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Ricardo Alberto MARQUES PEREIRA
Dipartimento di Informatica e Studi Aziendali
Università degli Studi di Trento
Via Inama 5, TN 38122 Trento ITALIA
Tel +39-0461-282147 Fax +39-0461-282124
E-mail: ricalb.marper@unitn.it

The dynamics of consensus in group decision making: investigating the pairwise interactions between fuzzy preferences

Mario Fedrizzi, Michele Fedrizzi, R. A. Marques Pereira, and Matteo Brunelli

Abstract In this paper we present an overview of the soft consensus model in group decision making and we investigate the dynamical patterns generated by the fundamental pairwise preference interactions on which the model is based.
The dynamical mechanism of the soft consensus model is driven by the minimization of a cost function combining a collective measure of dissensus with an individual mechanism of opinion changing aversion. The dissensus measure plays a key role in the model and induces a network of pairwise interactions between the individual preferences.
The structure of fuzzy relations is present at both the individual and the collective levels of description of the soft consensus model: pairwise preference intensities between alternatives at the individual level, and pairwise interaction coefficients between decision makers at the collective level.
The collective measure of dissensus is based on non linear scaling functions of the linguistic quantifier type and expresses the degree to which most of the decision makers disagree with respect to their preferences regarding the most relevant alternatives. The graded notion of consensus underlying the dissensus measure is central to the dynamical unfolding of the model.
The original formulation of the soft consensus model in terms of standard numerical preferences has been recently extended in order to allow decision makers to express their preferences by means of triangular fuzzy numbers. An appropriate notion of distance between triangular fuzzy numbers has been chosen for the construction of the collective dissensus measure.
In the extended formulation of the soft consensus model the extra degrees of freedom associated with the triangular fuzzy preferences, combined with non linear na-

Mario Fedrizzi, Michele Fedrizzi, Ricardo Alberto Marques Pereira
University of Trento, Department of Computer and Management Science, via Inama 5, 38122 Trento, Italy e-mail: mario.fedrizzi@unitn.it, michele.fedrizzi@unitn.it, ricalb.marper@unitn.it

Matteo Brunelli
IAMSR, Turku Centre for Computer Science, Abo Akademi University, Joukahainengatan 3-5A, FIN-20520 Abo, Finland and University of Trento, Department of Computer and Management Science, via Inama 5, 38122 Trento, Italy e-mail: matteo.brunelli@abo.fi

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ture of the pairwise preference interactions, generate various interesting and suggestive dynamical patterns. In the present paper we investigate these dynamical patterns which are illustrated by means of a number of computer simulations.

1 Introduction

In the study of aggregational models of group decision making the central notions of interaction and consensus have been the subject of a great deal of investigation. Fundamental contributions in this general area of research have been made by: Shapley (1953) on cooperative game theory [65]; French (1956) and Harary (1959) on social power theory [31] [44]; DeGroot (1974), Chatterjee and Seneta (1977), Berger (1981), Kelly (1981), and French (1981) on DeGroot's consensus formation model [16] [13] [7] [53] [32]; Sen (1982) on models of choice and welfare [64]; Wagner (1978, 1982) and Lehrer and Wagner (1981) on the rational choice model [73] [55] [74]; Anderson and Graesser (1976), Anderson (1981, 1991), and Graesser (1991) on the information integration model [4] [2] [3] [42]; Davis (1973, 1996) on the social decision scheme model [14] [15]; and Friedkin (1990, 1991, 1993, 1998, 1999, 2001), Friedkin and Johnsen (1990, 1997, 1999), and Marsden and Friedkin (1993, 1994) on social influence network theory [33] [38] [34] [35] [59] [60] [39] [36] [37] [40] [41].
In the classical literature stream indicated above the notion of consensus has conventionally been understood in terms of strict and unanimous agreement. However, since decision makers typically have different and conflicting opinions to a lesser or greater extent, the traditional strict meaning of consensus is often unrealistic. The human perception of consensus is typically 'softer', and people are generally willing to accept that consensus has been reached when most actors agree on the preferences associated to the most relevant alternatives.
In this different perspective, and in parallel with the traditional approach mostly formulated on a probabilistic basis, Ragade (1976) and Bezdek, Spillman, and Spillman (1977, 1978, 1979, 1980) proposed to conceptualize consensus within the fuzzy framework [63] [8] [9] [10] [66] [67] [68]. A few years later, combining the fuzzy notion of consensus with the expressive power of linguistic quantifiers, Kacprzyk and Fedrizzi (1986, 1988, 1989) and Kacprzyk, Fedrizzi, and Nurmi (1992, 1993, 1997) developed the so-called soft consensus measure in the context of fuzzy preference relations [47] [48] [49] [50] [19] [51] and considered various interesting implications of the model in the context of decision support, see Fedrizzi, Kacprzyk, and Zadrozny (1988) and Carlsson et al. (1992) [18] [12].
The soft consensus paradigm proposed by Kacprzyk and Fedrizzi was subsequently reformulated by the Trento research group [20] [21] [22] [23] [25] [24] [26] [27] [61] [28] [29] [30] . The linguistic quantifiers in the original soft consensus measure were substituted by smooth scaling functions with an analogous role and a dynamical model was obtained from the gradient descent optimization of a soft consensus cost function, combining a soft measure of collective dissensus with an individual

mechanism of opinion changing aversion. The resulting soft consensus dynamics acts on the network of single preference structures by a combination of a collective process of diffusion and an individual mechanism of inertia.

Introduced as an extension of the crisp model of consensus dynamics described in [27], the fuzzy soft consensus model [29] substitutes the standard crisp preferences by fuzzy triangular preferences. The fuzzy extension of the soft consensus model is based on the use of a distance measure between triangular fuzzy numbers. In analogy with the standard crisp model, the fuzzy dynamics of preference change towards consensus derives from the gradient descent optimization of the new cost function of the fuzzy soft consensus model.

In the meantime a number of different fuzzy approaches have been proposed. The linguistic approach [79] is applicable when the information involved either at individual level or at group level present qualitative aspects that cannot be effectively represented by means of precise numerical values. Innovative approaches to the modelling of consensus in fuzzy environments were developed under linguistic assessments and the interested reader is referred, among others, to [45] [46] [5] [62] [11] [77]. The typical problem addressed is that in which decision makers have different levels of knowledge about the alternatives and use linguistic term sets with different cardinality to assess their preferences. This is the so-called group decision making problem in a multigranular fuzzy linguistic context.

Another different approach to the analysis of consensus under fuzziness, based on a distance from consensus, has been proposed in [69] using intuitionistic fuzzy preferences. In that paper, taking into account Atanasov's hesitation margin, the approach to consensus in [9] [10] and [68] has been extended to individual preferences represented by interval values. This approach has been further developed in [70] introducing a similarity measure to compare the distances between intuitionistic fuzzy relations. More recently, a new and more effective similarity measure has been introduced and applied to consensus analysis in the context of interval-valued intuitionistic fuzzy set theory [78].

