LAGRANGE-HAMILTON MODEL FOR CONTROL AFFINE SYSTEMS WITH POSITIVE HOMOGENEOUS COST

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Abstract: In this paper we give a new technique to obtain the Hamiltonian function in order to solve the driftless control affine systems (distributional systems) with positive homogeneous costs. The method consists by using the Lagrange multipliers and Legendre transformation associated to a singular Lagrangian. This method could be an alternative to the classical Pontryagin Maximum Principle.

JEL classification: C02, C6

Key words: control affine systems, positive homogeneous cost, Lagrange-Hamilton formalism, Pontryagin Maximum Principle.

1. INTRODUCTION

The model offers an intuitive picture, but rigorous study of the phenomenon and allows finding a link between the various sizes that characterize the economic process. Along the models of macro and microeconomic type, econometric, the mathematical models are characterized by finding the optimal solution or as close to optimum [2], [3], [6].

It is well-known that the solution of a control affine system is provided by Pontryagin's Maximum Principle [1]: that is, the curve $c(t)=(x(t),u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t),p(t))$ satisfying the Hamilton's equations.

In this paper we give a new formula which permit us to find the Hamiltonian on the dual space, using the Lagrange multipliers and Legendre transformation. The paper is organized as follows. In the second section are presented the preliminaries on driftless control affine systems and is given the expresion of the Hamiltonian function. In the last part, using the new formula for the Hamiltonian, some illustrative examples are given. Other point of view involving Lie algebroids is given in [4].

2. CONTROL AFINE SYSTEMS

Let us consider the drift-less control affine system (called also distributional systems) in the space R^n on the form

$$\dot{X}(t)=\sum_{i=1}^{m}u^i(t)X_i(x(t)) \qquad (1)$$

with X_i, $i=1,...,m$ vector fields in R^n and the controls $u=(u_1,u_2,...,u_m)$ take values in an open subset $\Omega \subset R^n$. The vector fields X_i generate a distribution $D \subset R^n$ such that the rank of *D* is assumed to be constant.

Let x_0 and x_1 be two points of R^n. An optimal control problem consists of finding those trajectories of the distributional system which connect x_0 and x_1, while minimizing the cost

$$\min_{u(.)} \int_I F(u(t))dt \quad ,$$

where F is a positive homogeneous cost (Minkowski norm) on D.

We consider the Lagrangian function of the form

$L=\frac{1}{2}F^2$ and it results that is 2-homogeneous function.

Considering $\left(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},...,\frac{\partial}{\partial x^n}\right)$ the canonical basis in R^n, we have the following relation

$$X_i = \sum_{r=1}^{n} a_i^r(x)\frac{\partial}{\partial x^r}$$

and it results

$$\dot{X}(t)=\sum_{r=1}^{n}\sum_{i=1}^{m} u^i(t)a_i^r(x)\frac{\partial}{\partial x^r} \qquad (2)$$

But, on the other hand, we have

$$\dot{X}(t)=\sum_{r=1}^{n}\dot{x}^r(t)\frac{\partial}{\partial x^r} \qquad (3)$$

and from (2) and (3) we obtain

$$\dot{x}^r=\sum_{i=1}^{m} u^i a_i^r(x), \qquad (4)$$

or, in the equivalent form

$$\begin{cases} u^1a_1^1+...+u^m a_m^1=\dfrac{dx^1}{dt} \\ .... \\ u^1a_1^m+...+u^m a_n^m=\dfrac{dx^m}{dt} \\ .... \\ u^1a_1^n+...+u^m a_m^n=\dfrac{dx^n}{dt} \end{cases}$$

The system (4) is a linear system in u^i, $i=1,...,m$ and *rankA=rankD=m*.

We suppose, without lose the generality (can by changed the lines into the system) that the first *m* lines are linearly independent. Let m_j^i be the matrix built from the initial a_j^i preserving first *j* lines $i,j=1,...,m$, and we obtain

$$\dot{x}^j = \sum_{i=1}^{m} u^i m_i^j ,$$

which yields

$$u^i = \sum_{i=1}^{m} \dot{x}^j (m_j^i)^{-1}, \qquad (5)$$

But from the system (4) remains n-m equations, i.e. we have n-m constraints of the control system, in the form (Einstein sumation)

$$\Phi^\alpha = \dot{x}^\alpha - u^i a_i^\alpha (x), \qquad \alpha = m+1,...,n$$

From (5) follows

$$\Phi^\alpha = \dot{x}^\alpha - \dot{x}^j (m_j^i)^{-1} a_i^\alpha (x), \qquad i, j = 1,...m$$

