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Rational Preferences under Ambiguity*

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Abstract

This paper analyzes preferences in the presence of ambiguity that are *rational* in the sense of satisfying the classical ordering condition as well as monotonicity. Under technical conditions that are natural in an Anscombe-Aumann environment, we show that even for such general preference model it is possible to identify a set of priors, as first envisioned by Ellsberg (1961). We then discuss ambiguity attitudes, as well as unambiguous acts and events, for the class of rational preferences we consider.

KEYWORDS: Rational Preferences, Ambiguity, Unambiguous Acts and Events

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*Section 5 of this paper subsumes an earlier paper of Ghirardato, Maccheroni, and Marinacci presented and circulated under the title "Revealed Ambiguity and Its Consequences: Unambiguous Acts and Events." We are grateful to the participants in the RUD 2005 conference for comments on that paper.

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1 Introduction

Daniel Ellsberg's seminal paper (1961) ignited a large and growing literature aimed at developing decision models that accommodate a concern for ambiguity. Among the first, and most prominent contributions, Schmeidler (1989)'s axiomatization of Choquet-expected utility (CEU) and Gilboa and Schmeidler (1989)'s foundations for maxmin-expected utility (MEU) with multiple priors occupy a special place. Furthermore, applications in several areas of economic theory have demonstrated their usefulness.

More recently, several influential contributions have proposed decision models that overcome specific perceived limitations of the CEU and MEU models. Two behavioral aspects have received special attention. First, both the CEU and the MEU model satisfy *Certainty Independence*: the main implication of this axiom is that preferences and, in particular, ambiguity attitudes are unaffected by changes in the "scale" and "location" of utilities. To fix ideas, suppose the individual is risk-neutral, and assume that an individual is just indifferent between receiving \$3 dollars for sure, and participating in a bet that yields \$10 dollars if a certain ambiguous event obtains, and 0 otherwise. Then, Certainty Independence also implies that the individual would be indifferent: (i) between receiving \$300 for sure, and participating in a bet that yields \$1,000 if the event obtains and 0 otherwise; and also (ii) between receiving \$1,003 for sure and participating in a bet that yields \$1,010 if the event obtains and \$1,000 otherwise. Analogies with choice under risk suggest that subjects may reasonably violate either one or both of these conclusions.

Second, the MEU model is characterized by a specific form of dislike for ambiguity, formalized by the "Uncertainty Aversion" axiom due to Schmeidler (1989). This axiom delivers quasi-concavity of the functional representing preferences, and hence ensures a convenient mathematical structure as shown by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008). At the same time, this axiom imposes restrictions on preferences which one may want to dispense with (see Ghirardato and Marinacci (2002) for a theoretical discussion, or Baillon, L'Haridon, and Placido (forthcoming) for an experimental perspective).

Recent decision-theoretic models relax the Certainty Independence and Uncertainty Aversion axioms in specific ways. For instance, variational preferences (Maccheroni, Marinacci, and

Rustichini, 2006) relax invariance to the scale of utilities, but retain invariance to their location, as well as Uncertainty Aversion; the model studied by Cerreia-Vioglio et al. (2008) drops Certainty Independence entirely, but retains Uncertainty Aversion; Grant and Polak (2007) instead drop Certainty Independence, and weaken Uncertainty Aversion. Siniscalchi (2009) retains invariance to the location of utilities, but drops scale invariance, as well as Uncertainty Aversion entirely.

This paper drops *both* Certainty Independence and Uncertainty Aversion. We consider preferences that only satisfy what in our view are the basic tenets of rationality under ambiguity: *weak order* and *monotonicity*. We call these preferences *rational*. Since they are a weak order, they are rational in the usual sense of utility theory. At the same time, the monotonicity assumption guarantees consistency with state-wise dominance, which in turn annihilates the relative effects of ambiguity. All the models discussed above, and several others, belong to this class of preferences. In particular, this class includes the MBC preferences introduced by Ghirardato and Siniscalchi (2010, GS henceforth) and the Uncertainty Averse Preferences introduced by Cerreia-Vioglio et al. (2008).

We first show that, for such preferences, a set of priors can be obtained following the approach of Ghirardato, Maccheroni, and Marinacci (2004, GMM henceforth); i.e., as a representation of the derived unambiguous preference relation.[1] Thus, in a specific behavioral sense, one can identify probabilities that are significant for the decision maker's choices, regardless of the representation of her preferences, which following GS we call "relevant priors." We carry on this task in an Anscombe-Aumann setting, and under two additional assumptions: Risk Independence and Archimedean continuity. We call the rational preferences satisfying these additional axioms *MBA preferences* (for **M**onotonic, **B**ernoullian, and **A**rchimedean). We thus directly generalize the results of GMM and Nehring (2002), and provide a basis over which both the analysis of GS and most results of Cerreia-Vioglio et al. (2008, C3M henceforth) rest.

We then leverage this general representation result to analyze the individual's perception of ambiguity and her attitudes toward it. MBA preferences provide a relatively "neutral" ground

[1] Nehring (2001) and Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) derive a set of priors from a *separate* relation, which they interpret as embodying "objective rationality," and impose consistency conditions between such relation on the decision maker's preference relation.

for the study of these issues, precisely because they do not incorporate any *specific* assumption about invariance and/or attitudes toward ambiguity. We show that MBA preferences admit a "generalized Hurwicz (or α-MEU) representation," thus extending an analogous result established by GMM for preferences satisfying Certainty Independence. This representation provides a useful tool to study, for instance, comparative ambiguity attitudes. Then, we discuss two different notions of ambiguity aversion and the relations between them. Finally, we propose a behavioral definition of unambiguous acts, and characterize it in terms of the set of priors we identify. We then define unambiguous events, and again provide a functional characterization.

Related literature

As outlined above, the main contributions of this paper are: 1) showing that the (arguably) most general rationality assumptions for choice under ambiguity guarantee the existence of a set of priors, first envisioned by Ellsberg (1961) and modeled in the seminal papers of Gilboa and Schmeidler (1989) and Bewley (2002); 2) the discussion of ambiguity attitudes in such general context; 3) the characterization of unambiguous acts and events and some consequences thereof.

In respect to the first contribution, our debt to the GMM paper is obvious. The added contribution here is clearly in showing how (most of) the representation results of that paper generalize to rational preferences which do not satisfy Certainty Independence, but only Risk Independence. The GS paper is complementary to the present one. Its main focus is the characterization of the set of relevant priors for popular preference models. Such characterizations hinge on a differential result which requires a stronger continuity condition, and thus applies only to a subset of MBA preferences, which GS dub MBC (where C stands for "[Cauchy] continuous"). The C3M paper is also complementary to the present one, because its main focus is the analysis of rational preferences which also satisfy Schmeidler's "Uncertainty Aversion" axiom. C3M also characterize the set of relevant priors in several ways (but, their differential characterization is different from the one in GS).

The discussion on ambiguity attitudes is also related to earlier work. We show how the ideas in Ghirardato and Marinacci (2002) can be extended to the MBA class of preferences. We refer to that paper for detailed discussion on the relation of such vision of ambiguity aversion to those spoused in other papers, in particular Schmeidler (1989) and Epstein (1999).

As to this paper's third contribution, this paper comes within a well established literature. Early attempts to characterize behaviorally ambiguity were focussed on ambiguity of events in specific preference models. Such is the case of Nehring (1999) and Zhang (2002), which consider CEU preferences. Subsequently, Epstein and Zhang (2001) and Nehring (2001) offered definitions of unambiguous event which apply in principle to any preference, providing a characterization over rich state spaces. Nehring's proposal is particularly relevant to our paper since it can be shown to be equivalent to the one offered here. The Epstein-Zhang definition, on the other hand, is markedly different from ours. We refer the reader to section 5.3, and especially to Nehring (2006) and Amarante and Filiz (2007) for discussion on the relations between the definition of unambiguous event presented here, Zhang's and Epstein-Zhang's. To the best of our knowledge, the only previous paper that provides a definition of unambiguous *act* as primitive, and events as derivative, is Ghirardato and Marinacci (2002). However, their definition only applies to preferences which are ambiguity *averse* (or loving) according to the definition in that paper. For such preferences, the definition of unambiguous act offered in the two papers can be shown to coincide.

Finally, some of the consequences that we draw from our definitions of ambiguity owe to previous work, and our debts and contributions are clearly identified in the respective sections.

2 Notation and preliminaries

We consider a state space S, endowed with an algebra Σ. The notation $B_0(\Sigma,\Gamma)$ indicates the set of simple Σ–measurable real functions on S with values in the interval[2] $\Gamma \subset \mathbb{R}$, endowed with the topology induced by the supremum norm; for simplicity, write $B_0(\Sigma,\mathbb{R})$ as $B_0(\Sigma)$.

The set of finitely additive probabilities on Σ is denoted $ba_1(\Sigma)$. The (relative) weak* topology on $ba_1(\Sigma)$ is the topology induced by $B_0(\Sigma)$ or, equivalently, by $B(\Sigma)$.

A functional $I : B_0(\Sigma,\Gamma) \to \mathbb{R}$ is:

[2] Which may be open or closed on the left or right, and may also be unbounded on one or both sides.

- **monotonic** if $I(a) \geq I(b)$ for all $a \geq b$
- **continuous** if it is sup-norm continuous
- **normalized** if $I(\alpha 1_S) = \alpha$ for all $\alpha \in \Gamma$

Next, fix a convex subset X of a vector space. (Simple) acts are Σ-measurable functions $f : S \to X$ such that $f(S) = \{f(s) : s \in S\}$ is finite; the set of all (simple) acts is denoted by $\mathscr{F}$. We define mixtures of acts pointwise: for any $\alpha \in [0,1]$, $\alpha f + (1-\alpha)g$ is the act that delivers the prize $\alpha f(s) + (1-\alpha)g(s)$ in state s. Given $f, g \in \mathscr{F}$ and $A \in \Sigma$, we denote by $f\,A\,g$ the act in $\mathscr{F}$ which yields $f(s)$ for $s \in A$ and $g(s)$ for $s \in A^c \equiv S \setminus A$.

3 Rational preferences and relevant priors: characterizations

In this section we first briefly introduce our basic assumptions on preferences, characterizing what we earlier dubbed the "MBA" model. (We refer the reader to GS and C3M for more detailed discussion of the axioms.) Then we show that for MBA preferences the unambiguous preference relation introduced by GMM can be used to obtain a set of possible probabilistic models of the decision problem that might be employed by the decision maker the *relevant priors*.

3.1 Axioms

The main object of interest is a bynary relation $\succcurlyeq$ on $\mathscr{F}$. As usual, $\succ$ (resp. $\sim$) denotes the asymmetric (resp. symmetric) component of $\succcurlyeq$, and we abuse notation by identifying the prize x and the constant act that delivers x for every s.

Axiom 1 (Weak Order) *The relation $\succcurlyeq$ is nontrivial, complete, and transitive on $\mathscr{F}$.*

Axiom 2 (Monotonicity) *If $f, g \in \mathscr{F}$ and $f(s) \succcurlyeq g(s)$ for all $s \in S$ then $f \succcurlyeq g$.*

These two axioms define *rational preferences*. Next two axioms are tailored to the Anscombe-Aumann setup we are considering.

Axiom 3 (Risk Independence) *If $x, y, z \in X$ and $\lambda \in (0,1]$ then $x \succ y$ implies $\lambda x + (1-\lambda)z \succ \lambda y + (1-\lambda)z$.*

Axiom 4 (Archimedean) *If $f, g, h \in \mathscr{F}$ and $f \succ g \succ h$ then there are $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.*

As it is well-known, the above two axioms, in addition to the ones characterizing rational preferences, imply the existence of:

- a *Bernoulli utility index* on X; that is, $u : X \to \mathbb{R}$ which is affine and represents the restriction of $\succcurlyeq$ to X;
- the existence of *certainty equivalents* x_f for all acts $f \in \mathscr{F}$.

A binary relation $\succcurlyeq$ on $\mathscr{F}$ that satisfies Axioms 1–4 will henceforth be called an **MBA preference** (for **M**onotonic, **B**ernoullian, **A**rchimedean).

We now provide a basic representation result for the preferences satisfying the above axioms. It generalizes previous results of Gilboa and Schmeidler (1989), GMM, GS and C3M, which all impose more stringent axiomatic requirements on preferences.

