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On the sub-optimality cost of immediate annuitization in DC pension funds*

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Abstract

We consider the position of a member of a defined contribution (DC) pension scheme having the possibility of taking programmed withdrawals at retirement. According to this option, she can defer annuitization of her fund to a propitious future time, that can be found to be optimal according to some criteria. This option, that adds remarkable flexibility in the choice of pension benefits, is not available in many countries, where immediate annuitization is compulsory at retirement. In this paper, we address and try to answer the questions: “Is immediate annuitization optimal? If it is not, what is the cost to be paid by the retiree obliged to annuitize at retirement?”. In order to do this, we consider the model by [7] and extend it in two different ways.

In the first extension, we prove a theorem that provides necessary and sufficient conditions for immediate annuitization being always optimal. The – not surprising – result is that compulsory immediate annuitization turns out to be sub-optimal. We then quantify the extent of sub-optimality, by defining the *sub-optimality cost* as the loss of expected present value of consumption from retirement to death and measuring it in many typical situations. We find that it varies in relative terms between 6% and 40%, depending on the risk aversion.

In the second extension, we make extensive numerical investigations of the model and seek the optimal annuitization time. We find that the optimal annuitization time depends on personal factors such as the retiree’s risk aversion and her subjective perception of remaining lifetime. It also depends on the financial market, via the Sharpe ratio of the risky asset. Optimal annuitization should occur a few years after retirement with high risk aversion, low Sharpe ratio and/or short remaining lifetime, and many years after retirement with low risk aversion, high Sharpe ratio and/or long remaining lifetime.

This paper supports the availability of programmed withdrawals as an option to retirees of DC pension schemes, by giving an idea about the extent of loss in wealth suffered by a retiree who cannot choose programmed withdrawals, but is obliged to annuitize immediately on retirement.

J.E.L. classification: D91, G11, G23, J26.

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1 Introduction

In defined contribution (DC) pension schemes, the financial risk is borne by the member: contributions are fixed in advance and the benefits provided by the scheme depend on the investment

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performance experienced during the active membership and on the price of the annuity at retirement, in the case that the benefits are given in the form of an annuity. Therefore, the financial risk can be split into two parts: investment risk, during the accumulation phase, and annuity risk, focused at retirement. In order to limit the annuity risk – which is the risk that high annuity prices (driven by low bond yields) at retirement can lead to a lower than expected pension income – in many schemes the member has the possibility of deferring the annuitization of the accumulated fund. This possibility consists of leaving the fund invested in financial assets as in the accumulation phase, and allows for periodic withdrawals by the pensioner, until annuitization occurs (if ever). In UK this option is named “income drawdown option”, in US the periodic withdrawals are called “phased withdrawals” or “programmed withdrawals”. There are a number of countries where programmed withdrawals are an option to retirees of DC schemes. These include Argentina, Australia, Brazil, Canada, Chile, Denmark, El Salvador, Japan, Peru, UK, US. To the best of our knowledge, there are even substantially more countries where this option is not available. These include Austria, Bulgaria, Colombia, Germany, Hong Kong, India, Ireland, Italy, Luxembourg, Netherlands, Poland, Portugal, South Africa, Sweden, Switzerland. An exhaustive recent survey regarding the list of several forms of payment offered by pension schemes can be found in [2].

A pensioner who takes the income drawdown option has three degrees of freedom:

1. she can decide what investment strategy to adopt in investing the fund at her disposal;
2. she can decide how much of the fund to withdraw at any time between retirement and ultimate annuitization (if any);
3. she can decide when to annuitize (if ever).

The first two choices represent a classical intertemporal decision making problem, which can be dealt with using optimal control techniques in the typical Merton framework ([10]), whereas the third choice can be tackled by defining an optimal stopping time problem.

Examples of works dealing with these three choices via the formulation of a combined stochastic control and optimal stopping problem are [7, 12, 14, 16].

Differently from the other mentioned papers, in [7] the authors solve the problem of optimal annuitization in the presence of quadratic, target-depending loss functions. By so doing, they extend [5] and [6], where the optimal couple investment-consumption is found in the presence of quadratic loss function. The main novelty in the literature is that the optimal annuitization time depends on the level of the wealth, that is linked to the stochastic evolution of the market returns.

In this paper, we extend [7] by addressing some of the issues left open.

From the theoretical point of view, we prove a theorem that provides necessary and sufficient conditions for immediate annuitization being always optimal. A first application of this theorem shows the not surprising result that compulsory immediate annuitization is sub-optimal within this model. We then quantify the extent of sub-optimality, by defining the *sub-optimality cost* as the loss of expected present value of consumption from retirement to death, and measuring it in many typical situations. We find that it varies in relative terms between 6% and 40%, depending on the risk aversion.

From the applicative point of view, we carry out a broad sensitivity analysis of the model. The main aim we have in mind is to provide a wide range of results regarding optimal annuitization time and consequent size of annuity in most common situations.

The remainder of the paper is organized as follows. Section 2 presents a review of the actuarial literature on the decumulation phase of DC pension funds. Section 3 presents briefly the model exploited. In section 4 we prove the theorem that states the necessary and sufficient conditions for optimality of immediate annuitization and we interpret these conditions. The sub-optimality of compulsory immediate annuitization comes as a consequence. In section 5 the practical implications of the theorem are illustrated: in section 5.1 we present four typical scenarios which will be used in the simulations and we check the condition of sub-optimality; in section 5.2 the cost of sub-optimality is defined and measured in those scenarios. Section 6 completes the analysis, by reporting the results of the broad sensitivity analysis of the model, i.e. optimal annuitization time and other related issues in the selected scenarios. Section 7 concludes.

2 Review of actuarial literature on decumulation phase

In the countries where the possibility of deferring ultimate annuitization is an option, the pensioner who takes this option can decide what investment strategy to adopt in investing the fund at her disposal, how much of the fund to withdraw at any time between retirement and ultimate annuitization, and when to annuitize. The evidence that the option is actually taken by many pensioners is in apparent contrast with [18] fundamental work, according to which retirees should annuitize immediately at retirement. By addressing the three choices outlined above, the actuarial literature tries also to extend Yaari’s Theorem and to give convincing explanations to the so-called “annuity puzzle”. Whereas the income drawdown option adds flexibility in the choices of the pensioner, and gives her the hope of being able to buy later on a better annuity than the one purchasable at retirement, the main drawback is that with self-annuitization the member bears the longevity risk, i.e. the risk of outliving her own assets. Therefore, another relevant issue that arises when the income drawdown option is chosen is the ruin probability, i.e. the probability that the pensioner runs out of money when she is still alive.

A number of authors have dealt with the problem of managing the financial resources of a pensioner after retirement and have investigated the alternatives available to a retiree other than immediate annuitization. All these papers approach some or all of the four important issues outlined above, namely, investment and consumption strategies, optimal annuitization time, ruin probability. Papers that address mainly the ruin probability are, e.g., [1] and [13]. Papers that explore different investment strategies and/or different consumption paths, possibly analyzing also the ruin issue are, e.g., [4, 5, 6, 9, 17]. Papers that add to their analysis also investigations on the optimal annuitization time are, e.g., [3, 8, 11, 12, 14, 16].

In this paper we focus on the optimal time of annuitization. It is then important to review briefly the other contributions to this relevant topic. In [8] and [11] the basic idea is that since the annuity price is calculated with the riskfree rate, in the first years after retirement the equity risk premium pays more than the mortality credits (due to annuitants who die earlier than average); therefore in the first years after retirement the individual should invest and consume, and should annuitize when the mortality credits become so large that it becomes worthwhile annuitize (“do-it-yourself-and-then-switch” strategy). According to their simulations, in UK the maximum annuitization age of 75 (which is even 10 years greater than NRA⁽¹⁾) is at least 5 years too low, and a Canadian female aged 65 has 90% probability of beating the interest rate return until the age of 80. In [3] the authors find that the optimal annuitization age is sensitive to the degree of risk aversion and varies from NRA (for very high risk aversion) to 79–80 (for very low risk aversion). [5] and [6] find

⁽¹⁾ *NRA stands for “normal retirement age”.*

that income drawdown option has to be preferred to immediate annuitization when risk aversion is not too high and the risky asset is sufficiently good compared to the riskfree one. In [14] the authors solve an optimization problem and find optimal investment/consumption strategies and optimal annuitization time using logarithmic utility function. They find that optimal annuitization age, which depends on the relative risk aversion and on the wealth status, is typically higher than 70. In [7] the authors solve a similar problem with quadratic loss functions. They find that optimal annuitization is mainly driven by risk profile of the retiree, level of the fund and market conditions, and in some typical situations should occur 6–7 years after retirement, but may occur also 10–15 years after it.

