < -\delta$

$$f_i\left(t_\delta+\delta\right) - f_i\left(t_\delta\right) = b_i\delta.$$

Then, $\sup_{i\in I}\left(f_i\left(t_\delta+\delta\right) - f_i\left(t_\delta\right)\right) = \infty$, which contradicts condition (39). ■

Lemma 25 *Let C be a convex subset of a vector space and $f_1, f_2 : C \to \mathbb{R}$ be quasiconvex functions. If $f_1 \geq 0$, $f_2 \leq 0$, and $f_1 f_2 = 0$, then $f_1 + f_2$ is quasiconvex.*

Proof. Let $f = f_1 + f_2$. Set $C^- = \{x \in C : f_2\left(x\right) < 0\}$. The set C^- is convex, and we can assume $C^- \neq \emptyset$ (otherwise, $f_1 + f_2 = f_1$ is quasiconvex). As f_1 and f_2 are quasiconvex, we have

$$f_1\left(\lambda x_1 + (1-\lambda)\,x_2\right) \leq f_1\left(x_1\right) \vee f_1\left(x_2\right) \quad \text{and} \quad f_2\left(\lambda x_1 + (1-\lambda)\,x_2\right) \leq f_2\left(x_1\right) \vee f_2\left(x_2\right)$$

and so $f\left(\lambda x_1 + (1-\lambda)\,x_2\right) \leq f_1\left(x_1\right) \vee f_1\left(x_2\right) + f_2\left(x_1\right) \vee f_2\left(x_2\right)$ for all $x_1, x_2 \in C$ and all $\lambda \in [0,1]$. We want to show that $f\left(\lambda x_1 + (1-\lambda)\,x_2\right) \leq f\left(x_1\right) \vee f\left(x_2\right)$. Consider the following cases:

Case (a): $x_1, x_2 \in C^-$. The convexity of C^- implies $\lambda x_1 + (1-\lambda)\,x_2 \in C^-$, and over C^- we have $f_1 = 0$, then $f_{|C^-} = f_2$ delivers the result.

Case (b): $x_1 \in C^-$ and $x_2 \notin C^-$. We have $f_1\left(x_1\right) = 0$, $f_2\left(x_1\right) < 0$, $f_2\left(x_2\right) = 0$, and $f_1\left(x_2\right) \geq 0$. Therefore,

$$f\left(\lambda x_1 + (1-\lambda)\,x_2\right) \leq f_1\left(x_1\right) \vee f_1\left(x_2\right) + f_2\left(x_1\right) \vee f_2\left(x_2\right) = f_1\left(x_2\right)$$

and $f\left(x_1\right) \vee f\left(x_2\right) = \left(0 + f_2\left(x_1\right)\right) \vee \left(f_1\left(x_2\right) + 0\right) = f_1\left(x_2\right)$, as wanted.

Case (c): $x_1, x_2 \notin C^-$. We have $f_2\left(x_1\right) = f_2\left(x_2\right) = 0$, $f_1\left(x_1\right) \geq 0$, $f_1\left(x_2\right) \geq 0$. Hence,

$$f\left(\lambda x_1 + (1-\lambda)\,x_2\right) \leq f_1\left(x_1\right) \vee f_1\left(x_2\right) + f_2\left(x_1\right) \vee f_2\left(x_2\right) = f_1\left(x_1\right) \vee f_1\left(x_2\right) = f\left(x_1\right) \vee f\left(x_2\right)$$

which concludes the proof. ■

The proof of Theorem 10 relies on the following two lemmas. Here, given a subset $K \subset X^*$, its positive polar cone is $K^\oplus = \{x \in X : \langle \xi, x\rangle \geq 0 \text{ for all } \xi \in K\}$.

Lemma 26 *A function $G : \mathbb{R}\times\Delta \to (-\infty, \infty]$ belongs to $\mathcal{L}_{ms}\left(\mathbb{R}\times\Delta\right)$ only if it belongs to $\mathcal{L}_{qcx}\left(\mathbb{R}\times\Delta\right)$ and $G\left(\cdot,\xi\right)$ is positively homogeneous and such that $G\left(1,\xi\right) \neq 0$ for each $\xi \in \Delta$. Moreover, $G \in \mathcal{E}_{ms}\left(\mathbb{R}\times\Delta\right)$ only if $G \in \mathcal{E}\left(\mathbb{R}\times\Delta\right)$ and $G\left(\cdot,\xi\right)$ is positively homogeneous and such that $G\left(1,\xi\right) \neq 0$ for each $\xi \in \Delta$.*

Proof. (i) G satisfies A.1. Since $c_1(\xi), c_2(\xi) \geq 0$ for each $\xi \in \widetilde{\Delta}$, we have that $G(\cdot, \xi)$ is nondecreasing for each $\xi \in \widetilde{\Delta}$, while $G(\cdot, \xi)$ is constant for each $\xi \in \Delta \backslash \widetilde{\Delta}$.

(ii) G satisfies A.2. Since $c_1(\xi) \geq 0$ for each $\xi \in \widetilde{\Delta}$,

$$\lim_{t \to +\infty} G(t, \xi) = \lim_{t \to +\infty} \frac{t}{c_1(\xi)} = \infty,$$

while $G(\cdot, \xi) = \infty$ for each $\xi \in \Delta \backslash \widetilde{\Delta}$.

Therefore, (i) and (ii) imply that $G \in \mathcal{M}(\mathbb{R} \times \Delta)$. Notice that

$$G(t, \xi) = \frac{t^+}{c_1(\xi)} - \frac{t^-}{c_2(\xi)}, \qquad \forall (t, \xi) \in \mathbb{R} \times \widetilde{\Delta}. \tag{40}$$

(iii) G satisfies A.4. We first prove lower semicontinuity of G, and then its quasiconvexity. Since $\widetilde{\Delta}$ is closed, the set $\mathbb{R} \times \widetilde{\Delta}$ is closed in $\mathbb{R} \times \Delta$. Since $G(t, \xi) = \infty$ outside $\mathbb{R} \times \widetilde{\Delta}$, it suffices to check that G is lower semicontinuous on $\mathbb{R} \times \widetilde{\Delta}$, where G is given by (40). It is convenient to study first separately the two functions $(t, \xi) \mapsto t^+/c_1(\xi)$ and $(t, \xi) \mapsto t^-/c_2(\xi)$.