Proposition 1 *A preference relation $\succcurlyeq$ satisfies Axioms 1–4 if and only if there exists a non-constant, affine function $u : X \to \mathbb{R}$ and a monotonic, normalized, continuous functional $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that for all $f, g \in \mathscr{F}$*

$$f \succcurlyeq g \Longleftrightarrow I(u \circ f) \geq I(u \circ g). \tag{1}$$

Moreover, if (I_v, v) also satisfies Eq. (1), and $I_v : B_0(\Sigma, v(X)) \to \mathbb{R}$ is normalized, then there are $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$ such that $v(x) = \lambda u(x) + \mu$ for all $x \in X$, and $I_v(b) = \lambda I(\lambda^{-1}[b-\mu]) + \mu$ for all $b \in B_0(\Sigma, v(X))$.

Observe that differently from Lemma 1 in GMM, the functional I is not necessarily constant-linear.[3] I therefore depends upon the normalization chosen for the utility function (see Ghirardato, Maccheroni, and Marinacci (2005)). On the other hand, thanks to normalization, I is uniquely determined by u and the equality $I\left(u\left(f\right)\right) = I\left(u\left(x_f\right) 1_S\right) = u\left(x_f\right)$.

[3] I is **constant-linear** if and only if $I(\alpha a + \beta 1_S) = \alpha I(a) + \beta$ for all $a \in B_0(\Sigma, u(X))$, $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$, such that $\alpha a + \beta 1_S \in B_0(\Sigma, u(X))$.

3.2 Relevant priors and unambiguous preferences

We now recall GMM's notion of "unambiguous preference" relation (see also Nehring, 2007). The more general preference setting notwithstanding, such relation has the same interpretation as in GMM: since ambiguity sensitivity may lead to violations of the Anscombe-Aumann independence axiom, we look for rankings that are not reversed by mixtures.

Definition 1 Let $f, g \in \mathscr{F}$. We say that f is **unambiguously preferred to** g, denoted $f \succcurlyeq^* g$, if and only if, for all $h \in \mathscr{F}$ and all $\lambda \in (0,1]$, $\lambda f + (1-\lambda)h \succcurlyeq \lambda g + (1-\lambda)h$.

The relation $\succcurlyeq^*$ enjoys the properties identified by GMM (see their Props. 4 and 5), and hence, as in GMM, it admits a representation à la Bewley (2002) (cf. e.g. GMM Prop. A.2):

Proposition 2 *For any MBA preference $\succcurlyeq$, there exists a non-empty, unique, convex and weak*-closed set $C \subset ba_1(\Sigma)$ such that, for all $f, g \in \mathscr{F}$,*

$$f \succcurlyeq^* g \quad \Longleftrightarrow \quad \int u \circ f \, dP \geq \int u \circ g \, dP \quad \text{for all } P \in C,$$

where u is the function obtained in Proposition 1. Moreover, C is independent of the choice of normalization of u.

The last sentence —which follows from the structure of the Bewley-style representation and the uniqueness of C given u— shows that C is cardinally invariant, even though I is not.

Thus, the unambiguous preference gives rise to a set of priors, which GMM interpret as the (subjective) ambiguity revealed by the decision maker's preferences. We refer the reader to that paper for discussion of the appropriateness of such interpretation.

GS propose a behavioral definition of the set of priors that are *relevant* for the individual's primitive preference relation $\succcurlyeq$; they then show that the resulting set is precisely C, and also show that the arguments provided by GMM in support of their interpretation of C as revealed ambiguity extend to the preferences they study. We refer the interested reader to GS for details; we shall sometimes implicitly invoke GS' equivalence result and thus refer to C as the set of "relevant priors."

4 A generalized Hurwicz representation

We now turn to the first consequence of the general representation results of the previous section. We show that that the generalized α-MEU representation suggested by GMM, which is in the spirit of Hurwicz's "pessimism index" model Hurwicz (1951), extends to MBA preferences, and so does its interpretation in terms of comparative ambiguity. Thus, throughout this section, $\succcurlyeq$ is an MBA preference, represented by the pair (I, u) as per Proposition 1 and with relevant priors C as per Proposition 2.

We first introduce convenient notation. For any measure $Q \in ba_1(\Sigma)$ and function $a \in B(\Sigma)$, let $Q(a) = \int a\, dQ$. Also, given a weak* closed set $D \subset ba_1(\Sigma)$ and function $a \in B(\Sigma)$, let $\underline{D}(a) = \min_{Q \in D} Q(a)$ and $\overline{D}(a) = \max_{Q \in D} Q(a)$; note that $\underline{D}$ (resp. $\overline{D}$) is a monotonic, normalized, constant-linear and concave (resp. convex) functional on $B(\Sigma)$. We then get the following immediate Corollary of the previous representation results.

Corollary 3 *For every $a \in B_b(\Sigma, u(X))$,*

$$\underline{C}(a) \equiv \min_{P \in C} P(a) \leq I(a) \leq \max_{P \in C} P(a) \equiv \overline{C}(a).$$

A second piece of terminology is useful. GMM deem an act *crisp* if, intuitively, it cannot be used to hedge the ambiguity of any other act. GMM formalize this intuition via a behavioral condition that indirectly relies upon Certainty Independence; since MBA preferences do not necessarily satisfy this property, we require a slightly stronger definition: we deem an act crisp if it is unambiguously indifferent to a constant.[4] Formally, denote by $\sim^*$ the symmetric component of $\succcurlyeq^*$. Then, the act $f \in \mathscr{F}$ is **crisp** if there is $x \in X$ such that $f \sim^* x$ (that is, for all $g \in \mathscr{F}$ and $\lambda \in [0,1]$, $\lambda f + (1-\lambda) g \sim \lambda x + (1-\lambda) g$). The characterization of crispness in terms of C follows.

Corollary 4 *An act $f \in \mathscr{F}$ is crisp if and only if $\underline{C}(u \circ f) = \overline{C}(u \circ f)$.*

We can now provide the sought generalized α-MEU representation. Given a normalized representation (I, u) of an MBA preference $\succcurlyeq$, define an **ambiguity index** $\alpha : B_b(\Sigma, u(X)) \to \mathbb{R}$ by

[4] For GMM's preferences the two conditions are equivalent: this follows immediately from Corollary 4 below and GMM's Prop. 10.

letting

$$\alpha(a)=\frac{\overline{C}(a)-I(a)}{\overline{C}(a)-\underline{C}(a)} \tag{2}$$

for every non-crisp function $a\in B_b(\Sigma,u(X))$; by convention, let $\alpha(a)=\frac{1}{2}$ for every crisp function a. The following result is then immediately proved.

Proposition 5 *Let $\succcurlyeq$ be an MBA preference. Then there exist a non-empty, weak*–closed, and convex set $C\subset ba_1(\Sigma)$, a non-constant, affine function $u:X\to\mathbb{R}$, and a function $\alpha:B_b(\Sigma,u(X))\to[0,1]$ such that (i) for all $f,g\in\mathscr{F}$,*

$$f\succcurlyeq g \quad\Longleftrightarrow\quad \alpha(u\circ f)\,\underline{C}(u\circ f)+[1-\alpha(u\circ f)]\,\overline{C}(u\circ f)\geq\alpha(u\circ g)\,\underline{C}(u\circ g)+[1-\alpha(u\circ g)]\,\overline{C}(u\circ g)$$

and (ii) u and C represent $\succcurlyeq^*$ in the sense of Prop. 2. Furthermore, for all non-crisp functions $a,b\in B_b(\Sigma,u(X))$, if $P(a)=P(b)$ for all $P\in C$, then $\alpha(a)=\alpha(b)$.

Finally, if (u',C',α') also satisfy (i) and (ii), then $C'=C$, $u'(x)=\lambda u(x)+\mu$ for some $\lambda,\mu\in\mathbb{R}$ with $\lambda>0$, and $\alpha'(\lambda a+\mu)=\alpha(a)$ for all non-crisp $a\in B_b(\Sigma,u(X))$.

Remark 4.1 The uniqueness statement in Proposition 5 may be paraphrased as follows: C is unique, u is cardinally unique, and if the ambiguity index $\alpha(\cdot)$ is viewed as a function of *acts*, rather than of utility profiles, then it is also unique (for non-crisp acts). More precisely, it is invariant to cardinal transformations of the utility function u. It is worth recalling that GMM define the ambiguity index $\alpha(\cdot)$ over (equivalence classes of) acts, rather than functions.

Since the functional I derived in Proposition 1 is not necessarily constant-linear, the functional α does not have the same structure as in GMM. There, it is shown that, for any two acts $f,g\in\mathscr{F}$, $\alpha(u\circ f)=\alpha(u\circ g)$ holds if, for every $P,Q\in C$, $P(u\circ f)\geq Q(u\circ f)$ if and only if $P(u\circ g)\geq Q(u\circ g)$. For MBA preferences such equality only obtains under the more restrictive condition that $P(u\circ f)=P(u\circ g)$ for every $P\in C$.

4.1 Ambiguity aversion

Here we consider the characterization of ambiguity attitudes for MBA preferences. We first show that, as it transpired from our choice of terminology, and consistently with the analysis in GMM, the function α can be interpreted as an index of ambiguity aversion: The higher $\alpha(u\circ f)$

is, the more averse to the ambiguity entailed by f is the decision maker. "More averse to ambiguity" here is in the sense of Ghirardato and Marinacci (2002, to which the reader is referred for explanation and discussion; GM henceforth): We say that preference $\succcurlyeq_1$ is **more averse to ambiguity than** $\succcurlyeq_2$ if for all $f \in \mathscr{F}$ and all $x \in X$, $f \succcurlyeq_1 x$ implies $f \succcurlyeq_2 x$. The comparison is made between preferences which display the same relevant priors C and utility u, or equivalently (see GMM, Proposition 6, which generalizes immediately to our case), for any $f, g \in \mathscr{F}$,

$$f \succcurlyeq_1^* g \Longleftrightarrow f \succcurlyeq_2^* g \tag{3}$$

We then immediately obtain:[5]

Proposition 6 (GMM, Proposition 12) *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be MBA preferences, and suppose that $\succcurlyeq_1$ and $\succcurlyeq_2$ reveal identical ambiguity. Then $\succcurlyeq_1$ is more ambiguity averse than $\succcurlyeq_2$ if and only if for any common utility u, $\alpha_1(u \circ f) \geq \alpha_2(u \circ f)$ for any noncrisp $f \in \mathscr{F}$.*

Notice that, since as observed the function α may *not* be independent of the choice of the normalization of utility, here we first normalize the two utility functions to be identical,[6] and then perform the comparison of the α functions.

Turning to an *absolute* notion of ambiguity aversion, we recall that GM (in this differing from Epstein (1999), see the discussion in their paper) suggest using subjective expected utility preferences as a benchmark for ambiguity neutrality, and propose the following axiomatic definition of ambiguity aversion:

Axiom 5 (Ambiguity Aversion) *There exists a SEU preference $\geqslant$ that agrees with $\succcurlyeq$ on X and such that, for all $f \in \mathscr{F}$ and $x \in X$,*

$$f \succcurlyeq x \Longrightarrow f \geqslant x$$

That is, a preference is ambiguity averse if it is more ambiguity averse than some SEU preference that displays the same risk attitudes.[7]

[5]Here and henceforth, for results which are straightforward extension of existing results we omit the proof and provide a reference to the existing result.

[6]Eq. (3) implies that the Bernoullian utilities are cardinally equivalent, thus equality of utility is w.l.o.g.

[7]To further clarify, we consider SEU preferences à la Anscombe-Aumann, rather than à la Savage.

The characterization of ambiguity aversion given by GM immediately generalizes to MBA preferences. A piece of terminology first. Given an MBA preference with a representation (I, u), define

$$Core(I) = \{P \in ba_1(\Sigma) : \forall a \in B_0(\Sigma, u(X)),\ I(a) \leq P(a)\} \quad \text{and}$$
$$Eroc(I) = \{P \in ba_1(\Sigma) : \forall a \in B_0(\Sigma, u(X)),\ I(a) \geq P(a)\}.$$

These correspond to the game-theoretic notions when the preference is CEU, but not otherwise. Absolute ambiguity aversion corresponds to non-emptiness of $Core(I)$. (The symmetric property of ambiguity love is analogously characterized as nonemptiness of $Eroc(I)$.)

Proposition 7 (GM, Theorem 12) *Let $\succcurlyeq$ be an MBA preference and (I, u) a representation in the sense of Prop. 1. Then $\succcurlyeq$ is ambiguity averse if and only if $Core(I) \neq \emptyset$.*

The GM proposal is not the most popular definition of ambiguity aversion in the literature. The following notion, proposed by Schmeidler (1989), claims that title. It imposes convexity of preferences.[8]

Axiom 6 (Convexity) *If $f, g \in \mathscr{F}$ and $\alpha \in (0,1)$ then*

$$f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succcurlyeq f.$$

These two notions of aversion to ambiguity are *a priori* different. Indeed, GM present an example (Example 25) of an ambiguity averse MBA preference which is not convex, while the following is an example of a convex MBA preference which is not ambiguity averse.

