Economica, **57**, 239–47

Separability and Aggregation

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Final version received 11 April 1989. Accepted 18 July 1989.

The paper examines the conditions for homothetic separability of a technology which is an aggregate of technologies of individual firms or industries. Given that the primitive (industry or firm) technologies exhibit homothetic separability, we establish necessary and sufficient conditions for the aggregate (sectoral) technology to also exhibit homothetic separability. These conditions are expressed in terms of the cost functions of the primitive technologies. They may then be utilized to obtain an econometric test for separability of the aggregate technology, using data only on the primitive technologies.

Introduction

Many authors in both the theoretical and the empirical literature make the assumption that a sub-group of inputs is weakly separable from all other inputs and outputs in the production process. As is well known, this assumption implies restrictions upon the functional structure for the cost function. For example, homothetic weak separability of a sub-group of inputs implies that the cost share equations for these inputs are independent of the prices or quantities of all the other inputs and outputs. This is very convenient, since the share equations for this sub-group of inputs may then be estimated econometrically without requiring any data on the other inputs and outputs.

The homothetic separability assumption has been effectively used in many econometric studies, following an initial suggestion by Arrow (1972). Examples include Fuss (1977), Griffin (1977) and Pindyck (1979). Griffin assumes separability to enable concentration upon substitution possibilities between fuels, without considering other inputs and outputs. Fuss and Pindyck make the same separability assumption and exploit it by estimating the complete technology in a two-stage process.

Various procedures have been developed to test the hypothesis of separability. Berndt and Christensen (1973, 1974) test the separability hypothesis directly by testing the parametric restrictions corresponding to a homothetically separable production function. An alternative procedure is to undertake the test on the cost function, which is weakly separable in the sub-group of factor prices if the production function is weakly separable in the inputs for that sub-group. This approach is taken by Berndt and Wood (1975), Griffin (1977) and Magnus (1979), who test parametric restrictions corresponding to a weakly separable cost function. A third approach suggested and used by Woodland (1978) is to test separability of the production function indirectly by testing the implied strong separability of the variable profit function. This approach avoids some of the problems with the earlier approaches pointed out by Blackorby, Primont and Russell (1977).

The purpose of this paper is to take a quite different approach, which is applicable when the technology subject to testing for separability is an aggregate of technologies of individual firms or industries. Given that the primitive (industry or firm) technologies exhibit homothetic separability, we establish necessary and sufficient conditions for the aggregate (sectoral) technology also to exhibit homothetic separability. These conditions are expressed in terms of the cost functions of the primitive technologies. They may then be utilized to obtain an econometric test for separability of the aggregate technology, using only data on the primitive technologies.

The problem of establishing the conditions under which aggregation over inputs and outputs is valid has received much attention over the last 50 years. Recently, this literature has been elegantly summarized and extended by Blackorby and Schworm (1988), who emphasize several important features of the aggregation literature. The first is that there is an important distinction to be made between aggregation over inputs and outputs that are efficiently allocated among firms and those inputs and outputs that are arbitrarily allocated. Second, the aggregation problem involves not only the separability of one group of inputs or outputs from other inputs or outputs in the aggregate technology, but also the requirement that the aggregate technology should indeed be derived from the technologies of individual firms. Blackorby and Schworm (Theorem 1) derive necessary and sufficient conditions on the firms' profit functions for there to exist an aggregate of efficiently allocated inputs and outputs when other efficiently and arbitrarily allocated inputs and outputs occur.

A special case of Blackorby and Schworm's result occurs when the firms' aggregator functions are required to be homothetic. A further special case occurs if it is also assumed that the inputs and outputs not being aggregated are all arbitrarily allocated among firms. This special case was considered by Fisher (1968, Theorem 7.1), who showed that aggregation of inputs in the sectoral technology occurs if and only if the aggregator function for each firm can be chosen to be the same. The aggregation theorem obtained in the present paper may be regarded as a dual version of Fisher's theorem, which is expressed in terms of the production function. In addition to providing a dual representation of this result, our theorem is more general than Fisher's since he assumed the differentiability of the production functions whereas we do not.

