

Employing Extended Kalman Filter in a Simple Macroeconomic Model

Levent Özbek*, Ümit Özlale** and
Fikri Öztürk*

**Ankara University, Faculty of Science, Department of Statistics*

System Modelling and Simulation Laboratory

06100 Tandoğan/Ankara

ozbek@science.ankara.edu.tr

URL: <http://science.ankara.edu.tr/~ozbek>

*** Bilkent University, Department of Economics*

06800, Bilkent, Ankara

ozlale@bilkent.edu.tr

Abstract

In this study, the estimation power of Extended Kalman Filter is tested within a simple Keynesian macroeconomic model. After the model is written in a non-linear state space form, Extended Kalman Filter emerges as the appropriate methodology to estimate both state variables and the parameters. The simulation results suggest that such a methodology can also be employed in explaining more complex macroeconomic dynamics.

JEL Classification: C15, C16

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1. Introduction

Kalman Filter has been extensively used in recent economics literature as a recursive estimation technique. It is a powerful algorithm, which can be easily employed in linear state space models, as noted in (Harvey 1990). Recently, (Ljungqvist and Sargent 2000) make usage of this method in various dynamic macroeconomic models. However, Kalman Filter fails to be appropriate in cases of non-linear state space forms. In this context, Extended Kalman Filter (EKF henceforth) has been proposed as the only possible algorithm. Although powerful, EKF has only been employed in a few studies such as (Grillenzoni 1993) and (Tanizaki 2000), where the main motivation was to compare the effectiveness of EKF with other possible solution algorithms.

This paper employs the above-mentioned EKF within a simple ad-hoc Keynesian model with no microeconomic foundations. Although the model is highly stylized, it is the first attempt of its rank for Turkish economy, which presents encouraging results of EKF algorithm to be used in future studies. In addition, such a preliminary exercise also allows us to test the estimation power of EKF.

The simulation results show that the parameters of the model are very close to their expected values and all of the simulated series are successfully estimated. Such a result implies that EKF can be viewed as a promising estimation technique to be employed in more realistic models with real-time data applications.

The outline of the paper is as follows: The next section introduces non-linear state space models and EKF in details. Next, the simple macroeconomic model along with its implications is discussed. Then, the simulation results are displayed. The final section concludes.

2.1. Discrete-Time Linear State Space Model

Discrete-time linear state space models have been employed in 1960's mostly in controlling and signalling processes in defence industry. The extension and application of such models in other fields have taken place in the beginning of 1990s. Some of these studies include (Chui and Chen 1991), (Efe and Ozbek 1999), (Ozbek 2000,2001), and finally (Durbin and Koopman 2001).

A general state space model takes the following form:

$$x_{k+1} = \Phi_k x_k + G_k w_k \quad (1)$$

$$y_k = H_k x_k + v_k \tag{2}$$

Here, $x_k \in \mathfrak{R}^n$ represents the state vector while $y_k \in \mathfrak{R}^m$ represents the observation vector. Φ_k is the $n \times n$ system transition matrix, H_k is the $m \times n$ observation matrix. $w_k \in \mathfrak{R}^n$ and $v_k \in \mathfrak{R}^m$ are white noises with zero mean, for which the following assumptions can be made for each k, j values:

$$E[v_k] = 0 \tag{3}$$

$$E[w_k] = 0 \tag{4}$$

$$E[v_k v_j'] = R_k \delta_{kj} \tag{5}$$

$$E[w_k w_j'] = Q_k \delta_{kj} \tag{6}$$

$$E[v_k w_j'] = 0 \tag{7}$$

$$E[x_0] = \bar{x}_0 \tag{8}$$

$$E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'] = P_0 \tag{9}$$

$$E[x_0 w_k'] = 0 \tag{10}$$

$$E[x_0 v_k'] = 0 \tag{11}$$

Moreover, for $k = 0, 1, 2, \dots$ Φ_k, H_k, G_k, Q_k and R_k are assumed to be known. As introduced in (Jazwinski 1970), the filtering problem is to estimate the state vector x_k , given the observation vector $Y_k = \{y_0, y_1, \dots, y_k\}$, which can be denoted as:

$$\hat{x}_{k|k} = E[x_k | y_0, y_1, \dots, y_k] = E[x_k | Y_k]$$

with the covariance matrix:

$$P_{k|k} = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})' | Y_k]$$

Let the observation matrix take the form: $Y_{k-1} = \{y_0, y_1, \dots, y_{k-1}\}$, then estimating the state vector x_k will be as

$$\hat{x}_{k|k-1} = E[x_k | y_0, y_1, \dots, y_{k-1}] = E[x_k | Y_{k-1}]$$

with the covariance matrix

$$P_{k|k-1} = E\left[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})' \middle| Y_{k-1}\right]$$

