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ON STOCHASTIC PROPERTIES BETWEEN SOME ORDERED RANDOM VARIABLES*

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Abstract

A great number of articles have dealt with stochastic comparisons of ordered random variables in the last decades. In particular, distributional and stochastic properties of ordinary order statistics have been studied extensively in the literature. Sequential order statistics are proposed as an extension of ordinary order statistics. Since sequential order statistics models unify various models of ordered random variables, it is interesting to study their distributional and stochastic properties. In this work, we consider the problem of comparing sequential order statistics according to magnitude and location orders.

Keywords: stochastic orderings, reliability, order statistics

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1 Introduction.

Models of ordered random variables are widely used in statistical modelling and inference. If the random variables $X_1, \ldots, X_n$ are arranged in ascending order of magnitude, then the i'th smallest of X_i's is denoted by $X_{i:n}$. The ordered quantities

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}\,, \tag{1.1}$$

are called *order statistics* (OS), and $X_{i:n}$ is the i'th order statistic. These random variables are of great interest in many areas of statistics, in particular, there is a very interesting application of OS's in reliability theory. The $(n-k+1)$'th OS in a sample of size n represents the life length of a k-out-of-n system which is an important technical structure. It consists of n components of the same kind with independent and identically distributed life lengths. All n components start working simultaneously, and the system works, if at least k components function; i.e. the system fails, if $(n-k+1)$ or more components fail. Special cases of k-out-of-n systems are series and parallel systems.

Kamps [4] introduced the concept of *sequential order statistics* (SOS) as an extension of OOS model. The SOS model is closely connected to several other models of ordered random variables and, in particular it unifies type II censored order statistics, k-th record values and k_n records from nonidentical distributions. Sequential order statistics model the reliability of certain k-out-of-n systems without the assumption of independence of the lifetime of the components. In this model, the lifetime distribution of the remaining components in the system may change after each failure of the components. At the beginning, the lifetimes of the components are iid with a common distribution function F_1. After the first component fails, the distribution of the residual lifetimes of the remaining $(n-1)$ components changes to that of the residual lifetime distribution of a second distribution F_2. If we observe the i'th failure at time t, the remaining $(n-i)$ components are now supposed to have a possibly different distribution. Proceeding in this way we obtain a triangular scheme of random variables where the i'th line containing $n-i+1$ random variables with distribution function F_i, $1 \leq i \leq n$, indicating that $i-1$ components previously failed.

In its general form the SOS model is linked with nonhomogeneous pure birth (NHPB) processes. In this field, there are several papers which study ageing notions of epoch times under conditions on the parameters of the NHPB process. Pellerey et al. [12] give conditions for the log-concavity of the density function of epoch times and inter-epoch times. Shaked et al. [15] highlight the relationship between l_∞-spherical densities and NHPB processes and provide applications to load sharing models, noting that studying the first n epoch times of a NHPB process is equivalent to studying the lifetimes of n components of a load sharing system. Results about multivariate stochastic comparisons of epoch times of two NHPB process have been given by Belzunce et al. [2]. They illustrate their results with applications to generalized Yule processes, load-sharing models, and minimal repairs in reliability theory.

Distributional and stochastic properties of ordinary order statistics have been studied extensively in the literature. Since SOS models unify various models of ordered random variables, it is interesting to study their distributional and stochastic properties. Cramer and Kamps [3] give an expression for marginal distributions of SOS in terms of the so-called relevation transform (cf. Krakowski [8]). Zhuang and Hu [16] present some results on multivariate stochastic comparisons of SOS models and in particular, investigate conditions on the underlying distributions on which the SOS models are based.

The purpose of this article is to present some results on univariate stochastic comparisons of SOS in order to establish stochastic ordering of the epoch times of NHPB processes.

The article is organized as follows. In Section 2, we review various types of stochastic orders and in Section 3, we recall the marginal distributions of SOS models and give some important auxiliary results. In Section 4, we discuss stochastic ordering of SOS models, respectively. Examples of the underlying distributions, on which the SOS models are based, which satisfy these conditions are given. Finally, some applications of the main results are presented in Section 5.

2 Definitions and useful lemmas.

In this section we review some definitions and well-known notions of stochastic orders and also give some useful lemmas which will be used later. Throughout this article "increasing" means "non-decreasing" and "decreasing" means "non-increasing".