It is our opinion that the problem of finding optimal investment and consumption strategies and optimal annuitization time should be rigorously formulated as a combined stochastic control and optimal stopping problem. Up to our knowledge, in the literature this has been done only by [7, 12, 14, 16]. While in [12] the authors minimize the probability of financial ruin, in [16] the author finds optimal choices in a very general expected utility setting, distinguishing between utility pre-annuitization and utility post-annuitization and selecting as a special case the power utility function, and in [14] the authors maximize expected utility of lifetime consumption and bequest, with age-dependent force of mortality and power utility function. Differently from these papers, in [7] the authors solve the problem of optimal investment and consumption strategies as well as optimal annuitization time by selecting a quadratic loss function. We briefly present their model in the next section.

3 The model

A retiree has a lump sum of size x_0 which can be invested either in a riskless asset paying interest at fixed rate or in a risky asset, whose price evolves randomly following a log-normal process. We assume that the remaining lifetime of the pensioner is exponentially distributed with constant force of mortality. The retiree can choose the proportion of the fund to be invested in the risky asset and the withdrawal rate from the fund until the time of annuitization. She is also able to select the time of eventual annuitization. The size of the annuity purchasable with sum x is kx , where $k > r$. If the amount of money in the fund is ever exhausted, no further investment or withdrawal is permitted, that means that occurrence of ruin is prevented by the model's design.

We use this notation:

- $y(t)$ is the proportion of the fund invested in the risky asset at time t ;
- $b(t) dt$ is the income withdrawn from the fund between time t and time $t + dt$.
- T is the time of annuitization;
- T_0 is the time when the fund goes below 0;
- T_D is the retiree's time of death, as measured from the time when the lump sum is received;
- $x(t)$ is the size of the fund at time t (where $t < \min(T, T_D, T_0)$).

This model investigates the problem using $y(\cdot)$ and $b(\cdot)$ as control variables, and T as stopping time. The proportion invested in the fund, the income withdrawn, and the annuitization time are

chosen in such a way as to minimize the following quadratic cost criterion

$$J^{b,y,T}(x) = E_x \left[v \int_0^\tau e^{-(\rho+\delta)t} (b_0 - b(t))^2 dt + \frac{w e^{-(\rho+\delta)\tau}}{\rho + \delta} (b_1 - kx(\tau))^2 \right], \quad (1)$$

where:

- $E_x(\cdot) = E(\cdot | x(0) = x)$, i.e. the expectation value is conditioned to the current size of the fund;
- $\tau = \min(T, T_0)$;
- v and w are positive weights which determine the relative importance in the cost criterion of the payment before and after annuitization, respectively;
- ρ is a subjective discount factor;
- δ is the force of mortality which is assumed to be constant;
- b_0 is the income target before purchasing the annuity;
- b_1 is the amount that represents the targeted income after ultimate annuitization;
- k is the amount of annuity which can be purchased with one unit of money.

The intertemporal wealth equation is

$$dx(t) = [x(t)(y(t)(\lambda - r) + r) - b(t)] dt + \sigma x(t)y(t)dB(t),$$

where r is the instantaneous rate of return from the riskless investment and λ is the instantaneous rate of return from the risky investment with volatility σ . The standard Brownian motion $B(\cdot)$ is the source of uncertainty that represents market risk.

The amount b_0 , the income target until the annuity is purchased, will in many cases be equal to kx_0 , the size of the annuity which could have been purchased if the retiree had annuitized immediately on retirement. The process x evolves until either it is advantageous to annuitize or the fund falls to a negative value, in which case no further trading is permitted. The loss associated with annuitization when the level of the fund is $x \geq 0$, so that the annuity pays kx per unit time, is

$$K(x) = \frac{w}{\rho + \delta} (b_1 - kx)^2. \quad (2)$$

The ratio $\frac{b_1}{b_0}$ is a measure of risk propensity: the higher $\frac{b_1}{b_0}$, the higher the target, the lower the risk aversion and vice versa. Obviously the targeted annuity b_1 is always assumed greater than income withdrawn b_0 , hence $\frac{b_1}{b_0} > 1$.

Should the fund hit $\frac{b_1}{k}$, the pensioner would be able to buy a lifetime annuity with income rate b_1 and the penalty from that moment on would be null, which is what we expect to be.

Clearly, if the fund is equal to $\frac{b_1}{k}$ the optimal decision is to annuitize. However, the main novelty of the model is that optimal annuitization occurs also with a fund size x^* that is *lower* than $\frac{b_1}{k}$. The intuition behind that is the following. When one reaches a certain level close enough to the desired target, it is better to stop the self-annuitization strategy and accept the low penalty given by (2), rather than keeping on investing and facing the risk of departing even more from the desired level.

The region where it is optimal the self-annuitization strategy turns out to be

$$U = [0, x^*) \cup \left(\frac{b_1}{k}, +\infty \right) \quad (3)$$

that in the optimal stopping theory terminology is called “continuation region”. The name of the region U is intuitive: if the wealth x is in the continuation region, then the loss paid in case of annuitization is higher than that paid in case of continuation of the optimization program, so that it is optimal to keep playing the game; vice versa, if one’s wealth is outside that region, then the loss paid in case of annuitization is lower than that paid playing the game, thus the game is over and the pensioner annuitizes.

A complete solution of this optimal control model is given in [7]. For a complete analysis of the mathematical model we refer to the original paper.

4 Sub-optimality of immediate annuitization

In this section, we prove the main theoretical result of this paper. In section 4.1, we state and prove a theorem that gives necessary and sufficient conditions under which immediate annuitization is optimal for every initial wealth $x \in [0, b_1/k]$. In section 4.2 we illustrate the applicability of the theorem and analyze the impact of different key parameters on the criterion provided by the theorem.

4.1 The criterion

Before stating the criterion, we notice that the region of interest for the initial wealth is $\left[0, \frac{b_1}{k}\right]$. Namely, for $x_0 < 0$ the problem is not defined; for $x_0 > \frac{b_1}{k}$ the optimal policy is to keep playing the game with consumption rate $b = b_0$ up until the fund falls to $\frac{b_1}{k}$.⁽²⁾ What happens for the much more interesting case $x_0 \in \left[0, \frac{b_1}{k}\right]$ depends on the form of the continuation region (3) via the value of the model’s parameters.

Theorem 1 *In the model outlined in section 3, it is optimal to annuitize immediately at retirement for any initial wealth $x \in \left[0, \frac{b_1}{k}\right]$ if and only if*

$$\phi \leq 2rD \frac{k}{b_1}, \quad (4)$$

where $D = \frac{b_0}{r} - \frac{b_1}{k} > 0$, $\phi = \rho + \delta + \beta^2 - 2r + k^2 \frac{w}{v(\rho+\delta)}$, and $\beta = \frac{\lambda-r}{\delta}$ is the Sharpe ratio of the risky asset.

Proof

(\Leftarrow) Denoting by T^* the optimal annuitization time, by application of the Corollary 2.4 of [7], it is $T^* = 0$ if $LK(x) \geq 0$ for all $x \in \left[0, \frac{b_1}{k}\right]$, where $K(x)$ is given by (2) and

$$\begin{aligned} LK(x) &= \inf_{b,y} \left\{ v(b_0 - b)^2 - (\rho + \delta)K + [-b + (\lambda + r)yx + rx] K' + \frac{1}{2} \sigma^2 y^2 x^2 K'' \right\} = \\ &= w(b_1 - kx)[2krD - \phi(b_1 - kx)]. \end{aligned}$$

⁽²⁾Indeed, with this simple strategy, the value function is identically null.