The paper is organized as follows. In section 2 we briefly review the soft consensus model proposed in [27] and we show how to derive the soft consensus dynamics on the basis of a cost function W combining a soft measure of collective dissensus with an individual mechanism of opinion changing aversion. In section 3, assuming fuzzy triangular preferences as in [29], we describe the new distance measure and introduce the cost function W of the fuzzy soft consensus model. In section 4 we derive the dynamical laws of the fuzzy soft consensus model as applied to fuzzy triangular preferences. Section 5 contains the main contribution of the paper: we present and discuss a number of computer simulations in order to illustrate the complex and suggestive dynamical patterns generated by the dynamics of the fuzzy soft consensus model. Finally, in section 6 we present some concluding remarks and notes on future research.

2 The soft dissensus measure and the consensus dynamics

In this section we present a brief review of the original soft consensus model introduced in [27]. Our point of departure is a set of individual fuzzy preference relations. If $A = \{a_1, \ldots, a_m\}$ is a set of decisional alternatives and $I = \{1, \ldots, n\}$ is a set of individuals, then the fuzzy preference relation R_i of individual i is given by its membership function $R_i : A \times A \to [0,1]$ such that

$$
\begin{array}{lll}
R_i(a_k, a_l) = & 1 & \text{if } a_k \text{ is definitely preferred over } a_l \\
R_i(a_k, a_l) \in & (0.5, 1) & \text{if } a_k \text{ is preferred over } a_l \\
R_i(a_k, a_l) = & 0.5 & \text{if } a_k \text{ is considered indifferent to } a_l \\
R_i(a_k, a_l) \in & (0, 0.5) & \text{if } a_l \text{ is preferred over } a_k \\
R_i(a_k, a_l) = & 0 & \text{if } a_l \text{ is definitely preferred over } a_k,
\end{array}
$$

where $i = 1, \ldots, n$ and $k, l = 1, \ldots, m$. Each individual fuzzy preference relation R_i can be represented by a matrix $[r^i_{kl}]$, $r^i_{kl} = R_i(a_k, a_l)$ which is commonly assumed to be reciprocal, that is $r^i_{kl} + r^i_{lk} = 1$. Clearly, this implies $r^i_{kk} = 0.5$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m$.
The general case $A = \{a_1, \ldots, a_m\}$ for the set of decisional alternatives is discussed in [27] and [29]. Here, for the sake of simplicity, we assume that the alternatives available are only two ($m = 2$), which means that each individual preference relation R_i has only one degree of freedom, denoted by $x_i = r^i_{12}$.
In the framework of the soft consensus model, assuming $m = 2$, the degree of dissensus between individuals i and j as to their preferences between the two alternatives is measured by

$$V(i, j) = f((x_i - x_j)^2) \,, \tag{1}$$

where f is a scaling function defined as

$$f(x) = -\frac{1}{\beta}\, ln(1 + e^{-\beta(x-\alpha)}) \,. \tag{2}$$

In the scaling function formula above, $\alpha \in (0,1)$ is a threshold parameter and $\beta \in (0, \infty)$ is a free parameter. The latter controls the polarization of the sigmoid function $f' : [0,1] \to (0,1)$ given by

$$f'(x) = 1/(1 + e^{\beta(x-\alpha)}) \,. \tag{3}$$

In the soft consensus model [27] each decision maker $i = 1, \ldots, n$ is represented by a pair of connected nodes, a primary node (dynamic) and a secondary node (static). The n primary nodes form a fully connected subnetwork and each of them encodes the individual opinion of a single decision maker. The n secondary nodes, on the other hand, encode the individual opinions originally declared by the decision makers, denoted $s_i \in [0,1]$, and each of them is connected only with the associated primary node.

The dynamical process of preference change corresponds to the gradient descent optimization of a cost function W, depending on both the present and the original network configurations. The value of W combines a measure V of the overall dissensus in the present network configuration with a measure U of the overall change from the original network configuration.
The various interactions involving node i are modulated by interaction coefficients whose role is to quantify the strength of the interaction. The consensual interaction between primary nodes i and j is modulated by the interaction coefficient $v_{ij} \in (0,1)$, whereas the inertial interaction between primary node i and the associated secondary node is modulated by the interaction coefficient $u_i \in (0,1)$. In the soft consensus model the values of these interaction coefficients are given by the derivative f' of the scaling function according to

$$v_{ij} = f'((x_i - x_j)^2), \quad v_i = \sum_{j(\neq i)=1}^{n} v_{ij}/(n-1), \quad u_i = f'((x_i - s_i)^2). \tag{4}$$

The average preference $\bar{x}_i$ is given by

$$\bar{x}_i = \sum_{j(\neq i)=1}^{n} v_{ij} x_j / \sum_{j(\neq i)=1}^{n} v_{ij} \tag{5}$$

and represents the average preference of the remaining decision makers as seen by decision maker $i = 1, ..., n$.
The construction of the cost function W that drives the dynamics of the soft consensus model is as follows. The individual dissensus cost $V(i)$ is given by

$$V(i) = \sum_{j(\neq i)=1}^{n} V(i,j)/(n-1), \quad V(i,j) = f((x_i - x_j)^2) \tag{6}$$

and the individual opinion changing cost $U(i)$ is

$$U(i) = f((x_i - s_i)^2). \tag{7}$$

Summing over the various decision makers we obtain the collective dissensus cost V and inertial cost U,

$$V = \frac{1}{4} \sum_{i=1}^{n} V(i), \quad U = \frac{1}{2} \sum_{i=1}^{n} U(i) \tag{8}$$

with conventional multiplicative factors of $1/4$ and $1/2$. The full cost function W is then $W = (1-\lambda)V + \lambda U$ with $0 \leq \lambda \leq 1$.
The consensual network dynamics, which can be regarded as an unsupervised learning algorithm, acts on the individual opinion variables x_i through the iterative process

$$x_i \rightsquigarrow x_i' = x_i - \gamma \frac{\partial W}{\partial x_i} \,. \tag{9}$$

Analyzing the effect of the two dynamical components V and U separately we obtain

$$\frac{\partial V}{\partial x_i} = v_i(x_i - \bar{x}_i) \tag{10}$$

where the coefficients v_i were defined in (4) and the average preference $\bar{x}_i$ was defined in (5), and therefore

$$x_i' = (1 - \gamma v_i)x_i + \gamma v_i \bar{x}_i \,. \tag{11}$$

On the other hand, we obtain

$$\frac{\partial U}{\partial x_i} = u_i(x_i - s_i) \,, \tag{12}$$

where the coefficients u_i were defined in (4), and therefore

$$x_i' = (1 - \gamma u_i)x_i + \gamma u_i s_i \,. \tag{13}$$

The full dynamics associated with the cost function $W = (V + U)/2$ acts iteratively according to

$$x_i' = (1 - \gamma(v_i + u_i))x_i + +\gamma v_i \bar{x}_i + \gamma u_i s_i \,. \tag{14}$$

and the decision maker i is in dynamical equilibrium, in the sense that $x_i' = x_i$, if the following stability equation holds,

$$x_i = (v_i \bar{x}_i + u_i s_i)/(v_i + u_i) \tag{15}$$

that is, if the present opinion x_i coincides with an appropriate weighted average of the original opinion s_i and the average opinion value $\bar{x}_i$.