Let X and Y be univariate random variables with cumulative distribution functions (c.d.f.'s) F and G, survival functions $\overline{F}\,(=1-F)$ and $\overline{G}\,(=1-G)$, p.d.f.'s f and g, hazard rate functions $h_F\left(=f/\overline{F}\right)$ and $h_G\left(=g/\overline{G}\right)$, and reversed hazard rate functions $r_F\,(=f/F)$ and $r_G\,(=g/G)$, respectively. The following definitions introduce the stochastic orders that we consider in this article.

Definition 1. *X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if $\overline{F}(t) \leq \overline{G}(t)$ for all t.*

Definition 2. *X is said to be smaller than Y in the hazard rate order, denoted by $X \leq_{hr} Y$, if $G(t)/F(t)$ is increasing in t for which the ratio $G(t)/F(t)$ is well defined.*

When the failure rate function exists, it is easy to see that $X \leq_{hr} Y$, if and only if $h_F(t) \geq h_G(t)$ for all t.

Definition 3. *X is said to be smaller than Y in the reversed hazard rate order, denoted by $X \leq_{rh} Y$, if $\overline{G}(t)/\overline{F}(t)$ is increasing in t for which the ratio $\overline{G}(t)/\overline{F}(t)$ is well defined.*

When the reversed hazard rate function exists, it is easy to see that $X \leq_{rh} Y$, if and only if $r_F(t) \leq r_G(t)$ for all t.

Definition 4. *X is said to be smaller than Y in the likelihood ratio ordering, denoted by $X \leq_{lr} Y$, if $g(t)/f(t)$ is increasing in $t \in (l_X, u_X) \cup (l_Y, u_Y)$.*

Likelihood ratio ordering implies hazard rate ordering and reversed hazard rate ordering which in turn imply usual stochastic ordering. For more details on stochastic orderings see Shaked and Shanthikumar [14].

Definition 5. *X is said to be smaller than Y in the dispersive ordering, denoted by $X \leq_{\text{disp}} Y$, if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$, for all $0 < \alpha < \beta < 1$.*

We recall that a function ϕ defined on $[0, \infty)$, which satisfies $\phi(0) = 0$, is said to be star-shaped (anti star-shaped) if $\phi(t)/t$ is increasing (decreasing) in t.

Definition 6. *X is said to be smaller than Y in the star ordering, denoted by $X \leq_{*} Y$, if $G^{-1}F(t)$ is star-shaped in t when the two random variables are non-negative.*

We shall be using the following known results to prove our results in this paper. The following lemma, regarding the preservation of the hazard rate and reversed hazard rate orders under monotone increasing transformations, can be found in Keilson and Sumita [6].

Lemma 1. *Let X and Y be two random variables. If $X \leq_{hr} (\leq_{rh}) Y$, and if ϕ is any increasing function, then $\phi(X) \leq_{hr} (\leq_{rh}) \phi(Y)$.*

Shaked [13] also established the following relation between star ordering and dispersion ordering.

Lemma 2. *Let X and Y be two non-negative random variables, then*

$$X \leq_* Y \Leftrightarrow \ln X \leq_{\text{disp}} \ln Y.$$

The next lemma due to Bartoszewicz [1] lists some relations between the dispersion order and other orders.

Lemma 3. *Let X and Y be two random variables. Then:*

i) if X and Y are non-negative and $X \leq_{hr} Y$ and X or Y is DHR, then $X \leq_{\text{disp}} Y$;

ii) if $X \leq_{rh} Y$ and X or Y is IRHR, then $X \geq_{\text{disp}} Y$.

3 Preliminary results.

Sequential order statistics were introduced by Kamps [4] as a modification of order statistics. The SOS model is more flexible than the model of order statistics in the sense that, after the failure of some component, the distribution of the residual lifetime of the components may change. Cramer and Kamps [3] inspired the following definition of SOS given by Lenz [10].

Definition 7 (Lenz[10])**.** *Let $G_1, \ldots, G_n$ be continuous distributions with $G_1^{-1}(1) \leq \ldots \leq G_n^{-1}(1)$ and let $X_{0,n}^* = -\infty$. Suppose that $U_i, i = 1, \ldots, n$ are independent random variables with $U_i \sim U(0,1)$. Then, the random variables*

$$X_{i,n}^* = G_i^{-1}\left(1 - U_i \overline{G}_i(X_{i-1,n}^*)\right)$$

are called SOS based on $\{G_1, \ldots, G_n\}$.