Consider a pair $(t, \xi) \in \mathbb{R} \times \widetilde{\Delta}$. If $c_1(\xi) = 0$ and $t^+ > 0$, then $t^+/c_1(\xi) > \alpha$ and $t^+ - \alpha c_1(\xi) > \alpha$ for each $\alpha \geq 0$. Otherwise, if $c_1(\xi) = 0$ and $t^+ = 0$ or $c_1(\xi) > 0$ and $t^+ \geq 0$, then $t^+/c_1(\xi) \leq \alpha$ if and only if $t^+ - \alpha c_1(\xi) \leq 0$. Therefore, for each $\alpha \geq 0$, we have

$$\left\{(t, \xi) \in \mathbb{R} \times \widetilde{\Delta} : \frac{t^+}{c_1(\xi)} \leq \alpha\right\} = \left\{(t, \xi) \in \mathbb{R} \times \widetilde{\Delta} : t^+ - \alpha c_1(\xi) \leq 0\right\}.$$

The latter set is closed and convex since the functions $(t, \xi) \mapsto t^+$ and $(t, \xi) \mapsto -\alpha c_1(\xi)$ are convex and lower semicontinuous. For $\alpha < 0$, the set $\left\{(t, \xi) \in \mathbb{R} \times \widetilde{\Delta} : \frac{t^+}{c_1(\xi)} \leq \alpha\right\}$ is empty. We thus obtain that $(\xi, t) \mapsto t^+/c_1(\xi)$ is lower semicontinuous and quasiconvex. Likewise, if $\alpha > 0$ we have:

$$\begin{aligned}
\left\{(t, \xi) \in \mathbb{R} \times \widetilde{\Delta} : \frac{t^-}{c_2(\xi)} \geq \alpha\right\} &= \left\{(t, \xi) \in \mathbb{R} \times \widetilde{\Delta} : \alpha c_2(\xi) - t^- \leq 0\right\} \\
&= \left\{(t, \xi) \in (-\infty, 0) \times \widetilde{\Delta} : \alpha c_2(\xi) + t \leq 0\right\} \cup \left\{(t, \xi) \in \mathbb{R}_+ \times \widetilde{\Delta} : \alpha c_2(\xi) \leq 0\right\} \\
&= \left\{(t, \xi) \in (-\infty, 0) \times \widetilde{\Delta} : \alpha c_2(\xi) + t \leq 0\right\} \cup \emptyset \\
&= \left\{(t, \xi) \in (-\infty, 0) \times \widetilde{\Delta} : \alpha c_2(\xi) + t \leq 0\right\} \cup \left\{(t, \xi) \in \mathbb{R}_+ \times \widetilde{\Delta} : \alpha c_2(\xi) + t \leq 0\right\} \\
&= \left\{(t, \xi) \in \mathbb{R} \times \widetilde{\Delta} : \alpha c_2(\xi) + t \leq 0\right\}.
\end{aligned}$$

The latter set is closed and convex since the functions $(t, \xi) \mapsto t$ and $(t, \xi) \mapsto \alpha c_2(\xi)$ are convex and lower semicontinuous. For $\alpha \leq 0$, $\left\{(t, \xi) \in \mathbb{R} \times \widetilde{\Delta} : \frac{t^-}{c_2(\xi)} \geq \alpha\right\} = \mathbb{R} \times \widetilde{\Delta}$. Therefore, $(t, \xi) \mapsto \frac{t^-}{c_2(\xi)}$ is upper semicontinuous and quasiconcave. It follows that G restricted to $\mathbb{R} \times \widetilde{\Delta}$ is lower semicontinuous, being G the sum of two lower semicontinuous functions on $\mathbb{R} \times \widetilde{\Delta}$.

We now prove quasiconvexity of G. Define as C the domain of the mapping $(t, \xi) \mapsto t^+/c_1(\xi)$. Since such mapping is quasiconvex, C is convex. Given (38) and (40), we have that $\operatorname{dom} G = C$ and therefore it is sufficient to prove that G is quasiconvex on C. Define as f_1 the mapping $(t, \xi) \mapsto t^+/c_1(\xi)$ and f_2 the mapping $(t, \xi) \mapsto -t^-/c_2(\xi)$. By Lemma 25 and (40), it follows that G is quasiconvex on C.

(iv) For each $\xi \in \Delta$ it is clear by (38) that $G(\cdot, \xi)$ is positive homogeneous; while $G(1, \xi) = \infty$ or $G(1, \xi) = 1/c_1(\xi) > 0$ since $c_1(\xi) \in [0, \infty)$ for each $\xi \in \widetilde{\Delta}$.

(i), (ii), (iii), and (iv) prove the first part of the statement.

For the second part of the statement, by the previous part of the proof, if $G \in \mathcal{E}_{ms}(\mathbb{R} \times \Delta)$ we can conclude that $G \in \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ and that $G(\cdot, \xi)$ is positively homogeneous and such that $G(1, \xi) \neq 0$ for each $\xi \in \Delta$.

By assumption $c_1(\xi), c_2(\xi) > 0$ for each $\xi \in \widetilde{\Delta}$, therefore we have that $G(\cdot, \xi) = \infty$ or $G(\cdot, \xi)$ is such that

$$G(t, \xi) = \begin{cases} \frac{t}{c_1(\xi)} & \text{if } t \geq 0 \\ \frac{t}{c_2(\xi)} & \text{if } t \leq 0 \end{cases},$$

where $1/c_1(\xi), 1/c_2(\xi) \in [0, \infty)$. This implies that $\operatorname{dom} G(\cdot, \xi) \in \{\emptyset, \mathbb{R}\}$. Since $\widetilde{\Delta}$ is nonempty, there exists $\bar{\xi}$ such that $1/c_1(\bar{\xi}), 1/c_2(\bar{\xi}) \in [0, \infty)$. Hence, $\operatorname{dom} G(\cdot, \bar{\xi}) = \mathbb{R}$ and we can conclude that G satisfies A.6.

Finally, since $c_2 : \widetilde{\Delta} \to (0, \infty]$ is lower semicontinuous and $\widetilde{\Delta} \subset \Delta$ is nonempty and closed we have that $\min_{\xi \in \widetilde{\Delta}} c_2(\xi) > 0$. This, with the assumption that $\inf_{\xi \in \widetilde{\Delta}} c_1(\xi) > 0$ and Lemma 24, implies that the nonempty family of functions $\{G(\cdot, \xi)\}_{\xi \in \Delta : \operatorname{dom} G(\cdot, \xi) = \mathbb{R}}$ is a family of uniformly equicontinuous functions, showing that G satisfies A.7. ∎

Lemma 27 *A function $g : X \to [-\infty, \infty]$ is non-degenerate, with $g > -\infty$, quasiconcave, monotone, positively homogeneous, and lower semicontinuous if and only if there exists a $G \in \mathcal{L}_{ms}(\mathbb{R} \times \Delta)$ such that*

$$g(x) = \min_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi), \qquad \forall x \in X. \tag{41}$$

Moreover, g is real valued and uniformly continuous if and only if $G \in \mathcal{E}_{ms}(\mathbb{R} \times \Delta)$.