3 The fuzzy soft dissensus measure

Let us now assume that the decision makers preferences are expressed by means of fuzzy numbers, see for instance [17] [80], in particular by means of triangular fuzzy numbers. Then, in order to measure the differences between the decision makers preferences, we need to compute the distances between the fuzzy numbers representing those preferences. Let

$$\mathbf{x} = \{\varepsilon_L, x, \varepsilon_R\} \quad \mathbf{y} = \{\theta_L, y, \theta_R\} \tag{16}$$

be two triangular fuzzy numbers, where x is the central value of the fuzzy number $\mathbf{x}$ and ε_L, ε_R are its left and right spread, respectively. Analogously for the triangular fuzzy number $\mathbf{y}$.

Various definitions of distance between fuzzy numbers are considered in the literature [43] [52] [71] [72]. Moreover, the question has been often indirectly addressed in papers regarding the ranking of fuzzy numbers, see [75] [76] for a detailed review. In our model we refer to a distance, indicated by $D^*(\mathbf{x},\mathbf{y})$, which belongs to a family of distances introduced in [43]. This distance is defined as follows.
For each $\alpha \in [0,1]$, the α–level sets of the two fuzzy numbers $\mathbf{x}$ and $\mathbf{y}$ are respectively

$$[x_L(\alpha), x_R(\alpha)] = [x - \varepsilon_L + \varepsilon_L \alpha \ , \ x + \varepsilon_R - \varepsilon_R \alpha] \tag{17}$$

$$[y_L(\alpha), y_R(\alpha)] = [y - \theta_L + \theta_L \alpha \ , \ y + \theta_R - \theta_R \alpha] \, . \tag{18}$$

The distance $D^*(\mathbf{x},\mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$ is defined by means of the differences between the left boundaries of (17), (18) and the differences between the right boundaries of (17), (18). More precisely, the left integral I_L is defined as the integral, with respect to α, of the squared difference between the left boundaries of (17) and (18),

$$I_L = \int_0^1 (x_L(\alpha) - y_L(\alpha))^2 d\alpha \tag{19}$$

and the right integral I_R is defined as the integral, with respect to α, of the squared difference between the right boundaries of (17), (18),

$$I_R = \int_0^1 (x_R(\alpha) - y_R(\alpha))^2 d\alpha \, . \tag{20}$$

Finally, the distance $D^*(\mathbf{x},\mathbf{y})$ is defined as

$$D^*(\mathbf{x},\mathbf{y}) = \left(\frac{1}{2}(I_L + I_R) \right)^{1/2} . \tag{21}$$

The distance (21) is obtained by choosing $p = 2$ and $q = 1/2$ in the family of distances introduced in [43]. In order to avoid unnecessarily complex computations, we skip the square root and we use, in our model, the simpler expression

$$D(\mathbf{x},\mathbf{y}) = (D^*(\mathbf{x},\mathbf{y}))^2 \ = \ \frac{1}{2}(I_L + I_R) \, . \tag{22}$$

Note that expression (22), except for the numerical factor $1/2$, has been introduced, independently from [43], also in [57]. It has been then pointed out in [1] that (22) is not a distance, as it does not always satisfy the triangular inequality. Nevertheless, as long as optimization is involved, expression (22) can be equivalently used in place of the distance (21) [58]. In any case, for simplicity, in the following we shall use the term distance when referring to (22). Developing (19) and (20), we obtain

$$D(\mathbf{x},\mathbf{y}) = d^2 + \frac{1}{6}\delta_L^2 + \frac{1}{6}\delta_R^2 + \frac{d}{2}(\delta_R - \delta_L), \tag{23}$$

where $d = x - y$, $\delta_L = \varepsilon_L - \theta_L$ and $\delta_R = \varepsilon_R - \theta_R$.

As explained in the previous section, the preferences of the n decision makers are expressed by pairwise comparing the alternatives $a_1, a_2, \ldots, a_m$. Given a pair of alternatives, we assume that the preference of the first over the second alternative is represented, for decision maker i, by a triangular fuzzy number indicated by

$$\mathbf{r}^i = \{\varepsilon_L^i, r^i, \varepsilon_R^i\}, \tag{24}$$

where, as in (16), r^i is the central value of the fuzzy number $\mathbf{r}^i$, whereas ε_L^i and ε_R^i are its left and right spreads respectively. Analogously, let $\mathbf{r}^j$ be the triangular fuzzy number of type (24) representing the preference of the first alternative over the second given by decision maker j.
Following definition (22), the distance between the fuzzy preference of decision maker i and the one of decision maker j becomes

$$D(\mathbf{r}^i, \mathbf{r}^j) = d^2 + \frac{1}{6}\delta_L^2 + \frac{1}{6}\delta_R^2 + \frac{d}{2}(\delta_R - \delta_L), \tag{25}$$

where $d = r^i - r^j$, $\delta_L = \varepsilon_L^i - \varepsilon_L^j$ and $\delta_R = \varepsilon_R^i - \varepsilon_R^j$.

As assumed in the previous section, we consider a problem with $m = 2$ alternatives and we define the dissensus measure between two decision makers by applying the scaling function f to $D(\mathbf{r}^i, \mathbf{r}^j)$,

$$V(i, j) = f(D(\mathbf{r}^i, \mathbf{r}^j)). \tag{26}$$

The dissensus measure of decision maker i with respect to the rest of the group is given by the arithmetic mean of the various dissensus measures $V(i, j)$,

$$V(i) = \sum_{j(\neq i)=1}^{n} V(i, j)/(n-1). \tag{27}$$

Finally, the global dissensus measure of the group is defined by

$$V = \frac{1}{4}\sum_{i=1}^{n} V(i), \tag{28}$$

thus obtaining

$$V = \frac{1}{4}\sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} f(D(\mathbf{r}^i, \mathbf{r}^j))/(n-1). \tag{29}$$

Denoting by $\mathbf{s}^i = \{\theta_L^i, s^i, \theta_R^i\}$ the triangular fuzzy number describing the initial preference of decision maker i, the cost for changing the initial preference $\mathbf{s}^i$ into the actual preference $\mathbf{r}^i$ is given by

$$U(i) = f(D(\mathbf{r}^i, \mathbf{s}^i)). \tag{30}$$

The global opinion changing aversion component U of the group is given by

$$U = \frac{1}{2}\sum_{i=1}^{n} U(i)\,. \tag{31}$$

As mentioned before, the global cost function W is defined as a convex combination of the components V and U,

$$W = (1-\lambda)V + \lambda U, \tag{32}$$

and the parameter $\lambda \in [0,1]$ represents the relative importance of the inertial component U with respect to the dissensus component V.