The marginal distribution functions $F_{*,1}, \dots, F_{*,n}$ of the SOS $X^*_{1,n}, \dots, X^*_{n,n}$ based on $\{G_1, \dots, G_n\}$ are given by:

$$\begin{aligned} F_{*,1}(t) &= G_1(t), \\ F_{*,i}(t) &= \begin{cases} F_{*,i-1}(t) - \displaystyle\int_{-\infty}^{t} \frac{\overline{G}_i(t)}{\overline{G}_i(z)}\, dF_{*,i-1}(z) & \text{if} \quad G_i(t) < 1, \\ 1 & \text{if} \quad G_i(t) = 1. \end{cases} \end{aligned} \tag{3.2}$$

From now on we shall assume that the distribution function of the i'th SOS is absolutely continuous with density function:

$$f_{*,i}(t) = h_i(t)\left(\overline{F}_{*,i}(t) - \overline{F}_{*,i-1}(t)\right), \tag{3.3}$$

where $h_i(t) = g_i(t)/\overline{G}_i(t)$, for all t. Cramer and Kamps [3] noted that the corresponding distribution functions of SOS can be viewed as relevation transforms (Krakowski [8]). The relevation transform $\overline{F}\#\overline{G}$ of the survival functions $\overline{F}$ and $\overline{G}$ is defined by the Lebesgue-Stieltjes integral

$$\left(\overline{F}\#\overline{G}\right)(t) = \overline{F}(t) - \int_{-\infty}^{t} \frac{\overline{G}(t)}{\overline{G}(z)}\, d\overline{F}(z), \qquad \text{for all } t.$$

Assuming that the supports of F and G are positive, then the relevation transform may be interpreted as the survival function of the time to failure of the second of two components when the second component with life distribution G is placed in service on the failure of the first component with life distribution F, assuming that the replacement component has the same age as the failed component (Lau and Prakasa Rao [9]). From (3.2), we have the representation

$$\overline{F}_{*,i}(t) = \overline{F}_{*,i-1}(t) - \int_{-\infty}^{t} \frac{\overline{G}_i(t)}{\overline{G}_i(z)}\, d\overline{F}_{*,i-1}(z), \qquad \text{for all } t. \tag{3.4}$$

Hence, we can write the survival function of the i-th SOS as relevation transform

$$\overline{F}_{*,i} = \overline{F}_{*,i-1}\#\overline{G}_i.$$

Let us define,

$$A_i(t) = \int_{-\infty}^{t} \frac{1}{\overline{G}_i(z)}\, dF_{*,i-1}(z), \tag{3.5}$$

then, from (3.2) and (3.4) we have,

$$\overline{F}_{*,i}(t) = \overline{F}_{*,i-1}(t) + \overline{G}_i(t)A_i(t), \tag{3.6}$$

and

$$F_{*,i}(t) = F_{*,i-1}(t) - \overline{G}_i(t)A_i(t), \tag{3.7}$$

for $i = 2, \ldots, n$.

Now, in order to prove our main results we first need to derive some preliminary results which are also of independent interest. In the following two lemmas, we show some stochastic orderings between SOS and their underlying distribution functions.

Lemma 4. *Let $X^*_{1,n}, \ldots, X^*_{n,n}$ be SOS based on absolutely continuous distribution functions $\{G_1, \ldots, G_n\}$, then*

i) $G_i \leq_{hr} F_{*,i}$ *;*

ii) $G_i \leq_{rh} F_{*,i}$ *;*

iii) $G_i \leq_{lr} F_{*,i}$ *.*

Proof.

i) By definition, $G_i \leq_{hr} F_{*,i}$ if and only if $h_i(t) \geq h_{*,i}(t)$ for all t. From (3.3) we have

$$h_{*,i}(t) = h_i(t)\left(\frac{\overline{F}_{*,i}(t) - \overline{F}_{*,i-1}(t)}{\overline{F}_{*,i}(t)}\right).$$

Then, $h_{*,i}(t) \leq h_i(t) \Leftrightarrow \overline{F}_{*,i}(t) - \overline{F}_{*,i-1}(t) \leq \overline{F}_{*,i}(t)$.

