The Relation between Monotonicity and Strategy-Proofness

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Abstract

The Muller-Satterthwaite Theorem (Muller and Satterthwaite, 1977) establishes the equivalence between Maskin monotonicity and strategy-proofness, two cornerstone conditions for the decentralization of social choice rules. We consider a general model that covers public goods economies as in Muller and Satterthwaite (1977) as well as private goods economies. For private goods economies we use a weaker condition than Maskin monotonicity that we call unilateral monotonicity. We introduce two *easy-to-check* domain conditions which separately guarantee that (i) unilateral/Maskin monotonicity implies strategy-proofness (Theorem 1) and (ii) strategy-proofness implies unilateral/Maskin monotonicity (Theorem 2). We introduce and discuss various classical single-peaked domains and show which of the domain conditions they satisfy (see Propositions 1 and 2 and an overview in Table 1). As a by-product of our analysis, we obtain some extensions of the Muller-Satterthwaite Theorem as summarized in Theorem 3. We also discuss some new "Muller-Satterthwaite domains" (e.g., Proposition 3).

Keywords: Muller-Satterthwaite Theorem, restricted domains, rich domains, single-peaked domains, strategy-proofness, unilateral/Maskin monotonicity.

JEL classification: D71

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1 Introduction

The Muller-Satterthwaite Theorem (Muller and Satterthwaite, 1977) states the equivalence between strategy-proofness and Maskin monotonicity, two cornerstone conditions for the decentralization of (social choice) rules.[1] As a consequence of the Muller-Satterthwaite Theorem, the class of Maskin monotonic rules is fairly small: only dictatorial rules are strategy-proof.[2] However, it is by now well-understood that the aforementioned theorem strongly relies on the assumption of an unrestricted domain of strict preferences –what we refer to as the Arrovian domain. In many situations though, it is natural to work with more structured preference domains. For instance, consider a group of agents who have to choose the location of a public facility on their street. A natural domain restriction is to assume that agents have single-peaked preferences over the possible locations (Black, 1948). We know that the class of strategy-proof rules for this type of economies is large (Moulin, 1980); and a natural question is whether the same conclusion holds for the class of Maskin monotonic rules. So, despite the equivalence provided by the Muller-Satterthwaite Theorem, it seems that for many domains and models of interest, the logical relation between Maskin monotonicity and strategy-proofness is not fully understood. In addition, notice that in public goods models, a rule selects an alternative at each preference profile, whereas in private goods models, an allocation will be selected –i.e., a bundle for each agent. An allocation is an object whose nature is different from an alternative in several aspects. For instance, the bundle that an agent (or a group of agents) receives at some preference profile may be conditional on the shape of preferences of some other agents. Rules that have this feature violate the well-known non-bossiness condition (Satterthwaite and Sonnenschein, 1981). Because of this difference between the two models, it is not clear whether there is a "direct" logical relation between Maskin monotonicity and strategy-proofness in private goods models.

Our contribution: Our goal is to provide a better understanding of the logical relation between monotonicity conditions and strategy-proofness. We consider a model that covers public goods as well as private goods economies.[3] In addition to Maskin monotonicity, we introduce a weaker condition called unilateral monotonicity which pertains to unilateral changes in preferences.[4] The use of this condition is pertinent when we refer to private goods models.

We introduce two easy-to-check domain conditions. Condition R1 is a domain richness condition, whereas Condition R2 is a domain restriction condition. A rule defined on a domain satisfying

[1]Both conditions are central in the mechanism design literature. Strategy-proofness is a necessary condition for implementation in dominant strategies, whereas Maskin monotonicity is a necessary condition for implementation in Nash equilibrium.

[2]The Muller-Satterthwaite Theorem has as well-known corollary the Gibbard–Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975): any efficient and strategy-proof rule defined on a domain of unrestricted linear orderings must be dictatorial.

[3]For private goods economies, our model covers both the infinitely divisible goods case as well as the indivisible goods case.

[4]As far as we know, unilateral monotonicity was first introduced by Takamiya (2001).