Let $\phi \leq 2rD\frac{k}{b_1}$. Then,

$$LK(x) \geq \frac{2wk^2rD}{b_1}(b_1 - kx)x \geq 0 \quad \text{for } 0 \leq x \leq \frac{b_1}{k}.$$

(\Rightarrow) If $T^* = 0$ for all $x \in \left[0, \frac{b_1}{k}\right]$, then $V(x) = K(x)$, where V is the value function associated to the quadratic cost criterion (1) and K is given by (2). Therefore, $LK(x) \geq 0$ for all $x \in \left[0, \frac{b_1}{k}\right]$. This, in turn implies that given the set U_0 defined as

$$U_0 = \{x \in \mathbf{R} : LK(x) < 0\},$$

we have

$$U_0 \cap \left[0, \frac{b_1}{k}\right] = \emptyset.$$

On the other hand, if $\phi > 2rD\frac{k}{b_1}$, then

$$U_0 \cap \left[0, \frac{b_1}{k}\right] = \left[0, \frac{b_1}{k} - \frac{2rD}{\phi}\right) \neq \emptyset.$$

Then, necessarily

$$\phi \leq 2rD\frac{k}{b_1}.$$

□

4.2 Some intuition and interpretation behind the criterion

We believe that the practical implications of Theorem 1 are worth exploring. In fact, the meaning of this theorem is the following.

If the condition (4) is satisfied, then it is optimal to annuitize immediately *for every initial fund* $x_0 \in \left[0, \frac{b_1}{k}\right]$. This means that compulsory immediate annuitization turns out to be optimal, in the sense that retirees are obliged to behave optimally by the legislation.

If the criterion (4) is violated, then it is not true that *for each initial fund* x_0 the optimal annuitization time is $T^* = 0$. This does *not* mean that $T^* = 0$ is *never* optimal, for there are cases in which $\phi > \frac{2krD}{b_1}$ and x_0 stays in the stopping region, leading to $T^* = 0$.⁽³⁾ Clearly, if (4) is not satisfied, *compulsory* immediate annuitization is not optimal, for there are values of the initial wealth that fall in the continuation region, for which it is optimal to take programmed withdrawals.⁽⁴⁾

For this reason, from now on we will associate the condition (4) to the optimality of a pension system where immediate annuitization is compulsory. In other word, Theorem 1 can be stated equivalently by saying that compulsory immediate annuitization is optimal if and only if (4) holds. Thus, in a pension system where immediate full or partial annuitization is the only option, the lack of flexibility in choosing timing of annuitization can be shown to be sub-optimal by verification of the condition (4).

We can rewrite the condition (4) in several ways, in order to study the role and the impact of each key factor on the criterion.

⁽³⁾This happens namely for all $x_0 \in \left[x^*, \frac{b_1}{k}\right]$.

⁽⁴⁾This happens indeed for all $x_0 \in [0, x^*]$.

1. A first equivalent way is

$$\beta^2 \leq 2k \frac{b_0}{b_1} - \left(\rho + \delta + k^2 \frac{w}{v} \frac{1}{\rho + \delta} \right). \quad (5)$$

Inequality (5) indicates that the value of the Sharpe ratio cannot be too high for having optimality of compulsory immediate annuitization. This is intuitive, as the choice of programmed withdrawals makes sense only in the hope of being better off than immediate annuitization. This is possible only in the presence of a sufficiently good risky asset. If the Sharpe ratio is too small, then immediate annuitization should be preferred, as the benefits of mortality credits provided by the insurance products would enhance the return, making immediate annuitization dominant with respect to investment in financial market. This is the classic result of [18], and has also been found in [5] and [15].

If we now consider the expression of k as given by $k = \frac{(r+\delta)}{(1+L)}$, with L insurance loading factor, and we set the simplifying and not unreasonable values $\rho = r, L = 0$ (notice that the former is a typical assumption in this kind of literature, see e.g. [14]), we have

$$k = \rho + \delta \quad (6)$$

and we obtain the simple condition

$$\beta^2 \leq (\rho + \delta) \left[2 \frac{b_0}{b_1} - \left(1 + \frac{w}{v} \right) \right] \quad (7)$$

from which we can gather useful information. Optimality of compulsory annuitization is not possible if $\frac{b_1}{b_0}$ is too high. In particular, if the targeted pension income b_1 is higher than twice the annuity b_0 purchasable at retirement, then the r.h.s. of (7) is negative, rendering the inequality impossible to hold. The factor $\frac{w}{v}$ also cannot be too high: considering that $\frac{b_0}{b_1} < 1$ for the problem to make sense, if $\frac{w}{v} \geq 1$, compulsory immediate annuitization can never be optimal. Interestingly, from (7) we can also notice the marginality of the factor $\rho + \delta$ with respect to $\frac{b_0}{b_1}$ and $\frac{w}{v}$ whenever the term in squared brackets is negative: namely, in this case optimality of compulsory annuitization can never occur, no matter what the value of $\rho + \delta$ is. On the other hand, if the term in squared brackets is positive, then the value of $\rho + \delta$ can affect the validity of the criterion: in fact, a too low value of $\rho + \delta$ will make impossible the criterion to hold, while a very high value of it can lead to optimality of compulsory annuitization. This is consistent with intuition: $\rho + \delta$ is a measure of the intolerance towards future income (this will be explained clearly in the next section). Thus, a very low value of $\rho + \delta$, that is associated to high tolerance for the future and high expectation of future remaining lifetime, is likely to render immediate annuitization sub-optimal.

2. A second equivalent way is

$$\frac{w}{v} \leq \frac{2b_0(\rho + \delta)}{kb_1} - \frac{(\rho + \delta + \beta^2)(\rho + \delta)}{k^2}. \quad (8)$$

Inequality (8) says that in order to have optimality of compulsory immediate annuitization the weight w given to loss in case of annuitization needs to be sufficiently low with respect to that given to loss for running consumption v , or, equivalently, the weight given to loss in case of running consumption v needs to be sufficiently high with respect to that given to annuitization w .

If we make the simplifying assumption (6) on k , we obtain

$$\frac{w}{v} \leq \frac{2b_0}{b_1} - \left(1 + \frac{\beta^2}{\rho + \delta}\right). \quad (9)$$

Notice that, again, the risk aversion of the pensioner plays a role. As before, if the targeted pension income b_1 is higher than twice the annuity b_0 purchasable at retirement, then the r.h.s. of (9) is negative, rendering the above inequality impossible to hold. Again, also the financial market has an effect: with too high values of the Sharpe ratio β , the r.h.s. can become very small and even negative, and the same can happen with a too low value of $\rho + \delta$.

3. A third equivalent way is

$$\frac{b_0}{b_1} \geq \frac{\rho + \delta + \beta^2}{2k} + \frac{k}{2} \frac{w}{v} \frac{1}{\rho + \delta}. \quad (10)$$

Inequality (10) indicates that the ratio $\frac{b_0}{b_1}$ cannot be too low for compulsory immediate annuitization to be optimal. This is equivalent to say that the targeted pension b_1 cannot be too high with respect to the pension b_0 purchasable at retirement. This is again intuitive, as immediate annuitization can turn out to be optimal only if the retiree is not too much unsatisfied with it, and this can happen only if her desired income does not deviate too much from the pension provided at retirement.

If we make the simplifying assumption (6) on k , we obtain

$$\frac{b_0}{b_1} \geq \frac{1}{2} + \frac{1}{2} \left(\frac{\beta^2}{\rho + \delta} + \frac{w}{v} \right).$$

Consistently with the analysis made before, we see that β and $\frac{w}{v}$ cannot be too high and $\rho + \delta$ cannot be too low for the condition to hold. In particular, we see that if

$$\frac{\beta^2}{\rho + \delta} + \frac{w}{v} \geq 1,$$

then compulsory immediate annuitization cannot be optimal, given that $\frac{b_0}{b_1} < 1$.

4. A simple condition on $\rho + \delta$ is not easy to interpret.⁽⁵⁾ However, if we apply the simplifying assumption (6) on k , from (7) we get

$$(\rho + \delta) \left[2 \frac{b_0}{b_1} - \left(1 + \frac{w}{v}\right) \right] \geq \beta^2.$$

Expectedly, this implies that the factor $\rho + \delta$ has to be high enough to render possible optimality of compulsory immediate annuitization. In addition, as commented in point 1 above, the value of $\rho + \delta$ has no impact whenever the other factors do not behave nicely. In fact, if $\frac{w}{v}$ and/or $\frac{b_1}{b_0}$ are too high, then compulsory immediate annuitization is not optimal no matter what the value of $\rho + \delta$ is.

⁽⁵⁾The condition is $z_1 \leq \rho + \delta \leq z_2$, where, for $i = 1, 2$, we have

$$z_i = \frac{1}{2} \left[2k \frac{b_0}{b_1} - \beta^2 + (-1)^i \sqrt{\left(\beta^2 - 2k \frac{b_0}{b_1}\right)^2 - 4k^2 \frac{w}{v}} \right]$$

Remark 2 Notice that the occurrence of condition (4) is linked to the simultaneous occurrence of the following:

- sufficiently low value of β ;
- sufficiently low value of $\frac{w}{v}$;
- sufficiently low value of $\frac{b_1}{b_0}$;
- sufficiently high value of $\rho + \delta$.