ii) Again, by definition, $G_i \leq_{rh} F_{*,i}$ if and only if $r_i(t) \leq r_{*,i}(t)$ for all t. First, we write the reversed hazard rate of the i-th SOS

$$r_{*,i}(t) = h_i(t)\left(\frac{\overline{F}_{*,i}(t) - \overline{F}_{*,i-1}(t)}{F_{*,i}(t)}\right) = r_i(t)\frac{G_i(t)}{\overline{G}_i(t)}\left(\frac{F_{*,i-1}(t) - F_{*,i}(t)}{F_{*,i}(t)}\right)$$

and then, we have that $r_{*,i}(t) \geq r_i(t)$ if and only if

$$\begin{aligned}
& 1 \leq \left(\frac{1-\overline{G}_i(t)}{\overline{G}_i(t)}\right)\left(\frac{F_{*,i-1}(t)-F_{*,i}(t)}{F_{*,i}(t)}\right) \\
\Leftrightarrow \quad & 0 \leq \frac{F_{*,i-1}(t)-F_{*,i}(t)-\overline{G}_i(t)F_{*,i-1}(t)}{\overline{G}_i(t)F_{*,i}(t)} \\
\Leftrightarrow \quad & F_{*,i}(t) = F_{*,i-1}(t) - \overline{G}_i(t)A_i(t) \leq G_i(t)F_{*,i-1}(t) \\
\Leftrightarrow \quad & F_{*,i-1}(t)\left(1-G_i(t)\right) \leq \overline{G}_i(t)A_i(t) \\
\Leftrightarrow \quad & F_{*,i-1}(t) \leq A_i(t).
\end{aligned}$$

The last condition holds since $\overline{G}_i(t) \leq 1$ and from (3.5).

iii) By definition, $G_i \leq_{lr} F_{*,i}$ if and only if $f_{*,i}(t)/\, g_i(t)$ is increasing for all t. From (3.3) and (3.5) we have

$$f_{*,i}(t) = g_i(t)A_i(t) \Leftrightarrow \frac{f_{*,i}(t)}{g_i(t)} = A_i(t).$$

Clearly $A_i(t)$ is increasing, then $G_i \leq_{lr} F_{*,i}$ holds.

□

Now, we present a connection in the star ordering between the SOS and their underlying distribution functions. First, let us define

$$u_i(t) = t \cdot h_i(t) \quad \text{and} \quad v_i(t) = t \cdot r_i(t).$$

Lemma 5. *Under the same assumptions as Lemma 4, if the support of G_i is non-negative for all i and*

i) $u_i(t)$ is decreasing, then $G_i \leq_* F_{*,i}$;

ii) $v_i(t)$ is increasing, then $G_i \geq_* F_{*,i}$.

Proof.

i) From Lemma 4(i) and Lemma 1 we have that $\ln G_i \leq_{hr} \ln F_{*,i}$. Now, the hazard rate of $\ln G_i$ is decreasing in t if and only if $u_i(t)$ is decreasing (see Theorem 2.3. in [7]). From Lemma 3(i), if $\ln G_i$ is DHR and $\ln G_i \leq_{hr} \ln F_{*,i}$, then $\ln G_i \leq_{\text{disp}} \ln F_{*,i}$. Finally, from Lemma 2 we have $\ln G_i \leq_{\text{disp}} \ln F_{*,i} \Leftrightarrow G_i \leq_* F_{*,i}$.

ii) From Lemma 4(ii) and Lemma 1 we have that $\ln G_i \leq_{rh} \ln F_{*,i}$. Now, it is easy to check that $v_i(t)$ is increasing if and only if the reversed hazard rate of $\ln G_i$ is increasing in t. From Lemma 3(ii), if $\ln G_i$ is IRHR and $\ln G_i \leq_{rh} \ln F_{*,i}$, then $\ln G_i \geq_{\text{disp}} \ln F_{*,i}$. Finally, from Lemma 2 we have $\ln G_i \geq_{\text{disp}} \ln F_{*,i} \Leftrightarrow G_i \geq_* F_{*,i}$.

□

It is worth noting that the condition that $u_i(t)$ is decreasing in Lemma 5(i) can be rewritten in the form $u_i'(t) = t \cdot h_i'(t) + h_i(t) \leq 0$. Therefore, it is clear that the condition that $h_i(t)$ be decreasing is a necessary but not sufficient condition for $u_i(t)$ to be decreasing. Similarly, the condition that $v_i(t)$ is increasing in Lemma 5(ii) can be rewritten as $v_i'(t) = t \cdot r_i'(t) + r_i(t) \geq 0$ and thus, it is clear that if $r_i(t)$ is increasing (i.e., X is IRHR) then $v_i(t)$ is also increasing (i.e., $\ln(X)$ is IRHR). However, the converse is not true as is illustrated by the following counterexample.