4 The dynamics of the fuzzy soft consensus model

In [29] the original consensus dynamics described in section 2 was extended to the case in which preferences are expressed by means of triangular fuzzy numbers. In the consensus dynamics, the global cost function $W = W(\mathbf{r}^i) = W(\varepsilon_L^i, r^i, \varepsilon_R^i)$ is minimized through the gradient descent method. This implies that in every iteration the new preference $\mathbf{r}'$ is obtained from the previous preference $\mathbf{r}$ in the following way (we skip the index i for simplicity)

$$\mathbf{r} \to \mathbf{r}' = \mathbf{r} - \gamma\,\nabla W\,. \tag{33}$$

The consensus dynamics (33) will gradually update the three preference values $(\varepsilon_L, r, \varepsilon_R)$ according to

$$r \to r' = r - \gamma\frac{\partial W}{\partial r}, \quad \varepsilon_L \to \varepsilon_L{}' = \varepsilon_L - \gamma\frac{\partial W}{\partial \varepsilon_L}, \quad \varepsilon_R \to \varepsilon_R{}' = \varepsilon_R - \gamma\frac{\partial W}{\partial \varepsilon_R} \tag{34}$$

We can consider separately the effect of the two components V and U of W, since ∇W is a convex combination of ∇V and ∇U,

$$\nabla W = (1-\lambda)\nabla V + \lambda \nabla U\,. \tag{35}$$

Let us first consider the component V. Taking again into account the index i, we have

$$\frac{\partial V}{\partial r^i} = v_i \left(\left(r^i - \bar{r}^i\right) + \frac{1}{4}\left(\varepsilon_R^i - \bar{\varepsilon}_R^i - \varepsilon_L^i + \bar{\varepsilon}_L^i\right) \right) \tag{36}$$

where

$$v_i = \sum_{j(\neq i)=1}^{n} v_{ij}/(n-1)\,; \quad v_{ij} = f'(D(\mathbf{r}^i, \mathbf{r}^j)) \tag{37}$$

$$\bar{r}^{\,i} \quad = \quad \frac{\sum_{j(\neq i)=1}^{n} v_{ij}\, r^j}{\sum_{j(\neq i)=1}^{n} v_{ij}}, \quad \bar{\varepsilon}_L^{\,i} = \frac{\sum_{j(\neq i)=1}^{n} v_{ij}\, \varepsilon_L^j}{\sum_{j(\neq i)=1}^{n} v_{ij}}, \quad \bar{\varepsilon}_R^{\,i} = \frac{\sum_{j(\neq i)=1}^{n} v_{ij}\, \varepsilon_R^j}{\sum_{j(\neq i)=1}^{n} v_{ij}}, \tag{38}$$

Analogously, we compute

$$\frac{\partial V}{\partial \varepsilon_L^{\,i}} = v_i\,(\frac{1}{6}(\varepsilon_L^{\,i} - \bar{\varepsilon}_L^{\,i}) - \frac{1}{4}(r^i - \bar{r}^{\,i})), \quad \frac{\partial V}{\partial \varepsilon_R^{\,i}} = v_i\,(\frac{1}{6}(\varepsilon_R^{\,i} - \bar{\varepsilon}_R^{\,i}) + \frac{1}{4}(r^i - \bar{r}^{\,i}))\,. \tag{39}$$

Let us now consider the inertial component U. We obtain

$$\frac{\partial U}{\partial r^i} = u_i\,((r^i - s^i) + \frac{1}{4}(\varepsilon_R^i - \theta_R^{\,i} - \varepsilon_L^i + \theta_L^{\,i})) \tag{40}$$

where

$$u_i = f'(D(\mathbf{r}^i, \mathbf{s}^i))\,, \tag{41}$$

$$\frac{\partial U}{\partial \varepsilon_L^{\,i}} = u_i\,(\frac{1}{6}(\varepsilon_L^{\,i} - \theta_L^{\,i}) - \frac{1}{4}(r^i - s^i)) \tag{42}$$

and

$$\frac{\partial U}{\partial \varepsilon_R^{\,i}} = u_i\,(\frac{1}{6}(\varepsilon_R^{\,i} - \theta_R^{\,i}) + \frac{1}{4}(r^i - s^{\,i}))\,. \tag{43}$$

At this point we can summarize the effects of the two components obtaining

$$\frac{\partial W}{\partial r^i} = ((1-\lambda)v_i + \lambda u_i)\Delta r^i - (1-\lambda)v_i \Delta \bar{r}^{\,i} - \lambda u_i \Delta s^i \tag{44}$$

where

$$\Delta r^i = r^i + \frac{1}{4}(\varepsilon_R^{\,i} - \varepsilon_L^{\,i}), \quad \Delta \bar{r}^{\,i} = \bar{r}^{\,i} + \frac{1}{4}(\bar{\varepsilon}_R^{\,i} - \bar{\varepsilon}_L^{\,i}), \quad \Delta s^i = s^i + \frac{1}{4}(\theta_R^{\,i} - \theta_L^{\,i})\,. \tag{45}$$

The derivative of W with respect to the left spread becomes

$$\frac{\partial W}{\partial \varepsilon_L^{\,i}} \;=\; ((1-\lambda)v_i + \lambda u_i)\Delta \varepsilon_L^{\,i} - (1-\lambda)v_i \Delta \bar{\varepsilon}_L^{\,i} - \lambda u_i \Delta \theta_L^{\,i} \tag{46}$$

where

$$\Delta \varepsilon_L^{\,i} = \frac{1}{6}\,\varepsilon_L^{\,i} - \frac{1}{4}\,r^i, \quad \Delta \bar{\varepsilon}_L^{\,i} = \frac{1}{6}\,\bar{\varepsilon}_L^{\,i} - \frac{1}{4}\,\bar{r}^{\,i}, \quad \Delta \theta_L^{\,i} = \frac{1}{6}\,\theta_L^{\,i} - \frac{1}{4}\,s^i\,. \tag{47}$$

The derivative of W with respect to the right spread becomes

$$\frac{\partial W}{\partial \varepsilon_R^{\,i}} \;=\; ((1-\lambda)v_i + \lambda u_i)\Delta \varepsilon_R^{\,i} - (1-\lambda)v_i \Delta \bar{\varepsilon}_R^{\,i} - \lambda u_i \Delta \theta_R^{\,i} \tag{48}$$

where

$$\Delta\varepsilon_R^i = \frac{1}{6}\,\varepsilon_R^i + \frac{1}{4}\,r^i, \quad \Delta\bar{\varepsilon}_R^i = \frac{1}{6}\,\bar{\varepsilon}_R^i + \frac{1}{4}\,\bar{r}^i, \quad \Delta\theta_R^i = \frac{1}{6}\,\theta_R^i + \frac{1}{4}\,s^i\,. \tag{49}$$

Let us now present some numerical simulations in order to illustrate the dynamical behaviour of the fuzzy soft consensus model in some interesting cases.