While our theorem is a special case of the general results of Blackorby and Schworm, there are several reasons why a separate treatment is desirable. First, while it is clearly important to have the general conditions, the context in which applied researchers are forced to operate often imposes constraints that focus attention upon special cases. The problem considered in this paper arose in such a context. With the available data-set consisting only of observations on energy inputs at the individual industry level, the assumption of homothetic separability of energy inputs at this level is required to undertake empirical estimation of the technology. Under this assumption, it is natural to determine the conditions under which a similar homothetic separability assumption would also be valid at the sectoral level of aggregation. Second, we are able to provide a proof of our main result that exploits the particular features of this special case.

The plan of the paper is as follows. In Section I the assumptions regarding the cost functions for the individual industries are presented and discussed, with emphasis on the implications of homothetic separability. In Section II the aggregate or sectoral cost function is defined and the conditions under which it is also separable in energy prices are established. Section III concludes the paper with some general remarks.

I. The Industry Cost Function

In this section the economic model of production for an individual industry and the notion of homothetic separability in a sub-group of inputs are briefly discussed. For the purpose of concreteness, we will consider separability in energy inputs, which we refer to as 'fuels'. However, the results are obviously of general applicability.

Consider a production sector consisting of q industries, each with its own technology. For industry i the net production vector $(-x^i, y^i)$ is feasible if it is contained in the production set T^i; that is,

$$(1) \qquad (-x^i, y^i) \in T^i,$$

where $x^i = (x_1^i, \ldots, x_n^i)'$ is a vector of n fuel inputs and y^i is a vector of net outputs of non-energy commodities (both inputs and outputs). As is well known (for example Diewert, 1974, and McFadden, 1978), under fairly general conditions the technology may be equivalently described by the cost function

$$(2) \qquad C^i(p, y^i) \equiv \min \{p'x^i : x^i \in X^i(y^i)\},$$

where

$$(3) \qquad X^i(y^i) \equiv \{x^i : (-x^i, y^i) \in T^i\}$$

is the set of fuel input vectors that are capable of producing net output vector y^i, and $p = (p_1, \ldots, p_n)'$ is the vector of fuel input prices.

The following assumption is made about the cost function.

Assumption 1. The industry cost functions $C^i(p, y^i)$ satisfy the following five regularity conditions:

(4) *Conditions on the cost function*
(a) The cost function $C^i(p, y^i)$ is defined for $p \in P$ and $y^i \in Y^i$.
(b) Y^i is non-empty and convex.
(c) $P = \{p : p > 0\}$.
(d) For each $y^i \in Y^i$, $C^i(p, y^i)$ is a concave, positively linearly homogenous, closed function of $p \in P$.
(e) $C^i(p, y^i) > 0$ for all $p \in P$, $y^i \in Y^i$ and $C^i(p, y^i) > 0$ if $y^i \neq 0$.

Assumption 1 ensures that the cost function can be derived from a technology that satisfies fairly general and standard conditions.[1] To establish our main result, additional structure on the industry cost functions is required, so we make the following assumption.

Assumption 2 (Separability). The industry cost functions satisfy

$$(5) \qquad C^i(p, y^i) = c^i(p)h^i(y^i) \quad \text{for all } p \in P, y^i \in Y^i.$$

Assumption 2 contains structure that is not normally imposed upon a general model of the firm. The cost function described by (5) is the product of a function $c^i(p)$ of fuel prices and a function $h^i(y^i)$ of net outputs of non-energy commodities. This is equivalent to assuming that the vector of fuel inputs x^i is *homothetically separable* from all other inputs and outputs in the technology.[2] It is well known that the assumption of homothetic separability is necessary and sufficient for a consistent two-stage budgeting or optimization process, whereby in the first stage the optimal mix of fuel inputs is chosen and in the second stage the optimal amount of 'aggregate energy' is chosen along with other variable inputs and outputs.[3] This result is extremely useful, since it implies that a researcher can investigate the substitution possibilities between the various fuel inputs without concern for substitution between fuel inputs and other commodities. Homothetic separability is a standard assumption made in the literature to enable concentration of attention upon a subset of inputs or outputs.