Counterexample 1. The reversed hazard rate of the uniform distribution on $[-1, 1]$ is given by

$$r(t) = \frac{1}{1+t}\,, \quad t \in [-1, 1].$$

Clearly, r is decreasing but the corresponding reversed hazard rate of the logarithm is

$$r_{\ln X}(t) = e^t \cdot r\left(e^t\right) = \frac{e^t}{1+e^t}\,, \qquad \text{for all } t,$$

and it is easy to verify that $r_{\ln X}$ is increasing.

4 Stochastic properties.

In this section, we investigate conditions on the underlying distribution functions on which the SOS are based, in order to obtain stochastic comparisons of SOS with various other univariate orders. Zhuang and Hu [16] presented some results on multivariate stochastic comparisons of SOS. They showed in their Theorem 3.7. that if the underlying distribution functions are ordered in the univariate hazard rate order, i.e., $G_1 \leq_{hr} G_2 \leq_{hr} \cdots \leq_{hr} G_n$, then

$$\left(X^*_{1,n}, \ldots, X^*_{n-1,n}\right) \leq_{st} \left(X^*_{2,n}, \ldots, X^*_{n,n}\right). \tag{4.8}$$

Since the usual multivariate stochastic order is closed under marginalization, we can get univariate comparisons of SOS from (4.8). However, in the univariate case, these results can be given without conditions, as we show below.

Theorem 1. *Let $X^*_{1,n}$,...,$X^*_{n,n}$ be SOS based on absolutely continuous distribution functions $\{G_1, \ldots, G_n\}$, then*

$$X^*_{i-1,n} \leq_{st} X^*_{i,n} \quad \textit{for } i = 2, \ldots, n.$$

Proof. From (3.2) we obtain the survival function of the i'th SOS

$$\overline{F}_{*,i}(t) = \overline{F}_{*,i-1}(t) + \int_{-\infty}^{t} \frac{\overline{G}_i(t)}{\overline{G}_i(z)}\, dF_{*,i-1}(z), \tag{4.9}$$

for $i = 2$,...,n. Then,

$$\overline{F}_{*,i}(t) - \overline{F}_{*,i-1}(t) = \overline{G}_i(t) A_i(t), \tag{4.10}$$

is positive, where $A_i(t)$ is defined in (3.5). □

Therefore, the successive SOS are increasing in the usual stochastic ordering. We now proceed to stochastic comparisons of the first SOS and the others in the univariate hazard rate and likelihood ratio ordering.

Theorem 2. *Under the same assumptions than in theorem 1, if $G_1 \leq_{hr(lr)} G_i$ for $i \geq 2$, then*

i) $X^*_{1,n} \leq_{hr} X^*_{i,n}$ *and*

ii) $X^*_{1,n} \leq_{lr} X^*_{i,n}$,

for $i = 2, \ldots, n$.

Proof.

i) By definition we know that $X^*_{1,n} \leq_{hr} X^*_{i,n} \Leftrightarrow \overline{F}_{*,i}(t)/\,\overline{F}_{*,1}(t)$ is increasing in t. To do this we will use induction. It is immediately that $F_{*,1} \leq_{hr} F_{*,2}$ since from Lemma 4 we know that $G_2 \leq_{hr} F_{*,2}$ and by the assumptions $F_{*,1} = G_1 \leq_{hr} G_2$. We assume that $F_{*,1} \leq_{hr} F_{*,i-1}$, so we need to show that it is true for i. We get from (3.6) that

$$\frac{\overline{F}_{*,i}(t)}{\overline{F}_{*,1}(t)} = \frac{\overline{F}_{*,i-1}(t)}{\overline{F}_{*,1}(t)} + \frac{\overline{G}_i(t) A_i(t)}{\overline{F}_{*,1}(t)},$$

which is increasing in t since $A_i(t)$ and $\overline{G}_i(t)/\,\overline{F}_{*,1}(t)$ are increasing.

ii) In this case, $X^*_{1,n} \leq_{lr} X^*_{i,n} \Leftrightarrow f_{*,i}(t)/\,f_{*,1}(t)$ is increasing in t. We have, from (3.3), that

$$\frac{f_{*,i}(t)}{f_{*,1}(t)} = \frac{g_{*,i}(t)}{f_{*,1}(t)} A_i(t),$$

which is increasing in t since $A_i(t)$ and $g_i(t)/\,f_{*,1}(t)$ are increasing.