Condition R1 is unilaterally monotonic/Maskin monotonic if it is strategy-proof (Theorem 1 and Corollary 1). Examples of rich domains include the Arrovian domain as well as various single-peaked (single-plateaued) domains, but exclude domains in which preferences are both single-peaked and symmetric. More generally, the domain of convex star-shaped preferences satisfies Condition R1 (Proposition 1). Next, for public goods models, Condition R2 entails that strategy-proofness implies Maskin monotonicity; and for private goods models, strategy-proofness implies unilateral monotonicity (Theorem 2). Indeed, in private goods models, there exist rules that are strategy-proof but not Maskin monotonic. As argued above, an important difference between public goods and private goods models turns out to be the existence of rules that violate non-bossiness in the latter.[5] As a consequence, for several domains, the "set-inclusion connection" between the class of Maskin monotonic rules and the class of strategy proof rules may be lost for private goods models.[6] However, when Condition R2 is satisfied, a logical relation between strategy-proofness and Maskin monotonicity can be recovered thanks to non-bossiness: strategy-proofness and non-bossiness together imply Maskin monotonicity (Corollary 2). Examples of domains satisfying Condition R2 include the Arrovian domain as well as some (symmetric) single-peaked (single-plateaued) preference domains but exclude larger domains like the single-peaked preference domain.[7] More generally, any convex norm induced preference domain satisfies Condition R2 (Proposition 2).

Next we come to the Muller-Satterthwaite Theorem and its extensions. As a by-product of our results, we obtain an extended version of the Muller-Satterthwaite Theorem that applies to the model at hand (Theorem 3). A straightforward corollary is the standard version of the theorem (Muller and Satterthwaite, 1977) for the public goods case, along with a new and direct proof. We then discuss some new "Muller-Satterthwaite domains" of interest (Proposition 3). This shows that the conclusion of the Muller-Satterthwaite Theorem can also spread to restricted domains.

Relation to the literature: The investigation of the relation between monotonicity conditions and strategy-proofness is not new. A seminal paper dealing with the relation between Maskin monotonicity and strategy-proofness is Dasgupta et al. (1979). They introduce a domain richness condition and prove that any Maskin monotonic rule defined on a rich domain is strategy-proof. More recently, Takamiya (2001, 2003) studies the relation between coalition strategy-proofness and Maskin monotonicity for a broad class of economies with indivisible goods. Takamiya (2007) generalizes the results obtained in his former two papers. Finally, in a paper independent of ours, Berga and Moreno (2009) study the relation between strategy-proofness, Maskin monotonicity, and non-bossiness for the single-peaked and single-plateaued domain for the provision of a pure public good.

[5]For preference domains satisfying Condition R2, unilateral monotonicity and non-bossiness imply Maskin monotonicity (Lemma 1).

[6]For example, in private goods models, the symmetric single-peaked domain admits rules that are strategy-proof but not Maskin monotonic, as well as rules that are Maskin monotonic but not strategy-proof.

[7]However, the domain of strict single-peaked preferences satisfies Condition R2. In fact, any domain composed only of strict preference relations satisfies Condition R2.

In addition to Dasgupta et al. (1979), domain richness conditions are used in papers close to ours, Fleurbaey and Maniquet (1997) and Le Breton and Zaporozhets (2009). Note that the richness condition (Condition R1) that we introduce differs from the conditions uncovered in the aforementioned papers, and it does not imply any "cross-profile" requirements. We discuss in the Appendix the logical relations between the latter conditions and our Condition R1.

The plan of the paper is the following. In Section 2, we introduce a general model that encompasses public goods as well as private goods economies, and we present the definitions and preference domains necessary for the paper. In Section 3, we define our two domain conditions and we prove our main results. In Section 4 we check both these conditions for well-known preference domains. We also provide an extended version of the Muller-Satterthwaite Theorem that applies to the model at hand. Finally, in the Appendix, we compare our domain richness condition (Condition R1) to the ones introduced in related papers.