Example 1 Suppose $X = \mathbb{R}$ and consider $S = \{s_1, s_2\}$. Further, suppose Σ is the power set. Then, we can identify each element $P \in ba_1(\Sigma)$ with the number $P(\{s_1\})$. For this reason, without loss of generality, we use P for either the number and the probability distribution. Next, consider the preference $\succcurlyeq$ over $\mathscr{F}$ represented by the functional $V : \mathscr{F} \to \mathbb{R}$ defined by

$$V(f) = \min_{P \in ba_1(\Sigma)} \left(\frac{\left(\int f\, dP\right)^+}{c_1(P)} - \frac{\left(\int f\, dP\right)^-}{c_2(P)} \right)$$

[8] Schmeidler calls this property "uncertainty aversion," while GM call it "ambiguity hedging."

where $c_1, c_2 : ba_1(\Sigma) \to \mathbb{R}$ are such that $c_1(P) = \frac{P+1}{2}$ and $c_2(P) = 1 + P$. It is immediate to see that c_1 is affine and continuous and c_2 is affine and continuous. Note also that u does not appear because it is the identity. Moreover, $\min_{P \in ba_1(\Sigma)} c_1(P) = \frac{1}{2} > 0$ and $\max_{P \in ba_1(\Sigma)} c_1(P) = 1$ while $\min_{P \in ba_1(\Sigma)} c_2(P) = 1$. By C3M (Corollary 22), $\succcurlyeq$ is an MBA preference that satisfies convexity. However, in light of the discussion in Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2009), $\succcurlyeq$ is ambiguity averse only if $\arg\max c_1 \cap \arg\min c_2 \neq \varnothing$, which is clearly not satisfied in our case.

However, the next result shows that a connection exists between the two notions of aversion to ambiguity: convexity amounts to ambiguity aversion holding "locally" for every act. For convenience, we restrict attention to MBA preferences for which there is **no worst consequence**: that is, for every $x \in X$ there is $y \in X$ such that $x \succ y$. Given a representation (I, u) as in Prop. 1, this is equivalent to the condition that $\inf_{x \in X} u(x) \notin u(X)$.[9]

Theorem 8 *For an MBA preference $\succcurlyeq$ that has no worst consequence, the following conditions are equivalent:*

(i) $\succcurlyeq$ is convex

(ii) for each $f \in \mathscr{F}$, there is a SEU preference $\geqslant_f$ such that, for all $g \in \mathscr{F}$,

$$f \geqslant_f g \Longrightarrow f \succcurlyeq g$$

In view of Theorem 8, Axiom 6 implies the following weak version of Axiom 5: at each $x \in X$ there is a SEU preference $\geqslant_x$ such that, for all $g \in \mathscr{F}$,

$$x \geqslant_x g \Longrightarrow x \succcurlyeq g.$$

Relative to Axiom 5, here the SEU preference $\geqslant_x$ depends on x. Hence, Axiom 6 actually implies Axiom 5 for all preferences where this dependence can be removed. It is useful to reformulate this condition by introducing the sets

$$S_{\succcurlyeq}(x) = \{\geqslant : \text{ for all } g \in \mathscr{F},\ x \geqslant g \Rightarrow x \succcurlyeq g\} \qquad \forall x \in X.$$

[9]This does *not* imply that $u(X)$ must be unbounded below: e.g. consider $X = (0,1)$ and $u(x) = x$.

In other words, $S_{\succcurlyeq}(x)$ is the collection of all SEU preferences that are more uncertainty averse than $\succcurlyeq$ at x. Using these sets, we can say that Axiom 6 implies Axiom 5 provided

$$\bigcap_{x\in X} S_{\succcurlyeq}(x) \neq \emptyset$$

Such is the case for CEU preferences, for which as observed by Ghirardato and Marinacci Axiom 6 implies Axiom 5.

4.2 Does B stand for Biseparable?

MBA preferences share some of the properties of what Ghirardato and Marinacci (2001) call "biseparable" preferences. In our context, a preference $\succcurlyeq$ is **biseparable** if there exists a unique capacity $\rho : \Sigma \to \mathbb{R}$ such that, given any representation (I, u) of $\succcurlyeq$, with I normalized, we have for any binary act $x\,A\,y$ with $x \succ y$,

$$I(u \circ (x\,A\,y)) = u(x)\rho(A) + u(y)(1-\rho(A)). \tag{4}$$

Biseparability thus requires that the "decision weight" attached to the event A in the evaluation of any bet $x\,A\,y$ be independent of the prizes x and y (provided $x \succ y$). Also observe that biseparability is a property of preferences, not of their representation: Eq. (4) is equivalent to the requirement that $x\,A\,y \sim \rho(A)x + [1-\rho(A)]y$, where the r.h.s. of this indifference is a mixture of the prizes x and y. Hence, the capacity ρ is also independent of the choice of u.[10]

It is not hard to see that in general, MBA preferences may fail to be biseparable, even though they induce a cardinal and affine utility u. The following example illustrates.

Example 2 On an arbitrary state space S and $X = \mathbb{R}_+$, consider a smooth-ambiguity preference with $u(x) = x$, $\mu(\{Q_1\}) = \mu(\{Q_2\}) = \frac{1}{2}$ with $Q_1(\{A\}) = Q_2(\{A\}) = \frac{3}{4}$ for some event $A \in \Sigma$, and $\phi(\alpha) = \log(\alpha)$ (so that ambiguity aversion is decreasing in α). Denote by I_u the normalized functional representing these preferences: that is, $I_u(a) = e^{\frac{1}{2}\log Q_1(a) + \frac{1}{2}\log Q_2(a)}$ for all $a \in B_b(\Sigma, \mathbb{R}_+)$.

[10] Consequently, under biseparability, the restriction of the normalized functional I to binary acts is also independent of u, even though, for general acts, this is not generally the case. As it is argued in Ghirardato et al. (2005), I is invariant with respect to u for all acts f in our Anscombe-Aumann framework only if we assume that the preference satisfies Certainty Independence.

Finally, consider now the bet $1\,A\,0$ that pays 1 USD if A obtains and 0 otherwise. We have

$$I_u(1\,A\,0) = e^{\frac{1}{2}\log(Q_1(A))+\frac{1}{2}\log(Q_2(A))} \approx 0.43301.$$

If on the other hand we consider the bet $2\,A1$, then

$$I_u(2\,A\,1) = e^{\frac{1}{2}\log(1+Q_1(A))+\frac{1}{2}\log(1+Q_2(A))} \approx 1.47902.$$

Thus, if we apply Eq. (4) to the bet $1\,A\,0$, we conclude that $\rho(A)$ equals 0.43301; however, if we consider the bet $2\,A\,1$ instead, Eq. (4) implies that $\rho(A)$ should be 0.47902: contradiction.

We therefore see that in this case $\rho(A)$ cannot be defined independently of the choice of $x \succ y$, a violation of biseparability. Intuitively, since $\phi(\alpha) = \log(\alpha)$ displays decreasing absolute ambiguity aversion, as we increase the prizes involved, we get a less conservative willingness to bet on the ambiguous event A.

While invariance of I and ρ to transformations of the utility function does not obtain, for MBA preferences we can still obtain a "locally" biseparable representation of $\succcurlyeq$, in the following sense. Fix a pair (I, u) that represents $\succcurlyeq$, with I normalized. Given a bet $x\,A\,y$ on an event $A \in \Sigma$ (with $x \succ y$), define

$$\rho_{x,y}(A) \equiv \alpha(u \circ x\,A\,y)(\underline{C}(1\,A\,0) - \overline{C}(1\,A\,0)) + \overline{C}(1\,A\,0); \tag{5}$$

The uniqueness properties of the ambiguity index $\alpha(\cdot)$ ensure that the quantity $\rho_{x,y}$ is independent of the utility function adopted (cf. Proposition 5). It is then easy to verify that, when restricted to binary acts (bets) of the form $x\,A\,y$ (for arbitrary $A \in \Sigma$), the preference $\succcurlyeq$ has the representation

$$I(u \circ (x\,A\,y)) = u(x)\rho_{x,y}(A) + u(y)(1 - \rho_{x,y}(A)) \tag{6}$$

With this notation, an MBA preference is biseparable if $\rho_{x,y}$ does not depend upon x and y; we may call such a preference **MBis**, for **M**onotone and **Bis**eparable. It is natural to ask if an additional axiom identifies the MBis subclass of MBA preferences. Ghirardato and Marinacci (2001) describe and axiomatize a model of preferences that turns out to have exactly the type of separability we need. The main axiom is the following; recall that an act $f \in \mathscr{F}$ is *binary* iff it is of the form $f = x\,A\,y$ for some $A \in \Sigma$ and $x, y \in X$ (not necessarily distinct or strictly ranked).

Axiom 7 (Binary Certainty Independence) *For all $f, g \in \mathscr{F}$, with f, g binary acts, $x \in X$, and $\lambda \in (0,1]$: $f \succ g$ if and only if $\lambda f + (1-\lambda)x \succ \lambda g + (1-\lambda)x$.*

We then have the following characterization; see also Theorem 9 in Ghirardato and Marinacci (2001).

Proposition 9 *An MBA preference $\succcurlyeq$ satisfies Axiom 7 if and only if it is biseparable.*

5 Ambiguity of acts and events

This section contains the main contributions of this paper. We first propose a notion of unambiguous acts which strengthens that of crisp acts (cf. §4), and characterize it for MBA preferences. Second, we employ this notion to define unambiguous events, and again provide characterizations. Armed with the characterization of unambiguous acts and events for MBA preferences, we proceed to investigate some consequences of such characterizations. In particular, we observe how, in the spirit of Epstein and Zhang (2001), the derived set of unambiguous events can be used to provide a "fully subjective" theory of expected utility (different from the one they propose). We finally generalize Marinacci (2002)'s result on the consistency of probabilistic sophistication and ambiguity aversion to non (α-)MEU preferences.

Throughout this section, it is convenient to adopt an explicit notation for simple acts. Fix a finite partition $\{E_1, \ldots, E_n\}$ of S in Σ, and corresponding prizes $x_1, \ldots, x_n \in X$. The act that delivers prize x_i in states $s \in E_i$, for $i = 1, \ldots, n$, will be denoted by $\{x_1, E_1; \ldots; x_n, E_n\}$. As before, if $n = 2$, then $\{x_1, E; x_2, S \setminus E\}$ will be denoted simply by $x_1\, E\, x_2$

5.1 Unambiguous acts

We begin by motivating our definition of unambiguous acts. In keeping with the intuition that ambiguity is revealed by non-neutral attitudes toward *hedging*, a starting point is to require that unambiguous acts be crisp. To elaborate, we surely want the set of unambiguous acts to include all constant acts; it then seems plausible to require that this set also include acts that, like constants, are revealed not to provide any hedging opportunities.

However, we would like the notion of unambiguous acts to capture an additional intuition. Consider the three-color Ellsberg urn, containing 30 red balls and 60 green and blue balls, in unspecified proportions. It is natural to regard a "bet on red" as an unambiguous act, because the partition of the state space $S = \{r, g, b\}$ it induces—the winning event $\{r\}$ and the losing event $\{g, b\}$—consists of events whose relative likelihood is intuitively clear. But, by the same token, a "bet on *not* red" should also be regarded as unambiguous.

More broadly, if two acts f, g induce the same partition of the state space S, in the sense that, as usual, for all states $s, s' \in S$, $f(s) = f(s')$ if and only if $g(s) = g(s')$, then either they are both ambiguous, or else they are both unambiguous. In other words, the property of being ambiguous or unambiguous depends upon the partition an act induces, rather than on the specific assignment of distinct prizes to different elements of the induced partition. The following example demonstrates that this additional, natural requirement has bite.

Example 3 Let $S = \{s_1, s_2, s_3\}$, and consider the set C generated by the priors $P = [1/3, 1/4, 5/12]$ and $Q = [1/4, 5/12, 1/3]$ and the act $f = \{x, \{s_1\}; y, \{s_2\}; z, \{s_3\}\}$, with $u(x) = 1$, $u(y) = 4$, $u(z) = 7$. Observe that $P(u \circ f) = Q(u \circ f)$, so f is crisp (cf. Corollary 4). However, the act $g = \{y, \{s_1\}; z, \{s_2\}; x, \{s_3\}\}$, which "permutes" the payoffs delivered by f but is measurable with respect to the same partition, satisfies $P(u \circ g) \neq Q(u \circ g)$: hence, it is not crisp.