□

Next, we discuss the likelihood ratio order. First, let us recall the definition of a TP_2 function. A nonnegative function h of two variables, x and y, say, is called TP_2 if $h(x',y)/h(x,y)$ is increasing in y whenever $x \leq x'$.

Lemma 6. *Under the same assumptions than in theorem 1, if $g_{i-1}(t)/g_i(t)$ and $h_i(t)$ are TP_2 in (i,t), and $G_{i-1} \leq_{hr} G_i$ for all i, then $A_i(t)$ is TP_2 in (i,t) for $i = 3, \ldots, n$.*

Proof. We will see, by induction on $i \geq 3$, that

$$A_i(t) = \int_{-\infty}^{t} \frac{1}{\overline{G}_i(z)} dF_{*,i-1}(z) = \int_{-\infty}^{t} \frac{g_{i-1}(z)}{g_i(z)} h_i(z) A_{i-1}(z) dz = \int_{-\infty}^{t} q_i(z) h_i(z) A_{i-1}(z) dz,$$

is TP_2 in (i,t), where $q_i(z) = \frac{g_{i-1}(z)}{g_i(z)}$. By the assumptions, we have

$$\frac{q_3(t)}{q_2(t)} \frac{h_3(t)}{h_2(t)} A_2(t),$$

is increasing in t, which implies that $A_3(t)/A_2(t)$ is increasing in t. Let now $i \geq 4$. Again

$$\frac{q_i(t)\, h_i(t)\, A_{i-1}(t)}{q_{i-1}(t)\, h_{i-1}(t)\, A_{i-2}(t)},$$

is increasing in t, by the assumptions and by the induction hypothesis, which implies that $A_i(t)/A_{i-1}(t)$ is increasing in t. Hence, $A_i(t)$ is TP_2 in (i,t). □

The following result gives conditions under which the SOSs are comparable in the univariate likelihood ratio order.

Theorem 3. *Under the same assumptions than in Lemma 6, then*

$$X^*_{i-1,n} \leq_{lr} X^*_{i,n},$$

for $i = 3, \ldots, n$.

Proof. By definition and from (3.3) we know that $X^*_{i-1,n} \leq_{lr} X^*_{i,n}$ iff

$$\frac{f_{*,i}(t)}{f_{*,i-1}(t)} = \frac{g_i(t)A_i(t)}{g_{i-1}(t)A_{i-1}(t)}, \tag{4.11}$$

is increasing in t. From the previous Lemma we know that $A_i(t)$ is TP_2 in (i,t), and from Theorem 1.C.4(a) in [14] we get that $G_{i-1} \leq_{lr} G_i$, then, it follows that $f_{*,i}(t)/f_{*,i-1}(t)$ is increasing in t for $i = 3, \dots, n$. □

Note that $g_{i-1}(t)/g_i(t)$ is TP_2 in (i,t) can be written as

$$\frac{(g_{i-1}(t))^2}{g_i(t)g_{i-2}(t)}, \tag{4.12}$$

is increasing in t. Zhuang and Hu [16] proved that if $G_{i-1} \leq_{lr} G_i$ and

$$\frac{\left(\overline{G}_{i-1}(t)\right)^2}{\overline{G}_i(t)\overline{G}_{i-2}(t)}, \tag{4.13}$$

is increasing in $t \in \Re_+$ and $i = 1, \dots, n-2$, then $X^*_{i-1,n} \leq_{lr} X^*_{i,n}$ for $i = 1, \dots, n-1$. Our previous result is equivalent to this of Zhuang and Hu [16] in the sense that both have the same result and almost the same assumptions, except condition (4.12) and (4.13), respectively. Note that the condition (4.12) is useful when we have not an analytical expression of the survival functions.

When $G_i(t) = 1 - (1 - F(t))^{\gamma_i}$ for some distribution function F and γ_i are positive numbers for $i = 1, \dots, n$, then $G_{i-1} \leq_{hr} G_i$ if and only if $\gamma_{i-1} \geq \gamma_i$, and the condition (4.12) holds if and only if $2\gamma_{i-1} \leq \gamma_i + \gamma_{i-2}$.