2 The Model, Key Properties, and Preference Domains

2.1 The Model

Let $N = \{1, \ldots, n\}$ be a *set of agents*. Let $A = A_1 \times \ldots \times A_n$ be a *set of alternatives*. We assume that for all $i, j \in N$, $A_i = A_j$. Furthermore, we assume that if $A_i \subseteq \mathbb{R}^m$ and $|A_i| = \infty$, then A_i is convex. Let $x = (x_1, \ldots, x_n) \in A$ be an alternative and $\mathbb{1} \equiv (1, \ldots, 1) \in \mathbb{R}^n$. If alternative x is such that for all $i, j \in N$, $x_i = x_j = \alpha$, then we denote $x = \alpha\mathbb{1}$. Next, let $F \subseteq A$ be the *set of feasible alternatives*. If for all $x \in F$ there exists α such that $x = \alpha\mathbb{1}$, then the set of feasible alternatives models a public goods economy. Otherwise, we model an economy with at least one private goods component. Hence, our model encompasses public and private goods economies.

To fix ideas, let us give two examples. It will be clear from these examples that given the set A of alternatives, the set F of feasible alternatives fully determines whether we are working with a public or a private goods model. Note that the Cartesian product notation we use for the set of alternatives is for notational convenience only; none of our results require it.

Example 1. Let $A = \{a_1, \ldots, a_n\} \times \ldots \times \{a_1, \ldots, a_n\}$. Suppose that the agents have to choose one candidate out of the set $\{a_1, \ldots, a_n\}$ of possible candidates. Then, $F = \{x \in A : \text{for all } i, j \in N,\ x_i = x_j\}$. On the other hand, if agents have to allocate the set of indivisible objects or tasks $\{a_1, \ldots, a_n\}$ among themselves, then $F = \{x \in A : \text{for all } i, j \in N,\ x_i \neq x_j\}$. ◇

Example 2. Let $A = [0, 1] \times \ldots \times [0, 1]$. Suppose that the agents have to choose a single point in the interval [0,1] that everyone will consume without rivalry, e.g., a public facility on a street (see Moulin, 1980). Then, $F = \{x \in A : \text{for all } i, j \in N,\ x_i = x_j\}$. On the other hand, if agents have to choose a division of one unit of an infinitely divisible good among themselves (see Sprumont, 1991), then feasibility is determined by the size of the resource and $F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$. $\diamond$

For all $i \in N$, *preferences* are represented by a complete, reflexive, and transitive binary relation R_i over A_i. As usual, for all $x, y \in A$, $x_i \, R_i \, y_i$ is interpreted as "i weakly prefers x to y", $x_i \, P_i \, y_i$ as "i strictly prefers x to y", and $x_i \, I_i \, y_i$ as "i is indifferent between x and y". Whenever our model captures a private goods component, we assume that agents only care about their own consumption. Therefore, for several of our results, we use both notations $x \, R_i \, y$ and $x_i \, R_i \, y_i$. This is done for convenience only and it should cause no confusion.

For all $i \in N$, let $\mathcal{R}_i = \mathcal{R}$ be a set of preferences on A_i. Thus, we assume that all agents have the same preference domain $\mathcal{R}$. Let $\mathcal{R}^N$ denote the set of *preference profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$.

For all $i \in N$, all preference relations $R_i \in \mathcal{R}$, and all alternatives $x \in A$, the *lower contour set of* R_i *at* x is $L(R_i, x) \equiv \{y \in A : x \, R_i \, y\}$; the *strict lower contour set of* R_i *at* x is $SL(R_i, x) \equiv \{y \in A : x \, P_i \, y\}$; the *upper contour set of* R_i *at* x is $U(R_i, x) \equiv \{y \in A : y \, R_i \, x\}$; and the *strict upper contour set of* R_i *at* x is $SU(R_i, x) \equiv \{y \in A : y \, P_i \, x\}$.

Let A, F, and $\mathcal{R}$ be given. Then, a *rule* φ is a function that assigns to every preference profile $R \in \mathcal{R}^N$ a feasible alternative $\varphi(R) \in F$.