Proof. "Only if." Define as G the mapping such that $(t, \xi) \mapsto G_\xi(t)$. Since $g > -\infty$, G takes values in $(-\infty, \infty]$. Since $g \in \mathcal{M}_{lsc}(X)$, by Theorem 3, $G \in \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ and satisfies (41). Then, observe that $G(\cdot, \xi)$ is positively homogeneous for each $\xi \in \Delta$. In fact, for all $t \in \mathbb{R}$ and $\lambda > 0$

$$G_\xi(\lambda t) = \sup\{g(x) : \langle \xi, x \rangle \leq \lambda t\} = \sup\{g(\lambda y) : \langle \xi, y \rangle \leq t\} = \sup\{\lambda g(y) : \langle \xi, y \rangle \leq t\} = \lambda G_\xi(t).$$

Since $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is such that $t \mapsto G(t, \xi)$ is positively homogeneous, we have that if $G(\bar{t}, \xi) = \infty$ for some $\bar{t} > 0$, then $G(t, \xi) = \infty$ for each $t > 0$. Likewise, if $G(\bar{t}, \xi) < \infty$ for some $\bar{t} > 0$, then $G(t, \xi) < \infty$ for each $t > 0$. Similarly, if $G(\bar{t}, \xi) = \infty$ for some $\bar{t} < 0$, then $G(t, \xi) = \infty$ for each $t < 0$ and, since $G(\cdot, \xi)$ is nondecreasing, $G(\cdot, \xi) = \infty$. Likewise, if $G(\bar{t}, \xi) < \infty$ for some $\bar{t} < 0$, then $G(t, \xi) < \infty$ for each $t < 0$. In this case, for each $t < 0$ we have that $G(t, \xi) = -tG(-1, \xi)$. Since $G(\cdot, \xi)$ is nondecreasing in t, then $G(t, \xi) \leq 0$ for each $t < 0$. Furthermore, since $G(\cdot, \xi)$ is nondecreasing and lower semicontinuous in t,

$$0 = \lim_{t \to 0^-} G(t, \xi) = \lim_{t \to 0^-} \inf G(t, \xi) \geq G(0, \xi).$$

Monotonicity of $G(\cdot, \xi)$ implies the other inequality, granting $0 = G(0, \xi)$. By the previous discussion, we can conclude that $\operatorname{dom} G(\cdot, \xi) \in \{\emptyset, (-\infty, 0], \mathbb{R}\}$.

Define $\widetilde{\Delta} = \{\xi \in \Delta : \operatorname{dom} G(\cdot, \xi) \in \{(-\infty, 0], \mathbb{R}\}\}$. If $\widetilde{\Delta}$ is empty, (38) is trivially satisfied and $G = \infty$ belongs to $\mathcal{L}_{ms}(\mathbb{R} \times \Delta)$. Otherwise, define $\rho_1 : \widetilde{\Delta} \to [0, \infty]$ such that $\rho_1(\xi) = G_\xi(1)$ and $\rho_2 : \widetilde{\Delta} \to [0, \infty)$ such that $\rho_2(\xi) = -G_\xi(-1)$. It follows that

$$G(t, \xi) = \begin{cases} \rho_1(\xi) t & \text{if } t \geq 0 \\ \rho_2(\xi) t & \text{if } t \leq 0 \end{cases} \tag{42}$$

for $\xi \in \widetilde{\Delta}$. At the same time, we have that $G(t,\xi) = \infty$ if $\xi \in \Delta \setminus \widetilde{\Delta}$. Note that $\rho_1(\xi) \in (0,\infty]$, otherwise $G\left(t,\overline{\xi}\right) = 0$ for some $\overline{\xi} \in \widetilde{\Delta}$ and for each $t \geq 0$. Since G satisfies (41), we would have that for each $x \in X^+$ there would exist $\xi_x \in \Delta$ such that $-\infty < g(x) = G(\langle \xi_x, x\rangle, \xi_x) \leq G\left(\left\langle \overline{\xi}, x\right\rangle, \overline{\xi}\right) = 0$. Therefore, $\xi_x \in \widetilde{\Delta}$. Since $\langle \xi_x, x\rangle \geq 0$, we could then conclude that for each $x \in X^+$

$$0 = G(0,\xi_x) \leq G(\langle \xi_x, x\rangle, \xi_x) = g(x) \leq G\left(\left\langle \overline{\xi}, x\right\rangle, \overline{\xi}\right) = 0.$$

Thus g would be degenerate, a contradiction. Furthermore, notice that if $x \notin \widetilde{\Delta}^{\oplus}$, then $g(x) \leq 0$. If $x \notin \widetilde{\Delta}^{\oplus}$ then $\left\langle \overline{\xi}, x\right\rangle < 0$ for some $\overline{\xi} \in \widetilde{\Delta}$, by (41) and (42), it follows that

$$g(x) = \min_{\xi \in \Delta} G(\langle \xi, x\rangle, \xi) \leq G\left(\left\langle \overline{\xi}, x\right\rangle, \overline{\xi}\right) \leq G\left(0, \overline{\xi}\right) = 0.$$

This last observation allows us to conclude that $G_\xi(1) = \sup\left\{g(x) : \langle \xi, x\rangle \leq 1 \text{ and } x \in \widetilde{\Delta}^{\oplus}\right\}$ for each $\xi \in X^*_+ \setminus \{0\}$. Clearly, $G_\xi(1) \geq \sup\left\{g(x) : \langle \xi, x\rangle \leq 1 \text{ and } x \in \widetilde{\Delta}^{\oplus}\right\}$. Vice versa, consider a sequence $\{x_n\}_n$ such that $\langle \xi, x_n\rangle \leq 1$ for each $n \in \mathbb{N}$ and $g(x_n) \uparrow G_\xi(1)$. By contradiction, assume that $G_\xi(1) > \sup\left\{g(x) : \langle \xi, x\rangle \leq 1 \text{ and } x \in \widetilde{\Delta}^{\oplus}\right\}$, then $\{x_n\}_n$ is eventually in the complement of $\widetilde{\Delta}^{\oplus}$. It follows that for n large enough $g(x_n) \leq 0$ and hence, $G_\xi(1) \leq 0$. But $\|\xi\|^{-1} e \in \widetilde{\Delta}^{\oplus}$ and $\left\langle \xi, \|\xi\|^{-1} e\right\rangle = 1$, since $\left\langle \widetilde{\xi}, \|\xi\|^{-1} e\right\rangle = \|\xi\|^{-1}\left\langle \widetilde{\xi}, e\right\rangle = \|\xi\|^{-1} > 0$ for each $\widetilde{\xi} \in \widetilde{\Delta}$. Hence, given that g is non-degenerate

$$G_\xi(1) \geq \sup\left\{g(x) : \langle \xi, x\rangle \leq 1 \text{ and } x \in \widetilde{\Delta}^{\oplus}\right\} \geq g\left(\|\xi\|^{-1} e\right) = \|\xi\|^{-1} g(e) > 0,$$

a contradiction.

Step 1. $\widetilde{\Delta}$ is convex and closed. Clearly, $\widetilde{\Delta} = \{\xi \in \Delta : G(-1,\xi) \leq 0\}$. Since $G \in \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$, it follows that $\widetilde{\Delta}$ is convex and closed.