In view of (5), (4d) and (4e), the function $c^i(p)$ is a positive, continuous, positively linearly homogeneous, concave function of $p \in P$. In other words, the function $c^i(p)$ satisfies all of the conditions required of a unit cost function. Thus, duality theory results may be used to show that it is the unit cost function corresponding to a valid constant-returns-to-scale production function $f^i(x^i)$.[4] The latter may be interpreted as a quantity index for 'energy', and $c^i(p)$ may be interpreted as a price index for 'energy'.

II. Aggregation and Separability

In the economic model discussed in the previous section, the cost functions for the q industries in the manufacturing sector are allowed to be different, and these cost functions may be estimated using data for each industry. An alternative procedure is to use aggregate data only to estimate a cost function for the production sector as a whole. This is the common practice of researchers dealing with the demand for energy.[5] Accordingly, it is of considerable interest to establish the theoretical conditions under which this practice is valid and a procedure to investigate empirically whether these conditions are met. This section deals with the theoretical conditions for aggregation of fuel inputs in the sectoral technology, given that homothetic separability occurs in the individual industry technologies.

Let

$$(6) \qquad x = \sum_{i=1}^{q} x^i \quad \text{and} \quad y = (y^1, \ldots, y^q)$$

be the vector of total (sectoral) fuel inputs, and the vector of net outputs of all non-energy commodities in the q industries, respectively. Also, let the set of sectoral fuel inputs that are capable of producing the net output vector y be denoted by $X(y)$, where

$$(7) \qquad y \in Y = \prod_{i=1}^{q} Y^i.$$

The sectoral cost function is

$$(8) \qquad C(p, y) = \min \{p'x \colon x \in X(y)\},$$

indicating the minimum expenditure on fuels required to produce the vector y of net outputs of all other commodities (inputs and outputs).

The sectoral cost function and the industry cost functions are not, of course, unrelated. If it is assumed that there are no production externalities, then the sectoral production set is simply the sum of the industry production sets; that is,

$$X(y)=\sum_{i=1}^{q} X^i(y^i).$$

Equivalently, the sectoral cost function is the sum of the industry cost functions in the case where inter-industry externalities and jointness are ruled out. These considerations lead to the following assumption.

Assumption 3 (Additivity). The sectoral cost function satisfies

$$(9) \qquad C(p,y)=\sum_{i=1}^{q} C^i(p,y^i), \qquad y^i\in Y^i, i=1,\dots,q; y\in Y; p\in P.$$

A researcher interested in using sectoral (aggregate) data on fuel consumption and fuel prices to estimate the substitution possibilities between fuels, without using data on other commodities, would invoke the assumption that the sectoral technology is homothetically separable in fuels. This is equivalent to assuming that the sectoral cost function factors as

$$(10) \qquad C(p,y)=c(p)h(y), \qquad p\in P, y\in Y.$$

That is, the researcher would make the same separability assumption about the sectoral cost function as we made about the industry cost functions in Section I.

This raises a very interesting question about aggregation: if the sectoral cost function is additive as in (9), and if the individual industry cost functions are separable as in (5), under what conditions is it valid to assume that the sectoral cost function is also separable as in (10)? The answer to this question is given in the theorem below.

If conditions (5), (9) and (10) hold, then equation (9) may be rewritten as

$$(11) \qquad c(p)h(y)=\sum_{i=1}^{q} c^i(p)h^i(y^i), \qquad y^i\in Y^i, i=1,\dots,q; y\in Y; p\in P.$$

The problem thus becomes one of solving this functional equation. To do this we impose one further restriction.

Assumption 4. The functions $h^i(y^i)$ are non-constant on $Y^i, i=1,\dots,q$.