It is worth noting that the $i-1$'th SOS is not greater than the i'th SOS in the hazard rate and reversed hazard rate ordering as we will show in the following theorem. From (3.6) and (3.7), we get that

$$\frac{F_{*,i}(t)}{F_{*,i-1}(t)} = 1 - \frac{\overline{G}_i(t)A_i(t)}{F_{*,i-1}(t)}\,, \tag{4.14}$$

and

$$\frac{\overline{F}_{*,i}(t)}{\overline{F}_{*,i-1}(t)} = 1 + \frac{\overline{G}_i(t)A_i(t)}{\overline{F}_{*,i-1}(t)}\,, \tag{4.15}$$

for $i = 2, \dots, n$.

Theorem 4. *Let $X^*_{1,n}, \dots, X^*_{n,n}$ be SOS based on absolutely continuous distribution functions $\{G_1, \dots, G_n\}$, then* *i)* $X^*_{i-1,n} \not\geq_{hr} X^*_{i,n}$ *and*

ii) $X^*_{i-1,n} \not\geq_{rh} X^*_{i,n}$,

for $i = 2, \ldots, n$.

Proof.

i) We suppose, by reduction to the absurd that $X^*_{i-1,n} \geq_{hr} X^*_{i,n}$. By definition we know

$$X^*_{i-1,n} \geq_{hr} X^*_{i,n} \Leftrightarrow \frac{\overline{F}_{*,i}(t)}{\overline{F}_{*,i-1}(t)} \text{ is decreasing in } t,$$

and from (4.15), we have

$$X^*_{i-1,n} \geq_{hr} X^*_{i,n} \Leftrightarrow \frac{\overline{G}_i(t)A_i(t)}{\overline{F}_{*,i-1}(t)} \text{ is decreasing in } t.$$

Note that

$$\frac{\overline{G}_i(t)A_i(t)}{F_{*,i-1}(t)} = \frac{\overline{G}_i(t)A_i(t)}{\overline{F}_{*,i-1}(t)} \frac{\overline{F}_{*,i-1}(t)}{F_{*,i-1}(t)},$$

which is decreasing in t when $\frac{\overline{G}_i(t)A_i(t)}{\overline{F}_{*,i-1}(t)}$ is decreasing in t, since $\frac{\overline{F}_{*,i-1}(t)}{F_{*,i-1}(t)}$ is decreasing in t. Now, from (4.14)

$$\frac{\overline{G}_i(t)A_i(t)}{F_{*,i-1}(t)} \text{ is decreasing in } t \Leftrightarrow X^*_{i-1,n} \leq_{rh} X^*_{i,n} \ ,$$

i.e., if $X^*_{i-1,n} \geq_{hr} X^*_{i,n}$ then $X^*_{i-1,n} \leq_{rh} X^*_{i,n}$. Thus, $X^*_{i-1,n} =^{st} X^*_{i,n}$, which is a contradiction, since $X^*_{i-1,n} \leq_{st} X^*_{i,n}$ from Theorem 1 . Hence $X^*_{i-1,n} \not\geq_{hr} X^*_{i,n}$.

ii) As before, we suppose, by reduction to the absurd that $X^*_{i-1,n} \geq_{rh} X^*_{i,n}$. By definition we know

$$X^*_{i-1,n} \geq_{rh} X^*_{i,n} \Leftrightarrow \frac{F_{*,i}(t)}{F_{*,i-1}(t)} \text{ is decreasing in } t,$$

and from (4.14), we have

$$X^*_{i-1,n} \geq_{rh} X^*_{i,n} \Leftrightarrow \frac{\overline{G}_i(t)A_i(t)}{F_{*,i-1}(t)} \text{ is increasing in } t.$$

Note that

$$\frac{\overline{G}_i(t)A_i(t)}{\overline{F}_{*,i-1}(t)} = \frac{\overline{G}_i(t)A_i(t)}{F_{*,i-1}(t)} \frac{F_{*,i-1}(t)}{\overline{F}_{*,i-1}(t)},$$

which is increasing in t when $\frac{\overline{G}_i(t)A_i(t)}{F_{*,i-1}(t)}$ is increasing in t, since $\frac{F_{*,i-1}(t)}{\overline{F}_{*,i-1}(t)}$ is increasing in t. Now, from (4.15)

$$\frac{\overline{G}_i(t)A_i(t)}{\overline{F}_{*,i-1}(t)} \text{ is increasing in } t \Leftrightarrow X^*_{i-1,n} \leq_{hr} X^*_{i,n} \ ,$$

i.e., if $X^*_{i-1,n} \geq_{rh} X^*_{i,n}$ then $X^*_{i-1,n} \leq_{hr} X^*_{i,n}$. Thus, $X^*_{i-1,n} =^{st} X^*_{i,n}$, which is again a contradiction. Hence $X^*_{i-1,n} \ngeq_{rh} X^*_{i,n}$.