Therefore, since these four key factors cannot be controlled – as some are linked to the financial market, some to mortality conditions, and some to personal preferences – it is evident that a pension system that imposes compulsory immediate annuitization to the whole universe of retirees is bound to be sub-optimal, according to this model. This simple conclusion evidently is not a great discovery. Clearly, giving more flexibility to the decision maker has the effect of increasing her individual utility, and this holds in every context. However, here we state that, even if immediate annuitization might turn out to be optimal for the single retiree, it cannot be optimal for the universe of retirees in its globality. More importantly, this model does provide what is the optimal annuitization time, and therefore it enables us to quantify sub-optimality of compulsory immediate annuitization, by simple computation of the cost of being sub-optimal. This is the topic of the next section.

5 Sub-optimality of compulsory immediate annuitization: practical implications

In this section, we illustrate the practical implications of Theorem 1. In section 5.1 we present four typical scenarios which will be used in the simulations and, in each of them, check the violation of criterion (4); in section 5.2 the cost of sub-optimality is defined and measured in those scenarios.

5.1 Basic assumptions and scenario generation

We consider a male retiree aged 60 and time horizon equal to $T = 30$. In fact, accordingly to the existing actuarial literature (see section 2), ages of optimal annuitization range typically between 70 and 80, and only rarely up to 85-90. It is therefore reasonable to assume that if the pensioner has not annuitized at the age of 90, he will not do it later. His initial wealth is $x_0 = 100$. The riskfree asset in each scenario will be chosen at 3%. For the annuity calculation, we make use of an Italian projected mortality tables (RG48). This set of assumptions corresponds to an immediate annuity value equal to $b_0 = 6.22$, and the conversion factor from lump sum in annuity is $k = 0.085$. The parameters of the problem are

$$r, \quad \lambda, \quad \sigma, \quad \delta, \quad k, \quad v, \quad w, \quad b_1, \quad \rho. \quad (11)$$

In a realistic setting r, λ, σ characterize the investment opportunities and depend on the financial market, δ depends on the demographic assumptions, and k depends on both the financial and demographic hypotheses.

Parameters that can be chosen are the weights given to penalty for running consumption, v , and to penalty for final annuitization, w , although it turns out that the relevant quantity is the ratio of these weights, $\frac{w}{v}$. Another parameter chosen by the retiree is the targeted level of annuity, b_1 , while

it is reasonable to assume that the level of interim consumption b_0 is given and depends on the size of the fund at retirement. A typical choice for b_0 is the size of annuity purchasable at retirement with the initial fund x_0 . Thus, typically b_1 is multiple of b_0 , and the relevant quantity is $\frac{b_1}{b_0} > 1$.

A parameter that is somehow arbitrary and somehow given is ρ , the intertemporal discount factor: although subjective by its own nature, in typical situations cannot differ too much from the riskfree rate of return r and in all our simulations it will be assumed $\rho = r$. We notice that the discount factor ρ is always associated to the mortality rate δ . This is intuitive: $\rho + \delta$ is the global discount factor of the individual, that takes into account the subjective tolerance towards future income, ρ , and the expected remaining lifetime, driven by δ . In this way the relevant quantity measuring the patience of the retiree for future events, the value of time, is given by the sum $\rho + \delta$.

In [7] the authors show that, expectedly, what really counts in the applications, are some relevant ratios of these values, that are

$$\beta, \quad \frac{b_1}{b_0}, \quad \frac{w}{v}, \quad \rho + \delta. \quad (12)$$

This is indeed consistent with the fact that, as shown before, the validity of the criterion (4) only depends on the value of these key quantities. This allows us in the following to fix some of the values (11) and change some others to get different values of the relevant quantities (12). In fact, similarly to the results found in the previous section, they find that equal sign variations of the first three quantities and opposite sign variation of the fourth one produce equal qualitative variations to some relevant features of the problem solution (not reported here because too technical and not useful in this context). In particular, they find that everything else being equal, the ratio $\frac{x^*}{(\frac{b_1}{k})}$, i.e. the width of the continuation region:

1. increases by increasing β ;
2. increases by increasing $\frac{w}{v}$;
3. increases by increasing $\frac{b_1}{b_0}$;
4. generally slightly decreases by increasing $\rho + \delta$.

Indeed, it is reasonable to accept that a high Sharpe ratio can well be coupled with low risk aversion, and also with high penalty in case of annuitization w.r.t. that paid in case of running consumption. Moreover, it is natural to expect that a low $\rho + \delta$ is consistent with a high $\frac{w}{v}$, because these choices are both led by high tolerance toward future income. And vice versa. Notice finally that these relationships between the key parameters and the width of the continuation region are consistent with the analysis made in section 4.2. In fact, we have commented above that $T^* = 0$ for every initial wealth, meaning $x^*/(\frac{b_1}{k}) = 0$, is possible only with low enough values of β , $\frac{w}{v}$ and $\frac{b_1}{b_0}$ and with high enough value of $\rho + \delta$.

In a first set of simulations, here not reported, we let all the relevant quantities vary accordingly to the description. We have found out that the result do not differ very much when the ratio $\frac{w}{v}$ increases. For this reason, we have finally fixed the ratio $\frac{w}{v}$ equal to 1. Results have turned out to be more sensible to the choice of $\rho + \delta$ that, as mentioned, measures the weight given to future and present flows, i.e. the time value of money for the retiree.

We fix four different scenarios, by starting with low values of the first two relevant quantities and high value of the third one, and then progressively augmenting the first two of them and reducing

the third one, at the same time. The values of the relevant quantities of the four scenarios, which we call A, B, C, and D, are reported in Table 1.⁽⁶⁾

	A	B	C	D
β	0.25	0.33	0.40	0.50
$\frac{b_1}{b_0}$	1.50	1.75	2.00	2.25
$\rho + \delta$	0.09	0.07	0.05	0.03

Table 1: Value of key parameters in scenarios A, B, C, D.

Although, clearly, the set of combinations of different values of the relevant ratios is potentially unlimited, and the model can be run with different combinations, here we focus on these four scenarios, which we find representative for four different kinds of individuals:

- Scenario A would be suitable for individuals quite risk averse, who desire a pension target only 50% higher than b_0 . They do not expose themselves too much to financial risk, and gain a small value of β on the market. These individuals strongly prefer current income to future income, probably due to a high estimate of the subjective mortality rate.
- Scenario B reports preferences for individuals moderately risk averse, who target a final pension 75% higher than b_0 . They find a $\beta = 0.33$ on the market, and give more weight to the present rather than the future.
- Scenario C would be suitable for individuals with low risk aversion, who aim to double their final annuity via programmed withdrawals. They are able to gain $\beta = 40\%$ on the financial market and give approximately the same importance to future and present income, taking into account also the mortality rate.
- Scenario D would be suitable to risk lovers, who aim to more than double the immediate annuity b_0 , and get a very high value of β on the market. They put substantial weight to future income, compared to present one, either due to low estimate of their own mortality rate or due to strict preference for the future.

Straight application of the definitions allows us to check that, as expected, in the four scenarios chosen the criterion (4) is not satisfied.

In each of the four scenarios, we have also tried to see if it was possible to obtain the validity of the criterion (4) just by changing one parameter out of the four relevant ones (that are $\beta^2, \frac{w}{v}, \frac{b_1}{b_0}, \rho + \delta$), leaving the other three fixed. This turns out to be impossible: in all cases the value of the key parameter that allows optimality of compulsory immediate annuitization is not admissible. In particular, in each of the four scenarios selected,

⁽⁶⁾In particular, the values of the full set of parameters is the following:

Scenario A: $r = 0.03, \lambda = 0.06, \sigma = 0.12, \delta = 0.06, k = 0.085, w = 1, v = 1, \rho = 0.03, b_0 = 6.22, b_1 = 9.33$.

Scenario B: $r = 0.03, \lambda = 0.08, \sigma = 0.15, \delta = 0.04, k = 0.085, w = 1, v = 1, \rho = 0.03, b_0 = 6.22, b_1 = 10.80$.