Now, if unambiguous acts must be crisp (as we wish to assume), then g must be deemed ambiguous. Since f and g induce the same partition of S, the preceding argument then implies that we must deem f ambiguous as well.

Observe that, in Example 3, the prizes delivered by the acts f and g are the same; this is the sense in which g is a "permutation" of f. We formalize this notion of permutation below.

The discussion so far suggests the following loose provisional definition: *an act is unambiguous if all its "permutations" are crisp*. However, a final difficulty must be overcome. Acts map states to *consequences*; on the other hand, hedging considerations involve *utility* tradeoffs. Hence, if we deem f unambiguous, and $f(s) \sim g(s)$ for all $s \in S$, we should deem g unambiguous, too. Indeed, it turns out that, in the approach we pursue, this is *necessary*, not just natural, in order to avoid paradoxical conclusions:

Example 4 Consider again the 3-color Ellsberg urn, with $S = \{r, g, b\}$; consider prizes x, y, z with $x \neq y \neq z$ and $u(x) = 1 > 0 = u(y) = u(z)$, and let $f = \{x, \{r\}; y, \{g\}; z, \{b\}\}$, so f is, intuitively, a bet on red, even though strictly speaking it is not a binary act. Finally, consider the set C generated by $P = [1/3, 2/3, 0]$ and $Q = [1/3, 0, 2/3]$. In keeping with the Ellsbergian intuition, we wish to deem f unambiguous; however, consider the act $f' = \{y, \{r\}; x, \{g\}; z, \{b\}\}$, which delivers the same prizes as f and is measurable with respect to the same partition. Then $P(u \circ f') = \frac{2}{3} > 0 = Q(u \circ f')$, so f' is not crisp.

As in the previous example, f' must be deemed ambiguous, and hence our provisional definition would deem f ambiguous as well, which seems counterintuitive.

Our definition of unambiguous act takes care of the difficulty illustrated in Example 4 by defining permutations in terms of utility levels instead of payoffs.

Definition 2 An act $g \in \mathscr{F}$ is a $\succcurlyeq$**-permutation** of another act $f \in \mathscr{F}$ if:

(i) $\forall s \in S$ there is $s' \in S$ such that $f(s) \sim g(s')$;

(ii) $\forall s \in S$ there is $s' \in S$ such that $g(s) \sim f(s')$;

(iii) for all $s, s' \in S$, $f(s) \sim f(s')$ if and only if $g(s) \sim g(s')$.

An act $f \in \mathscr{F}$ is **unambiguous** if every $\succcurlyeq$-permutation of f is crisp. The class of all unambiguous acts is denoted by $\mathscr{U}$.

Note that, if preferences are represented by a Bernoulli utility u on X, then conditions (i) and (ii) above are equivalent to the statement that $u \circ f(S) = u \circ g(S)$.

The following result shows that the set $\mathscr{U}$ is the largest set of crisp acts which is "closed" with respect to $\succcurlyeq$-permutations.

Proposition 10 *Given an MBA preference $\succcurlyeq$, $\mathscr{U}$ is the largest set of crisp acts such that if $f \in \mathscr{U}$ and $g \in \mathscr{F}$ is a $\succcurlyeq$-permutation of f, then $g \in \mathscr{U}$.*

The main result of this section shows that unambiguous acts have a sharp characterization in terms of their expected utility with respect to probabilities in the set C:

Theorem 11 *For any $f \in \mathscr{F}$, the following statements are equivalent:*

(i) $f \in \mathscr{U}$.

(ii) $P(\{s : f(s) \sim x\}) = Q(\{s\ f(s) \sim x\})$ *for all* $x \in X$, $P, Q \in C$.

(iii) $P(\{s : u \circ f(s) \geq \gamma\}) = Q(\{s : u \circ f(s) \geq \gamma\})$ *for all* $\gamma \in \mathbb{R}$, $P, Q \in C$.

(iv) $P(\{s\ u \circ f(s) = \gamma\}) = Q(\{s : u \circ f(s) = \gamma\})$ *for all* $\gamma \in \mathbb{R}$, $P, Q \in C$.

Statement (ii) is possibly the most useful, and powerful, characterization of unambiguous acts. In words, an act is unambiguous if and only if the events in the partition it induces have the same probability according to all members of the set C. This in particular implies that, if f is unambiguous and g induces the same partition as f, but possibly delivers entirely different prizes, then g is also unambiguous.

5.2 Unambiguous events

It is natural to define unambiguous any event with respect to which unambiguous acts are measurable (a similar approach to defining unambiguous events was earlier advocated by Ghirardato and Marinacci (2002)).

Definition 3 The class of **unambiguous events** is

$$\Lambda = \big\{\{s : f(s) \sim x\} : f \in \mathscr{U}, x \in X\big\}.$$

Analogously to what we had for unambiguous acts, we can offer two characterization results for unambiguous events. The first is a behavioral result:

Proposition 12 *For any* $A \in \Sigma$, $A \in \Lambda$ *if and only if for any* $x \succ y$, *the act* $x\,A\,y$ *is crisp.*

By part (ii) of Theorem 11, arguing as we did after the statement of that Theorem, the quantifier "for all $x \succ y$" could be changed to "for some $x \nsim y$" without invalidating the result. This makes the behavioral identification of the set Λ conceptually easier, and it also conforms with our intuition that ambiguity is a property of the event partition the act is based on.

Thus, an event A is unambiguous if it is such that any *bet on* such event —i.e., any act of the form $x\,A\,y$ for $x \succ y$— cannot be used to hedge the ambiguity in another act (Nehring (2001) proposes a different definition which turns out to be equivalent to Def. 3, and hence also to an earlier one he presented in Nehring (1999)). Conversely, A is ambiguous if $x\,A\,y \not\sim^* z$ for any $z \in X$; that is, if $x\,A\,y \sim z$, then there exist $\lambda \in (0,1]$, $g \in \mathscr{F}$ such that $\lambda x\,A\,y + (1-\lambda)g \not\sim \lambda z + (1-\lambda)g$.

The second result shows that unambiguous events have a simple and intuitive characterization in terms of the probabilities in C. (Notice that this is independent of the normalization chosen for u.) There is also a natural connection with the "local" willingness to bet $\rho_{x,y}$ defined in Eq. (6).

Proposition 13 *For any $A \in \Sigma$, $A \in \Lambda$ if and only if $P(A) = Q(A) = \rho_{x,y}(A)$ for all $P, Q \in C$ and $x, y \in X$.*

As a consequence, for all MBA preferences, the collection Λ has a simple and intuitive structure (cf. Zhang (2002) and Nehring (1999)).

Corollary 14 *Λ is a (finite) λ-system. That is: 1) $S \in \Lambda$; 2) if $A \in \Lambda$ then $A^c \in \Lambda$; 3) if $A, B \in \Lambda$ and $A \cap B = \emptyset$ then $A \cup B \in \Lambda$.*

It is natural to surmise that *any* act whose upper level sets are unambiguous events should be deemed unambiguous (cf., e.g., Epstein and Zhang (2001)). Proposition 13, paired with Theorem 10, allows us to show that this is indeed the case.

Corollary 15 *For any act $f \in \mathscr{F}$, $f \in \mathscr{U}$ if and only if its upper preference sets $\{s \in S : f(s) \succcurlyeq x\}$ belong to Λ for all $x \in X$.*

Nehring (1999) shows that, if S is finite and I is a Choquet integral (so that the set C can be simply characterized; see Example 17 in GMM), the set Λ can be further characterized as follows:

$$\Lambda = \{A \in \Sigma : \rho(B) = \rho(B \cap A) + \rho(B \cap A^c) \text{ for all } B \in \Sigma\},$$

where $\rho = \rho_{x,y}$, which in the CEU case is independent of the choice of x and y. It follows that for CEU preferences Λ is an algebra, a result that shows that such preferences *cannot* be used to model some potentially interesting ambiguity situations (see for instance the 4-color example in Zhang (2002)).

5.2.1 Ambiguity and willingness to bet

Ghirardato and Marinacci (2002) propose a behavioral notion of unambiguous event for a subclass of the biseparable preferences mentioned in Section 4.2, showing that it has a simple characterization terms of the willingness to bet set-function ρ of Eq. (4): an event B is unambiguous in their sense if and only if $\rho(B)+\rho(B^c)=1$.

The definition given above enjoys two main advantages over this earlier proposal: it is more general, because it applies to any MBA preference, and, more importantly, it is more accurate, as it allows to distinguish between events which are truly (perceived) unambiguous and those that appear to be because of the behavior of the decision maker's ambiguity attitude. The following result illustrates this point. Recall that $\rho_{x,y}(\cdot)$ of Eq. (6) is the local willingness-to-bet index defined in Eq. (6), and $\alpha(\cdot)$ is the the ambiguity index of Eq. (2).

Proposition 16 *Given an MBA preference with normalized representation (I,u) and any $x,y\in X$ such that $x\succ y$, the following are equivalent for any $A\in\Sigma$:*

(i) $\rho_{x,y}(A)+\rho_{x,y}(A^c)=1$ *($\rho_{x,y}$ is* ***complement-additive****)*

(ii) *either $A\in\Lambda$, or $A\in\Sigma\setminus\Lambda$ and $\alpha(u\circ x\,A\,y)+\alpha(u\circ x\,A^c\,y)=1$*

To interpret, an event satisfies the condition $\rho_{x,y}(A)+\rho_{x,y}(A^c)=1$ for some x and y (which is the natural generalization of the Ghirardato-Marinacci condition to MBA preferences) in exactly two cases: either 1) A is unambiguous, or 2) A is not unambiguous, but the decision maker's ambiguity index in evaluating the bets $x\,A\,y$ and $x\,A^c\,y$ behaves so as to perfectly compensate the ambiguity aversion (resp. appeal) revealed in evaluating $x\,A\,y$ by evaluating the complementary bet $x\,A^c\,y$ in an ambiguity seeking (resp. averse) fashion. That is, $\rho_{x,y}(A)+\rho_{x,y}(A^c)=1$ could be satisfied by a pure mathematical accident, if the decision maker's ambiguity attitude is "inconsistent" in just the right way.

On the other hand, suppose that the preference satisfies (for the given x and y) for every $A\in\Sigma\setminus\Lambda$,

$$\alpha(u\circ x\,A\,y)+\alpha(u\circ x\,A^c\,y)\neq 1 \tag{7}$$

Then $\rho_{x,y}(A)+\rho_{x,y}(A^c)=1$ if and only if A is unambiguous. For instance, this is the case of a decision maker for whom $\alpha > 1/2$ uniformly. The following example shows one case of such consistency of ambiguity aversion with CEU preferences.

Example 5 Consider the following variant of the Ellsberg "3-color" paradox. An urn contains 120 balls, 30 of which are red, while the remaining 90 are either blue, green or yellow. A decision maker facing this problem has a CEU preference $\succcurlyeq$ represented by a (non-constant and convex-ranged) utility u and a capacity ρ on $S = \{r, g, b, y\}$,[11] where

$$\rho = \frac{1}{4} 1_r + \frac{3}{4} v,$$

with v a capacity on $\{g, b, y\}$ defined as follows: $v(\emptyset) = 0$, $v(\{g, b, y\}) = 1$ and

$$v(\{g\}) = v(\{b\}) = v(\{y\}) = \frac{7}{24}, \quad v(\{g, b\}) = v(\{g, y\}) = v(\{b, y\}) = \frac{1}{2}.$$

Observe first that Core(ρ) contains (at least) the uniform probability on S. Therefore, $\succcurlyeq$ is ambiguity averse in the sense of Ghirardato and Marinacci (2002), though ρ is not supermodular. Observe next that $\rho(\{r\}) = 1/4$ and $\rho(\{g, b, y\}) = 3/4$. That is $\{\{r\}, \{g, b, y\}\}$ is a candidate for being an unambiguous partition. According to Proposition 16, this will be the case if $\alpha(\{r\}) + \alpha(\{g, b, y\}) \neq 1$. Using Example 17 of GMM it can be checked after some tedious calculation that for $\succcurlyeq$

$$C = \mathrm{Conv}\left\{[1/4, x, y, z] \in \mathbb{R}^4 : [x, y, z] \in \mathrm{Per}(\{5/32, 7/32, 12/32\})\right\}.$$

It follows that $\Lambda = \{\emptyset, S, \{r\}, \{g, b, y\}\}$ as expected. Moreover,

$$\alpha(\{y\}) = \alpha(\{b\}) = \alpha(\{g\}) = 5/7, \quad \alpha(\{r, g, b\}) = \alpha(\{r, g, y\}) = \alpha(\{r, b, y\}) = 1,$$

$$\alpha(\{r, g\}) = \alpha(\{r, b\}) = \alpha(\{r, y\}) = 5/8, \quad \alpha(\{b, y\}) = \alpha(\{g, y\}) = \alpha(\{g, b\}) = 1.$$

That is, $\succcurlyeq$ satisfies $\alpha(A) + \alpha(A^c) \neq 1$ for any $B \in \Sigma \setminus \Lambda$.