2.2 Properties of Rules

We discuss in turn two central properties of the mechanism design literature. First, strategy-proofness is an incentive property which requires that no agent ever benefits from misrepresenting his preference relation. In game theoretical terms, a rule is strategy-proof if in its associated direct revelation game form, it is a weakly dominant strategy for each agent to announce his true preference relation. By the revelation principle, strategy-proofness is a necessary condition for dominant strategy implementability.

For agent $i \in N$, preference profile $R \in \mathcal{R}^N$, and preference relation $R'_i \in \mathcal{R}$, we obtain preference profile (R'_i, R_{-i}) by replacing R_i at R by R'_i.

Strategy-Proofness: A rule φ is *strategy-proof* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi(R) \, R_i \, \varphi(R'_i, R_{-i})$.

Next, Maskin monotonicity is a property which requires the robustness (or invariance) of a rule with respect to specific preference changes. A rule φ is Maskin monotonic if an alternative x that is chosen at preference profile R remains chosen at a preference profile R' at which x is considered (weakly) better by all agents. An important result of the mechanism design literature is that Maskin monotonicity is a necessary condition for Nash implementability of a rule (see Maskin, 1977, 1999). Apart from its importance for Nash implementability, we consider Maskin monotonicity to be an appealing property in itself.

In order to introduce Maskin monotonicity, we first define monotonic transformations. Loosely speaking, for any alternative x and any preference profile R, if at a preference profile R' all agents $i \in N$ consider alternative x to be (weakly) better, then R' is a monotonic transformation of R at x. For preferences $R_i, R'_i \in \mathcal{R}$ and alternative $x \in A$, R'_i is a *monotonic transformation of R_i at x* if $L(R_i, x) \subseteq L(R'_i, x)$. By $MT(R_i, x)$ we denote the *set of all monotonic transformations of R_i at x* and by $MT(R, x)$ we denote the *set of all monotonic transformations of R at x*, i.e., $R' \in MT(R, x)$ if for all $i \in N$, $R'_i \in MT(R_i, x)$.

Maskin Monotonicity: A rule φ is *Maskin monotonic* if for all $R, R' \in \mathcal{R}^N$, $\varphi(R) = x$ and $R' \in MT(R, x)$ imply $\varphi(R') = x$.

For one of our "private goods results" we use the following weaker monotonicity property: a rule φ is unilaterally Maskin monotonic if given that alternative x is chosen at preference profile R, agent i's component x_i remains chosen at a unilateral deviation profile $R' = (R'_i, R_{-i})$ at which agent i considers x_i to be (weakly) better.

Unilateral Monotonicity: A rule φ is *unilaterally monotonic* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi(R) = x$ and $R'_i \in MT(R_i, x)$ imply $\varphi_i(R'_i, R_{-i}) = x_i$.

Note that Maskin monotonicity implies unilateral monotonicity. To be more precise, for public goods economies, Maskin monotonicity and unilateral monotonicity are equivalent and for private goods economies Maskin monotonicity implies unilateral monotonicity.

We close this section by introducing non-bossiness (in allocations) (see Satterthwaite and Sonnenschein, 1981), an auxiliary property that we use for some of our "private goods results". The property states that by changing his preference relation, an agent cannot change components of the allocation for the other agents without affecting his own. Obviously, this property is vacuous in a public goods model.

Non-Bossiness: A rule φ is *non-bossy* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$ implies that $\varphi(R) = \varphi(R'_i, R_{-i})$.

For private goods models, Maskin monotonicity implies non-bossiness under our richness domain condition (Condition R1), while the converse is not true. On the other hand, the conjunction of strategy-proofness and non-bossiness is equivalent to Maskin monotonicity under our domain restriction condition (Condition R2). These relations will be made clear in Section 3.