In this case, the Kalman Filter, depending on the starting values

$$P_{0|-1} = P_0$$

$$\hat{x}_{0|-1} = \bar{x}_0$$

is characterized by the following algorithms:

$$\hat{x}_{k|k-1} = \Phi_{k-1} \hat{x}_{k-1|k-1} \quad (12)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k [y_k - H_k \hat{x}_{k|k-1}] \quad (13)$$

$$K_k = P_{k|k-1} H_k' [H_k P_{k|k-1} H_k' + R_k]^{-1} \quad (14)$$

$$P_{k|k} = [I - K_k H_k] P_{k|k-1} \quad (15)$$

$$P_{k|k-1} = \Phi_{k-1} P_{k-1|k-1} \Phi_{k-1}' + G_{k-1} Q_{k-1} G_{k-1}' \quad (16)$$

As described in (Anderson and Moore 1979) and (Chen 1993), equation (14) is also known as the “Kalman Gain”.

2.2. Non-Linear State Space Models and EKF

A non-linear state space model takes the form of

$$x_{k+1} = f_k(x_k) + H_k(x_k) \xi_k \quad (17)$$

$$y_k = g_k(x_k) + \eta_k \quad (18)$$

Here, f_k and g_k are vector-valued functions, while ξ_k and η_k represent white noise processes with the covariance matrices, Q_k and R_k , respectively. The starting values for the EKF algorithm are:

$$P_0 = \text{cov}(x_0)$$

$$\hat{x}_0 = E(x_0)$$

As mentioned in (Chui and Chen 1991) and (Chen 1993), for $k = 1, 2, \dots$

$$P_{k|k-1} = \left[\frac{\partial f_{k-1}}{\partial x_{k-1}}(\hat{x}_{k-1}) \right] P_{k-1} \left[\frac{\partial f_{k-1}}{\partial x_{k-1}}(\hat{x}_{k-1}) \right]' + H_{k-1}(\hat{x}_{k-1}) Q_{k-1} H_{k-1}'(\hat{x}_{k-1}) \quad (19)$$

$$\hat{x}_{k|k-1} = f_{k-1}(\hat{x}_{k-1}) \tag{20}$$

$$K_k = P_{k|k-1} \left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1}) \right] \left[\left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1}) \right] P_{k|k-1} \left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1}) \right]' + R_k \right]^{-1} \tag{21}$$

$$P_k = \left[I - K_k \left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1}) \right] \right] P_{k|k-1} \tag{22}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k [y_k - g_k(\hat{x}_{k|k-1})] \tag{23}$$

represent the EKF updating equations.

In order to apply EKF, the matrices in the state space model above should be written as the functions, which depend on the unknown parameter vector, θ . That is, let the matrices be represented as $\Phi_k(\theta)$, $G_k(\theta)$, $H_k(\theta)$. Furthermore, let θ be a random walk process. In this case the following equations,

$$x_{k+1} = \Phi_k(\theta_k)x_k + G_k(\theta_k)w_k \tag{24}$$

$$y_k = H_k(\theta_k)x_k + v_k \tag{25}$$

and the parameter vector

$$\theta_{k+1} = \theta_k + \zeta_k \tag{26}$$

form the new state space model:

$$\begin{bmatrix} x_{k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k(\theta_k)x_k \\ \theta_k \end{bmatrix} + \begin{bmatrix} G_k(\theta_k)w_k \\ \zeta_k \end{bmatrix} \tag{27}$$

$$y_k = \begin{bmatrix} H_k(\theta_k) & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \theta_k \end{bmatrix} + v_k \tag{28}$$