This assumption is very weak, and means that each industry is capable of changing the minimum cost of fuel inputs at any given $p\in P$ by altering the vector of net outputs of non-energy commodities, y^i. In other words, y^i is moveable within Y^i, and at least one move will alter the value of h^i and hence the cost of fuels.

The main result of this section is contained in the following theorem.

Theorem. Let the industry cost functions $C^i(p,y^i)$ satisfy Assumptions 1, 2 and 4, and let the sectoral cost function $C(p,y)$ satisfy Assumptions 1 and 3. Then $C(p,y)$ is separable as in (10) if and only if there exist positive constants $\alpha_1,\dots,\alpha_q$ such that

$$(12) \qquad c^i(p)=\alpha_i c(p) \quad \text{for all } p\in P.$$

Proof.[6] By assumption, equations (4), (5) and (9) and Assumption 4 hold. The proof of the 'if' part of the theorem is easily demonstrated and is therefore omitted. To prove the 'only if' part, assume that (10) holds and define

$$(13) \qquad \gamma^i(p) \equiv c^i(p)/c(p).$$

(Notice that $c(p) > 0$ for $p \in P$ by (4e) and (10).) The functional equation (11) may now be rewritten as

$$(14) \qquad \sum_{i=1}^{q} h^i(y^i)\gamma^i(p) = h(y), \qquad y^i \in Y^i, p \in P, y \in Y.$$

By Assumption 4, there exist two net output vectors, $y^i_{(0)} \in Y^i$ and $y^i_{(1)} \in Y^i$, such that

$$(15) \qquad \delta^i \equiv h^i(y^i_{(1)}) - h^i(y^i_{(0)}) > 0, \qquad i = 1, \ldots, q.$$

We now construct q *sectoral* net output vectors $y_{(i)}$ by changing the net output vector in *industry* i from $y^i_{(0)}$ to $y^i_{(1)}$, keeping all other industries $j \neq i$ at $y^j_{(0)}$. That is,

$$(16) \qquad y_{(i)} = (y^1_{(0)}, \ldots, y^i_{(1)}, \ldots, y^q_{(0)}), \qquad i = 1, \ldots, q.$$

Since each of these q sectoral net output vectors will satisfy (14), we obtain

$$\sum_{j \neq i} h^j(y^j_{(0)})\gamma^j(p) + h^i(y^i_{(1)})\gamma^i(p) = h(y_{(i)}), \qquad i = 1, \ldots, q,$$

or, in view of (15),

$$(17) \qquad \delta^i \gamma^i(p) + \sum_{j=1}^{q} h^j(y^j_{(0)})\gamma^j(p) = h(y_{(i)}), \qquad i = 1, \ldots, q.$$

We divide both sides of (17) by δ^i (recall that $\delta^i > 0$ by (15)), and express the resulting q equations as one vector equation:

$$(18) \qquad (I_q + dg')\gamma(p) = b,$$

where

$$\gamma(p) = \{\gamma^1(p), \gamma^2(p), \ldots, \gamma^q(p)\}',$$

$$d = (\delta_1^{-1}, \delta_2^{-1}, \ldots, \delta_q^{-1})',$$

$$g = \{h^1(y^1_{(0)}), h^2(y^2_{(0)}), \ldots, h^q(y^q_{(0)})\}',$$

and

$$b = \{h(y_{(1)})/\delta^1, h(y_{(2)})/\delta^2, \ldots, h(y_{(q)})/\delta^q\}'.$$

The $q \times q$ matrix $I_q + dg'$ is non-singular if and only if $1 + g'd \neq 0$. But since each component of d is positive (by (15)) and each component of g is non-negative (in view of (4e) and (15)), we have $g'd > 0$ and therefore $1 + g'd \neq 0$ (in fact, > 1). Hence the $I + dg'$ is non-singular, and we obtain from (18)

$$(19) \qquad \gamma(p) = (I_q + dg')^{-1}b.$$

Now, since the right side of (18) does not depend on p, neither does the left side. Hence the function γ is constant on P. In other words, there exist constants $\alpha_1, \ldots, \alpha_q$ such that

$$(20) \qquad \gamma^i(p) = \alpha_i, \qquad i = 1, \ldots, q,$$

for all $p \in P$. Substituting (20) into (13), and noting that $\alpha_i > 0$ (because of (4e)), completes the proof. □

