The expectation of products of quadratic forms in normal variables: the practice

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Summary A table is presented to simplify the computation of the expectation of a product of quadratic forms in normal variables. Some peculiarities of the table are discussed.

1 Introduction

The reader of this journal has recently been exposed to two papers on the expectation of the product of an arbitrary number of quadratic forms in normally distributed variables. Let $u \simeq N_n(0, I)$, and $\varepsilon \simeq N_n(0, V)$, then the problem is to determine the expectation of $\prod_{j=1}^{s} u'A_j u$, and $\prod_{j=1}^{s} \varepsilon' A_j \varepsilon$, where the A_j $(j = 1 \ldots s)$ are symmetric (n, n) matrices. In [3] I proved that this expectation is – what I called – an $A(s)$-polynomial. DON [1] presented an alternative proof of the main result, based on polarization. Also he showed that the covariance matrix of ε–V– may be singular, without affecting the results, and he derived a simple formula for the coefficients of the $A(s)$-polynomial. In an earlier paper (unknown to me until recently), KUMAR [2] studied the same problem.** He obtained an equivalent expression using the joint moment generating function of the s quadratic forms.

It may, however, not be all to clear for the practical statistician how to compute these expectations. In the present note I shall present a table by which the expectation of $\prod_{j=1}^{s} \varepsilon' A_j \varepsilon$ is straightforwardly computed up to $s = 8$. The unfortunate statistician who needs the expectation for $s \geqslant 9$ will have to extend the table, which is somewhat tedious but not difficult.

First the reader should understand the concept of an $A(s)$-polynomial, and I must repeat some definitions from my earlier article.

2 The A(s)-polynomial

Let $A_1, A_2, \ldots, A_s$ be real symmetric matrices of the same order. The following four definitions describe the $A(s)$-polynomial.

Definition 2.1 (A(s)-form)
Divide the index set $\{1, 2, \ldots, s\}$ into mutually exclusive and exhaustive subsets. Within each subset, take the trace of the matrix product of the A_j's corresponding

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with indices from the subset. The product of all these traces will be called an *A(s)-form.*

Examples of $A(3)$-forms:

$$\operatorname{tr}(A_1)\operatorname{tr}(A_2A_3), \quad (\operatorname{tr}A_1)(\operatorname{tr}A_2)(\operatorname{tr}A_3), \quad \operatorname{tr}(A_1A_2A_3).$$

Definition 2.2 (similarity class)

Two $A(s)$-forms belong to the same *similarity class* iff their corresponding subsets (see definition 2.1) differ only by a permutation of indices.

EXAMPLES: $\operatorname{tr}(A_1)\operatorname{tr}(A_2A_3)$ is equal to $\operatorname{tr}(A_3A_2)\operatorname{tr}(A)_1$, but not necessarily equal to $\operatorname{tr}(A_2)\operatorname{tr}(A_1A_3)$ or $\operatorname{tr}(A_3)\operatorname{tr}(A_1A_2)$. However, all four $A(3)$-forms belong to the same similarity class. On the other hand, $\operatorname{tr}(A_1A_2A_3)$ belongs to a different similarity class.

Definition 2.3 (A(s)-sum)

The sum of all non-equal (i.e. not necessarily equal) $A(s)$-forms within a similarity class is called an *A(s)-sum.*

EXAMPLES: The three $A(3)$-sums are: $\operatorname{tr}(A_1A_2A_3)$, $(\operatorname{tr}A_1)(\operatorname{tr}A_2)(\operatorname{tr}A_3)$, and $[(\operatorname{tr}A_1)\operatorname{tr}(A_2A_3)+(\operatorname{tr}A_2)\operatorname{tr}(A_1A_3)+(\operatorname{tr}A_3)\operatorname{tr}(A_1A_2)]$.

Definition 2.4 (A(s)-polynomial)

Any linear combination of $A(s)$-sums is called an *A(s)-polynomial.*

EXAMPLES: The $A(2)$- and $A(3)$-polynomial are:

$$A(2){:}\quad v_1(\operatorname{tr}A_1)(\operatorname{tr}A_2)+v_2\operatorname{tr}(A_1A_2),$$

$$A(3){:}\quad v_1(\operatorname{tr}A_1)(\operatorname{tr}A_2)(\operatorname{tr}A_3)+v_2[(\operatorname{tr}A_1)(\operatorname{tr}A_2A_3)+(\operatorname{tr}A_2)(\operatorname{tr}A_1A_3)+$$
$$+(\operatorname{tr}A_3)(\operatorname{tr}A_1A_2)]+v_3\operatorname{tr}(A_1A_2A_3).$$

The importance of the $A(s)$-polynomial is clear from the following theorem (see MAGNUS [3, p. 207] and DON [1, sec.5]).