The above model is non-linear for which EKF can be readily applied. ζ_k in equation (26) shows the white noise process for which the covariance matrix is assumed to be $\text{cov}(\zeta_k) = S_k = S > 0$. In the particular case where $S = 0$, the parameter vector is assumed to be time-invariant, where EKF cannot be operative. If EKF algorithm is applied to equations (27)-(28), depending on the following starting values

$$\begin{bmatrix} \hat{x}_0 \\ \hat{\theta}_0 \end{bmatrix} = \begin{bmatrix} E(x_0) \\ E(\theta_0) \end{bmatrix} \text{ and } P_0 = \begin{bmatrix} \text{cov}(x_0) & 0 \\ 0 & S_0 \end{bmatrix}$$

for $k = 1, 2, \dots$ we get:

$$\begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{\theta}_{k|k-1} \end{bmatrix} = \begin{bmatrix} \Phi_{k-1}(\hat{\theta}_{k-1})\hat{x}_{k-1} \\ \hat{\theta}_{k-1} \end{bmatrix} \quad (29)$$

$$P_{k|k-1} = \begin{bmatrix} \Phi_{k-1}(\hat{\theta}_{k-1}) & \frac{\partial}{\partial \theta}(\Phi_{k-1}(\hat{\theta}_{k-1}))\hat{x}_{k-1} \\ 0 & I \end{bmatrix} P_{k-1} \begin{bmatrix} \Phi_{k-1}(\hat{\theta}_{k-1}) & \frac{\partial}{\partial \theta}(\Phi_{k-1}(\hat{\theta}_{k-1}))\hat{x}_{k-1} \\ 0 & I \end{bmatrix}' + \begin{bmatrix} G_{k-1}(\hat{\theta}_{k-1})Q_{k-1}G'_{k-1}(\hat{\theta}_{k-1}) & 0 \\ 0 & S_{k-1} \end{bmatrix} \quad (30)$$

$$K_k = P_{k|k-1} \begin{bmatrix} H_k(\hat{\theta}_{k-1}) & 0 \end{bmatrix} \left[\begin{bmatrix} H_k(\hat{\theta}_{k-1}) & 0 \end{bmatrix} P_{k|k-1} \begin{bmatrix} H_k(\hat{\theta}_{k-1}) & 0 \end{bmatrix} + R_k \right]^{-1} \quad (31)$$

$$P_k = [I - K_k \begin{bmatrix} H_k(\hat{\theta}_{k-1}) & 0 \end{bmatrix}] P_{k|k-1} \quad (32)$$

$$\begin{bmatrix} \hat{x}_k \\ \hat{\theta}_k \end{bmatrix} = \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{\theta}_{k|k-1} \end{bmatrix} + K_k [y_k - \begin{bmatrix} H_k(\hat{\theta}_{k-1})\hat{x}_{k|k-1} \end{bmatrix}] \quad (33)$$

The algorithm above has the potential to be used in many non-linear processes. The previous studies that have used EKF both in statistics and economics include (Ljung and Söderström 1983), (McKiernan 1996), (Bacchetta and Gerlach 1997), (Ozbek and Efe 2000, 2003). It should also be mentioned that, convergence problem in EKF may exist, for which (Aliev and Ozbek 1999) and, (Reif et al. 1999) propose answers for.

3. The Macroeconomic Model and the State-Space Representation

The estimation methodology that has been introduced above has not been employed within the context of a macroeconometric model in previous studies. Therefore, in this section, a simple macroeconomic model will be used to test the effectiveness of EKF algorithm in such a setting. A highly stylized Keynesian

model without any microeconomic foundations is employed for this purpose. This framework views interest rates as primary policy variable and takes government expenditures along with taxes as given. The model can be presented as:

Let y_k : Output at time k

c_k : Consumption at time k

i_k : Investment at time k

g_k : Government expenditures at time k

Furthermore, let consumption expenditures be related to the lagged values of output, and assume that investment is a function of the change in the consumption for the previous year. These two assumptions make perfect sense: an increase in output will lead to an increase in income, which, in turn affects the consumption positively. Also, investments adjust to meet the new level of consumption demand. Finally, let government expenditures follow a random walk. Formally,

$$c_k = ay_{k-1} \quad , \quad a > 0$$

$$i_k = b(c_k - c_{k-1}) \quad , \quad b > 0$$

$$y_k = c_k + i_k + g_k$$

$$g_k = dg_{k-1} + w_k \quad , \quad d > 0$$

Consistent with the previous explanation about consumption and investment equations, the parameters “a”, which is a measure of marginal propensity to consume, and “b”, which shows the sensitivity of investment to lagged consumption change are expected to be positive. For simplicity, we employ a closed economy model.