Step 2. The function $c_1 : \widetilde{\Delta} \to [0,\infty)$ such that $c_1(\xi) = 1/\rho_1(\xi)$ is a concave function over $\widetilde{\Delta}$. Recall that $\rho_1(\xi) > 0$. Let $\xi_1, \xi_2 \in \widetilde{\Delta}$ and $\lambda \in (0,1)$. If $\rho_1(\xi_1) = \infty$ or $\rho_2(\xi_2) = \infty$, wlog suppose that $\rho_1(\xi_1) = \infty$ then for each $\lambda \in (0,1)$

$$\begin{aligned}
\rho_1(\xi_2) = G_{\xi_2}(1) &= \sup\left\{g(x) : \langle \xi_2, x\rangle \leq 1 \text{ and } x \in \widetilde{\Delta}^{\oplus}\right\} \\
&\geq \sup\left\{g(x) : \left\langle \frac{\lambda}{1-\lambda}\xi_1 + \xi_2, x\right\rangle \leq 1 \text{ and } x \in \widetilde{\Delta}^{\oplus}\right\} = G_{\frac{\lambda}{1-\lambda}\xi_1 + \xi_2}(1) \\
&= G_{\lambda\xi_1 + (1-\lambda)\xi_2}(1-\lambda) = (1-\lambda) G_{\lambda\xi_1 + (1-\lambda)\xi_2}(1) = (1-\lambda)\rho_1(\lambda\xi_1 + (1-\lambda)\xi_2).
\end{aligned}$$

This implies that

$$\frac{1}{\rho_1(\lambda\xi_1 + (1-\lambda)\xi_2)} \geq (1-\lambda)\frac{1}{\rho_1(\xi_2)} \geq 0 + (1-\lambda)\frac{1}{\rho_1(\xi_2)} = \lambda\frac{1}{\rho_1(\xi_1)} + (1-\lambda)\frac{1}{\rho_1(\xi_2)}$$

Otherwise $\rho_1(\xi_1), \rho_1(\xi_2) \in (0,\infty)$. In this case, choose $k_1, k_2 > 0$ such that $k_1\rho_1(\xi_1) = k_2\rho_1(\xi_2)$. As $G_\xi(t)$ is quasiconvex,

$$G_{\lambda\xi_1 + (1-\lambda)\xi_2}(\lambda k_1 + (1-\lambda)k_2) \leq \max\{G_{\xi_1}(k_1), G_{\xi_2}(k_2)\}. \tag{43}$$

In view of (42), (43) becomes

$$(\lambda k_1 + (1-\lambda)k_2)\rho_1(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \max\{k_1\rho_1(\xi_1), k_2\rho(\xi_2)\} = k_1\rho(\xi_1),$$

since $k_2 \backslash (k_1 \rho_1 (\xi_1)) = 1 \backslash \rho_1 (\xi_2)$, we have that

$$\frac{1}{\rho_1 (\lambda \xi_1 + (1-\lambda) \xi_2)} \geq \frac{\lambda k_1 + (1-\lambda) k_2}{k_1 \rho_1 (\xi_1)} = \lambda \frac{1}{\rho_1 (\xi_1)} + (1-\lambda) \frac{1}{\rho_1 (\xi_2)}.$$

This shows that $c_1 (\xi) = 1 \backslash \rho_1 (\xi)$ is concave. Consequently, in (42) we can write $\rho_1 (\xi) t = t / c_1 (\xi)$, where c_1 is concave on $\widetilde{\Delta}$.

Step 3. The region $A = \left\{ \xi \in \widetilde{\Delta} : \rho_2 (\xi) > 0 \right\}$ is convex. In fact, for all $\xi \in \widetilde{\Delta}$, $\rho_2 (\xi) = -G_\xi (-1)$ and thus

$$A = \left\{ \xi \in \widetilde{\Delta} : G_\xi (-1) < 0 \right\}$$

is convex by quasiconvexity of $G_\xi (t)$.

Step 4. The function $c_2 : A \to (0, \infty)$ defined by $c_2 (\xi) = 1/\rho_2 (\xi)$ is convex on the set A defined above. Let $\xi_1, \xi_2 \in A \subset \widetilde{\Delta}$ and $\lambda \in (0,1)$. Pick $k_1, k_2 < 0$ such that $k_1 \rho_2 (\xi_1) = k_2 \rho_2 (\xi_2)$. From the quasiconvexity of $G_\xi (t)$ we have (43). Hence, in view of (42)

$$(\lambda k_1 + (1-\lambda) k_2) \rho_2 (\lambda \xi_1 + (1-\lambda) \xi_2) \leq \max \{ k_1 \rho_2 (\xi_1), k_2 \rho_2 (\xi_2) \} = k_1 \rho_2 (\xi_1),$$

which implies

$$\frac{1}{\rho_2 (\lambda \xi_1 + (1-\lambda) \xi_2)} \leq \frac{\lambda k_1 + (1-\lambda) k_2}{k_1 \rho_2 (\xi_1)} = \lambda \frac{1}{\rho_2 (\xi_1)} + (1-\lambda) \frac{1}{\rho_2 (\xi_2)},$$

and $c_2 (\xi) = 1/\rho_2 (\xi)$ is convex and finite on A. Clearly $\rho_2 (\xi) = 1/c_2 (\xi)$ on A. Setting $c_2 (\xi) = \infty$ for $\xi \in \widetilde{\Delta} \setminus A$, convexity of $c_2 : \widetilde{\Delta} \to (0, \infty]$ is maintained and $\rho_2 (\xi) = 1/c_2 (\xi)$ for all $\xi \in \widetilde{\Delta}$.

Hence, G has the representation (38) with $\widetilde{\Delta}$ convex and closed, $c_1 : \widetilde{\Delta} \to [0, \infty)$ concave, and $c_2 : \widetilde{\Delta} \to (0, \infty]$ convex.

Step 5. c_1 is upper semicontinuous on $\widetilde{\Delta}$. The function $\xi \mapsto G_\xi (1)$ is lower semicontinuous on Δ. For any $\alpha > 0$, then $\{ \xi \in \Delta : G_\xi (1) \leq \alpha^{-1} \}$ is closed in Δ. That is, the sets $\left\{ \xi \in \widetilde{\Delta} : c_1 (\xi) \geq \alpha \right\} = \{ \xi \in \Delta : G_\xi (1) \leq \alpha^{-1} \} \cap \widetilde{\Delta}$ are closed in $\widetilde{\Delta}$. Finally, for any $\alpha \leq 0$, $\left\{ \xi \in \widetilde{\Delta} : c_1 (\xi) \geq \alpha \right\} = \widetilde{\Delta}$. Therefore, c_1 is upper semicontinuous.

Step 6. $c_2 : \widetilde{\Delta} \to (0, \infty]$ is lower semicontinuous. The map $\xi \mapsto G_\xi (-1)$ is lower semicontinuous. For any $\alpha > 0$, $\{ \xi \in \Delta : G_\xi (-1) \leq -\alpha^{-1} \}$ is closed. Consequently, the sets $\left\{ \xi \in \widetilde{\Delta} : c_2 (\xi) \leq \alpha \right\} = \{ \xi \in \Delta : G_\xi (-1) \leq -\alpha^{-1} \} \cap \widetilde{\Delta}$ are closed. If $\alpha \leq 0$ then $\left\{ \xi \in \widetilde{\Delta} : c_2 (\xi) \leq \alpha \right\} = \emptyset$. Therefore, c_2 is lower semicontinuous.