Then, using the Lagrange multipliers, we obtain the total Lagrangian (including the constraints) given by

$$L'(x, \dot{x}) = L(x, \dot{x}) + \lambda_\alpha (t) \Phi^\alpha (x, \dot{x}) \qquad (6)$$

where $L(x, \dot{x}) = \frac{1}{2} F^2 (x, \dot{x})$. It results

$$L'(x, \dot{x}) = L(x, \dot{x}) + \lambda_\alpha (t)(\dot{x}^\alpha - \dot{x}^\alpha \, b_j^\alpha (x))$$

where we have denoted

$$b_j^\alpha = (m_j^i)^{-1} a_i^\alpha , \; i, j = 1,...m , \alpha = m+1,...,n$$

Remark 1. *The Euler-Lagrange equations for the total Lagrangian have the expression*

$$\frac{\partial L'}{\partial x^r} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}^r} \right) = 0$$

or, in the equivalent form

$$\frac{\partial L}{\partial x^r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^r} \right) =$$

$$\lambda_\alpha (t) \left(\frac{\partial \Phi^\alpha}{\partial x^r} - \frac{d}{dt}\left(\frac{\partial \Phi^\alpha}{\partial \dot{x}^r} \right) \right) - \frac{d\lambda_\alpha}{dt} \frac{\partial \Phi^\alpha}{\partial \dot{x}^r} .$$

But, we observe that the Hessian matrix of *L’* is singular, that is

$$\frac{\partial^2 L'}{\partial \dot{x}^r \partial \dot{x}^s} = \frac{\partial^2 L}{\partial \dot{x}^r \partial \dot{x}^s} = \begin{pmatrix} \dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} & 0 \\ 0 & 0 \end{pmatrix}$$

for *r=1,...,n* and $i, j = 1,...,m$, so *L’* is a degenerate Lagrangian (singular).

Next, using the Legeandre transformation, we can find the Hamiltonian function on the dual space, on the form

$$H' = \dot{x}^s \, p_s - L'$$

where

$$p_s = \frac{\partial L'}{\partial \dot{x}^s}, \; s = 1,\dots,n$$

and we get

$$p_s = \frac{\partial L}{\partial \dot{x}^s} + \lambda_\alpha(t)\frac{\partial \Phi^\alpha}{\partial \dot{x}^s}$$

which leads to the following system

$$\begin{cases} p_1 = \dfrac{\partial L}{\partial \dot{x}^1} - \lambda_\alpha b_1^\alpha \\ \dots\dots\dots\dots\dots\dots \\ p_m = \dfrac{\partial L}{\partial \dot{x}^m} - \lambda_\alpha b_m^\alpha \\ p_{m+1} = \lambda_{m+1} \\ \dots\dots\dots\dots\dots\dots \\ p_n = \lambda_n \end{cases} \qquad (7)$$

Then, we obtain

$$H = \dot{x}^s\, p_s - L' = \dot{x}^i\, p_i - \dot{x}^\alpha\, p_\alpha - L'$$

$$= \dot{x}^i\left(\frac{\partial L}{\partial \dot{x}^i} - \lambda_\alpha b_i^\alpha\right) + \dot{x}^\alpha\, \lambda_\alpha - L - \dot{x}^\alpha\, \lambda_\alpha +$$

$$\lambda_\alpha\, \dot{x}^\alpha\, b_i^\alpha = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L = 2L - L = L$$

because L is 2-homogeneous function with respect to $\dot{x}^i$ and it results the equality

$$H(x,p) = L(x,\dot{x}^i).$$

Let us consider the Hamiltonian $\tilde{H}$ associated to the Lagrangian L on the distribution D on the form

$$\tilde{H} = \dot{x}^i\, p_i - L, \qquad \tilde{p}_i = \frac{\partial L}{\partial \dot{x}^i}, \qquad i = 1,\dots,m$$

with $\tilde{H} = \tilde{H}(x,\tilde{p}_i)$. But from the system (7) we have

$$p_i = \frac{\partial L}{\partial \dot{x}^i} - \lambda_\alpha b_i^\alpha, \qquad p_\alpha = \lambda_\alpha$$

for $i = 1,\dots,m$ and $\alpha = m+1,\dots,n$ and it results

$$p_i + p_\alpha b_i^\alpha = \frac{\partial L}{\partial \dot{x}^i}$$

Using the fact that

$$\tilde{p}_i = \frac{\partial L}{\partial \dot{x}^i},$$

we obtain the following formula

$$\tilde{p}_i = p_i + p_\alpha b_i^\alpha .$$

Now, using the previous notations and considerations, we can present the main result of the paper:

Theorem 1. *The Hamiltonian H on the dual space associated to the total Lagrangian L' has the form*

$$H(x,p) = \tilde{H}(x,\tilde{p}_i) = \tilde{H}(x, p_i + p_\alpha b_i^\alpha), \; i = 1,...,m, \quad \alpha = m+1,...,n .$$

3. Some examples

Example 1. *Let us consider the driftless control affine system*

$$\dot{X}(t) = u^1 X_1 + u^2 X_2 + u^3 X_3$$

with $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ x \\ 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ x \end{pmatrix}$

and minimizing the cost

$$\min_{u(.)} \int_I F(u(t))dt ,$$

where $F = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}$ *is the quadratic cost (Euclidian metric).*

The distribution $D =< X_1, X_2, X_3 >$ generated by X_1, X_2, X_3 has constant rank 3 and the system of restrictions has the form

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \\ \dot{x}^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

Let

$$m_i^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}$$

be the reduced matrix with $rank\, m_i^j = 3$ and it results

$$(m_i^j)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x & 1 \end{pmatrix}$$

which yields the following equations

$$\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x & 1 \end{pmatrix} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{pmatrix}$$

or equivalent

$$\begin{cases} u_1 = \dot{x}^1 \\ u_2 = \dot{x}^1 \\ u_3 = -x\dot{x}^1 + \dot{x}^3 \end{cases}$$

The Lagrangian has the form

$$L = \frac{1}{2}F^2 = \frac{1}{2}\left((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3 - x\dot{x}^2)^2 \right)$$

$$= (\dot{x}^1)^2 + (1 + x^2)(\dot{x}^2)^2 + (\dot{x}^3)^2 - 2x\dot{x}^2\dot{x}^3$$

or, in the equivalent form $L = g_{ij}\dot{x}^i\dot{x}^j$ with

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{pmatrix}$$

The inverse matrix of $g^{ij} = (g_{ij})^{-1}$ is given by

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1+x^2 \end{pmatrix}$$

and we obtain the Hamiltonian

$$\tilde{H} = \frac{1}{2} g^{ij} \tilde{p}_i \tilde{p}_j = \frac{1}{2}\left(\tilde{p}_1^2 + \tilde{p}_2^2 + (1+x^2)\tilde{p}_3^2 + 2x\tilde{p}_2 \tilde{p}_3 \right)$$

But

$$b_j^\alpha = (m_j^i)^{-1} a_i^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -x \\ 0 & -x & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ -x^2 \\ x \end{pmatrix},$$

and it results

$$\tilde{p}_i = p_i + p_\alpha b_i^\alpha \Leftrightarrow \begin{cases} \tilde{p}_1 = p_1 \\ \tilde{p}_2 = p_2 - p_4 x^2 \\ \tilde{p}_3 = p_3 + p_4 x \end{cases}$$

Using these relations we can find the Hamiltonian on the dual space

$$H(p) = \tilde{H}(\tilde{p}) = \frac{1}{2}\left(p_1^2 + (p_2 - p_4 x^2)^2 + (1+x^2)(p_3 + p_4 x)^2 + 2x(p_2 - p_4 x^2)(p_3 + p_4 x)\right)$$

and by direct computation it results

$$H(x,p) = \frac{1}{2}\left(p_1^2 + p_2^2 + (1+x^2)p_3^2 + x^2 p_4^2 + 2x p_2 p_3 + 2x p_3 p_4\right)$$

We have to remark that

$$H(x,p) = G^{ij}(x) p_i p_j$$

where

$$G^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & x & 1+x^2 & x \\ 0 & 0 & x & x^2 \end{pmatrix}$$

But $\det\left(G^{ij}\right) = 0$ and it results that *H* is a degenerate Hamiltonian.