It turns out that Eq. (7) has a simple behavioral characterization:

[11] Notice that such preference is biseparable, so that ρ does not depend on the choice of x and y and $\alpha(u \circ xAy) = \alpha(u \circ x'Ay') \equiv \alpha(A)$ for every $x \succ y$ and $x' \succ y'$.

Proposition 17 *Given an MBA preference $\succcurlyeq$ and $x, y \in X$ such that $x \succ y$ and given $A \in \Sigma \setminus \Lambda$, eq. (7) holds for some normalized (I, u) representing $\succcurlyeq$ if and only if*

$$\frac{1}{2} c_{xAy} + \frac{1}{2} c_{xA^c y} \not\sim \frac{1}{2} x + \frac{1}{2} y \tag{8}$$

where for any $f \in \mathscr{F}$ we denote by c_f one of its certainty equivalents.

We shall see that this result proves useful in characterizing situations in which complement additivity is a full "marker" for the lack of ambiguity (see, e.g., Proposition 20 below).

We conclude this discussion by observing that the definition of the set Λ and some of the notation and terminology introduced in the previous paragraphs, allow us to provide an alternative characterization of MBis preferences complementing Prop. 9. If there are "enough" unambiguous events, Savage's Postulate P4—which is in general weaker than Binary Certainty Independence—suffices to guarantee that the preference is biseparable. A piece of notation first: Given a set $D \subseteq ba_1(\Sigma)$ and a collection $\Gamma \subseteq \Sigma$, denote $D(\Gamma) \equiv \{P(A) : \exists P \in D, A \in \Gamma\}$.

Proposition 18 *Given an MBA preference $\succcurlyeq$ with relevant priors C and unambiguous events Λ, suppose that $C(\Lambda)$ is dense in $(0,1)$. Then the following are equivalent:*

(i) *there exists a unique capacity ρ such that eq. (4) holds for any binary act $x\,A\,y$ and any normalized representation (I, u) of $\succcurlyeq$*

(ii) *$\succcurlyeq$ satisfies Savage's P4 axiom. That is, for any $A, B \in \Sigma$ and any $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$, $x\,A\,y \succcurlyeq x\,B\,y$ iff $x'\,A\,y' \succcurlyeq x'\,B\,y'$*

5.3 A "fully subjective" Expected Utility model

As observed by Epstein and Zhang (2001), there is an important sense in which Savage's (1954) construction of subjective probability is not "fully subjective." In fact, Savage (and later Machina and Schmeidler (1992), in their extension of Savage's construction) assumes exogenously that the probability which represents the decision maker's beliefs is defined on the whole σ-algebra Σ. Examples like Ellsberg's paradox suggest that a natural extension of Savage's philosophy might be to define probabilities wherever the decision maker feels comfortable, and avoid doing so otherwise, thus making also the domain of the probability charge "subjective." Epstein and Zhang propose a definition of unambiguous event, and in the spirit of Machina and

Schmeidler (1992) provide an axiomatization of preferences whose induced likelihood relations are represented by a probability charge on the set of unambiguous events —which under such axiomatic restrictions (with a minor amendment, see Kopylov (2002)) is a λ-system. Kopylov (2002) provides an analogous result using a slightly different set of axioms, generating weaker structural restrictions on the set of unambiguous events (it is what he calls a "mosaic").

The results obtained thus far allows us to provide a different "fully subjective" version of Savage's model, summarized below (cf. also Nehring (2002, Proposition 1)):[12]

Proposition 19 *If $\succcurlyeq$ is an MBA preference on $\mathscr{F}$, then there is a λ-system of events $\Lambda \subseteq \Sigma$ such that $\succcurlyeq$ has an SEU representation (with utility u) on the set $\mathscr{U}$ of the Λ-measurable acts. That is, there exist a probability charge $P : \Sigma \to [0,1]$ such that for any $f, g \in \mathscr{U}$,*

$$f \succcurlyeq g \Longleftrightarrow \int_S u(f(s))\,dP(s) \geq \int_S u(g(s))\,dP(s)$$

Moreover, P is uniquely defined on Λ.

We thus conclude that the sets of unambiguous events and acts derived above provide us with natural "endogenous" domains for a theory of subjective expected utility maximization. The decision maker assigns sharply defined probabilities only to those events that are revealed unambiguous by his behavior, assigning interval-valued probabilities to all the other events. Observe that nothing in our analysis prevents the trivial case $\Lambda = \{\emptyset, S\}$, in which SEU maximization never really appears. This is a difference with Epstein and Zhang's analysis, in which the set of unambiguous events is very rich by axiomatic requirement on the preferences.

As it is apparent from the statement, there is a sense in which our requirement on preferences is more stringent than Epstein-Zhang's. We look for a set of acts on which the preference $\succcurlyeq$ satisfies the full-blown SEU model of Savage, rather than just being probabilistically sophisticated in the sense of Machina and Schmeidler. The difference has more than just theoretical significance: The Epstein-Zhang construction is based on a definition of unambiguous event which implies that $\Lambda = \Sigma$, i.e., every event in unambiguous, when the decision maker is probabilistically sophisticated. However, as discussed at length in Ghirardato and Marinacci (2002), a

[12]As observed by Kopylov (2002), one can use Zhang's (2002) definition of unambiguous event to obtain a "fully subjective" SEU model, similarly to what we do here. The axiomatics and the sets of unambiguous events being different, the results are not equivalent.

probabilistically sophisticated decision maker might still be reacting to the presence of ambiguity. The only way to make sure that he is not is to have a (rich enough) collection of events which are *exogenously known* to be unambiguous, as a calibration device. Therefore, the conclusion that all events are unambiguous to a probabilistically sophisticated decision maker hinges on an exogenous notion of ambiguity of events which we dispense with.

A problem that is common to all such "fully subjective" approaches is that the domain of the probability charge may be far from being unique. That is, while our set Λ is certainly unique, it is not true that one cannot find another set of events on which $\succcurlyeq$ has an SEU representation. Just to make a simple example, suppose that $\succcurlyeq$ is a CEU preference on a finite S, and consider any monotonic class like $\Gamma = \{\{s_1\}, \{s_1, s_2\}, \ldots, S\}$. Given the family of acts which are Γ-measurable, there is a probability P which represents $\succcurlyeq$, as all such acts are comonotonic. On the other hand, one would have a hard time arguing that Γ is a natural domain for a "fully subjective" theory. But even imposing structural requirements on the domain (e.g., that it be a λ-system) is not enough to uniquely identify it in general.[13] There might be a multiplicity of "endogenous domains" for subjective probability, so that the choice of one must be motivated by considerations other than just finding where the decision maker is capable of formulating sharp probabilities.

5.4 Unambiguous events and weak probabilistic sophistication

A result of Marinacci (2002) shows that preferences which have an α-MEU representation (with constant $\alpha \neq 1/2$) and are probabilistically sophisticated with respect to a nonatomic prior collapse to SEU as soon as the set of priors used in the representation induces a "nontrivial" Λ (see below). Indeed, the result requires an even weaker condition than probabilistic sophistication, as spelled out below. We us the following terminology: A probability $P \in \mathrm{ba}_1(\Sigma)$ is **convex-ranged** on Σ if for any $B \in \Sigma$ and any $\alpha \in [0, P(B)]$, there exists $A \subseteq B$, $A \in \Sigma$ such that $P(A) = \alpha$.

Definition 4 A binary relation $\succcurlyeq$ on $\mathscr{F}$ has **weak probabilistic beliefs** if there exists a convex-

[13]A similar observation is made by Kopylov (2002) about his results, although he uses the weaker notion of mosaic.

ranged $P^* \in ba_1(\Sigma)$ and $x \succ y$ such that, for all $A, B \in \Sigma$,

$$P^*(A) = P^*(B) \Longrightarrow x\,A\,y \sim x\,B\,y$$

Thus, a preference has weak probabilistic beliefs if the indifference sets of the likelihood relation obtained by considering bets on events (with fixed payoffs $x \succ y$) contain the level sets of the probability P^*. The condition is weaker than probabilistic sophistication, as it does not require full agreement between the ranking induced by P^* and the likelihood ordering.[14]

We show that Marinacci's result generalizes to a broad class of MBA preferences violating the constant ambiguity index assumption. It is only needed that ambiguity attitudes over bets do not fluctuate in an "inconsistent" fashion; that is, that condition (8) holds.

Proposition 20 *Let $\succcurlyeq$ be an MBA preference with relevant priors C and unambiguous events Λ. Suppose that $\succcurlyeq$ satisfies condition (8) for any $A \in \Sigma \setminus \Lambda$, and that C only contains probability measures and satisfies $C(\Lambda) \neq \{0, 1\}$. Then, the following statements are equivalent:*

(i) *$\succcurlyeq$ has weak probabilistic beliefs.*

(ii) *$\succcurlyeq$ is an SEU preference, whose beliefs are represented by a nonatomic probability measure P^*.*

Marinacci's original result is an impossibility statement: under the assumptions of his theorem, probabilistic sophistication is compatible with α-MEU preferences only in the degenerate case of EU preferences. Our extension shows that Marinacci's result is indeed much more sweeping than that. In particular, it applies also to CEU preferences. Of course, the discussion in Marinacci (2002) on the importance of the assumptions in the theorem still applies. In particular, we want to emphasize a simple example of a class of CEU preferences which is probabilistic sophisticated without being SEU.

[14] Moreover, probabilistic sophistication imposes further requirements beyond the existence of probabilistic beliefs. While the requirement that P^* be convex-ranged is not strictly speaking part of the definition of probabilistic sophistication, all the existing axiomatizations of probabilistic sophistication in a fully subjective setting —first and foremost Machina and Schmeidler (1992)— characterize preferences inducing convex-ranged beliefs.

Example 6 On a state space (S,Σ), with S at least countably infinite, consider a nonatomic probability measure P and a strictly convex transformation function $\varphi : [0,1] \to [0,1]$, increasing and satisfying $\varphi(0)=0$ and $\varphi(1)=1$. Then a CEU preference $\succcurlyeq$ with (some utility u and) capacity $\rho = \varphi(P)$ —a subjective Rank-Dependent EU preference— is probabilistically sophisticated and not SEU. Notice that $\succcurlyeq$ is MBA (indeed, invariant biseparable) and satisfies condition (7), since it has $\alpha \equiv 1$ by the strict convexity of φ. However, it can be checked that for $\succcurlyeq$ we have $\Lambda = \{\emptyset, S\}$, so that there is no nontrivial unambiguous event.

We close by recalling an axiom from GMM which can be employed to ensure that, as in the assumptions of Proposition 20, all the elements of the set C are probability *measures*, rather than charges:

Axiom 8 (Monotone Continuity) *For all $x, y \in X$, if (A_n) is a sequence in Σ such that $A_n \downarrow \emptyset$ and if $z \in X$ is such that $y \succ z$, then $y \succcurlyeq^* x\, A_n\, z$ for some n.*

It is immediate to see that Proposition B.1 in GMM extends to MBA preferences, showing that in the presence of the previous axioms, Monotone Continuity is necessary and sufficient for C to contain only probability measures.

A Proofs of the results in Section 3

A.1 Proof of Proposition 1

We just prove the necessity part of the statement. Sufficiency follows from routine arguments. Since $\succcurlyeq$ satisfies Weak Order, Risk Independence, Archimedean, and by Kreps (1988, Theorem 5.11), it follows that there exists a nonconstant and affine function $u : X \to \mathbb{R}$ such that $x \succcurlyeq y$ if and only if $u(x) \geq u(y)$. We next show that each f in $\mathscr{F}$ admits a certainty equivalent.

Claim. For each $f \in \mathscr{F}$ there exists $x_f \in X$ such that $x_f \sim f$.