5 Computer simulations

In this section we present a number of computer simulations of the fuzzy soft consensus dynamics as applied to a single pair of preferences represented by triangular fuzzy numbers. Our goal is that of illustrating the various interesting dynamical patterns generated by the non linear nature of the pairwise interactions between preferences, given that these pairwise interactions are the fundamental elements of the soft consensus model.

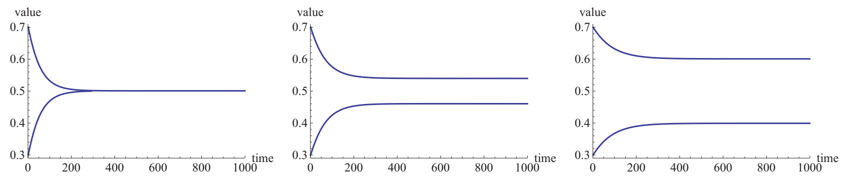
The first four figures associated with each computer simulation (except the first) depict four successive configurations of the preference pair of triangular fuzzy numbers, corresponding to the following moments in time: the initial configuration $t = 0$, two intermediate configurations $t = 25$ and $t = 100$, and the final (quasi-asymptotic) configuration $t = 1000$. The two dots appearing in each of the four figures indicate the positions of the centers as they vary in time according to the original crisp version of the soft consensus model. The other three figures associated with each computer simulation show the time plot of the preference centers plus that of the left and right spreads.

In general we observe in the computer simulations two distinct dynamical phases, clearly illustrated by the graphical plots of the preference changes over time: a short phase with fast dynamics followed by a much longer phase with slow dynamics. Interestingly, the preference changes over time in each of these two phases are not always monotonic. Moreover, the computer simulations show that the dynamics of the fuzzy soft consensus model is generally faster than that of the original crisp model. The final (quasi-asymptotic) values of the preference centers in the fuzzy model show moderate but significant differences with respect to the corresponding final preference values in the original crisp model.

The distance $D(\mathbf{x}, \mathbf{y})$ between two fuzzy numbers $\mathbf{x}$ and $\mathbf{y}$ defined in (22) and involved in the construction of the cost functions V, U, W plays a key role in the fuzzy extension of the soft consensus model. In particular, the two distinct phases (fast and slow) observed in the consensus dynamics of the model can be understood in terms of the different magnitudes of the coefficients associated with the various terms in the decomposition formula (23). The fact that the coefficient associated with the distance between centers is three times larger than the coefficient associated with the distance between spreads (left and right together) produces initially a fast consensus dynamics of the centers, followed by a much slower adjustment dynamics of the spreads. Roughly speaking, the fast phase leads to an overlapping of the two fuzzy triangular numbers, whose shape is then adjusted by the slow dynamical phase.

In all computer simulations (except partially the first) the parameter choices are as follows: $\alpha = 0.3$, $\beta = 10$, $\lambda = 1/3$, and $\gamma = 0.01$.

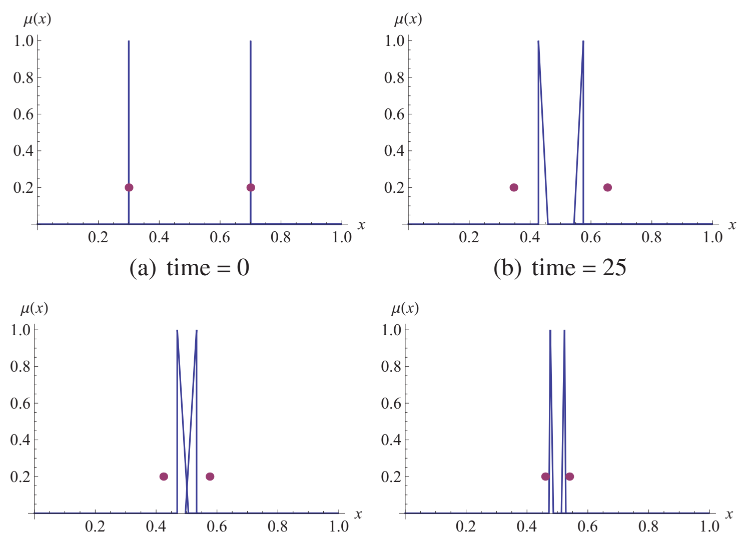
- These figures illustrate the dynamics of the original crisp soft consensus model as applied to two crisp initial preferences 0.3 and 0.7, for three different choices of the parameter λ. This parameter controls the relative strength of the mechanism of opinion changing aversion with respect to the consensual aggregation mechanism. In the case $\lambda = 0$ the dynamics is purely consensual and thus, over time, the two preferences converge exactly to a common final value.



(a) $\lambda = 0$ (b) $\lambda = 1/3$ (c) $\lambda = 2/3$

Fig. 1 Crisp dynamics acting on two crisp preferences, for different values of λ

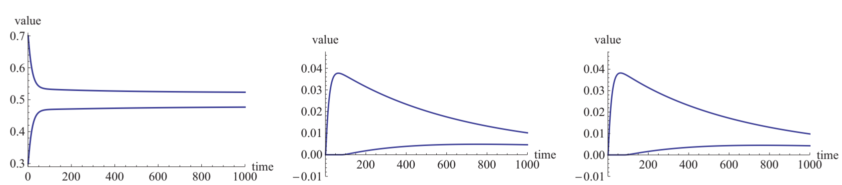
- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to the same two crisp initial preferences 0.3 and 0.7 as before. Notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the internal spreads. Initially, in the fast phase, the centers approach rapidly and the internal spreads increase significantly whereas the external spreads remain essentially null, a sort of cooperative opening to the opposing preference. Then, in the slow phase, the centers keep on approaching very slowly while the internal spreads gradually decrease and the external spreads increase slightly, converging towards a nearly common final value. In the final configuration the spreads are once again very small (they were initially null) even though they reach much larger values during the transient ”negotiation” process. This a suggestive reality effect of the non linear dynamics of the fuzzy soft consensus model.



(a) time = 0 (b) time = 25

(c) time = 100 (d) time = 1000

Fig. 2 Fuzzy dynamics acting on two crisp preferences



(a) Center plot (b) Left spread plot (c) Right spread plot

Fig. 3 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to two fuzzy initial preferences (isosceles triangles) centered at the usual values 0.3 and 0.7. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the internal and external spreads. Initially, in the fast phase, the centers approach rapidly while the internal (resp. external) spreads increase (resp. decrease) significantly, again a sort of cooperative opening to the opposing preference. Then, in the slow phase, the centers keep on approaching very slowly while the internal (resp. external) spreads gradually decrease (resp. increase), converging towards a nearly common final value.