Example 2. *Let us consider the driftless control affine system*

$$\dot{X}(t) = u^1 X_1 + u^2 X_2$$

with $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ x \\ \frac{x^2}{2} \end{pmatrix}$,

and minimizing the cost

$$\min_{u(.)} \int_I F(u(t))dt,$$

where $F=\sqrt{(u^1)^2+\left(u^2\right)^2}$ *is the quadratic cost (Euclidian metric).*

The distribution $D=<X_1,X_2>$ generated by X_1,X_2 has constant rank 2 and the system of restrictions has the form

$$\begin{pmatrix}\dot{x}^1\\ \dot{x}^2\\ \dot{x}^3\\ \dot{x}^4\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & x\\ 0 & x^2/2\end{pmatrix}\begin{pmatrix}u^1\\ u^2\end{pmatrix}$$

Let $m_i^j=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ be the reduced matrix with $rank\, m_i^j=2$ and it results $(m_i^j)^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ which yields the equations $u^1=\dot{x}^1$, $u^2=\dot{x}^2$ and it results

$$L=\frac{1}{2}F^2=\frac{1}{2}\left((u^1)^2+(u^2)^2\right)=\frac{1}{2}\left((\dot{x}^1)^2+(\dot{x}^1)^2\right)$$

The Hamiltonian has the form

$$\tilde{H}(x,p)=\frac{1}{2}\left(\tilde{p}_1^2+\tilde{p}_2^2\right)$$

where

$$b_j^\alpha=(m_j^i)^{-1}a_i^\alpha=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ x & x^2/2\end{pmatrix}=\begin{pmatrix}0 & 0\\ x & x^2/2\end{pmatrix},$$

and it results

$$\tilde{p}_i=p_i+p_\alpha b_i^\alpha \Leftrightarrow \begin{cases}\tilde{p}_1=p_1\\ \tilde{p}_2=p_2+p_3x+p_4x^2/2\end{cases}$$

Using these relations we can find the Hamiltonian on the dual space

$$H(x,p)=\tilde{H}(x,\tilde{p})=\frac{1}{2}\left(p_1^2+(p_2+p_3x+p_4x^2/2)^2\right).$$

Example 3. *Let us consider the driftless control affine system (Heisenberg group)*

$$\dot{X}(t)=u^1X_1+u^2X_2$$

with $X_1=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$, $X_2=\begin{pmatrix}0\\ 1\\ x\end{pmatrix}$,

and minimizing the cost

$$\min_{u(.)}\int_I F(u(t))dt\, ,$$

where $F=\sqrt{(u^1)^2+\left(u^2\right)^2}+\varepsilon u^1$, $\varepsilon<1$ *is the positive homogeneous cost (Randers metric).*

The distribution $D=<X_1,X_2>$ generated by X_1,X_2 has constant rank 2 and the system of restrictions has the form

$$\begin{pmatrix}\dot{x}^1\\ \dot{x}^2\\ \dot{x}^3\end{pmatrix}=\begin{pmatrix}1&0\\0&1\\0&x\end{pmatrix}\begin{pmatrix}u^1\\u^2\end{pmatrix}$$

Let $m_i^j=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ be the reduced matrix with $rank\, m_i^j=2$ and it results $(m_i^j)^{-1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ which yields the equations $u^1=\dot{x}^1$, $u^2=\dot{x}^2$ and

$$L=\frac{1}{2}F^2=\frac{1}{2}\left(\sqrt{(u^1)^2+\left(u^2\right)^2}+\varepsilon u^1\right)^2$$

The Hamiltonian has the form [5]

$$\tilde{H}(x,\tilde{p})=\frac{1`}{2}\left(\sqrt{\frac{\tilde{p}_1^{\,2}}{(1-\varepsilon^2)^2}+\frac{\tilde{p}_2^{\,2}}{1-\varepsilon^2}}-\frac{\varepsilon}{1-\varepsilon^2}\,p_1\right)^2$$

But, we have the equalities

$$b_j^\alpha=(m_j^i)^{-1}a_i^\alpha=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}0\\x\end{pmatrix}=\begin{pmatrix}0\\x\end{pmatrix},$$

and the equations

$$\tilde{p}_i=p_i+p_\alpha b_i^\alpha \Leftrightarrow \begin{cases}\tilde{p}_1=p_1\\ \tilde{p}_2=p_2+p_3x\end{cases}$$

lead to the following expression for the Hamiltonian function

$$H(x,p)=\frac{1`}{2}\left(\sqrt{\frac{p_1^{\,2}}{(1-\varepsilon^2)^2}+\frac{(p_2+p_3x)^2}{1-\varepsilon^2}}-\frac{\varepsilon}{1-\varepsilon^2}\,p_1\right)^2.$$

5. Conclusions

In this paper we give a new formula which permit us to find the Hamiltonian function on the dual space, using the Lagrange multipliers and Legendre transformation associated with a singular Lagrangian. This tehnique could be an alternative to the classical Pontryagin Maximum Principle in the case of distributional systems. In last part of the paper, some illustrative examples are given..

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