Proof of the Claim. Since $f(S)$ is a finite subset of X and since $\succcurlyeq$ is a Weak Order and it satisfies Monotonicity, it follows that there exist two consequences x_1 and x_0 in X such that $x_1 \succcurlyeq f \succcurlyeq x_0$. We denote by $x_\alpha = \alpha x_1 + (1-\alpha) x_0$ for all $\alpha \in [0,1]$. If either $x_0 \sim f$ or $x_1 \sim f$ then the statement follows. Otherwise, we have that $x_1 \succ f \succ x_0$. Define

$$\begin{aligned} U &= \{\alpha \in (0,1) : \alpha x_1 + (1-\alpha) x_0 \succ f\} \\ & \quad and \\ L &= \{\beta \in (0,1) : f \succ \beta x_1 + (1-\beta) x_0\}. \end{aligned}$$

Since $\succcurlyeq$ satisfies Archimedean, it follows that U and L are nonempty. Moreover, since $\succcurlyeq$ satisfies Weak Order and u is affine, we have that

$$\alpha > \beta \qquad \forall \alpha \in U, \forall \beta \in L. \tag{9}$$

Define $\bar{\alpha} = \inf_{\alpha \in U} \alpha$ and $\bar{\beta} = \sup_{\beta \in L} \beta$. By (9), it is immediate to see that $\bar{\alpha} \geq \bar{\beta}$. Since U and L are nonempty, we have that $1 > \bar{\alpha} \geq \bar{\beta} > 0$. Then, we have three cases:

1. $x_{\bar{\alpha}} \sim f$. The statement follows by imposing $x_f = x_{\bar{\alpha}}$.

2. $\bar{\alpha} \in U$. It follows that $x_{\bar{\alpha}} \succ f$. Since $\succcurlyeq$ satisfies Archimedean, it follows that there exists $\lambda \in (0,1)$ such that
$$x_{\lambda\bar{\alpha}} = \lambda x_{\bar{\alpha}} + (1-\lambda) x_0 \succ f,$$
thus $\lambda\bar{\alpha} \in U$ and $\lambda\bar{\alpha} < \bar{\alpha}$. This is a contradiction with $\bar{\alpha} = \inf_{\alpha \in U} \alpha$.

3. $\bar{\alpha} \notin U$ and $x_{\bar{\alpha}} \not\sim f$. Since $\succcurlyeq$ satisfies Weak Order, it follows that $f \succ x_{\bar{\alpha}}$, that is, $\bar{\alpha} \in L$. Since $\bar{\alpha} \geq \bar{\beta} = \sup_{\beta \in L} \beta$, this implies that $\bar{\alpha} = \bar{\beta}$. Since $\succcurlyeq$ satisfies Archimedean, it follows that there exists $\lambda \in (0,1)$ such that
$$f \succ \lambda x_1 + (1-\lambda) x_{\bar{\beta}} = x_{\lambda + (1-\lambda)\bar{\beta}},$$
thus $\lambda + (1-\lambda)\bar{\beta} \in L$ and $\bar{\beta} < \lambda + (1-\lambda)\bar{\beta}$. This is a contradiction with $\bar{\beta} = \sup_{\beta \in L} \beta$. $\square$

Notice that $B_0(\Sigma, u(X)) = \{u \circ f : f \in \mathscr{F}\}$. We define $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ by

$$I(a) = u\left(x_f\right) \text{ where } f \in \mathscr{F} \text{ and } u \circ f = a.$$

First, observe that I is well defined. Indeed, pick $a \in B_0(\Sigma, u(X))$. Consider $f, g \in \mathscr{F}$ such that $u \circ f = a = u \circ g$. It follows that $u\left(f(s)\right) = a(s) = u\left(g(s)\right)$ for all $s \in S$. Since u represents $\succcurlyeq$

over X, it follows that $f(s) \sim g(s)$ for all $s \in S$. By Monotonicity, we can conclude that $f \sim g$. Since $\succsim$ satisfies Weak Order, it follows that $x_f \sim x_g$. Thus, we have that

$$u\left(x_f\right) = I(a) = u\left(x_g\right).$$

Next, consider $a, b \in B_0(\Sigma, u(X))$ such that $a(s) \geq b(s)$ for all $s \in S$. It follows that there exists $f, g \in \mathscr{F}$ such that $u \circ f = a$ and $u \circ g = b$. Since $a \geq b$ and $\succsim$ satisfies Monotonicity, it follows that $f \succsim g$. Since $\succsim$ satisfies Weak Order and u represents $\succsim$ on X, we thus obtain that

$$x_f \succsim x_g \text{ and } I(a) = u\left(x_f\right) \geq u\left(x_g\right) = I(b).$$

Next, we show that I is normalized. Pick $k \in u(X)$. By assumption, there exists $x \in X$ such that $u(x) = k$. Moreover, if $a = k1_S$, then $a = u \circ f$ where $f = x$. Notice that x_f can be chosen to be equal to x. By definition of I, it follows that

$$I(a) = u\left(x_f\right) = u(x) = k.$$

Pick $f, g \in \mathscr{F}$. Since $\succsim$ satisfies Weak Order and u represents $\succsim$ restricted to X, we have that

$$f \succsim g \Leftrightarrow x_f \succsim x_g \Leftrightarrow u\left(x_f\right) \geq u\left(x_g\right) \Leftrightarrow I\left(u \circ f\right) \geq I\left(u \circ g\right). \tag{10}$$

Finally, we are left to prove the continuity of I. First, observe that $I(B_0(\Sigma, u(X))) = u(X)$. Consider $a, b \in B_0(\Sigma, u(X))$ such that $a \leq b$ and $I(b) > k$ where $k \in \mathbb{R}$. It follows that there exist f and g in $\mathscr{F}$ such that $a = u \circ f$ and $b = u \circ g$. We have two cases:

1. $I(a) > k$. In this case, $B_0(\Sigma, u(X)) \ni \alpha b + (1-\alpha) a \geq a$ for all $\alpha \in (0,1)$. Since I is monotonic, it follows that

$$I(\alpha b + (1-\alpha) a) \geq I(a) > k.$$

2. $I(a) \leq k$. Since $I(b) > k$, we have that there exists $k' \in u(X)$ such that $I(b) > k' > k \geq I(a)$. This implies that there exists $x' \in X$ such that $u(x') = k'$. By (10), we have that $g \succ x' \succ f$. Since $\succsim$ satisfies Archimedean, it follows that there exists $\alpha \in (0,1)$ such that $\alpha g + (1-\alpha) f \succ x'$. Since u is affine and by (10), we have that

$$I(\alpha b + (1-\alpha) a) = I\left(u \circ \left(\alpha g + (1-\alpha) f\right)\right) > I\left(u\left(x'\right)\right) = u\left(x'\right) = k' > k.$$

It follows that I satisfies condition (iv) of C3M Lemma 45. By Proposition 46 of C3M, it follows that I is lower semicontinuous. Upper semicontinuity follows by a symmetric argument.

The uniqueness part of the statement follows from routine arguments.

B Proofs of the results in Section 4

B.1 Proof of Theorem 8

Assume that $\succsim$ satisfies Weak Order, Risk Independence, Archimedean, Monotonicity. By Proposition 1, it follows that $\succsim$ satisfies Continuity as defined in C3M.

(i) implies (ii). By C3M (Theorem 3), if $\succsim$ satisfies Convexity then there exists a nonconstant affine function $u : X \to \mathbb{R}$ and a function $G^\star : u(X) \times ba_1(\Sigma) \to (-\infty, \infty]$ such that the functional $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ defined by

$$I(a) = \min_{P \in ba_1(\Sigma)} G^\star \left(\int a \, dP, P \right)$$

is well defined and such that

$$f \succsim g \Leftrightarrow I(u \circ f) \geq I(u \circ g).$$

Moreover, $G^\star(t, P) = \sup_{h \in \mathscr{F}} \left\{ u(x_h) : \int u \circ h \, dP \leq t \right\}$ for all $(t, P) \in u(X) \times ba_1(\Sigma)$. Fix an act $f \in \mathscr{F}$. Consider $P_f \in ba_1(\Sigma)$ such that $G^\star \left(\int u \circ f \, dP_f, P_f \right) = I(u \cdot f)$. Define $t = \int u \circ f \, dP$. Assume that $g \in \mathscr{F}$ is such that $\int u \circ f \, dP_f \geq \int u \circ g \, dP_f$. By the definition of $G^\star \left(t, P_f \right)$ and since $t \geq \int u \circ g \, dP_f$, it follows that $u \left(x_g \right) \leq G^\star \left(t, P_f \right) = I(u \circ f)$. Since I is normalized, it follows that $I(u \circ g) = I \left(u \left(x_g \right) \right) \leq I(u \circ f)$, that is, $f \succsim g$. Summing up, if we define the binary relation $\geq_f$ on $\mathscr{F}$ by

$$f_1 \geq_f f_2 \Leftrightarrow \int u \circ f_1 \, dP \geq \int u \circ f_2 \, dP$$

then we have that $f \geq_f g$ implies that $f \succsim g$. Since f was arbitrarily chosen, the statement follows.

(ii) implies (i). By Proposition 1, it follows that there exists a nonconstant affine function $u : X \to \mathbb{R}$ and a normalized, monotonic, and continuous functional $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that $f \succsim g$ if and only if $I(u \circ f) \geq I(u \circ g)$. We define $G^\star : u(X) \times ba_1(\Sigma) \to (-\infty, \infty]$ by

$$G^\star(t, P) = \sup_{h \in \mathscr{F}} \left\{ u(x_h) : \int u \circ h \, dP \leq t \right\} \qquad \forall (t, P) \in u(X) \times ba_1(\Sigma).$$

Notice that $G^\star(\cdot, P) : \mathbb{R} \to (-\infty, \infty]$ is an increasing function for all $P \in ba_1(\Sigma)$. Moreover, observe that $I(u \circ f) = u \left(x_f \right) \leq G^\star \left(\int u \circ f \, dP, P \right)$ for all $f \in \mathscr{F}$ and for all $P \in ba_1(\Sigma)$. It follows that

$$I(u \circ f) \leq \inf_{P \in ba_1(\Sigma)} G^\star \left(\int u \circ f \, dP, P \right) \qquad \forall f \in \mathscr{F}.$$

Pick $f \in \mathscr{F}$. By assumption, there exists a SEU preference $\geq_f$ such that

$$f \geq_f g \Rightarrow f \succcurlyeq g.$$

In other words, we have that there exists $\bar{P} \in ba_1(\Sigma)$ such that

$$\int u \circ f \, d\bar{P} \geq \int u \circ g \, d\bar{P} \Rightarrow I(u \circ f) \geq I(u \circ g).$$

By definition of $G^\star$, this implies that

$$G^\star\left(\int u \circ f \, d\bar{P}, \bar{P}\right) = I(u \circ f).$$

Since f was arbitrarily chosen, we can conclude that

$$I(u \circ f) = \min_{P \in ba_1(\Sigma)} G^\star\left(\int u \circ f \, dP, P\right) \qquad \forall f \in \mathscr{F}. \tag{11}$$

Consider $f, g \in \mathscr{F}$ such that $f \sim g$. Define $k = I(u \circ f) = I(u \circ g)$. Define

$$U_P(k) = \left\{h \in \mathscr{F} : G^\star\left(\int u \circ f \, dP, P\right) \geq k\right\}.$$

Since $G^\star(\cdot, P)$ is an increasing function for all $P \in ba_1(\Sigma)$, it follows that $U_P(k)$ is closed under convex combinations for all $P \in ba_1(\Sigma)$. By (11), it follows that $f, g \in U_P(k)$ for all $P \in ba_1(\Sigma)$. This implies that

$$G^\star\left(\int u \circ (\alpha f + (1-\alpha) g) \, dP, P\right) \geq k \qquad \forall \alpha \in (0,1), \forall P \in ba_1(\Sigma).$$

By (11), we can conclude that $I(u \circ (\alpha f + (1-\alpha) g)) \geq I(u \circ f)$, that is, $\alpha f + (1-\alpha) g \succcurlyeq f$. Since f and g were arbitrarily chosen, it follows that $\succcurlyeq$ satisfies Convexity.

B.2 Proof of Proposition 9

Suppose $\succcurlyeq$ is biseparable, so $\rho_{x,y}$ is independent of x, y. Then, for all $x, y \in X$ with $x \succ y$ and all $A \in \Sigma$, $I(u \circ xAy) = \rho(A)u(x) + [1-\rho(A)]u(y)$. Furthermore, if $x \sim y$, $I(u \circ xAy) = I(u(x)) = u(x) = \rho(A)u(x) + [1-\rho(A)]u(y)$; the first equality follows from monotonicity. Thus, $I(u \circ xAy) = \rho(A)u(x) + [1-\rho(A)]u(y)$ whenever $x \succcurlyeq y$.