2.3 Well-Known Preference Domains

2.3.1 The Arrovian Domain

We refer to the unrestricted domain of strict preferences $\mathcal{R}_A$ as the *Arrovian domain*, i.e., $\mathcal{R}_A$ is such that for all $i \in N$, all $R_i \in \mathcal{R}$, and all $x_i, y_i \in A_i$, $x_i \, R_i \, y_i$ implies $x_i \, P_i \, y_i$ or $x_i = y_i$.

DHM domain

$\subset \quad \supset$

single-plateaued domain *Cantala domain*

$\supset \quad \subset$

single-peaked domain

$\cup$

symmetric single-peaked domain

Figure 1: Set-relationships between one-dimensional single-peaked domains

2.3.2 One-Dimensional Single-Peaked and Single-Plateaued Preferences

Here we introduce the general single-peaked preference domain and several of its well-known subdomains. We start by defining the smallest domain we consider, the symmetric single-peaked preference domain introduced in Border and Jordan (1983). The domain of symmetric single-peaked preferences is induced by the Euclidean norm $\|\cdot\|_E$.

Symmetric Single-Peaked (Euclidean) Preferences on $\mathbb{R}$: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *symmetrically single-peaked* (or Euclidean) if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$, $x_i\, R_i\, y_i$ if and only if $\|p(R_i) - x_i\|_E \leq \|p(R_i) - y_i\|_E$.

By relaxing the symmetry assumption, one obtains the domain of single-peaked preferences introduced in Black (1948) and Moulin (1980).

Single-Peaked Preferences on $\mathbb{R}$: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *single-peaked* if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$ satisfying either $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, $x_i\, P_i\, y_i$.

We now introduce two superdomains of the single-peaked preference domain. First, consider again the location of a public facility on a street. As in Example 2, we assume that agents' preferences are single-peaked, but that in addition they have an outside option so that if the public facility is too far away, they will not use it. This class of preferences is introduced and analyzed by Cantala (2004). Here we give an ordinal representation of Cantala's (2004) class of preferences.

Single-Peaked Preferences on $\mathbb{R}$ with an Outside Option: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *single-peaked with an outside option* if there exists an interval $[a, b] \subseteq A_i$ and a point $p(R_i) \in (a, b)$ such that (i) R_i is single-peaked on $[a, b]$; (ii) for all $x_i \in (a, b)$ and $y_i \in A_i \setminus [a, b]$, $x_i\, P_i\, y_i$; and (iii) for all $x_i, y_i \in A_i \setminus (a, b)$, $x_i\, I_i\, y_i$.

The second superdomain of the single-peaked domain frequently encountered in the literature (see Moulin, 1984) is the so-called single-plateaued domain. For such a domain, we allow agents to have an interval of best points, so that instead of the peak we have a plateau.

Single-Plateaued Preferences on $\mathbb{R}$: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *single-plateaued* if there exists an interval $[\underline{p}(R_i), \overline{p}(R_i)] \subseteq A_i$ such that (i) for all $x_i, y_i \in [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i \, I_i \, y_i$; (ii) for all $x_i \in [\underline{p}(R_i), \overline{p}(R_i)]$ and all $y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i \, P_i \, y_i$; and (iii) for all $x_i, y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$ satisfying either $y_i < x_i \leq \underline{p}(R_i)$ or $\overline{p}(R_i) \leq x_i < y_i$, $x_i \, P_i \, y_i$.

Note that the definition above only allows for a unique plateau of best alternatives. Dasgupta, Hammond, and Maskin (1979), DHM for short, consider a more general single-plateaued domain (which they call the single-peaked domain) by allowing for additional plateaus left and right from the "top-plateau".

DHM Single-Plateaued Preferences on $\mathbb{R}$: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *DHM single-plateaued* if there exists an interval $[\underline{p}(R_i), \overline{p}(R_i)] \subseteq A_i$ such that (i) for all $x_i, y_i \in [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i I_i y_i$; (ii) for all $x_i \in [\underline{p}(R_i), \overline{p}(R_i)]$ and all $y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i P_i y_i$; and (iii) for all $x_i, y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$ satisfying either $y_i < x_i \leq \underline{p}(R_i)$ or $\overline{p}(R_i) \leq x_i < y_i$, then $x_i \, R_i \, y_i$.