Scenario C: $r = 0.03, \lambda = 0.102, \sigma = 0.18, \delta = 0.02, k = 0.085, w = 1, v = 1, \rho = 0.03, b_0 = 6.22, b_1 = 12.44$.

Scenario D: $r = 0.03, \lambda = 0.13, \sigma = 0.20, \delta = 0.005, k = 0.085, w = 1, v = 1, \rho = 0.03, b_0 = 6.22, b_1 = 14.00$.

- the value of β^2 such that $\phi = \frac{2krD}{b_1}$ (leaving unchanged the value of the other three key parameters) is negative,
- the value of $\rho + \delta$ such that $\phi = \frac{2krD}{b_1}$ (leaving unchanged the value of the other three key parameters) is negative,
- the value of $\frac{w}{v}$ such that $\phi = \frac{2krD}{b_1}$ (leaving unchanged the value of the other three key parameters) is negative,
- the value of $\frac{b_1}{b_0}$ such that $\phi = \frac{2krD}{b_1}$ (leaving unchanged the value of the other three key parameters) is lower than 1.

This is further evidence of the fact observed in Remark 2 that if the preferences and needs of pensioners can be represented by the model exploited, then a system where immediate annuitization is compulsory and programmed withdrawals are not an option is bound to be sub-optimal.

The next section is devoted to measuring the extent of sub-optimality for the single individual in the four scenarios selected previously.

5.2 Sub-optimality of immediate annuitization: cost of sub-optimality

In this section, we define two measures – one absolute and one in relative terms – for the cost of sub-optimality of immediate annuitization in terms of loss of expected present value (EPV) of consumption stream from retirement up to death. In particular, we calculate the EPV of consumption in the two cases of optimal annuitization (EPV_{OA}) and immediate annuitization (EPV_{IA}) and make the difference⁽⁷⁾. Henceforth, we will call the difference $EPV_{OA} - EPV_{IA}$ *cost of sub-optimality*, or *sub-optimality cost* (SC). In other words:

$$SC = EPV_{OA} - EPV_{IA} = \text{sub-optimality cost.}$$

To give a better idea of the improvement that can be achieved in case of optimal annuitization, we introduce a new quantity, called *relative sub-optimality cost* (RSC). This is defined as the ratio between the cost of sub-optimality and the EPV of immediate annuitization:

$$RSC = \frac{SC}{EPV_{IA}} = \text{relative sub-optimality cost.}$$

This quantity indicates by how much in percentage the pensioner can increase his reward (measured in terms of EPV of consumption) by adopting programmed withdrawals with respect to immediate annuitization.

In order to have an idea of the extent of the cost of sub-optimality in real applications, we perform numerical simulations for each scenario selected in section 5.1. For each scenario, we run 1000 Monte Carlo simulations for the risky asset. Across the four scenarios, we have fixed the 1000 trajectories of the Brownian motion.

It is important to underline that in order to calculate the sub-optimality cost, we focus only on those trajectories for which optimal annuitization has actually occurred between retirement and the time-horizon of 30 years. The trajectories where the optimal fund fails to reach the boundary of the continuation region x^* within the time-frame of 30 years have been assigned a 0 value to the SC , because the model does not require a finite time-horizon and, more importantly, because

⁽⁷⁾By expected present value of a future stream we mean the actuarial value at retirement of the future stream.

optimal annuitization at a later age would produce different results in terms of SC . Assignment of $SC = 0$ to all trajectories where optimal annuitization does not occur is the explanation for a bar around 0 in the histograms. However, in the statistics of SC (Table 2) we will not consider the zeroes and will present the statistics only for the relevant cases in which $T^* < 30$.

In each scenario and for each trajectory, the EPV of consumption stream has been calculated according to actuarial principles with the interest rate r for the financial basis, and with the mortality table RG48 for the demographic basis. For the immediate annuitization option the flow to be discounted is equal to b_0 at any time from retirement up until age 110. For the optimal annuitization case, the flow to be discounted is equal to the optimal consumption from retirement until age of optimal annuitization T^* , to the actual pension rate achieved from that time up to age 110.

Figures (1a), (1b), (1c), and (1d) report the histograms of the distribution of SC in scenarios A, B, C and D, respectively.

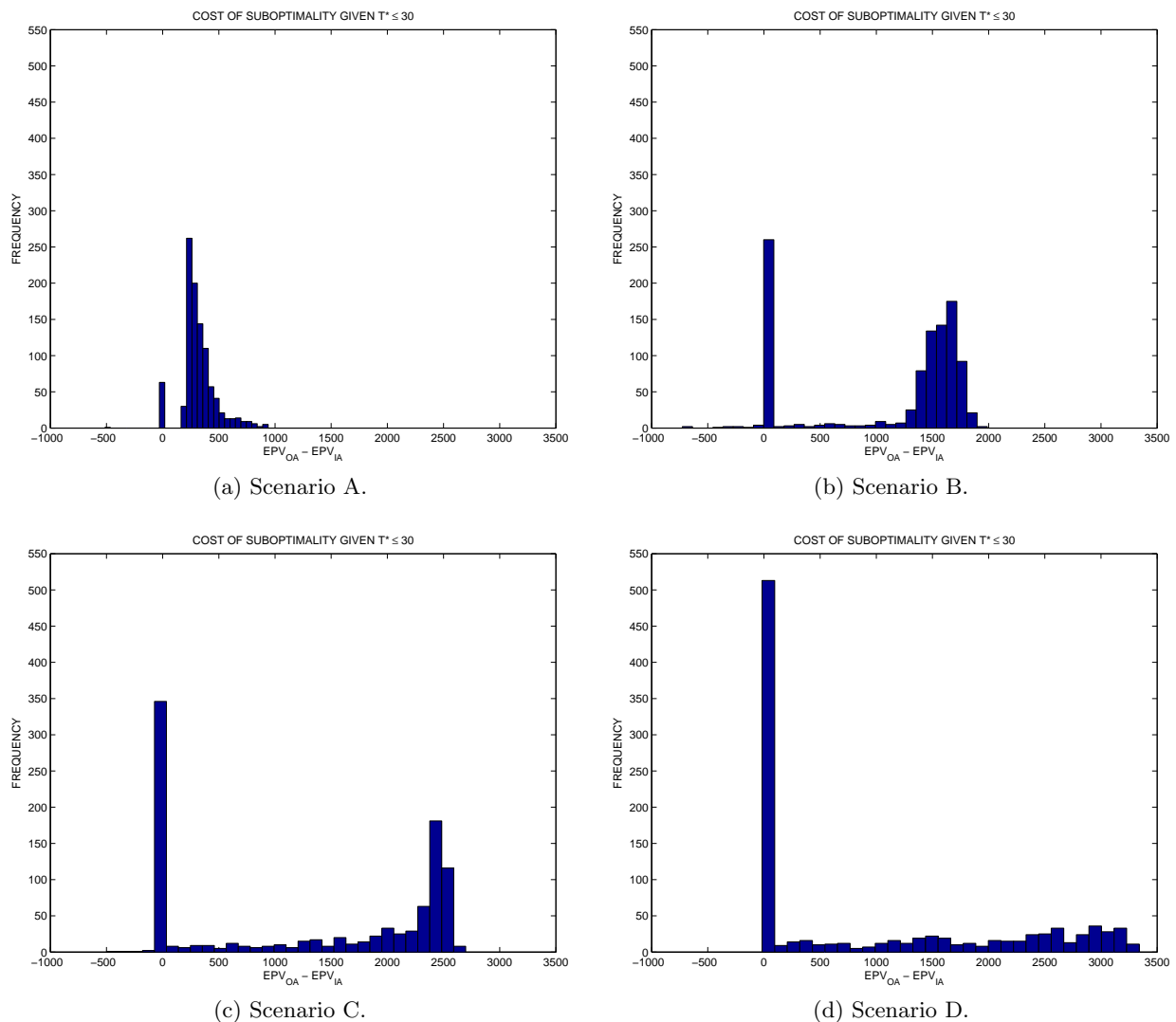


Figure 1: The sub-optimality cost deriving from scenarios A, B, C and D, respectively.

Table 2 is made by two parts. In what follows, by $\text{Prob}(E)$ we mean the frequency over the 1000 simulations of the event E .

1. The first part reports for each scenario the probability that optimal annuitization does not occur in the time-frame of 30 years, the probability that the sub-optimality cost SC is positive, the mean of the relative sub-optimality cost RSC , the mean of the optimal annuitization time T^* , and the mean of the final annuity A^* received upon annuitization at time T^* (more detailed statistics of T^* and A^* are provided by the next section).
2. The second part reports for each scenario some statistics of the sub-optimality cost.