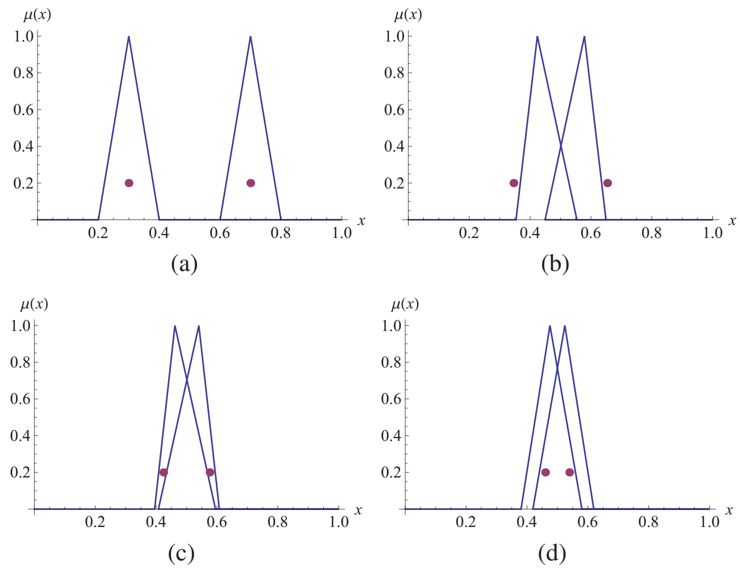
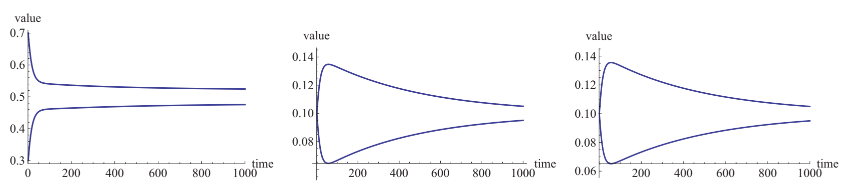


Fig. 4 Fuzzy dynamics acting on two isosceles triangles



(a) Center plot (b) Left spread plot (c) Right spread plot

Fig. 5 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to two fuzzy initial preferences (right triangles facing each other) centered at the usual values 0.3 and 0.7. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the internal spreads. Initially, in the fast phase, the centers approach rapidly and the internal spreads increase slightly whereas the external spreads remain essentially null, again a sort of cooperative opening to the opposing preference. Then, in the slow phase, the centers keep on approaching very slowly while the internal (resp. external) spreads gradually decrease (resp. increase), converging towards a nearly common final value.

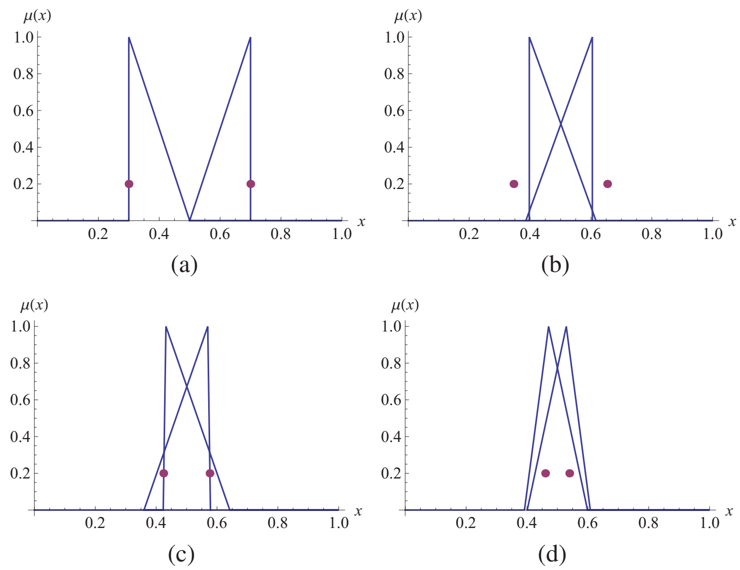
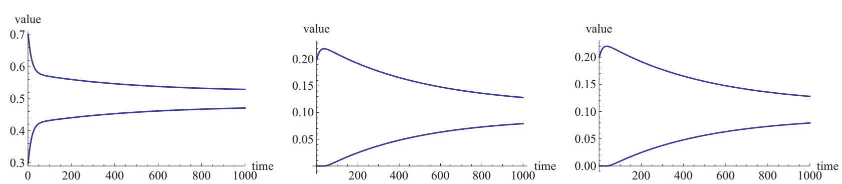


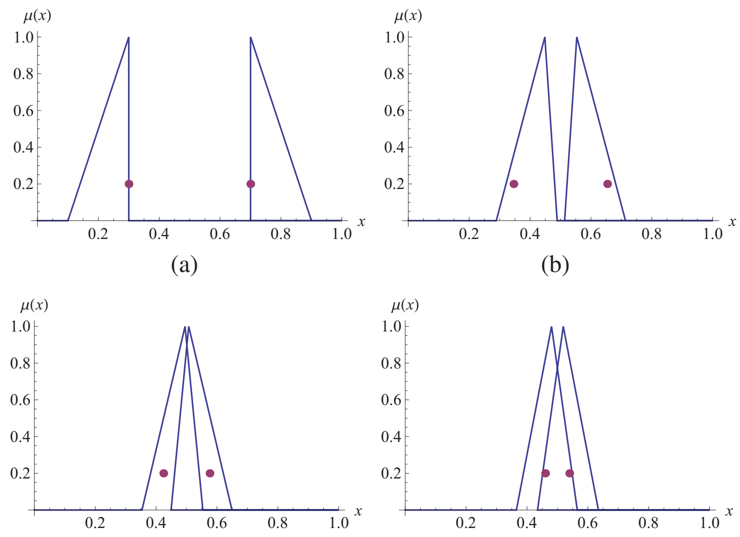
Fig. 6 Fuzzy dynamics acting on two right triangles facing each other



(a) Center plot (b) Left spread plot (c) Right spread plot

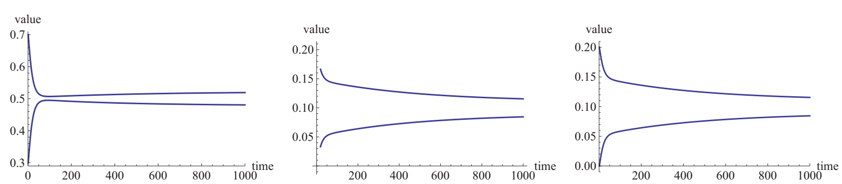
Fig. 7 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to two fuzzy initial preferences (right triangles facing opposite to each other) centered at the usual values 0.3 and 0.7. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the centers. Initially, in the fast phase, the centers approach rapidly (almost crossing) and the internal (resp. external) spreads increase (resp. decrease), again a sort of cooperative opening to the opposing preference. Then, in the slow phase, the centers adjust by moving away very slowly while the internal (resp. external) spreads keep on gradually increasing (resp. decreasing), converging towards a nearly common final value.