2.3.3 Higher-Dimensional Single-Peaked Preferences

There are various extensions of the one-dimensional single-peaked domains to higher dimensions. We start again by defining the smallest domains first. The first two domains are extensions of the one-dimensional symmetric single-peaked preferences introduced before (see Border and Jordan, 1983).

Symmetric Single-Peaked (Euclidean) Preferences on $\mathbb{R}^m$: Preferences R_i on $A_i \subseteq \mathbb{R}^m$ are *symmetrically single-peaked* (or Euclidean) if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$, $x_i \, R_i \, y_i$ if and only if $\|p(R_i) - x_i\|_E \leq \|p(R_i) - y_i\|_E$.

Note that for symmetric single-peaked preferences, upper contour sets are spheres. The following domain loosely speaking extends the symmetric preference domain to also allow for ellipsoids as upper contour sets (with axes that are parallel to the coordinate axes).

Separable Quadratic Preferences on $\mathbb{R}^m$: Preferences R_i on $A_i \subseteq \mathbb{R}^m$ are *separable quadratic* if there exists a point $p(R_i) \in A_i$, $\alpha_1, \ldots, \alpha_n > 0$, and a utility representation u_i of R_i such that for all $x_i \in A_i$, $u_i(x_i) = -\sum_{k=1}^{m} (\alpha_k (x_{i,k} - p_k(R_i))^2$. Note that if for all $i, j \in N$, $\alpha_i = \alpha_j$, then preferences are symmetric.

In order to introduce the next domain, we need some definitions and notation. We define the convex hull of two points $x_i, y_i \in \mathbb{R}^m$ by $\text{conv}(x_i, y_i) = \{z_i \in \mathbb{R}^m :$ there exists $t \in [0, 1]$ such that $z_i = tx_i + (1 - t)y_i\}$. Let $\| \cdot \|$ be a strictly convex norm, i.e.,

(i) for all $x_i \in \mathbb{R}^m$, $\|x_i\| \geq 0$, (positivity)
(ii) for all $x_i \in \mathbb{R}^m$, $\|x_i\| = 0$ if and only if $x_i = 0$, (positive definiteness)
(iii) for all $x_i \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, $\|\alpha x_i\| = |\alpha| \, \|x_i\|$, (positive homogeneity)
(iv) for all $x_i, y_i \in \mathbb{R}^m$, $\|x_i\| + \|y_i\| \geq \|x_i + y_i\|$, (triangular inequality)
(v) for all $x_i, y_i, z_i \in \mathbb{R}^m$, (strict convexity)
$\|x_i - y_i\| + \|y_i - z_i\| = \|x_i - z_i\|$ if and only if $y_i \in \text{conv}(x_i, z_i)$.

Note that the requirement of strict convexity means that any sphere of positive radius does not contain any line segment that is not reduced to a point. Our definition of strict convexity for norms is based on Papadopoulos (2005, Proposition 7.2.1), which also lists various equivalent conditions for the strict convexity of a norm. For instance, the so-called ℓ^p norm $\|\cdot\|_p$ on $\mathbb{R}^m$ is strictly convex for any $p > 1$ (see Papadopoulos, 2005, Proposition 7.3.2).[8]

The following domain includes the two previously introduced domains.

Preferences on $\mathbb{R}^m$ that are Induced by a Strictly Convex Norm $\|\cdot\|$: Preferences R_i on $A_i \subseteq \mathbb{R}^m$ are *induced by a strictly convex norm* $\|\cdot\|$ if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$, $x_i \; R_i \; y_i$ if and only if $\|p(R_i) - x_i\| \leq \|p(R_i) - y_i\|$.

Finally, we introduce the most general higher-dimensional single-peaked domain that we are aware of (see Border and Jordan, 1983).