In order to present the model in state-space form, we can rewrite the equations as

$$y_k = ay_{k-1} + b(ay_{k-1} - ay_{k-2}) + g_k$$

$$y_k - (a + ab)y_{k-1} + aby_{k-2} = g_k$$

$$c_{k+1} = ay_k$$

$$y_{k+1} = c_{k+1} + i_{k+1} + g_{k+1}$$

$$= c_{k+1} + b(c_{k+1} - c_k) + g_{k+1} = (1 + b)ay_k - bc_k + g_{k+1}$$

Next, we can form the state and observation matrices as

$$\begin{bmatrix} c_{k+1} \\ y_{k+1} \\ g_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ -b & (1+b)a & d \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} c_k \\ y_k \\ g_k \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} w_k \quad (34)$$

$$y_k = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_k \\ y_k \\ g_k \end{bmatrix} + v_k \quad (35)$$

In order to estimate the state variables and the unknown parameters in equation (34), we construct the parameter vector in equation (26) as $\theta_k = [a \ b \ d]$ and form the state-space model in equations (27) and (28). After taking the derivatives, EKF algorithm that is specified in equations (29) through (33) is applied.

4. Simulation

In simulation, to generate data from equations (34) and (35), the following starting values for parameters and variances of the disturbances are taken:

$$[c_0 \ y_0 \ g_0]' = [10 \ 10 \ 10]$$

$$a = 0.6, \ b = 0.6, \ d = 1.01$$

$$\text{var}(w_k) = 1$$

$$\text{var}(v_k) = 1$$

where $w_k : N(0,1)$ and $v_k : N(0,1)$ are generated from a Gaussian distribution. The values, which are necessary to employ the EKF updating equations (29) through (33), are taken as:

$$\begin{bmatrix} \hat{x}_0 \\ \hat{\theta}_0 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 10 \\ 0.5 \\ 0.5 \\ 0.7 \end{bmatrix}, \quad Q_k = 100, \quad R_k = 10, \quad S_0 = I_{3 \times 3} \quad S_k = 0.0001 I_{3 \times 3},$$

$$P_0 = \begin{bmatrix} 17^2 & 0 & 0 \\ 0 & 30^2 & 0 \\ 0 & 0 & 11^2 \end{bmatrix}$$

In figures 1, 2, and 3, the data that has been generated from the model are displayed along with the estimations that have been obtained via EKF. Also, the recursive estimates for the parameters can be seen in figures 4, 5 and 6. Figure 4 implies that the parameter “a” takes values between 0.45 and 0.65, which makes sense as a measure of marginal propensity to consume. The parameter “b”, on the other hand, can be viewed as being stable around 0.5 after a sharp drop in the first 10 periods. Finally, in figure 6, parameter “d” in the equation for government expenditure has been stable around 1, consistent with the random walk assumption. As a result, it will not be wrong to claim that the recursive estimates for the parameters are meaningful and the estimated state variables are very close to their simulated values.

5. Results

In this study, a simple ad-hoc Keynesian model has been employed to test the estimation power of Extended Kalman Filter, which is the appropriate methodology to be used in non-linear state space models. Although the model can be criticized for being too stylized, it is the first attempt to estimate a macroeconomic model for the Turkish economy using Extended Kalman Filter. The results obtained from the simulation exercise show that the estimated state variables are very close to the simulated series, and the recursive parameter estimates are fairly reasonable. These findings suggest that the estimation methodology that has been introduced in this study has the potential to be used in more complex and realistic models with real-time data applications.