"If." If $G \in \mathcal{L}_{ms} (\mathbb{R} \times \Delta)$ then, by Lemma 26, $G \in \mathcal{L}_{qcx} (\mathbb{R} \times \Delta)$. By Lemma 12, it follows that g, defined as in (41), is monotone, quasiconcave, and lower semicontinuous. Since $G (\mathbb{R} \times \Delta) \subset (-\infty, \infty]$, we have that $g > -\infty$. By Lemma 26, we have that $G (\cdot, \xi)$ is positively homogeneous and nondecreasing for each $\xi \in \Delta$ and furthermore $G (1, \xi) \neq 0$ for each $\xi \in \Delta$. It follows that $G (1, \xi) > 0$ for each $\xi \in \Delta$. This implies that g is non-degenerate. Indeed,

$$g (e) = \min_{\xi \in \Delta} G (\langle \xi, e \rangle, \xi) = \min_{\xi \in \Delta} G (1, \xi) > 0.$$

Finally, g is positively homogeneous, since for each $x \in X$ and for each $\lambda > 0$ we have that

$$g (\lambda x) = \min_{\xi \in \Delta} G (\langle \xi, \lambda x \rangle, \xi) = \min_{\xi \in \Delta} G (\lambda \langle \xi, x \rangle, \xi) = \lambda \min_{\xi \in \Delta} G (\langle \xi, x \rangle, \xi) = \lambda g (x).$$

Finally, given $g \in \mathcal{M}_{upo}(X)$ define as G the mapping $(t,\xi) \mapsto G_\xi(t)$. By the previous part of the proof $G \in \mathcal{L}_{ms}(\mathbb{R}\times\Delta)$ and by Theorem 5, $\widetilde{\Delta}$ is nonempty and it is equal to the set $\{\xi \in \Delta : dom\,(G(\cdot,\xi)) = \mathbb{R}\}$. By Theorem 5, the nonempty family of functions $\{G(\cdot,\xi)\}_{\xi\in\Delta:\mathrm{dom}G(\cdot,\xi)=\mathbb{R}}$ is a family of uniformly equicontinuous functions. Each function in such family is such that

$$G(t,\xi) = \begin{cases} \frac{t}{c_1(\xi)} & \text{if } t \geq 0 \\ \frac{t}{c_2(\xi)} & \text{if } t \leq 0 \end{cases},$$

where $c_1(\xi), c_2(\xi) \in (0,\infty)$ and $\xi \in \widetilde{\Delta}$. By Lemma 24, it follows that $\sup_{\xi\in\widetilde{\Delta}} \frac{1}{c_1(\xi)}, \sup_{\xi\in\widetilde{\Delta}} \frac{1}{c_2(\xi)} < \infty$. This implies that $\inf_{\xi\in\widetilde{\Delta}} c_1(\xi) > 0$, proving that $G \in \mathcal{E}_{ms}(\mathbb{R}\times\Delta)$.

Vice versa, if $G \in \mathcal{E}_{ms}(\mathbb{R}\times\Delta)$ then $G \in \mathcal{L}_{ms}(\mathbb{R}\times\Delta)$ and we have from the previous part of the proof that g, defined as in (41), is non-degenerate, $g > -\infty$, monotone, quasiconcave, and positively homogeneous. By Lemma 26, $G \in \mathcal{E}(\mathbb{R}\times\Delta)$ and, by Lemma 16, it follows that g is uniformly continuous and real valued. ■

Proof of Theorem 10. Consider $\mathbf{T} : \mathcal{M}_{po}(X) \to \mathcal{M}(\mathbb{R}\times\Delta)$. By the proof of Lemma 27, we have that $\mathbf{T}(\mathcal{M}_{po}(X)) \subset \mathcal{L}_{ms}(\mathbb{R}\times\Delta) \subset \mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$. Hence, $\mathbf{T} : \mathcal{M}_{po}(X) \to \mathcal{L}_{ms}(\mathbb{R}\times\Delta)$. By Proposition 1, $\mathbf{T}$ is injective on $\mathcal{M}_{po}(X) \subset \mathcal{M}_{eqc}(X)$. Let $G \in \mathcal{L}_{ms}(\mathbb{R}\times\Delta) \subset \mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$. By Lemma 27, $\mathbf{Q}G \in \mathcal{M}_{po}(X) \subset \mathcal{M}_{lsc}(X)$. By Lemma 13, $\mathbf{TQ}G = G$ therefore $\mathbf{T}$ is surjective. That is, $\mathbf{T}^{-1} = \mathbf{Q}$ on $\mathcal{L}_{ms}(\mathbb{R}\times\Delta)$. ■

Proof of Corollary 7. Clearly, $\mathcal{M}_{upo}(X) = \mathcal{M}_{uc}(X) \cap \mathcal{M}_{po}(X) \subseteq \mathcal{M}_{eqc}(X)$, since $\mathbf{T} : \mathcal{M}_{eqc}(X) \to \mathcal{M}_{sqc}(\mathbb{R}\times\Delta)$ is bijective, then by Theorems 6 and 10,

$$\mathbf{T}(\mathcal{M}_{upo}(X)) = \mathbf{T}(\mathcal{M}_{uc}(X)) \cap \mathbf{T}(\mathcal{M}_{po}(X)) = \mathcal{E}(\mathbb{R}\times\Delta) \cap \mathcal{L}_{ms}(\mathbb{R}\times\Delta).$$

Finally, $\mathcal{E}(\mathbb{R}\times\Delta) \cap \mathcal{L}_{ms}(\mathbb{R}\times\Delta) = \mathcal{E}_{ms}(\mathbb{R}\times\Delta)$ follows from Lemmas 24 and 26. ■

Remark. In light of Theorem 10 and of the proof of Lemma 27, it is immediate to see how Lemma 26 turns out to be an "if and only if" statement.

7.1 Representation

The next result, based on Theorem 10, is a representation theorem for positively homogeneous functions in terms of a pair (c_1, c_2) of functions, with c_1 concave and c_2 convex.