	Scenario A	Scenario B	Scenario C	Scenario D
Prob($T^* > 30$)	6.30%	25.30%	34.10%	50.90%
Prob($SC > 0$)	93.60%	73.50%	65.00%	49.00%
Mean RSC	6.62%	28.91%	38.80%	38.82%
Mean T^*	1.0312	8.1532	14.1807	19.6263
Mean A^*	6.8397	11.6771	18.0114	26.8969
Mean SC	338.9556	1,479.9263	1,985.7943	1,987.0308
Standard deviation SC	134.4206	378.5419	700.8871	935.0963
Minimum SC	-514.0536	-726.6107	-499.4168	-16.5548
Maximum SC	938.7281	1,988.1095	2,697.1238	3,340.5489
5th percentile SC	214.8806	603.5453	352.3035	301.9541
25th percentile SC	248.7041	1,451.6475	1,724.5226	1,254.0524
50th percentile SC	303.6802	1,568.8207	2,350.9992	2,200.6371
75th percentile SC	383.6738	1,675.8389	2,458.8257	2,841.1091
95th percentile SC	642.5026	1,789.9402	2,561.7912	3,183.1867

Table 2: Statistics of the cost of sub-optimality and of all relevant quantities deriving from all scenarios.

Figures (1a), (1b), (1c), (1d), and Table 2 allow a broad comparison among all scenarios regarding all types of quantities analyzed throughout the paper. Let us recall that the scenarios A, B, C and D are descriptive of different risk aversion attitudes, with the A scenario representing the highest and the D scenario reporting the lowest. We can gather the following information:

- considering that the sum of the number of cases in which optimal annuitization does not occur and number of cases characterized by $SC > 0$ (sum of the first two lines of Table 2) is almost approximately 99%, for each scenario we find that in all but 10 cases in which optimal annuitization occurs, the sub-optimality cost is positive, meaning that the pensioner is better off when programmed withdrawals are adopted;
- the few cases when optimal annuitization does occur but $SC < 0$ are motivated by the fact that the dynamic programming approach minimizes the expected loss (or maximizes expected utility) and fails to capture very rare unfavorable scenarios (e.g. extreme events);
- the extent of improvement with optimal annuitization (or the extent of loss with immediate annuitization) is measured by the mean RSC , in the third line of the table: it amounts to only 7% for scenario A, to 29% for scenario B, 39% for scenarios C and D. As expected, the margin of improvement increases when risk aversion of the individual decreases;

- the probability of unsuccessful use of this model, meaning absence of annuitization within 30 years from retirement, also increases when decreasing the risk aversion;
- the optimal annuitization time increases when decreasing the risk aversion;
- the size of annuity achieved upon optimal annuitization also increases on average when decreasing the risk aversion.

6 The optimal annuitization time: performance analysis

In order to make the analysis more complete, in this section we provide more detailed information gathered from the simulations, about the optimal annuitization time T^* and the final annuity size A^* achieved at time T^* .

In particular, for each scenario we provide the following (here, again, by $\text{Prob}(E)$ we mean the frequency over the 1000 simulations of the event E):

- the histogram with the distribution of the optimal annuitization time T^* , measured in years from retirement;
- the histogram with the distribution of the final annuity A^* achieved upon optimal annuitization; the final annuity A^* has been calculated each time through the actuarial fairness principle, using the RG48 mortality table and the interest rate $r = 0.03$; each time, the actual age of annuitization has been used, consistently with what would happen in the practice;
- statistics of the optimal annuitization time T^* and of the size of annuity A^* upon optimal annuitization, with consequent comparison with that achievable on immediate annuitization (Tables 3, 4, 5, and 6); note that the mean of T^* and of A^* are already reported in Table 2;
- some relevant information relative to extreme cases such as:
 - probability of optimal annuitization within one year from retirement, $\text{Prob}(T^* < 1)$,
 - probability of optimal annuitization after the time-frame considered, $\text{Prob}(T^* > 30)$,
 - probability that the final annuity A^* is lower or equal than b_1 , $\text{Prob}(A^* \leq b_1)$;
 - probability of negative optimal consumption, $\text{Prob}(b^* < 0)$, and average time of negative consumption, given that there is negative consumption (Tables 3, 4, 5, and 6).

Summarizing, Figures (2b), (2a) and Table 3 report results relative to scenario A; Figures (3b), (3a) and Table 4 those relative to scenario B; Figures (4b), (4a) and Table 5 those relative to scenario C; Figures (5b), (5a) and Table 6 those relative to scenario D.

Remark 3 *The statistics of optimal annuitization time and final annuity reported in Tables 3, 4, 5, and 6 are conditional on $T^* \leq 30$. When optimal annuitization does not occur within the 30-years time-frame, T^* is set equal to 0, but its statistics is not reported into the mentioned Tables, and the same for the statistics of the final annuity. In a similar way, in Figures (a) and (b) of each scenario, that report the histograms of distribution of T^* and final annuity A^* , the cases when optimal annuitization does not occur have not been reported.*

Remark 4 *Scenarios B, C and D are characterized by the same scale in all Figures, whereas scenario A differs from the others relatively to Figures (2a) and (2b). This is due to the much higher concentration of values around the mode of the distribution in scenario A than in the other scenarios.*

6.0.1 Scenario A

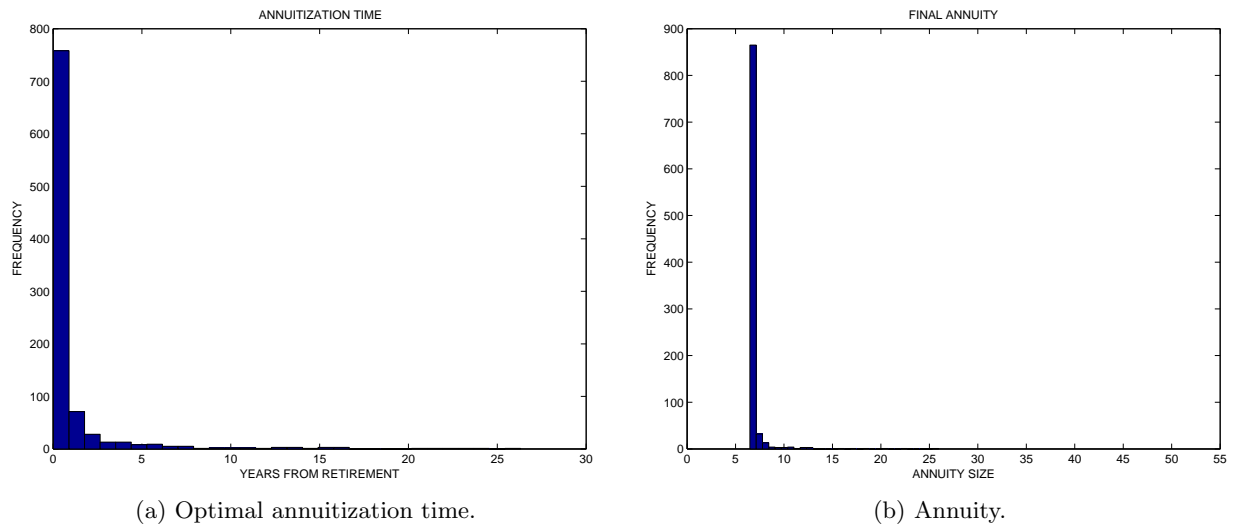


Figure 2: Scenario A.

Mean T^*	1.0312
Standard deviation T^*	2.8087
Mean A^*	6.8397
Standard deviation A^*	1.2638
$\text{Prob}(T^* < 1)$	77.00%
$\text{Prob}(T^* > 30)$	6.30%
$\text{Prob}(A^* \geq b_1)$	2.24%
$\text{Prob}(b^* < 0)$	2.40%
Mean time of $b^* < 0$ (given $b^* < 0$)	4.40 yrs

Table 3: Statistics of the optimal annuitization time, statistics of the size of annuity, and relevant information on extreme cases in scenario A.