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Fig. 1. Consumption and Its Estimate

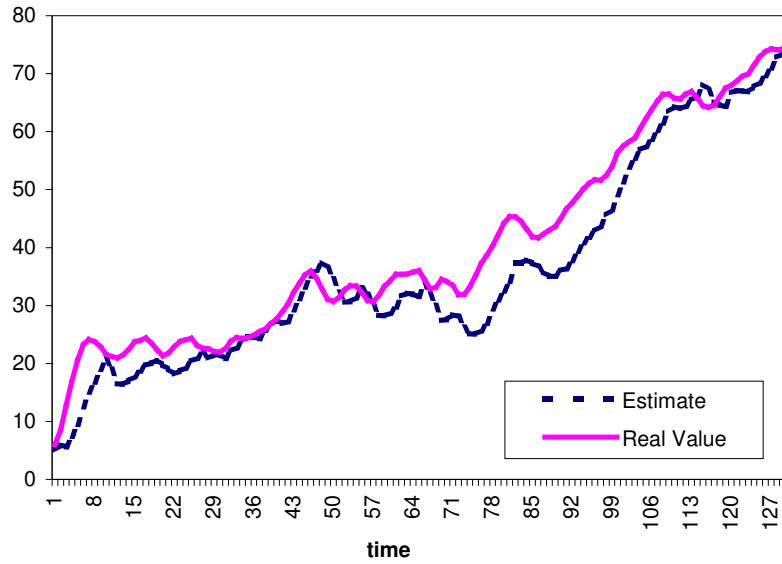


Fig. 2. Output and Its Estimate

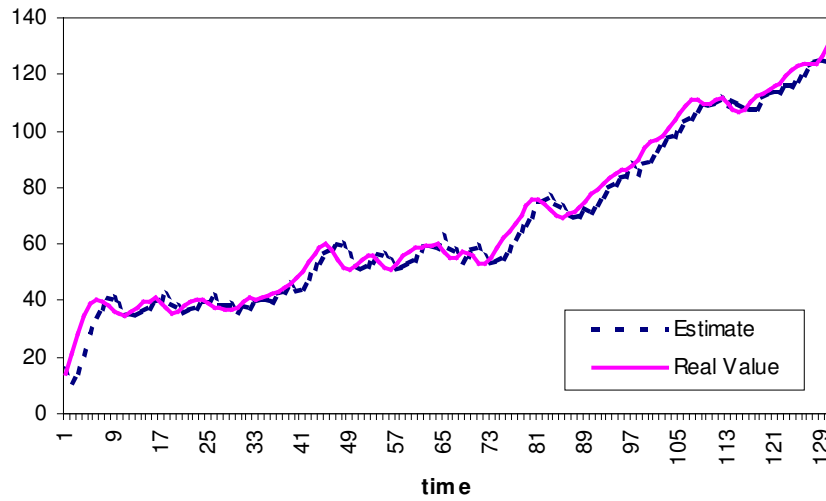


Fig. 3. Government Exp. and Its Estimate

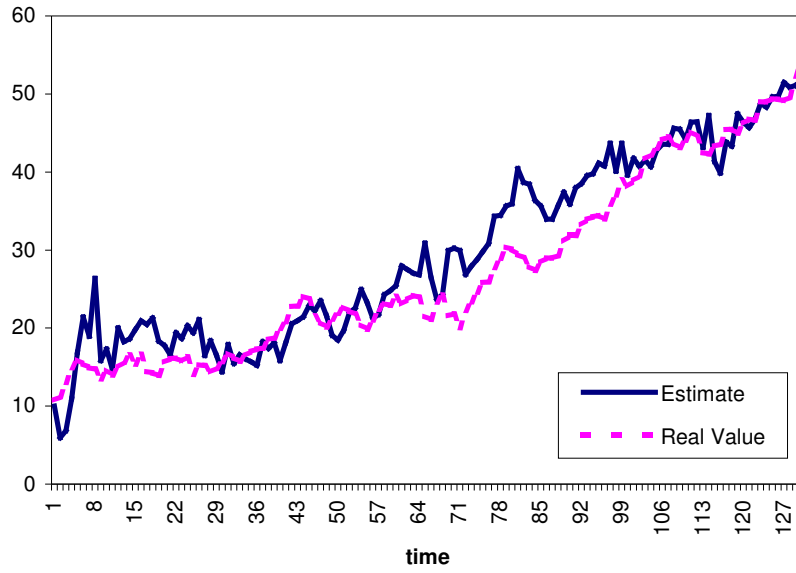


Fig. 4. Parameter 'a'