(a) (b) (c) (d)

Fig. 8 Fuzzy dynamics acting on two right triangles facing opposite to each other



(a) Center plot (b) Left spread plot (c) Right spread plot

Fig. 9 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to two fuzzy initial preferences (different isosceles triangles) centered at the usual values 0.3 and 0.7. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the internal and external spreads. Initially, in the fast phase, the centers approach rapidly and the internal (resp. external) spreads increase (resp. decrease) slightly on the right and significantly on the left. Then, in the slow phase, the centers keep on approaching very slowly while the internal (resp. external) spreads gradually decrease (resp. increase). In this case the dynamical pattern is more complex for the left spreads, with two crossings during the slow phase.

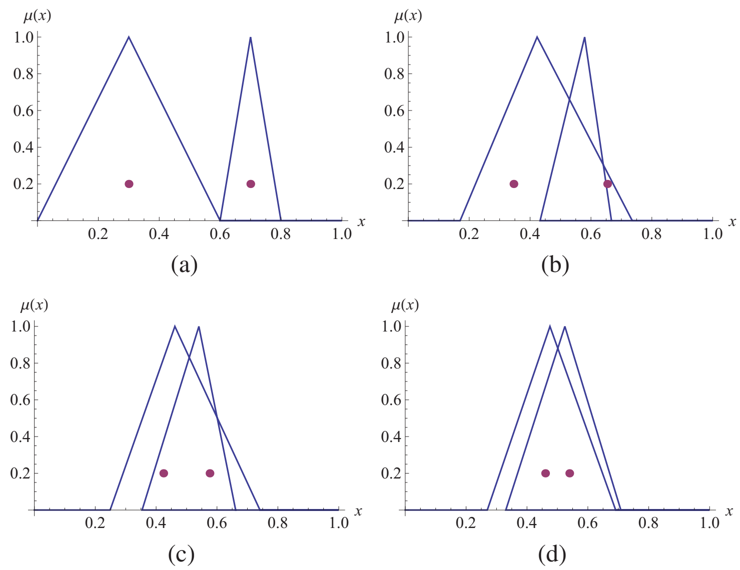


Fig. 10 Fuzzy dynamics acting on two different isosceles triangles

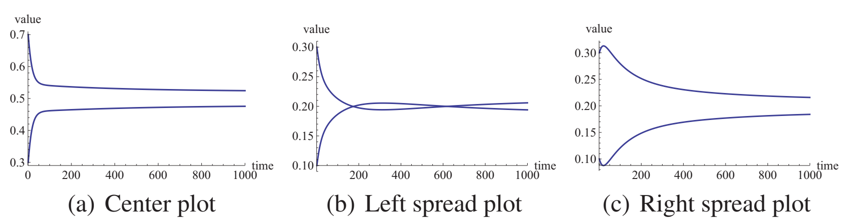


Fig. 11 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to one crisp and one fuzzy initial preferences (isosceles triangle the latter) centered at the usual values 0.3 and 0.7. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the internal and external spreads. Initially, in the fast phase, the centers approach rapidly and the internal (resp. external) spreads increase (resp. stay null or decrease) slightly on the right and significantly on the left. Then, in the slow phase, the centers keep on approaching very slowly while the internal (resp. external) spreads gradually decrease (resp. increase). In this case the dynamical pattern is more complex for the left spreads, with one crossing between the two phases and another one during the slow phase.

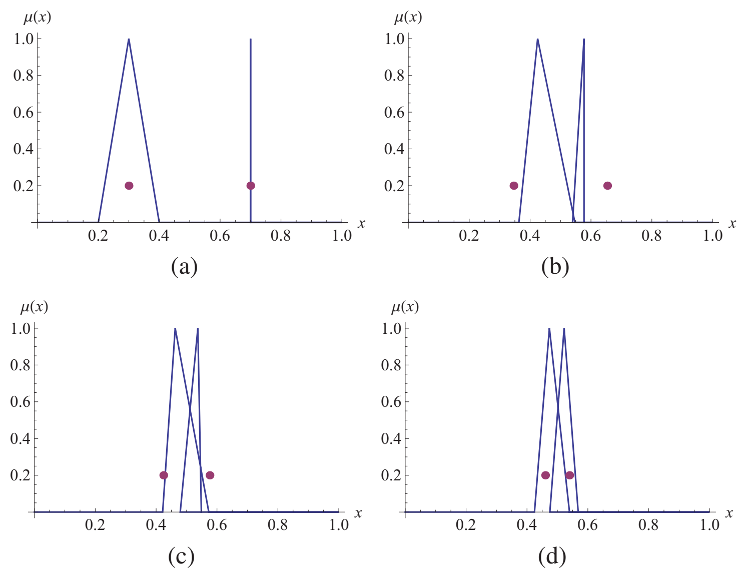
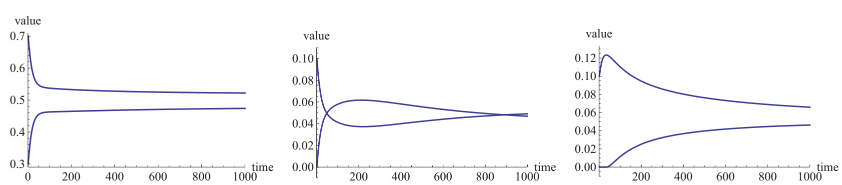


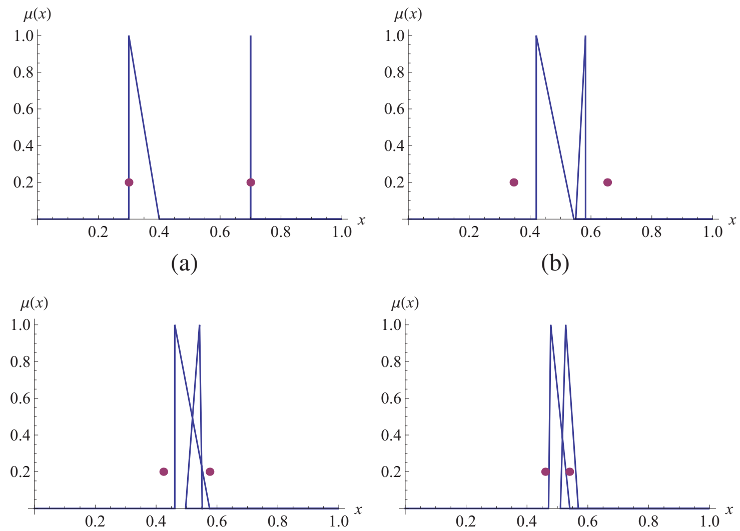
Fig. 12 Fuzzy dynamics acting on one crisp preference and one isosceles triangle



(a) Center plot (b) Left spread plot (c) Right spread plot

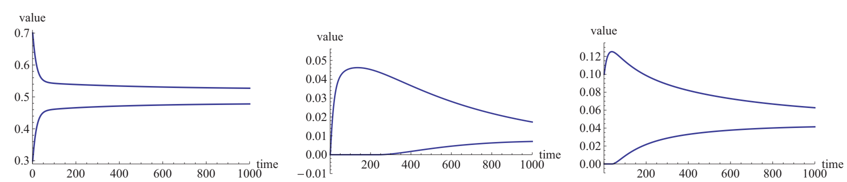
Fig. 13 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to one crisp and one fuzzy initial preferences (right triangle facing inward the latter) centered at the usual values 0.3 and 0.7. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the internal and external spreads. Initially, in the fast phase, the centers approach rapidly and the internal (resp. external) spreads increase (resp. stay null) slightly on the right and significantly on the left. Then, in the slow phase, the centers keep on approaching very slowly while the internal (resp. external) spreads gradually decrease (resp. increase).