□

A consequence of Theorem 4 is that $X^*_{i-1,n} \ngeq_{lr} X^*_{i,n}$ for $i = 2, \ldots, n$.

5 Applications in reliability.

In this section, some applications of the main results in Section 4 are presented. Specifically we give an application for nonhomogeneous pure birth processes.

Nonhomogeneous pure birth processes are called relevation counting processes in [12], where some applications of them in reliability theory are described. Another interpretation of these processes in reliability theory, by means of load sharing, is described in [15]. A counting process $\{N(t), t \geq 0\}$ is a nonhomogeneous pure birth process (NHPB) with intensity functions $\{\lambda_i(t), i \geq 0\}$ if the following hold:

i) $N(t), t \geq 0$ has the Markov property;

ii) $P\{N(t+\Delta t) = i+1 | N(t) = i\} = \lambda_i(t)\Delta t + \circ(\Delta t)$ for $i \geq 1$;

iii) $P\{N(t+\Delta t) > i+1 | N(t) = i\} = \circ(\Delta t)$ for $i \geq 1$,

the λ_i's are non-negative functions that satisfy

$$\int_t^\infty \lambda_i(x)dx = \infty, \qquad \text{for all } t \geq 0. \tag{5.16}$$

Condition (5.16) ensures that, with probability 1, the process has a jump after any time point t. When all the λ_i are identical, a nonhomogeneous pure birth process reduces to a nonhomogeneous Poisson process. We are especially interested in the coincidence (in distribution) of the epoch times of pure birth processes with certain models of ordered random variables such as record values, order statistics, generalized order statistics, Pfeifer record values, and SOS. In a distributional theoretical sense, there is one-to-one correspondence between SOS and the first n epoch times of a NHPB process, which is stated in the following proposition.

Proposition 1 (Corollary 3.3.4. in Lenz [10])**.** *Let $G_1, \dots, G_n$ be continuous distribution functions with $G_i(0) = 0$ and $G_i^{-1}(1) = c_i \in (0, \infty)$, $c_i \leq c_{i+1}$ and $X^*_{1,n}, \dots, X^*_{n,n}$ the corresponding SOS. Let $\{N(t), t \geq 0\}$ be a NHPB process with mean value function $\Lambda_i(t)$ and denote the epoch times by $S_i, i = 1, \dots, n$. Then S_i and $X^*_{i,n}$ coincide in distribution if and only if*

$$\Lambda_i(t) = -ln\, \overline{G}_i(t), \quad \text{for all } t \in [0, c_i).$$

Given this relationship and from Theorems 1-2, it is possible to derive the following result.

Corollary 1. *Let S_i, $i \geq 1$ denote the epoch times of a NHPB process $\{N(t), t \geq 0\}$ with intensity functions $\lambda_i(t)$ and mean value function $\Lambda_i(t)$. Then:*

i) $S_{i-1} \leq_{st} S_i$, *for* $i = 2, \dots, n$,

ii) *if* $\lambda_1(t) \geq \lambda_i(t)$ *for all* t *and for* $i = 2, \dots, n$, *then* $S_1 \leq_{hr} S_i$, *for* $i = 2, \dots, n$,

iii *if* $\lambda_1(t) \geq \lambda_i(t)$ *and* $\frac{\lambda_i(t)}{\lambda_1(t)}$ *is increasing in* t *for* $i = 2, \dots, n$, *then* $S_1 \leq_{lr} S_i$, *for* $i = 2, \dots, n$.

Proof. Define

$$h_i(t) = \lambda_i(t) \qquad \text{for} \quad i = 1, \dots, n.$$

Since (5.16) holds, $h_i(t)$ can be regarded as the hazard rate function of some distribution G_i. Let $X^*_{1,n}, \dots, X^*_{n,n}$ be the SOS based on distributions $\{G_1, \dots, G_n\}$. Then, the result follows from Proposition 1 and Theorems 1–2. □

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