Theorem

Let $A_1, A_2, \ldots, A_s$ be real symmetric (n,n)-matrices, $u \sim N_n(0, I)$, and $\varepsilon \sim N_n(0, V)$, V positive semi-definite (possibly singular). Then,

(i) $\mathrm{E}\prod_{j=1}^{s} u'A_ju$ is an $A(s)$-polynomial;

(ii) $\mathrm{E}\prod_{j=1}^{s} \varepsilon'A_j\varepsilon$ is the same $A(s)$-polynomial with each A_j replaced by A_jV.

It is no simple exercise to write down the $A(s)$-polynomial for $s \geq 4$. Furthermore, in order to compute the desired expectation one has to determine the coefficients of the $A(s)$-polynomial. The following table will prove useful.

structure of similarity class								c	number of non-equal $A(s)$-forms							
8	7	6	5	4	3	2	1		1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	(1)	0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	(2)		1		1	3	6	10	15	21	28
1	1	1	1	1	(3)			3			1	4	10	20	35	56
1	1	1	1	2	2			2				3	15	45	105	210
1	1	1	1	(4)				4				3	15	45	105	210
1	1	1	2	3				4					10	60	210	560
1	1	1	(5)					5					12	72	252	672
1	1	2	2	2				3						15	105	420
1	1	2	4					5						45	315	1260
1	1	3	3					6						10	70	280
1	1	(6)						6						60	420	1680
1	2	2	3					5							105	840
1	2	5						6							252	2016
1	3	4						7							105	840
1	(7)							7							360	2880
2	2	2	2					4								105
2	2	4						6								630
2	3	3						7								280
2	6							7								1680
3	5							8								672
4	4							8								315
(8)								8								2520

3 How to use the table

For $s \leq 8$ the table informs the statistician of:

(i) the number and structure of the similarity classes (first 8 columns);
(ii) the number of non-equal $A(s)$-forms within each similarity class (last 8 colunms);
(iii) the coefficient of each $A(s)$-sum (middle column).

The working of the table is best explained by means of an example. Let $s=4$, i.e. the desired expectation is $\mathrm{E}\prod_{j=1}^{4}\varepsilon' A_j\varepsilon$, where $\varepsilon \simeq N(0, V)$, V positive semi-definite (possibly singular).

Step 1: From the left part of the table (under "structure of similarity class"), copy the north-east rectangle with 4 (s, in general) columns and 5 rows. The last row thus consists of a single 4 (s, in general); from the middle part of the table, copy the 5 numbers under c; from the right side, copy the 5 numbers under 4 (s, in general). This gives:

similarity class				c	# $A(s)$-forms
1	1	1	1	0	1
1	1	2		1	6
1	3			3	4
2	2			2	3
4				4	3

Step 2: If the number k appears under "similarity class", this means that the typical $A(s)$-form contains the trace of a product of k matrices. This leads to the following derived table:

typical $A(s)$-form	2^c	# $A(s)$-forms
(tr A) (tr B) (tr C) (tr D)	1	1
(tr A) (tr B) (tr CD)	2	6
(tr A) (tr BCD)	8	4
(tr AB) (tr CD)	4	3
tr ($ABCD$)	16	3

Note that the second column displays 2^c rather than c.

Step 3: For $s=4$ there are 5 similarity classes and therefore 5 $A(s)$-sums. Of each $A(s)$-sum, we know the typical $A(s)$-form, the coefficient (2^c), and the number of non-equal $A(s)$-forms. Thus, the 5 $A(s)$-sums $\sigma_1, \sigma_2, \ldots, \sigma_5$ are