The theorem states that, under the maintained assumptions, the sectoral cost function is separable as in (10) if and only if the industry price indices for energy, $c^i(p)$, are identical up to a factor of proportionality. An implication of (12) is that the fuel share equations, obtained by logarithmic differentiation of the industry price indices with respect to prices, will be identical in each of the q industries in the sector. This condition, which is clearly very stringent, can be tested empirically.[7]

If the individual industry price indices $c^i(p)$ are not identical up to a factor of proportionality, then there is no justification to assume that the sectoral cost function factors as in equation (10); and hence it is not valid to specify share equations for fuels at the sectoral level that depend only upon fuel prices and not upon net outputs of non-energy commodities in every industry. In this case, the researcher must model the whole technology at the sectoral level, or disaggregate to the industry level. If the former option is chosen, the data and estimation requirements are substantial, since the sectoral cost function depends upon the distribution of all other inputs and outputs among all the industries as well as upon the prices of the energy inputs. Of course, if these other inputs and outputs are optimally distributed among industries, then they may be aggregated over industries in a straightforward way.

III. Conclusion

In this paper we have derived the necessary and sufficient condition for the existence of homothetic separability of one group of inputs from all other inputs and outputs in an aggregate or sectoral technology. The condition is expressed in terms of restrictions on the individual industry cost functions. These restrictions may be tested empirically using standard econometric techniques. Because we assume homothetic separability at the industry level, the test requires data on the inputs in question at the disaggregated (industry) level alone and does not require data at the aggregate or sectoral level or data on other inputs and outputs.

Our test for separability at the sectoral level is an alternative to the usual tests in the literature. The usual procedure is to test directly for separability of the sectoral cost function using sectoral (aggregate) data on fuels and other commodities, assuming that these are optimally distributed among industries, or that aggregates exist. *Our* procedure is to assume separability at the industry level and to test the separability of the sectoral cost function indirectly via the industry cost functions, without using any data on other commodities, and without assuming that they are optimally chosen or that aggregates exist. If information is available on the prices and quantities of the other inputs and outputs that are optimally allocated or on the actual distribution of arbitrarily allocated inputs and outputs, then more general testing procedures, such as those used by Blackorby, Schworm and Fisher (1986), may be used. Our test procedure should be useful for other applications where disaggregate data exist but are limited to a subset of inputs of special interest.

ACKNOWLEDGMENTS

Magnus' research was supported in part by the Netherlands Organization for the Advancement of Pure Research (ZWO). Woodland's research was supported by grants from the Social Services and Humanities Research Council of Canada and the University of Sydney. We are greatly indebted to Greg Defina, Mira Kempisty, Shyama Roy and David Ryan for exceptional research assistance, and to J. S. Cramer, Terence Gorman and Dilip Madan for helpful comments.