(a) (b) (c) (d)

Fig. 14 Fuzzy dynamics acting on one crisp preference and one right triangle facing inward



(a) Center plot (b) Left spread plot (c) Right spread plot

Fig. 15 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to one crisp and one fuzzy initial preferences (right triangle facing outward the latter) centered at the usual values 0.3 and 0.7. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the internal and external spreads. Initially, in the fast phase, the centers approach rapidly and the internal (resp. external) spreads increase (resp. stay null or decrease) significantly on both sides. Then, in the slow phase, the centers keep on approaching very slowly while the internal (resp. external) spreads gradually decrease (resp. increase). In this case the dynamical pattern is more complex for the left spreads, with one crossing between the two phases and another one during the slow phase.

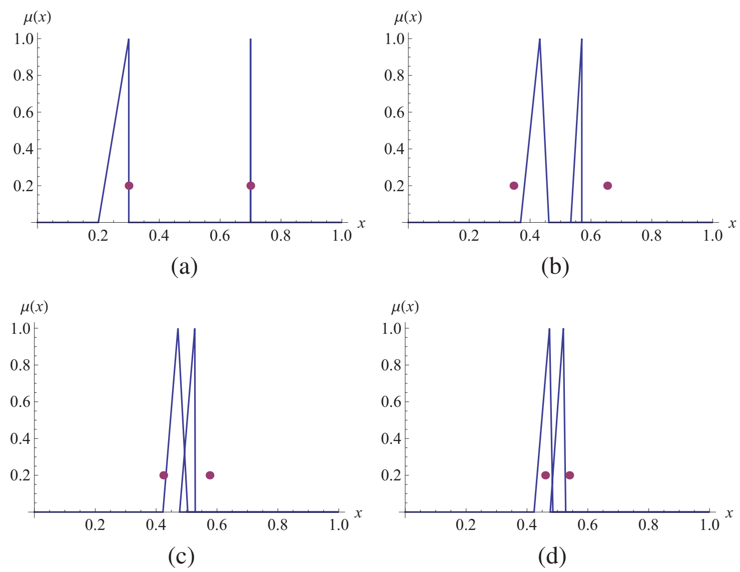
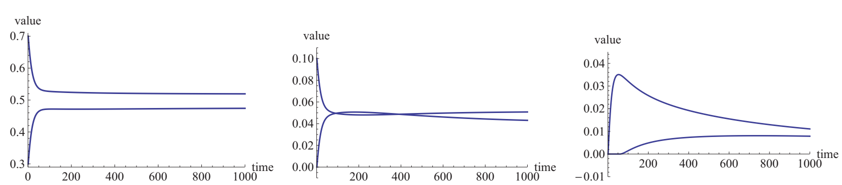


Fig. 16 Fuzzy dynamics acting on one crisp preference and one right triangle facing outward



(a) Center plot (b) Left spread plot (c) Right spread plot

Fig. 17 Corresponding time plots of centers and spreads

- These figures illustrate the dynamics of the fuzzy soft consensus model as applied to the special case of two fuzzy initial preferences (right triangles facing opposite to each other) centered at 0.5 and 0.6. Once again, notice the two dynamical phases, initially fast and then slow, and the suggestive non monotonic behaviour of the centers. Initially, in the fast phase, the centers move rapidly towards each other, crossing and then moving away from each other. Then, in the slow phase, the centers adjust by slowly re-approaching, converging towards a nearly common final value. This is another interesting effect of the non linear dynamics of the fuzzy soft consensus model, due to the combined effect of the two mechanisms of consensus reaching and opinion changing aversion as they act on centers and spreads of the fuzzy triangular preferences.

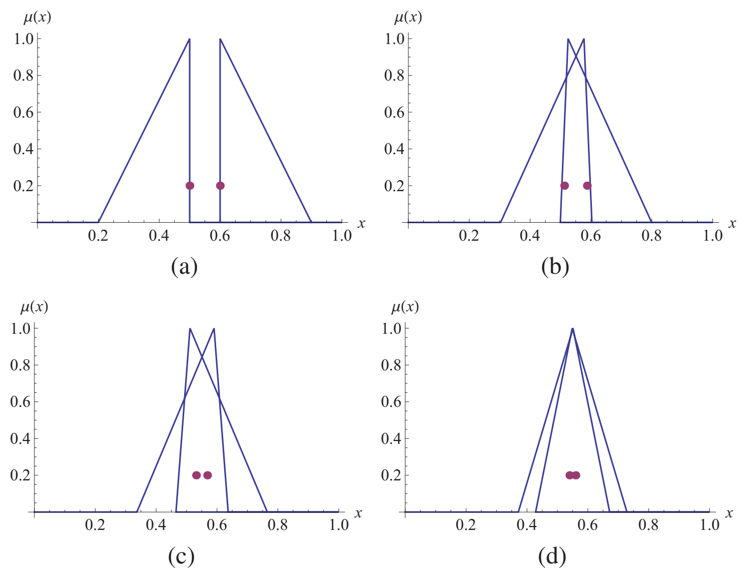


Fig. 18 Fuzzy dynamics acting on two right triangles facing opposite to each other

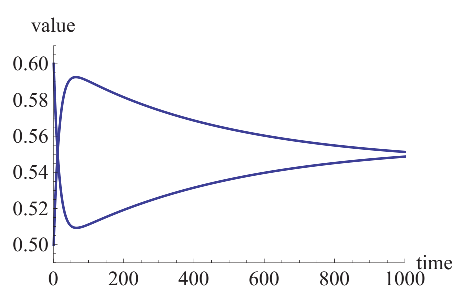


Fig. 19 Corresponding time plot of the centers

6 Concluding remarks

We have illustrated by means of numerical simulations the dynamical behaviour of the fuzzy soft consensus model, in which the individual preferences are represented by triangular fuzzy numbers. A selection of these simulations is presented in section 5. The computer simulations provide clear evidence that the fuzzy soft consensus model exhibits interesting non standard opinion changing behaviour in relation to the original crisp version of the model. Future research should explore the particular features of the fuzzy soft consensus model and demonstrate the potential of the methodology as an effective support for the modelling of consensus reaching in multicriteria and multiagent decision making.

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