$A(s)$-sum	coefficient
$\sigma_1 = (\mathrm{tr}\, A_1)(\mathrm{tr}\, A_2)(\mathrm{tr}\, A_3)(\mathrm{tr}\, A_4)$	1
$\sigma_2 = (\mathrm{tr}\, A_1)(\mathrm{tr}\, A_2)(\mathrm{tr}\, A_3A_4)+(\mathrm{tr}\, A_1)(\mathrm{tr}\, A_3)(\mathrm{tr}\, A_2A_4)+(\mathrm{tr}\, A_1)(\mathrm{tr}\, A_4)(\mathrm{tr}\, A_2A_3)$ $+(\mathrm{tr}\, A_2)(\mathrm{tr}\, A_3)(\mathrm{tr} A_1A_4)+(\mathrm{tr}\, A_2)(\mathrm{tr}\, A_4)(\mathrm{tr}\, A_1A_3)+(\mathrm{tr}\, A_3)(\mathrm{tr}\, A_4)(\mathrm{tr}\, A_1A_2)$	2
$\sigma_3 = (\mathrm{tr}\, A_1)(\mathrm{tr}\, A_2A_3A_4)+(\mathrm{tr}\, A_2)(\mathrm{tr}\, A_1A_3A_4)+(\mathrm{tr}\, A_3)(\mathrm{tr} A_1A_2A_4)+(\mathrm{tr}\, A_4)(\mathrm{tr}\, A_1A_2A_3)$	8
$\sigma_4 = (\mathrm{tr}\, A_1A_2)(\mathrm{tr}\, A_3A_4)+(\mathrm{tr}\, A_1A_3)(\mathrm{tr}\, A_2A_4)+(\mathrm{tr}\, A_1A_4)(\mathrm{tr}\, A_2A_3)$	4
$\sigma_5 = \mathrm{tr}\,(A_1A_2A_3A_4)+\mathrm{tr}\,(A_1A_2A_4A_3)+\mathrm{tr}\,(A_1A_3A_2A_4)$	16

Step 4: By the theorem of section 2, the desired expectation is

$$\mathrm{E}\prod_{j=1}^{4}(u'A_j u)=\sigma_1+2\sigma_2+8\sigma_3+4\sigma_4+16\sigma_5.$$

Moreover, the expectation of $\prod_{j=1}^{4}(\varepsilon' A_j \varepsilon)$ is the same expression with each A_j replaced by $A_j V$.

4 Some peculiarities of the table

The left part of the table gives all possible *partitions* (in DON's words) of the number s, in a logical ordering. For $s=4$, these partitions are (1, 1, 1, 1), (1, 1, 2), (1, 3), (2, 2), and (4). For any partition of s let n_j ($j=1\ldots s$) be the number of times that j appears in the partition. Again for $s=4$, this gives

partition	n_1	n_2	n_3	n_4	$c=s-n_1-n_2$	φ
1 1 1 1	4				0	1
1 1 2	2	1			1	6
1 3	1		1		3	4
2 2		2			2	3
4				1	4	3

Of course $\sum_{j=1}^{s} jn_j=s$. DON showed that the coefficients of the $A(s)$-polynomial are

$$2^{s-n_1-n_2},$$

which is c_λ in his terminology (see DON [1, p. 78]). Also he proved that the number of non-equal $A(s)$-forms in a similarity class (the number of standard tableaux in his words, see [1, p. 77]) equals

$$\varphi=s!\,2^{n_1+n_2}\Big/\prod_{j=1}^{s}n_j!\,(2j)^{n_j}$$

Note the special role of n_1 and n_2 in both formulas.

Because of the way the similarity classes are ordered, two peculiarities of the table arise. First, there is only one (rather than s) column c. Thus, for example, the partition (1,3) for $s=4$ and (1, 1, 3) for $s=5$ have the same coefficient, namely $2^3=8$. The reason is that n_1 increases with s, so that $s-n_1$ remains unaltered. Secondly, there is a simple connection between the numbers in each row of the right side of the table (with the exception of the first row which contains only ones). Let $\varphi_s(1)$ denote any of the underlined values in the right side of the table in column s, e.g. $\varphi_6(1)$ may be 15, 45, 10 or 60. Let $\varphi_s(2)$ be its right neighbour (105, 315, 70, or 420), etc. Then $\varphi_s(2)=(s+1)\varphi_s(1)$, and in general

$$\varphi_s(j+1) = \frac{s+j}{j}\,\varphi_s(j) = \binom{s+j}{j}\,\varphi_s(1).$$

where

$$\binom{s+j}{j} = \frac{(s+j)!}{s!\,j!}.$$

This provides a simple means for extending the table.

References

[1] DON, F. J. H., The expectation of products of quadratic forms in normal variables. Statistica Neerlandica (1979), **33**, 73–79.

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[3] MAGNUS, J. R., The moments of products of quadratic forms in normal variables. Statistica Neerlandica (1978), **32**, 201–210.