NOTES

1. Specifically, Assumption 1 implies that the conditions of Lemma 23 of McFadden (1978, p. 81) are satisfied, which proves the duality between a restricted profit function (the negative of our cost function) and the production set.
2. If $\tilde{F}^i(-x^i, y^i) = 0$ is the transformation frontier denoting the boundary of the production set T^i, then it is assumed that $\tilde{F}^i(-x^i, y^i) = F^i(-f^i(x^i), y^i)$ where $f^i(x^i)$ is a homothetic function which aggregates fuel inputs. The aggregator function $f^i(x^i)$ may be interpreted as a quantity index for the fuel inputs and hence be called 'energy'.
3. The sufficiency of this condition was established by Shephard (1970, pp. 143–6). That the condition is also necessary follows from the duality between production and cost functions.
4. See Diewert (1974) for details on the duality between production and cost functions under constant returns to scale.
5. See Pindyck (1979) and Griffin (1977), for example. On the other hand, Fuss (1977) disaggregates the Canadian manufacturing sector by region but not by industry.
6. The theorem may be proved using the more general results of Gorman (1982) and specializing to our case. Our proof, however, applies directly to the functional equation (11) and is instructive in that it indicates very clearly the role played by the assumptions. We are indebted to Dilip Madan for the suggestion of this proof.
7. We have undertaken an empirical test using time-series data on four fuels (coal, oil, gas and electricity) used in six manufacturing industries in the Netherlands. In Magnus and Woodland (1987), the results of estimation of translog functional forms for the industry cost functions are presented and discussed. In that framework, the condition for homothetic separability at the sectoral level given by (12) may be tested statistically by testing the null hypothesis that the parameters of the share equations are the same for each industry. Using both likelihood ratio and Wald tests, the null hypothesis was soundly rejected by the data. Further testing revealed that the data do not even support aggregation over any subset of industries. Accordingly, for this data-set there are no grounds for assuming that the sectoral technology is homothetically separable in energy inputs.

REFERENCES

ARROW, K. J. (1972). The measurement of real value added. Technical Report no. 60, Institute of Mathematical Studies in the Social Sciences, Stanford University.

BERNDT, E. R. and CHRISTENSEN, L. R. (1973). The translog function and the substitution of equipment, structures, and labor in US manufacturing, 1929–68. *Journal of Econometrics*, **1**, 81–114.

——, —— (1974). Testing for the existence of a consistent aggregate index of labor inputs. *American Economic Review*, **64**, 391–404.

—— and WOOD, D. O. (1975). Technology, prices and the derived demand for energy. *Review of Economics and Statistics*, **57**, 259–68.

BLACKORBY, C., PRIMONT, D. and RUSSELL, R. R. (1977). On testing separability restrictions with flexible functional forms. *Journal of Econometrics*, **5**, 195–209.

—— and SCHWORM, W. (1988). The existence of input and output aggregates in aggregate production functions. *Econometrica*, **56**, 613–43.

——, —— and FISHER, T. (1986). Testing the existence of input aggregates in an economy production function. University of British Columbia, Department of Economics, Discussion Paper no. 86-26.

DIEWERT, W. E. (1974). Applications of duality theory. In M. Intriligator and D. Kendrick (eds.), *Frontiers of Quantitative Economics*, Vol. 2, pp. 106–71. Amsterdam: North-Holland.

FISHER, F. M. (1968). Embodied technology and the existence of labour and output aggregates. *Review of Economic Studies*, **35**, 391–412.

FUSS, M. (1977). The demand for energy in Canadian manufacturing: an example of the estimation of production structures with many inputs. *Journal of Econometrics*, **5**, 89–116.

GORMAN, W. M. (1982). Aggregates, activities, and overheads. Technical Report no. 390, Institute for Mathematical Studies in the Social Sciences, Stanford University.

GRIFFIN, J. M. (1977). Inter-fuel substitution possibilities: a translog application to intercountry data. *International Economic Review*, **18**, 755–70.

MAGNUS, J. R. (1979). Substitution between energy and non-energy inputs in the Netherlands, 1950–76. *International Economic Review*, **20**, 465–84.

—— and WOODLAND, A. D. (1987). Inter-fuel substitution in Dutch manufacturing. *Applied Economics*, **19**, 1639–64.

MCFADDEN, D. (1978). Cost, revenue, and profit functions. Section 1.1 in M. Fuss and D. McFadden (eds.), *Production Economics: A Dual Approach to Theory and Applications*, Vol. 1, pp. 3–109. Amsterdam: North-Holland.

PINDYCK, R. S. (1979). Interfuel substitution and the industrial demand for energy: an international comparison. *Review of Economics and Statistics*, **61**, 169–79.

SHEPHARD, R. W. (1970). *Theory of Cost and Production Functions*. Princeton University Press.

WOODLAND, A. D. (1978). On testing weak separability. *Journal of Econometrics*, **10**, 361–83.