From Figures (2a), (2b) and from Table 3, we can gather the following information:

- the most striking feature that can be observed is the timing of optimal annuitization T^* : in most of the cases, T^* occurs only 1-2 years after retirement. In particular, in 770 cases out of 1000 optimal annuitization occurs within one year from retirement;
- in 63 cases out of 1000 optimal annuitization does not occur within the time-frame of 30 years after retirement;
- the size of final annuity A^* upon optimal annuitization is always higher than the pension achievable on immediate annuitization b_0 ; however, the improvement does not seem to be particularly significant, since in 75% of the cases the final annuity A^* lies between 6.48 and 6.71, vs $b_0 = 6.22$. This is due to the fact that annuitization occurs too early and the price of annuity at that age is still too high, compared to the value of k chosen, and this results into a low pension rate;

- in 98% of the cases the annuity value turns out to be lower than b_1 . On the contrary, in very few cases (2.24%) the annuity value turns out to be higher than $b_1 = 9.33$: this is due to annuitization occurring 20-25 years after retirement, when the old age of the pensioner pushes downwards the price of the annuity, leading to a high annuity value to be purchased with the wealth available at T^* ;
- optimal negative consumption occurs in 24 cases out of 1000, and on average consumption remains negative for 4 years.

6.0.2 Scenario B

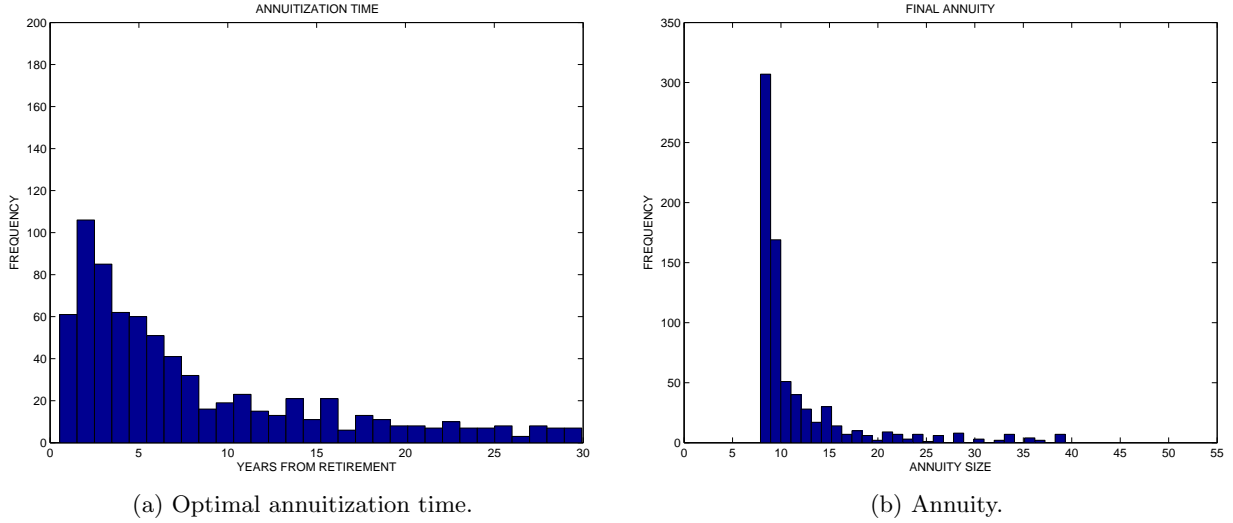


Figure 3: Scenario B.

Mean T^*	8.1532
Standard deviation T^*	7.2171
Mean A^*	11.6771
Standard deviation A^*	5.9555
$\text{Prob}(T^* < 1)$	2.20%
$\text{Prob}(T^* > 30)$	25.30%
$\text{Prob}(A^* \geq b_1)$	29.58%
$\text{Prob}(b^* < 0)$	5.30%
Mean time of $b^* < 0$ (given $b^* < 0$)	2.84 yrs

Table 4: Statistics of the optimal annuitization time, statistics of size of annuity, and relevant information on extreme cases in scenario B.

From Figures (3a), (3b) and from Table 4 we can gather the following information:

- the timing of optimal annuitization, T^* is much more spread out than in scenario A: on average, T^* occurs 8 years after retirement, and in 50% of the cases it occurs after 5 years. In

only 22 cases out of 1000 optimal annuitization occurs within one year from retirement and in 25% of the cases it occurs at a date later than 11 years after retirement;

- in 253 cases out of 1000 optimal annuitization does not occur within the time-frame of 30 years after retirement;
- the size of final annuity A^* upon optimal annuitization is always higher than the pension achievable on immediate annuitization b_0 ; this time, the improvement is more significant than in scenario A, since in 75% of the cases the final annuity A^* lies between 7.88 and 11.77, vs $b_0 = 6.22$. This is due to the fact that now optimal annuitization occurs at a later age than in scenario A and the price of annuity at that age is sufficiently low to guarantee a more than adequate improvement in the pension rate;
- in 30% of the cases the annuity value turns out to be higher than $b_1 = 10.89$: as before, this is due to annuitization occurring 15-25 years after retirement, when the relatively old age of the pensioner pushes downwards the price of the annuity, leading to a high annuity value to be purchased with the wealth available at T^* ;
- optimal negative consumption occurs in 53 cases out of 1000, and on average consumption remains negative for 2.8 years.

6.0.3 Scenario C

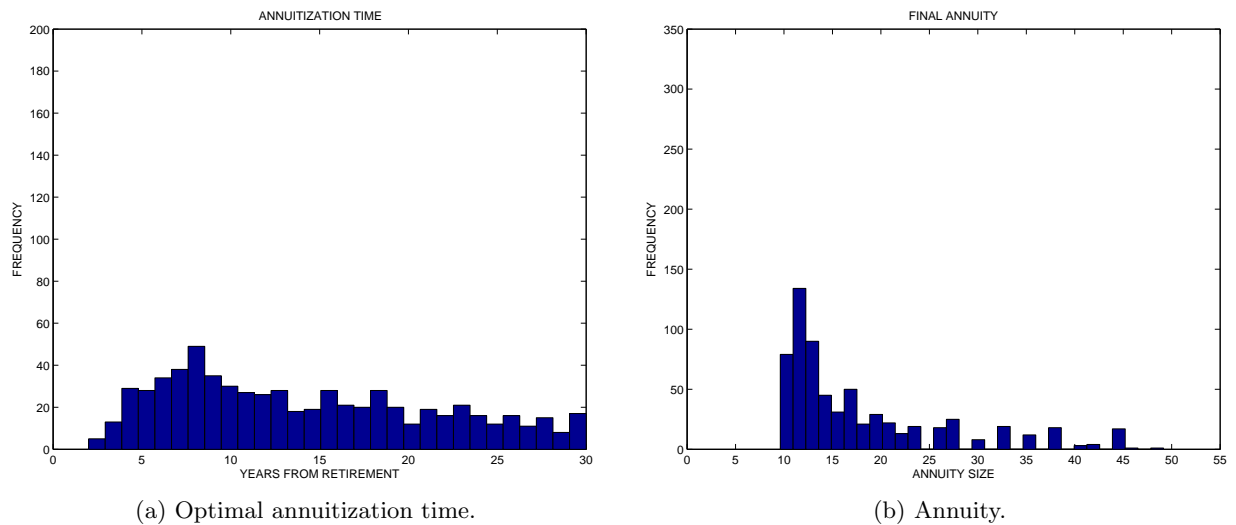


Figure 4: Scenario C.

Mean T^*	14.1807
Standard deviation T^*	7.3754
Mean A^*	18.0114
Standard deviation A^*	28.8639
Prob($T^* < 1$)	0.00%
Prob($T^* > 30$)	34.10%
Prob($A^* \geq b_1$)	63.13%
Prob($b^* < 0$)	3.20%
Mean time of $b^* < 0$ (given $b^* < 0$)	2.01 yrs

Table 5: Statistics of the optimal annuitization time, statistics of the size of annuity, and relevant information on extreme cases in scenario C

From Figures (4a), (4b) and from Table 5 we can gather the following information:

- the distribution of optimal annuitization, T^* , has more or less the same dispersion than in scenario B, but the mean is much higher: on average T^* occurs 14 years after retirement, and in 50% of the cases it occurs after 13 years. Optimal annuitization never occurs within one year from retirement and the minimum T^* is equal to 2 years. In 25% of the cases it occurs at a date later than 20 years after retirement;
- in 341 cases out of 1000 optimal annuitization does not occur within the time-frame of 30 years after retirement; this too high percentage of individuals not annuitizing before age 90 is the price to be paid when the target aimed is chosen to be very high;
- the size of final annuity A^* upon optimal annuitization is always significantly higher than the pension achievable on immediate annuitization b_0 ; the improvement with respect to b_0 is now remarkable: in 75% of the cases the final annuity A^* lies between 9.62 and 20.88, vs $b_0 = 6.22$. This is effect of the two combined facts that x^* is much higher than before and that optimal annuitization occurs at a much later age, with consequent very low price of the annuity;
- in 63% of the cases the annuity value turns out to be higher than $b_1 = 12.44$: as before, this is due to annuitization occurring 15-25 years after retirement, when the relatively old age of the pensioner pushes downwards the price of the annuity, leading to a high annuity value to be purchased with the wealth available at T^* ;
- optimal negative consumption occurs in 32 cases out of 1000, and on average consumption remains negative for 2 years.