Now, for any two binary acts f, g, we can always choose $A, A' \in \Sigma$ so that $f = xAy$ and $g = x'A'y'$, with $x \succcurlyeq y$ and $x' \succcurlyeq y'$. Then, for all $z \in X$ and $\lambda \in (0,1]$, $\lambda f + (1-\lambda) z = (\lambda x + (1-$

$\lambda)z)A(\lambda x' + (1-\lambda)z)$, and so $I(u \circ [\lambda f + (1-\lambda)z]) = \rho(A)u(\lambda x + (1-\lambda)z) + [1-\rho(A)]u(\lambda x' + (1-\lambda)z) = \lambda I(u \circ f) + (1-\lambda)u(z)$, and similarly for $\lambda g + (1-\lambda)z$. Axiom 7 follows.

In the opposite direction, suppose Axiom 7 holds. Fix $A \in \Sigma$ and consider the fictitious state space $S_A = \{s, t\}$ acts $\mathscr{F}_A = X^{S_A}$, and preferences $\succcurlyeq_A$ on $\mathscr{F}_A$ defined by $f_A \succcurlyeq_A g_A$ iff $f_A(s) A f_A(t) \succcurlyeq g_A(s) A g_A(t)$ for all $f_A, g_A \in \mathscr{F}_A$. Then $\succcurlyeq_A$ satisfies the GMM axioms and admits a representation (I_A, u_A), with I_A monotonic, constant-linear and normalized; furthermore, we can assume w.l.o.g. that $u_A = u$, because $\succcurlyeq_A$ and $\succcurlyeq$ agree on constant acts.

Now consider $x, y, x', y' \in X$ with $x \succ y$ and $x' \succ y'$. There exist $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$, such that $\alpha u(x) + \beta = u(x')$ and $\alpha u(y) + \beta = u(y')$: hence, if $f_A, g_A \in \mathscr{F}_A$ are defined by $f_A(s) = x$, $f_A(t) = y$, $g_A(s) = x'$ and $g_A(t) = y'$, we have $I_A(u \circ g_A) = \alpha I_A(u \circ f_A) + \beta$; therefore, if $c_{f_A}, c_{g_A} \in X$ are the $\succcurlyeq_A$–certainty equivalents of f_A and g_A respectively, $u(c_{g_A}) = \alpha u(c_{f_A}) + \beta$ as well.

Now $c_{f_A} \sim_A f_A$ iff $c_{f_A} \sim x A y$, and similarly $c_{g_A} \sim_A g_A$ iff $c_{g_A} \sim x' A y$. It follows that $I(u \circ x' A y') = u(c_{g_A}) = \alpha u(c_{f_A}) + \beta = \alpha I(u \circ x A y) + \beta$; Eq. (6) and the fact that $\alpha u(x) + \beta = u(x')$ and $\alpha u(y) + \beta = u(y')$ then imply that $\rho_{x,y}(A) = \rho_{x',y'}(A)$.

Hence, a set function $\rho : \Sigma \to [0,1]$ that satisfies Eq. (4) can be uniquely defined; it is then straightforward to verify that ρ is in fact a capacity.

C Proofs of the results in Sec. 5

Throughout this appendix we write $\underline{C}(A)$ (resp. $\overline{C}(A)$) in place of $\underline{C}(1 A 0)$ (resp. $\overline{C}(1 A 0)$). We also write $\alpha_u \circ f$ in lieu of $\alpha(u \circ f)$. Notice that for expositional reasons, the results are proved in a different order than that in the main text.

We also make a useful observation. Call **reduced** an act f such that $f(s) \sim f(s')$ implies $f(s) = f(s')$. Given any non-reduced act f, we observe that there is a reduced act which, while being state-by-state indifferent to f, "simplifies" it by restricting its range so that it only contains non-indifferent payoffs. A $\succcurlyeq$**-reduction** g of f is a reduced act $g = \{x_1, A_1; ...; x_n, A_n\}$, with $x_1 \succ x_2 \succ ... \succ x_n$ and $\{A_1, ..., A_n\}$ a partition of S in Σ, such that $g(s) \sim f(s)$ for all $s \in S$. Finally, given a reduced act $f = \{x_1, A_1; ...; x_n, A_n\}$, with $x_1 \succ x_2 \succ ... \succ x_n$ and $\{A_1, ..., A_n\}$ a partition of S in Σ, and a permutation σ of $\{x_1, x_2, ..., x_n\}$, define the **permuted act** f_σ as $f_\sigma = \{\sigma(x_1), A_1; ...; \sigma(x_n), A_n\}$. The following lemma is immediately verified:

Lemma 21 *Given an MBA preference $\succcurlyeq$, f is unambiguous if and only if there is some $\succcurlyeq$-reduction g of f for which g_σ is crisp for every permutation σ of g's payoffs.*

Proof: Note that a $\succcurlyeq$-reduction of an act f is a $\succcurlyeq$-permutation according to Def. 2. Hence, if f is unambiguous and g is a $\succcurlyeq$-reduction of g, every permutation of g is a $\succcurlyeq$-permutation of f, and therefore it is crisp. Conversely, let $\bar{f}$ be a $\succcurlyeq$-permutation of f, and let g be a $\succcurlyeq$-reduction of f for which g_σ is crisp for every permutation σ. In particular, there is a permutation $\bar{\sigma}$ such that $g_{\bar{\sigma}}(s) \sim \bar{f}(s)$ for all s. By assumption, $g_{\bar{\sigma}}$ is crisp, so $g_{\bar{\sigma}} \sim^* x$ for some $x \in X$. But then, by monotonicity of $\succcurlyeq^*$, also $\bar{f} \sim^* x$, i.e. $\bar{f}$ is crisp. Thus, f is unambiguous. ■

C.1 Proof of Proposition 10

Let $\mathscr{U}'$ be the set defined in the statement of the proposition. More precisely, let $\mathscr{U}'$ be the union of all sets $\mathscr{V}$ of crisp acts that are closed under $\succcurlyeq$-permutations. Notice that, if f is crisp, the set of all $\succcurlyeq$-permutations of f is one such set $\mathscr{V}$, because the $\succcurlyeq$-permutation relation is an equivalence. Furthermore, all constants are crisp; thus, $\mathscr{U}'$ is both well-defined and non-empty.

We will prove that $\mathscr{U} = \mathscr{U}'$. We begin with the observation that any act f whose $\succcurlyeq$-permutations are all crisp must belong to $\mathscr{U}'$. In fact, if $f \notin \mathscr{U}'$, one could add f and all its $\succcurlyeq$-permutations to $\mathscr{U}'$, thus obtaining a larger set and contradicting the definition of $\mathscr{U}'$. Conversely, if $f \in \mathscr{U}'$, then any $\succcurlyeq$-permutation of f must be in $\mathscr{U}'$, hence crisp. This proves that f is unambiguous.

C.2 Proofs of Propositions 12 and 13, and of Corollary 14

A lemma first:

Lemma 22 *Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, c \in \mathbb{R}$ be such that $\sum_{h=1}^n a_h b_{\sigma(h)} = c$ for all permutations $\sigma \in Per(n)$. Then either $a_1 = a_2 = ... = a_n$ or $b_1 = b_2 = ... = b_n$.*

Proof. By contradiction, assume that there exist $i, j \in \{1, \ldots, n\}$ such that $a_i \neq a_j$ and $k, l \in \{1, \ldots, n\}$ such that $b_k \neq b_l$. Consider a permutation σ such that $\sigma(i) = k$ and $\sigma(j) = l$, and the permutation $\sigma' = \sigma(kl)$ obtained applying σ and then switching around k and l. It follows that

$$a_i b_k + a_j b_l + \sum_{h \neq i,j} a_h b_{\sigma(h)} = \sum_{h=1}^{n} a_h b_{\sigma(h)} = c = \sum_{h=1}^{n} a_h b_{\sigma'(h)} = a_i b_l + a_j b_k + \sum_{h \neq i,j} a_h b_{\sigma(h)},$$

whence $a_i b_k + a_j b_l = a_i b_l + a_j b_k$. That is, $a_i(b_k - b_l) = a_j(b_k - b_l)$, which implies $a_i = a_j$, a contradiction. $\square$

C.2.1 Proofs of Propositions 12 and 13.

We prove the two Propositions by showing that the following statements are equivalent for any $A \in \Sigma$:

(i) $A \in \Lambda$.

(ii) $P(A) = Q(A) = \rho_{x,y}(A)$ for all $P, Q \in C$ and $x \succ y$.

(iii) For every $x \nsim y$, the act xAy is crisp.

(iv) For some $x \succ y$, the act xAy is crisp.

$(i) \Rightarrow (ii)$: Suppose that $A \in \Lambda$. Therefore, there is $f \in \mathscr{U}$ and $x \in X$ such that $A = \{s \in S : f(s) \sim x\}$. Since $f \in \mathscr{U}$, there exists a reduction $\{x_i, A_i\}$ of f (with $x_i \nsim x_j$ for every $i \neq j$) such that for every permutation $\sigma \in \mathrm{Per}(n)$, $\{x_{\sigma(i)}, A_i\}$ is crisp. Then

$$\sum_{i=1}^{n} u\left(x_{\sigma(i)}\right) P(A_i) = \sum_{i=1}^{n} u\left(x_{\sigma(i)}\right) Q(A_i)$$

and

$$\sum_{i=1}^{n} [P(A_i) - Q(A_i)]\, u\left(x_{\sigma(i)}\right) = 0. \tag{12}$$

Therefore, by the Lemma above, either $P(A_1) - Q(A_1) = P(A_2) - Q(A_2) = \cdots = P(A_n) - Q(A_n) = b$ or $u(x_1) = u(x_2) = ... = u(x_n)$. In the former case $1 = \sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{n} Q(A_i) + nb = 1 + nb$. Therefore $b = 0$ and A_i satisfies condition (ii) for any for all $i = 1, 2, ..., n$. As $A \in \{A_i : i = 1, \ldots, n\}$, the conclusion follows. In the latter case, $n = 1$ and $A = \{f \sim x\}$ is then either S or $\emptyset$ (depending on whether $x \sim x_1$ or not). Clearly $P(S) = Q(S) = 1$ and $P(\emptyset) = Q(\emptyset) = 0$ for any $P, Q \in C$, so that once again (ii) follows, also proving that $\{\emptyset, S\} \in \Lambda$. Notice finally that if $P(A) = Q(A)$ for all $P, Q \in C$, it then follows from the definition of $\rho_{x,y}$ that $\rho_{x,y}(A) = P(A) = Q(A)$.

$(ii) \Rightarrow (iii)$: Let $x \nsim y$. Then,

$$P(u(x\,A\,y)) = (u(x) - u(y))\,P(A) + u(y) = (u(x) - u(y))\,Q(A) + u(y) = Q(u(x\,A\,y))$$

for all $P, Q \in C$. That is, $x\,A\,y$ is crisp.

$(iii) \Rightarrow (iv)$: Obvious.

$(iv) \Rightarrow (i)$: Let $x \succ y$ be s.t. $x\,A\,y$ is crisp. We want to show that $f = x\,A\,y \in \mathscr{U}$. This is the case if f has a $\succcurlyeq$-reduction whose permutations are all crisp. But f is a reduced act, and the only permutation of f is $g = x\,A^c\,y$. Since f is crisp,

$$P(u(x\,A\,y)) = (u(x) - u(y))\,P(A) + u(y) = (u(x) - u(y))\,Q(A) + u(y) = Q(u(x\,A\,y))$$

which implies that $P(A) = Q(A)$. In turn, this implies $P(A^c) = Q(A^c)$, so that

$$P(u(x\,A^c\,y)) = (u(x) - u(y))\,P(A^c) + u(y) = (u(x) - u(y))\,Q(A^c) + u(y) = Q(u(x\,A^c\,y))$$

and g is also crisp. Notice that this argument also shows that if $A \in \Lambda$, then $A^c \in \Lambda$. □

C.2.2 Proof of Corollary 14.

We have proved properties 1 and 2 of a λ-system in the course of proving the previous two propositions, so we only need to show property 3. If $A, B \in \Lambda$ and $A \cap B = \emptyset$, for all $P, Q \in C$, $P(A \cup B) = P(A) + P(B) = Q(A) + Q(B) = Q(A \cup B)$, hence $A \cup B \in \Lambda$. □

C.3 Proofs of Theorem 11 and Corollary 15

Using the definition of Λ and the characterization of Proposition 12, the statements to be shown equivalent are reformulated as follows:

(i) $f \in \mathscr{U}$.

(ii) $\{s \in S : f(s) \succcurlyeq x\} \in \Lambda$ for all $x \in X$.

(iii) $\{s \in S : f(s) \sim x\} \in \Lambda$ for all $x \in X$.

(iv) $\{s \in S : u \circ f(s) \geq a\} \in \Lambda$ for all $a \in \mathbb{R}$.

(v) $\{s \in S : u \circ f(s) = a\} \in \Lambda$ for all $a \in \mathbb{R}$.