Proposition 6 *Let $g : X \to [-\infty,\infty]$. A function g belongs to $\mathcal{M}_{po}(X)$ if and only if*

$$g(x) = \min_{\xi\in\widetilde{\Delta}} \left(\frac{\langle \xi, x\rangle^+}{c_1(\xi)} - \frac{\langle \xi, x\rangle^-}{c_2(\xi)} \right), \tag{44}$$

with $c_1 : \widetilde{\Delta} \to [0,\infty)$ upper semicontinuous and concave, and $c_2 : \widetilde{\Delta} \to (0,\infty]$ lower semicontinuous and convex, $\widetilde{\Delta} \subset \Delta$ convex and closed. Moreover,

(i) g is non negative and concave on $\widetilde{\Delta}^{\oplus} = \left\{x \in X : \langle \xi, x\rangle \geq 0 \text{ for all } \xi \in \widetilde{\Delta}\right\}$;

(ii) if $g : X \to \mathbb{R}$, g is concave if and only if $c_2 \leq c_1$. In this case, g is uniformly continuous;

(iii) if $\widetilde{\Gamma}$ is a closed and convex subset of Δ, $d_1 : \widetilde{\Gamma} \to [0,\infty)$ is concave and upper semicontinuous, $d_2 : \widetilde{\Gamma} \to (0,\infty]$ is convex and lower semicontinuous, and

$$g(x) = \min_{\xi\in\widetilde{\Gamma}} \left(\frac{\langle \xi, x\rangle^+}{d_1(\xi)} - \frac{\langle \xi, x\rangle^-}{d_2(\xi)} \right), \qquad \forall x \in X, \tag{45}$$

then $\left(\widetilde{\Gamma}, d_1, d_2\right) = \left(\widetilde{\Delta}, c_1, c_2\right)$;

(iv) g is real valued and uniformly continuous if and only if $\widetilde{\Delta}$ is nonempty and $\inf_{\xi\in\widetilde{\Delta}} c_1(\xi) > 0$;

(v) g is normalized if and only if $\max_{\xi\in\widetilde{\Delta}} c_1(\xi) = \min_{\xi\in\widetilde{\Delta}} c_2(\xi) = 1$.

Proof. Necessity easily follows from Lemma 27 and (40). Sufficiency follows easily from Lemma 27, once we notice that the function $G : \mathbb{R}\times\Delta \to (-\infty,\infty]$ such that

$$G(t,\xi) = \begin{cases} \frac{t^+}{c_1(\xi)} - \frac{t^-}{c_2(\xi)} & \text{if } (t,\xi)\in\mathbb{R}\times\widetilde{\Delta} \\ \infty & \text{if } (t,\xi)\in\mathbb{R}\times\left(\Delta\setminus\widetilde{\Delta}\right) \end{cases} = \begin{cases} \frac{t}{c_1(\xi)} & \text{if } t\geq 0 \text{ and } \xi\in\widetilde{\Delta} \\ \frac{t}{c_2(\xi)} & \text{if } t\leq 0 \text{ and } \xi\in\widetilde{\Delta} \\ \infty & \text{if } \xi\in\Delta\setminus\widetilde{\Delta} \end{cases}, \tag{46}$$

is such that $g(x) = \min_{\xi\in\Delta} G(\langle\xi,x\rangle,\xi)$ and such that $G\in\mathcal{L}_{ms}(\mathbb{R}\times\Delta)$, by definition.

(i) If $\widetilde{\Delta}$ is empty then $\widetilde{\Delta}^{\oplus} = X$ and $g=\infty$. It follows that the statement is trivially true. If $\widetilde{\Delta}$ is nonempty then $\langle\xi,x\rangle\geq 0$ for each $x\in\widetilde{\Delta}^{\oplus}$ and for each $\xi\in\widetilde{\Delta}$. Given (44), it follows that

$$g(x) = \min_{\xi\in\widetilde{\Delta}}\left(\frac{\langle\xi,x\rangle^+}{c_1(\xi)} - \frac{\langle\xi,x\rangle^-}{c_2(\xi)}\right) = \min_{\xi\in\widetilde{\Delta}}\left(\frac{1}{c_1(\xi)}\langle\xi,x\rangle\right)$$

which clearly implies that g is concave and non-negative on the closed convex cone $\widetilde{\Delta}^{\oplus}$.

(ii) By Corollary 5, g is concave if and only if G_ξ is concave for each $\xi\in\Delta$. This is automatically true, if $\xi\in\Delta\setminus\widetilde{\Delta}$, while if $\xi\in\widetilde{\Delta}$ this amounts to say that the function

$$G(t,\xi) = \begin{cases} \frac{t}{c_1(\xi)} & \text{if } t\geq 0 \\ \frac{t}{c_2(\xi)} & \text{if } t\leq 0 \end{cases}$$

is concave, or equivalently

$$0 \leq \frac{1}{c_1(\xi)} \leq \frac{1}{c_2(\xi)}.$$

Since $0 < c_2(\xi) < \infty$, this is equivalent to $c_1(\xi)\geq c_2(\xi)$. Since $g : X\to\mathbb{R}$ then $\widetilde{\Delta}$ is nonempty and therefore, it follows that $\inf_{\xi\in\widetilde{\Delta}} c_1(\xi) \geq \min_{\xi\in\widetilde{\Delta}} c_2(\xi) > 0$. This implies that G defined as in (46) satisfies (44) and belongs to $G\in\mathcal{E}_{ms}(\mathbb{R}\times\Delta)$, implying the statement.

(iii) Suppose $\widetilde{\Gamma}$ is closed, and convex subset of Δ, $d_1 : \widetilde{\Gamma}\to[0,\infty)$ is concave and upper semicontinuous, $d_2 : \widetilde{\Gamma}\to(0,\infty]$ is convex and lower semicontinuous, and

$$g(x) = \min_{\xi\in\widetilde{\Gamma}}\left(\frac{\langle\xi,x\rangle^+}{d_1(\xi)} - \frac{\langle\xi,x\rangle^-}{d_2(\xi)}\right), \qquad \forall x\in X.$$

Set G as in (46) and define $H : \mathbb{R}\times\Delta\to(-\infty,\infty]$ to be such that,

$$H(t,\xi) = \begin{cases} \frac{t}{d_1(\xi)} & \text{if } t\geq 0 \text{ and } \xi\in\widetilde{\Gamma} \\ \frac{t}{d_2(\xi)} & \text{if } t\leq 0 \text{ and } \xi\in\widetilde{\Gamma} \\ \infty & \text{if } \xi\in\Delta\setminus\widetilde{\Gamma} \end{cases} = \begin{cases} \frac{t^+}{d_1(\xi)} - \frac{t^-}{d_2(\xi)} & \text{if } (t,\xi)\in\mathbb{R}\times\widetilde{\Gamma} \\ \infty & \text{if } (t,\xi)\in\mathbb{R}\times\left(\Delta\setminus\widetilde{\Gamma}\right) \end{cases}.$$

It is easy to see that $G, H\in\mathcal{L}_{ms}(\mathbb{R}\times\Delta)$. By assumption, we have that $\mathbf{Q}G = g = \mathbf{Q}H$, by Theorem 10, it follows that $G = H$, and hence the statement.