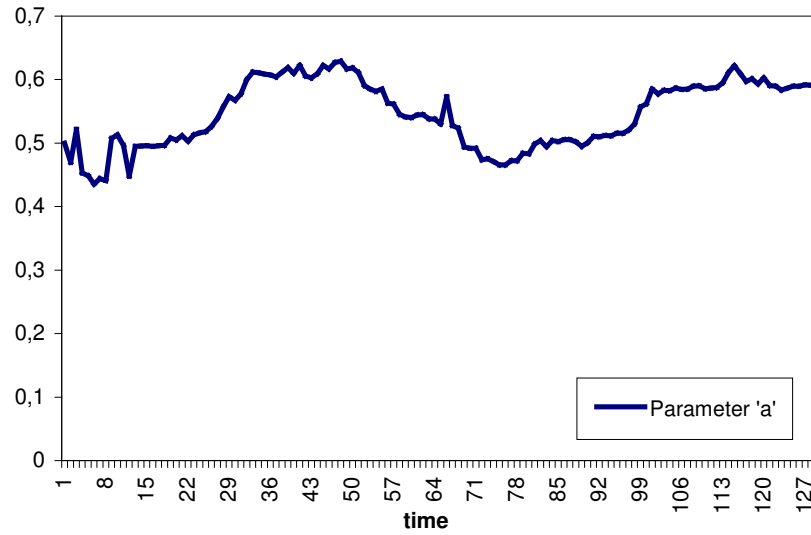


Fig. 5. Parameter 'b'

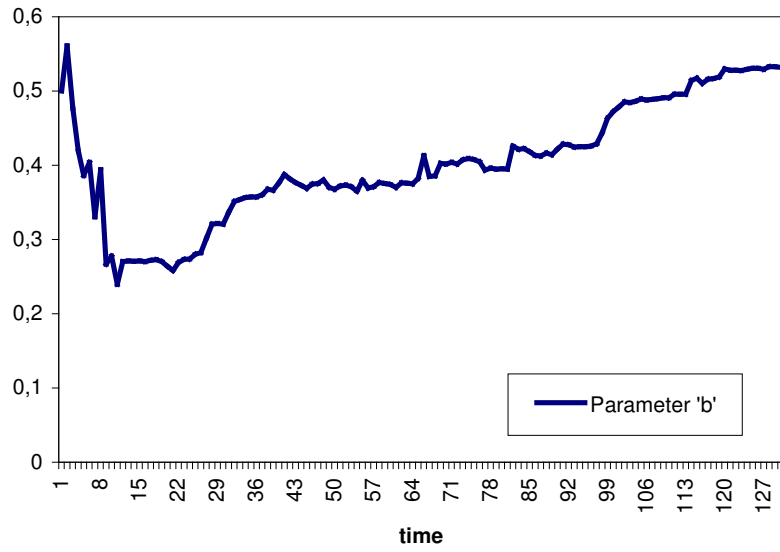
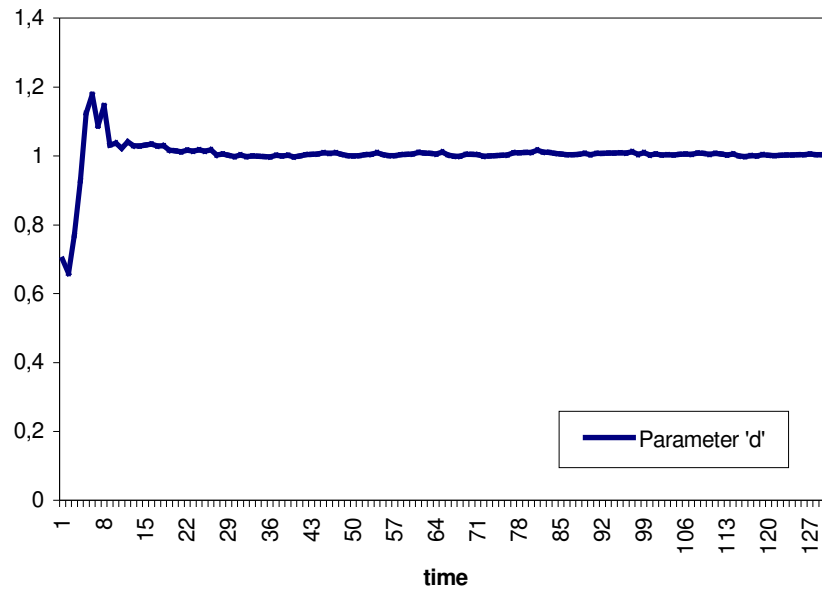


Fig. 6. Parameter 'd'



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