(vi) For every $\succcurlyeq$-reduction $\{x_i, A_i\}_{i=1}^n$ of f (with $x_i \not\sim x_j$ if $i \neq j$), $\{A_1, A_2, ..., A_n\}$ is a partition of S in Λ.

(vii) There exist a $\succcurlyeq$-reduction $\{x_i, A_i\}_{i=1}^n$ of f, with $\{A_1, A_2, ..., A_n\}$ a partition of S in Λ (and $x_i \not\sim x_j$ if $i \neq j$).

The equivalence of (i) and (vii) follows immediately from the argument used to show $(i) \Rightarrow (ii)$ in appendix C.2.1 and from Proposition 13. We shall now prove that statements $(ii) - (vii)$ are equivalent.

$(vii) \Rightarrow (ii)$: Given f, let $g = \{x_i, A_i\}_{i=1}^n$ be its $\succcurlyeq$-reduction with $\{A_1, A_2, ..., A_n\}$ a partition of S in Λ (and $x_i \not\sim x_j$ if $i \neq j$), so that $u \circ f = u \circ g = \Sigma_{i=1}^n u(x_i) 1_{A_i}$. For all $x \in X$, $\{s \in S : f(s) \succcurlyeq x\} = \{s \in S : u \circ f(s) \geq u(x)\}$. Hence, $\{s \in S : f(s) \succcurlyeq x\}$ is a disjoint union of elements of Λ, which is a λ-system.

$(ii) \Rightarrow (iv)$: Notice that $u(X)$ is an interval. Let $a \in \mathbb{R}$. If $a \in u(X)$, say $a = u(x')$, then $\{s \in S : u \circ f(s) \geq a\} = \{s \in S : f(s) \succcurlyeq x'\} \in \Lambda$. Else, either $a < t$ for all $t \in u(X)$, and then $\{s \in S : u \circ f(s) \geq a\} = S \in \Lambda$, or $a > t$ for all $t \in u(X)$, and then $\{s \in S : u \circ f(s) \geq a\} = \emptyset \in \Lambda$.

$(iv) \Rightarrow (v)$: Let $u \circ f = \Sigma_{i=1}^n a_i 1_{A_i}$, with $\{A_1, A_2, ..., A_n\}$ a partition of S in Σ and $a_1 > a_2 > ... > a_n$. If $a \notin \{a_1, a_2, ..., a_n\}$, then $\{s \in S : u \circ f(s) = a\} = \emptyset \in \Lambda$. The set $A_1 = \{s \in S : u \circ f(s) = a_1\} = \{s \in S : u \circ f(s) \geq a_1\} \in \Lambda$. For all $i \geq 2$, then $\Lambda \ni \{s \in S : u \circ f(s) \geq a_i\} = \{s \in S : u \circ f(s) \in \{a_1, a_2, ..., a_i\}\} = \bigcup_{j=1}^i \left\{s \in S : u \circ f(s) = a_j\right\} = A_1 \cup A_2 \cup ... \cup A_i$. Therefore, for all $i \geq 2$, $\{s \in S : u \circ f(s) = a_i\} = A_i = (A_1 \cup A_2 \cup ... \cup A_i) \setminus (A_1 \cup A_2 \cup ... \cup A_{i-1}) \in \Lambda$ (remember that if Λ is a λ-system, $B, C \in \Lambda$ and $C \subseteq B$ imply $B \setminus C \in \Lambda$).

$(v) \Rightarrow (iii)$: For all $x \in X$, $\{s \in S : f(s) \sim x\} = \{s \in S : u \circ f(s) = u(x)\} \in \Lambda$.

$(iii) \Rightarrow (vi)$: Given f, let $g = \{x_i, A_i\}_{i=1}^n$ be any one of its $\succcurlyeq$-reductions, with $\{A_1, A_2, ..., A_n\}$ a partition of S in Σ (and $x_i \not\sim x_j$ if $i \neq j$). W.l.o.g. set $x_1 \succ x_2 \succ ... \succ x_n$ so that $u \circ f = u \circ g = \Sigma_{i=1}^n u(x_i) 1_{A_i}$ and $u(x_1) > u(x_2) > ... > u(x_n)$. Therefore, $A_i = \{s \in S : u \circ f(s) = u(x_i)\} = \{s \in S : f(s) \sim x_i\} \in \Lambda$ for all $i = 1, ..., n$.

$(vi) \Rightarrow (vii)$: Trivial.

C.4 Proofs of Propositions 16, 17 and 18

C.4.1 Proposition 16

Given $x \succ y$, define $\rho_{x,y}$ via Eq. (5). Then $\rho_{x,y}(A)+\rho_{x,y}(A^c)=1$ iff

$$[\alpha_u(x\,A\,y)(\underline{C}(A)-\overline{C}(A))+\overline{C}(A)]+[\alpha_u(x\,A^c\,y)(\underline{C}(A^c)-\overline{C}(A^c))+\overline{C}(A^c)]=1$$

which, since $\overline{C}(A^c)=1-\underline{C}(A)$ and $\underline{C}(A^c)=1-\overline{C}(A)$, is equivalent to

$$\alpha_u(x\,A\,y)(\underline{C}(A)-\overline{C}(A))+\alpha_u(x\,A^c\,y)(\underline{C}(A)-\overline{C}(A))+(\overline{C}(A)-\underline{C}(A))=0$$

in turn equivalent to

$$(\overline{C}(A)-\underline{C}(A))=(\alpha_u(x\,A\,y)+\alpha_u(x\,A^c\,y))(\overline{C}(A)-\underline{C}(A))$$

Therefore, $\rho_{x,y}(A)+\rho_{x,y}(A^c)=1$ iff either $\overline{C}(A)=\underline{C}(A)$ or $\alpha_u(x\,A\,y)+\alpha_u(x\,A^c\,y)=1$.

C.4.2 Proposition 17

Notice that under the representation assumptions, for the given $x \succ y$ eq. (8) holds iff

$$\frac{1}{2}I(u\circ(x\,A\,y))+\frac{1}{2}I(u\circ(x\,A^c\,y))\neq\frac{1}{2}u(x)+\frac{1}{2}u(y)$$

If we recall eq. (6), the l.h.s. can be rewritten as follows:

$$u(y)+\frac{1}{2}\left[u(x)-u(y)\right]\left\{\alpha_u(x\,A\,y)(\underline{C}(A)-\overline{C}(A))+\overline{C}(A)+\alpha_u(x\,A^c\,y)(\underline{C}(A^c)-\overline{C}(A^c))+\overline{C}(A^c)\right\}$$

so that after rewriting and using the shorthand $\Delta C(A)=\overline{C}(A)-\underline{C}(A))$ we obtain

$$\begin{aligned}\frac{1}{2}I(u\circ(x\,A\,y))+\frac{1}{2}I(u\circ(x\,A^c\,y)) &= u(y)+\frac{1}{2}\left[u(x)-u(y)\right]\left\{\left(\alpha_u(x\,A\,y)+\alpha_u(x\,A^c\,y)-1\right)(-\Delta C(A))+1\right\}\\ &= u(y)+\frac{1}{2}\left[u(x)-u(y)\right]\left\{\left(1-\alpha_u(x\,A\,y)+\alpha_u(x\,A^c\,y)\right)\Delta C(A)+1\right\}\end{aligned}$$

We thus get

$$\frac{1}{2}I(u\circ(x\,A\,y))+\frac{1}{2}I(u\circ(x\,A^c\,y))=\frac{1}{2}u(x)+\frac{1}{2}u(y)+\frac{1}{2}\left[u(x)-u(y)\right]\left\{\left(1-\alpha_u(x\,A\,y)-\alpha_u(x\,A^c\,y)\right)\Delta C(A)\right\}$$

Therefore, eq. (8) holds iff

$$\frac{1}{2}u(x)+\frac{1}{2}u(y)\neq\frac{1}{2}u(x)+\frac{1}{2}u(y)+\frac{1}{2}\left[u(x)-u(y)\right]\left\{\left(1-\alpha_u(x\,A\,y)-\alpha_u(x\,A^c\,y)\right)\Delta C(A)\right\}$$

which, since $A\in\Sigma\setminus\Lambda$ implies $\Delta C(A)>0$ (and $x\succ y$ implies $u(x)>u(y)$), holds iff

$$1\neq\alpha_u(x\,A\,y)+\alpha_u(x\,A^c\,y)$$

concluding the proof.

C.4.3 Proposition 18

We begin by recalling that, given a normalized representation (I, u) and $x \succ y$, $x A y \succcurlyeq x B y$ iff

$$\alpha_u(x A y)(\underline{C}(A) - \overline{C}(A)) + \overline{C}(A) \geq \alpha_u(x B y)(\underline{C}(B) - \overline{C}(B)) + \overline{C}(B)$$

with the left-hand (resp. right-hand) side collapsing to $P(A) = \underline{C}(A) = \overline{C}(A)$ (resp. $P(B) = \underline{C}(B) = \overline{C}(B)$) if $A \in \Lambda$ (resp. $B \in \Lambda$). Clearly, if there is a unique ρ for which Eq. (4) holds, $\alpha_u(x A y)$ does not depend on x or y. Hence, the implication (i) ⇒ (ii) is trivial. We prove that (ii) ⇒ (i).

It is enough to show that $\alpha_u(x A y) = \alpha_u(x' A y')$ for every u and $x \succ y$, $x' \succ y'$: this implies that $\rho_{x,y}(A) = \rho_{x',y'}(A)$ whenever $x \succ y$, $x' \succ y'$, so a set function $\rho : \Sigma \to [0,1]$ that satisfies Eq. (4) can be uniquely defined; it is then straightforward to verify that ρ is a capacity.

Thus, argue by contradiction, and suppose w.l.o.g. that $\alpha_u(x A y) > \alpha_u(x' A y')$. By the richness assumption on $C(\Lambda)$, there exists $B \in \Lambda$ such that $P(B) = \underline{C}(B) = \overline{C}(B)$ satisfies

$$\alpha_u(x A y)(\underline{C}(A) - \overline{C}(A)) + \overline{C}(A) < P(B) < \alpha_u(x' A y')(\underline{C}(A) - \overline{C}(A)) + \overline{C}(A)$$

but then we have a violation of P4, since the first inequality implies $x B y \succ x A y$, and the second implies $x' A y' \succ x' B y'$. Thus, we must have $\alpha_u(x A y) = \alpha_u(x' A y')$. This completes the proof.

C.5 Proof of Proposition 20

The implication $(ii) \Rightarrow (i)$ being trivial, we prove $(i) \Rightarrow (ii)$. By weak probabilistic beliefs (assumption (i)), there exists $x \succ y$ and a convex-ranged probability charge P^* such that for all $A, B \in \Sigma$

$$P^*(A) = P^*(B) \Longrightarrow \rho_{x,y}(A) = \rho_{x,y}(B)$$

Consider now $A \in \Lambda$ such that $\underline{C}(A) = \overline{C}(A) = \rho_{x,y}(A) \in (0,1)$. It follows that $P^*(A) \in (0,1)$, since $P^*(A) = 0$ (resp. $P^*(A) = 1$) implies $P^*(A) = P^*(\emptyset)$ (resp. $P^*(A) = P^*(S)$), which in turn implies by (i) that $\rho_{x,y}(A) = \rho_{x,y}(\emptyset) = 0$ (resp. $\rho_{x,y}(A) = \rho_{x,y}(S) = 1$), a contradiction.

Let $B \in \Sigma$ be such that $P^*(B) = P^*(A)$, so that (i) implies $\rho_{x,y}(B) = \rho_{x,y}(A)$ and (since $P^*(B^c) = P^*(A^c)$ as well) $\rho_{x,y}(B^c) = \rho_{x,y}(A^c)$. It follows that

$$\rho_{x,y}(B) + \rho_{x,y}(B^c) = \rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$$

where the last equality follows from Proposition 16.

We also know that $\succcurlyeq$ satisfies condition (8) for any $A \in \Sigma \setminus \Lambda$. It therefore follows from Propositions 17 and 16 that $\rho_{x,y}(B) + \rho_{x,y}(B^c) = 1$ implies $B \in \Lambda$, so that $\rho_{x,y}(B) = P(B)$ for any $P \in C$. We can thus conclude that with the chosen $A \in \Lambda$ we have for every $B \in \Sigma$ and $P \in C$,

$$P^*(B) = P^*(A) \Longrightarrow P(B) = P(A)$$

so that $P^* = P$ follows from Theorem 2 of Marinacci (2002). Since this is true for any $P \in C$ —that is, $C = \{P^*\}$— we conclude that $\succcurlyeq$ is a SEU preference with probability P^*.

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