(iv) Necessity follows easily from Lemma 27 and (40). Sufficiency follows easily from Lemma 27, once we notice that the function $G : \mathbb{R}\times\Delta\to(-\infty,\infty]$ such that

$$G(t,\xi) = \begin{cases} \frac{t^+}{c_1(\xi)} - \frac{t^-}{c_2(\xi)} & \text{if } (t,\xi)\in\mathbb{R}\times\widetilde{\Delta} \\ \infty & \text{if } (t,\xi)\in\mathbb{R}\times\left(\Delta\setminus\widetilde{\Delta}\right) \end{cases} = \begin{cases} \frac{t}{c_1(\xi)} & \text{if } t\geq 0 \text{ and } \xi\in\widetilde{\Delta} \\ \frac{t}{c_2(\xi)} & \text{if } t\leq 0 \text{ and } \xi\in\widetilde{\Delta} \\ \infty & \text{if } \xi\in\Delta\setminus\widetilde{\Delta} \end{cases},$$

is such that $g(x) = \min_{\xi \in \Delta} G(\langle \xi, x \rangle, \xi)$ and such that $G \in \mathcal{E}_{ms}(\mathbb{R} \times \Delta)$, by definition.

(v) Finally, by Lemma 20, g is normalized if and only if, for each $t \in \mathbb{R}$,

$$t = g(te) = \inf_{\xi \in \Delta} G_\xi(t) = \begin{cases} \inf_{\xi \in \widetilde{\Delta}} \frac{t}{c_1(\xi)} & \text{if } t \geq 0 \\ \inf_{\xi \in \widetilde{\Delta}} \frac{t}{c_2(\xi)} & \text{if } t \leq 0 \end{cases} = \begin{cases} t \inf_{\xi \in \widetilde{\Delta}} \frac{1}{c_1(\xi)} & \text{if } t \geq 0 \\ t \sup_{\xi \in \widetilde{\Delta}} \frac{1}{c_2(\xi)} & \text{if } t \leq 0 \end{cases},$$

which is equivalent to $\max_{\xi \in \widetilde{\Delta}} c_1(\xi) = \min_{\xi \in \widetilde{\Delta}} c_2(\xi) = 1$ thanks to the semicontinuity properties of c_1 and c_2. ■

In Proposition 6 we saw that lower semicontinuous, positively homogeneous, and quasiconcave functions are concave on the cone $\widetilde{\Delta}^{\oplus}$ on which they are non-negative. In fact, this is a quite general property enjoyed by these functions. A first result of this type was stated by [22] and then reformulated (still in a finite dimensional setting) by [6, Proposition 2], with a simpler proof. Next we give a general result.

Proposition 7 *Let $g : X \to [-\infty, \infty)$ be an evenly quasiconcave and positively homogeneous function. Then:*

(i) g is concave on the cone $\overline{\{g > 0\}}$,

(ii) g is concave on any evenly convex cone $K \subset \{g \leq 0\}$.

Proof. (i) Suppose that $\{g > 0\} \neq \emptyset$, otherwise the claim is trivial. Clearly the set $\{g > 0\}$ is a convex cone. Therefore, $\overline{\{g > 0\}}$ is a closed convex cone. Consider the new function

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{if } x \in \overline{\{g > 0\}} \\ -\infty & \text{else} \end{cases}.$$

It is evenly quasiconcave and positively homogeneous. The functions $\widetilde{G}_\xi$ are clearly positively homogeneous and monotone. Consequently they are concave on $(-\infty, 0)$ and on $[0, \infty)$. Let $\overline{c} < 0$ and $\xi \in S^*$. Consider the half-space $\langle \xi, x \rangle \leq 0$. If $\{\langle \xi, x \rangle \leq \overline{c}\} \cap \overline{\{g > 0\}} = \emptyset$, then $\widetilde{G}_\xi(\overline{c}) = -\infty$. As $\widetilde{G}_\xi(t)$ is positively homogeneous, it follows that $\widetilde{G}_\xi(t) = -\infty$ for all $t < 0$. Hence, $\widetilde{G}_\xi(t)$ is concave on $\mathbb{R}$. Assume that $\{\langle \xi, x \rangle \leq \overline{c}\} \cap \overline{\{g > 0\}} \neq \emptyset$. By perturbing $\overline{c}$, we have $\{\langle \xi, x \rangle \leq \widetilde{c}\} \cap \{g > 0\} \neq \emptyset$ for some $0 > \widetilde{c} > \overline{c}$. Hence, $\widetilde{G}_\xi(\widetilde{c}) > 0$. If $\widetilde{G}_\xi(\widetilde{c}) < \infty$, the function $\widetilde{G}_\xi(t)$ would be strictly decreasing on $(-\infty, 0)$, a contradiction. Therefore $\widetilde{G}_\xi(\widetilde{c}) = \infty \Longrightarrow \widetilde{G}_\xi = \infty$. We conclude that in any case $\widetilde{G}_\xi(t)$ are concave and, by Proposition 3, $\widetilde{g}(x)$ is concave.

(ii) Like (i), define

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{if } x \in K \\ -\infty & \text{else} \end{cases}.$$

Even in this case, $\widetilde{g}$ is evenly quasiconcave and positively homogeneous and the functions $\widetilde{G}_\xi$ are positively homogeneous and monotone. Here, $\widetilde{G}_\xi(t) \leq 0$ for all t and ξ. If $-\infty < \widetilde{G}_\xi(c) < 0$ for some $c > 0$, $\widetilde{G}_\xi$ would be decreasing on $(0, \infty)$. Hence, either $\widetilde{G}_\xi(c) = 0$ or $\widetilde{G}_\xi(c) = -\infty$. In both cases we deduce that the functions $\widetilde{G}_\xi$ are concave. ■

8 Glossary and Concluding Remarks

8.1 Glossary of Notation

Throughout the paper we considered several functions spaces, which for convenience we now list. Here X is an M-space and $\Delta = \{\xi \in X_+^* : \|\xi\| = 1\}$.

(i) $\mathcal{M}_{qc}(X)$ is the set of all quasiconcave monotone functions $g : X \to [-\infty, \infty]$.

(ii) $\mathcal{M}_{eqc}(X) \subseteq \mathcal{M}_{qc}(X)$ is the collection of functions in $\mathcal{M}_{qc}(X)$ that are evenly quasiconcave.

(iii) $\mathcal{M}_{lsc}(X) \subset \mathcal{M}_{eqc}(X)$ is the collection of functions in $\mathcal{M}_{eqc}(X)$ that are lower semicontinuous.

(iv) $\mathcal{M}_{c}(X) \subset \mathcal{M}_{lsc}(X)$ is the collection of functions in $\mathcal{M}_{lsc}(X)$ that are extended-valued continuous (i.e., both lower and upper semicontinuous).

(v) $\mathcal{M}_{uc}(X) \subset \mathcal{M}_{c}(X)$ is the subset of $\mathcal{M}_{c}(X)$ of the real valued functions $g : X \to \mathbb{R}$ that are uniformly continuous.

(vi) $\mathcal{M}_{conc}(X) \subset \mathcal{M}_{qc}(X)$ is the subset of $\mathcal{M}_{qc}(X)$ of the real valued functions $g : X \to \mathbb{R}$ that are upper semicontinuous and concave.

(vii) $\mathcal{M}_{tr}(X) \subset \mathcal{M}_{conc}(X) \cap \mathcal{M}_{uc}(X)$ is the the collection of quasiconcave, monotone, normalized, and translation invariant functions $g : X \to \mathbb{R}$.