6.0.4 Scenario D

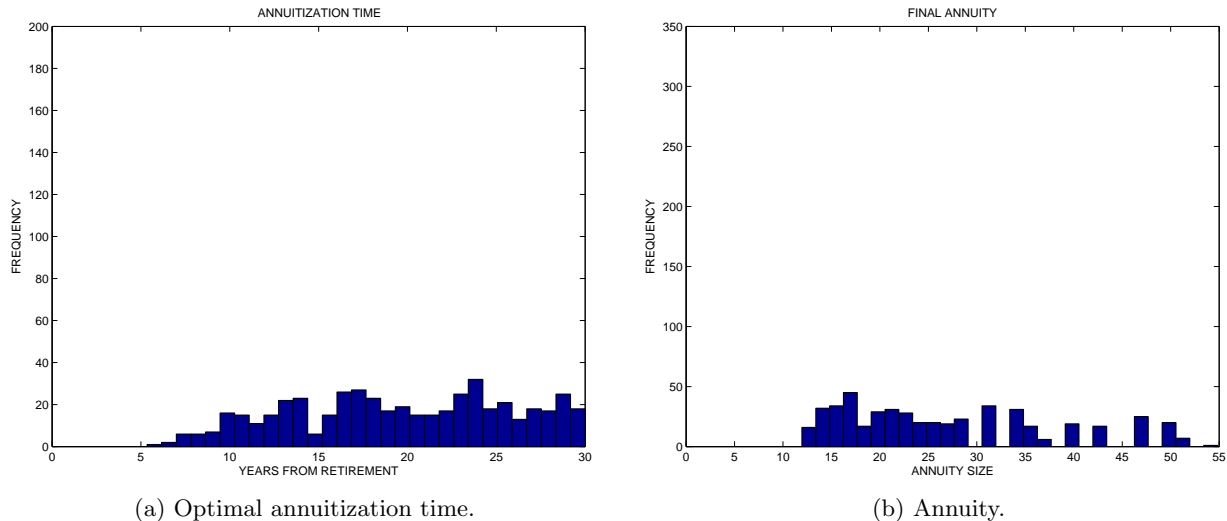


Figure 5: Scenario D.

Mean T^*	19.6263
Standard deviation T^*	6.1493
Mean A^*	26.8969
Standard deviation A^*	10.9322
$\text{Prob}(T^* < 1)$	0.00%
$\text{Prob}(T^* > 30)$	50.09%
$\text{Prob}(A^* \geq b_1)$	93.48%
$\text{Prob}(b^* < 0)$	1.10%
Mean time of $b^* < 0$ (given $b^* < 0$)	2.05 yrs

Table 6: Statistics of the optimal annuitization time, statistics of the size of annuity, and relevant information on extreme cases in scenario D.

From Figures (5a), (5b) and from Table 6 we can gather the following information:

- the most striking feature is the probability that optimal annuitization does not occur in the 30 years time-frame: indeed, it is as high as 50.09%. We believe that with such a high probability of failure to annuitizing in a reasonable time horizon, this model should not be used.
- when $T^* < 30$, it occurs on average after 19 years from retirement, with a moderately high dispersion; in 70% of the cases it occurs between 10 and 25 years from retirement. Moreover, optimal annuitization never occurs within 5 years from retirement. In 25% of the cases it occurs at a date later than 25 years after retirement;

- the size of final annuity A^* upon optimal annuitization is dramatically higher than the pension achievable on immediate annuitization b_0 : in case of optimal annuitization the minimum pension rate achieved is 11.94, that is almost the double than $b_0 = 6.22$, and the size of annuity between the fifth and the seventy-fifth percentiles ranges between 14 and 33, while in 25% of the cases it ranges between 33 and 50. This is mainly due to the too high selected b_1 , that leads to an extremely high value of the boundary of the continuation region x^* : in the most favorable scenarios, when the wealth reaches x^* , the annuity size is extremely high;
- the results on the size of annuity are confirmed also by the fact that in 93% of the cases the annuity value turns out to be higher than $b_1 = 14$;
- optimal negative consumption occurs in 11 cases out of 1000, and on average consumption remains negative for 2 years.

7 Conclusions

In this paper, we have considered the position of a retiree member of a DC pension scheme and have investigated the optimal annuitization time by application of the model introduced by [7]. In particular, we have focused on the appropriateness of immediate annuitization, that is compulsory in many countries where the option of programmed withdrawals is not available.

In a first part of the paper, we have assessed a necessary and sufficient condition for compulsory immediate annuitization to be optimal. We have found that the validity of this condition depends on some key factors, that are linked to the financial market (via the Sharpe ratio of the risky asset) to the demographic conditions (via the force of mortality of the retiree) and to individual preferences (via the risk aversion of the individual and the different weights given to consumption pre- and post-annuitization). In particular, we have found that the Sharpe ratio, the risk aversion and the weight given to loss for consumption post-annuitization, relative to that pre-annuitization, have to be sufficiently low in order to have sub-optimality of compulsory immediate annuitization. On the contrary, the intolerance towards future income as well as the force of mortality have to be sufficiently high for the condition to be satisfied. These are intuitive results. As a consequence, we have found that, since these key factors cannot be controlled a priori, a pension system that imposes compulsory immediate annuitization is bound to be sub-optimal.

In a second part of the paper, we have measured the extent of sub-optimality, in terms of loss of expected present value of consumption. We have defined the sub-optimality cost both in absolute and in relative terms as the difference between EPV of consumption from retirement to death in the case of optimal annuitization and in the case of immediate annuitization. We have run extensive simulations to find the sub-optimality cost in different representative scenarios. With a very few exceptions, the cost of sub-optimality turns out to be positive in all cases in which optimal annuitization occurs in the time-frame of 30 years from retirement considered. Quite remarkably, in relative terms the sub-optimality cost varies from 6% with high risk aversion up to 20 – 30% with medium risk aversion, and can be as high as 40% with low risk aversion. This clearly indicates by how much a pensioner gives up part of her future wealth when she annuitizes immediately, with respect to undertaking programmed withdrawals followed by optimal annuitization.

Finally, in order to complete the analysis, from the simulations we have collected and provided useful information regarding other relevant issues, like optimal annuitization time and size of annuity upon optimal annuitization. We have found the intuitive result that the optimal annuitization time decreases with the risk aversion. In particular, we find that it should occur on average 1-2

years after retirement when risk aversion of the retiree is very high, 8-9 years after retirement with medium risk aversion, 14-15 years after retirement when risk aversion is very low and should occur after 20 years or may not happen at all if the individual is strongly risk lover. Finally, we find the quite obvious result that the size of final annuity upon optimal annuitization is always higher than that obtainable on immediate annuitization. Expectedly, the size of annuity on average increases when the optimal annuitization time increases: in other words, it is seemingly worth to wait in order to obtain a higher reward.

We think that the main contribution of this paper are twofold. The first novelty is the proof of sub-optimality of a pension system where immediate annuitization is mandatory. The second and most important novelty is the precise quantification of the degree of sub-optimality of immediate annuitization. To this aim, the sub-optimality cost reports the loss in expected present value of consumption suffered by a pensioner who has to accept immediate annuitization, rather than taking phased withdrawals. The striking figures of the sub-optimality cost in relative terms clearly indicate the suitability of programmed withdrawals for retirees of DC pension schemes. One of the main concerns of each legislator should be the welfare of its pensioners, and this is obviously strictly linked to the affordability of a proper consumption stream. Therefore, the ambitious aim behind the evidence provided by this paper would be to suggest appropriate adjustment to the legislation of those countries where programmed withdrawals are not available: we believe that to some extent this option should be offered to retirees, so as increase flexibility in choosing a proper timing of benefits after retirement, and – not least – allow optimal choices for pensioners.

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