(viii) $\mathcal{M}_{po}(X) \subset \mathcal{M}_{lsc}(X)$ is the subset of $\mathcal{M}_{lsc}(X)$ of the nondegenerate functions $g : X \to (-\infty, \infty]$ that are positively homogeneous.

(ix) $\mathcal{M}_{upo}(X) \subset \mathcal{M}_{uc}(X)$ is the subset of $\mathcal{M}_{uc}(X)$ of the real valued functions $g : X \to \mathbb{R}$ that are positively homogeneous.

The "dual" functions spaces consist of functions $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$. In particular:

(i) $\mathcal{M}(\mathbb{R} \times \Delta)$ is the space of the functions $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$ such that: (i) $G(\cdot, \xi)$ is increasing for each $\xi \in \Delta$; (ii) $\lim_{t \to +\infty} G(t, \xi) = \lim_{t \to +\infty} G(t, \xi')$ for all $\xi, \xi' \in \Delta$.

(ii) $\mathcal{M}^{\Diamond}_{qcx}(\mathbb{R} \times \Delta) \subseteq \mathcal{M}(\mathbb{R} \times \Delta)$ is the subset of $\mathcal{M}(\mathbb{R} \times \Delta)$ of the functions such that $(t, \xi) \mapsto G(t, \xi)$ is $\Diamond$-evenly quasiconvex on $\mathbb{R} \times \Delta$.

(iii) $\mathcal{L}_{qcx}(\mathbb{R} \times \Delta) \subset \mathcal{M}(\mathbb{R} \times \Delta)$ is the subset of $\mathcal{M}(\mathbb{R} \times \Delta)$ of the functions such that $(t, \xi) \mapsto G(t, \xi)$ is lower semicontinuous and quasiconvex on $\mathbb{R} \times \Delta$ (it follows that $\mathcal{L}_{qcx}(\mathbb{R} \times \Delta) \subseteq \mathcal{M}^{\Diamond}_{qcx}$).

(iv) $\mathcal{C}(\mathbb{R} \times \Delta) \subset \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ the subset of $\mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ of the functions such that $G(\cdot, \xi)$ is extended-valued continuous on $\mathbb{R}$ for each $\xi \in \Delta$.

(v) $\mathcal{CO}(\mathbb{R} \times \Delta) \subset \mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ the subset of $\mathcal{L}_{qcx}(\mathbb{R} \times \Delta)$ of the functions G such that $\mathbf{Q}G^{+} = \mathbf{Q}G$ (G^{+} is the right-continuous regularization of G, with respect to the first variable).

(vi) $\mathcal{E}(\mathbb{R} \times \Delta) \subset \mathcal{C}(\mathbb{R} \times \Delta)$ is the subset of $\mathcal{C}(\mathbb{R} \times \Delta)$ of the functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ that have the following additional properties: (i) $\mathrm{dom}G(\cdot, \xi) \in \{\emptyset, \mathbb{R}\}$ for all $\xi \in \Delta$, and there exists at least one $\bar{\xi}$ such that $\mathrm{dom}G\left(\cdot, \bar{\xi}\right) = \mathbb{R}$; (ii) $G(\cdot, \xi)$ are uniformly equicontinuous on $\mathbb{R}$ for all $\xi \in \Delta$ such that $\mathrm{dom}G(\cdot, \xi) = \mathbb{R}$.

(vii) $Co(\mathbb{R} \times \Delta) \subset \mathcal{M}(\mathbb{R} \times \Delta)$ is the collection of all functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ in $\mathcal{M}(\mathbb{R} \times \Delta)$ such that: (i) $G(\cdot, \xi)$ is concave and closed for all $\xi \in \Delta$; (ii) $\gamma : X^{*} \to [-\infty, \infty)$ is proper, upper semicontinuous, cofinite, and concave (γ is defined in (36)).

(viii) $\mathcal{E}_{as}(\mathbb{R} \times \Delta) \subset \mathcal{E}(\mathbb{R} \times \Delta)$ is the collection of functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ in $\mathcal{E}(\mathbb{R} \times \Delta)$ that are additively separable, i.e., such that $G(t, \xi) = t + c(\xi)$, where $c : \Delta \to [0, \infty]$ is lower semicontinuous, convex, and $\min_{\xi \in \Delta} c(\xi) = 0$.

(ix) $\mathcal{L}_{ms}(\mathbb{R}\times\Delta) \subset \mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ is the collection of functions $G:\mathbb{R}\times\Delta\to(-\infty,\infty]$ in $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$ that are multiplicatively separable.

(x) $\mathcal{E}_{ms}(\mathbb{R}\times\Delta) \subset \mathcal{E}(\mathbb{R}\times\Delta)$ is the collection of functions $G:\mathbb{R}\times\Delta\to(-\infty,\infty]$ in $\mathcal{E}(\mathbb{R}\times\Delta)$ that are multiplicatively separable.

8.2 Concluding Remarks

In the paper we have introduced a notion of quasiconcave monotone duality, Definition 1), and then identified eight monotone duality pairs:

(i) $\left\langle \mathcal{M}_{eqc}(X), \mathcal{M}^{\Diamond}_{qcx}(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Theorem 2

(ii) $\left\langle \mathcal{M}_{lsc}(X), \mathcal{L}_{qcx}(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Theorem 3;

(iii) $\left\langle \mathcal{M}_{c}(X), \mathcal{CO}(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Theorem 4;

(iv) $\left\langle \mathcal{M}_{uc}(X), \mathcal{E}(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Theorem 6;

(v) $\left\langle \mathcal{M}_{conc}(X), Co(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Theorem 8;

(vi) $\left\langle \mathcal{M}_{tr}(X), \mathcal{E}_{as}(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Theorem 9;

(vii) $\left\langle \mathcal{M}_{po}(X), \mathcal{L}_{ms}(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Theorem 10;

(viii) $\left\langle \mathcal{M}_{upo}(X), \mathcal{E}_{ms}(\mathbb{R}\times\Delta)\right\rangle_{qc}$, in Corollary 7.

Among them, (i) is the most basic dual pair. The dual pairs (ii)-(iv) are specifications of the basic dual pair (i) with richer and richer continuity properties. Since $\mathcal{M}_{tr}(X) \subset \mathcal{M}_{uc}(X)$ and $\mathcal{M}_{upo}(X) \subset \mathcal{M}_{uc}(X)$, the pairs (vi)-(viii) are further specifications of the basic duality (i). In particular, $\mathcal{M}_{tr}(X)$ and $\mathcal{M}_{po}(X)$ are in duality with, respectively, the additively separable and the multiplicatively separable functions in $\mathcal{E}(\mathbb{R}\times\Delta)$ or in $\mathcal{L}_{qcx}(\mathbb{R}\times\Delta)$.

Finally, the interest of (v) lies mostly in the connections with Fenchel duality that arise during its derivation.

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