Star-Shaped Preferences on $\mathbb{R}^m$: Preferences R_i on $A_i \subseteq \mathbb{R}^m$ are *star-shaped* if there exists a point $p(R_i) \in A_i$ such that for all $x_i \in A_i \backslash \{p(R_i)\}$ and all $\lambda \in (0,1)$, $p(R_i) P_i [\lambda x_i + (1-\lambda) p(R_i)] P_i x_i$.

If in addition to star-shapedness we require convexity of preferences, we obtain the following class of preferences.[9]

Convex Star-Shaped Preferences on $\mathbb{R}^m$: Preferences R_i on $A_i \subseteq \mathbb{R}^m$ are *convex star-shaped* if they are star-shaped and for all $x \in A$, $U(R_i, x)$ is a convex set.

3 Monotonicity and Strategy-Proofness

3.1 Rich Domains: Monotonicity implies Strategy-Proofness

For $i \in N$ and $R_i \in \mathcal{R}$, by $b(R_i)$ we denote agent i's best alternatives in A, i.e., $b(R_i) \equiv \{x \in A : \text{for all } y \in A, \; x \; R_i \; y\}$. To establish our first result, we introduce the following domain "richness" condition.

[8]For $p > 1$ and $x \in \mathbb{R}^m$, $\|x\|_p = \left(\sum_{j=1}^m |x_j|\right)^{\frac{1}{p}}$.
[9]Preferences R_i on A_i are convex if for all $x_i, y_i \in A_i$ and $\lambda \in [0,1]$, $x_i \; R_i \; y_i$ implies $\lambda x_i + (1-\lambda) y_i \; R_i \; y_i$.

Condition R1: Let $i \in N$, $R_i \in \mathcal{R}$, and $x, y \in A$ be such that $y \; P_i \; x$. Then, there exists $R'_i \in \mathcal{R}$ such that $y \in b(R'_i)$ and $L(R_i, x) \subseteq L(R'_i, x)$.[10]

Remark 1. Note that Condition R1 is different from Dasgupta et al.'s (1979) or Fleurbaey and Maniquet's (1997) richness conditions. Our condition involves one preference relation R_i while the other two richness conditions are based on conditions involving two preference relations R_i and R'_i. The domain richness condition closest to ours seems to be the one introduced by Le Breton and Zaporozhets (2009). We briefly state and discuss the relation between these richness conditions in more detail in Appendix A. △

Examples of rich domains satisfying Condition R1 are the Arrovian domain, the single-peaked preference domain on $\mathbb{R}$, and more generally the convex star-shaped preference domain on $\mathbb{R}^m$ (see Proposition 1). We will check if the domains introduced above satisfy Condition R1 in Section 4 and give a short survey in Table 1.

Theorem 1. *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R1 and let rule φ be defined on $\mathcal{R}^N$. If φ is unilaterally monotonic, then it is strategy-proof.*

Proof. Suppose φ is unilaterally monotonic, but not strategy-proof. Then, there exist $R \in \mathcal{R}^N$, $i \in N$, and $\bar{R}_i \in \mathcal{R}$ such that $\varphi(\bar{R}_i, R_{-i}) \; P_i \; \varphi(R)$. Denote $\varphi(R) = x$ and $\varphi(\bar{R}_i, R_{-i}) = y$. Hence, $y_i \; P_i \; x_i$ and by Condition R1 there exists $R'_i \in \mathcal{R}$ such that $y \in b(R'_i)$ and $L(R_i, x) \subseteq L(R'_i, x)$. Thus, $R'_i \in MT(\bar{R}_i, y)$ and $R'_i \in MT(R_i, x)$. By unilateral monotonicity, $\varphi_i(R'_i, R_{-i}) = y_i$ and $\varphi_i(R'_i, R_{-i}) = x_i$. Hence, $x_i = y_i$; contradicting our assumption that $y_i \; P_i \; x_i$. □

Corollary 1. *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R1 and let rule φ be defined on $\mathcal{R}^N$. If φ is Maskin monotonic, then it is strategy-proof.*

We demonstrate for the public as well as for the private goods case that strategy-proofness does not necessarily imply unilateral/Maskin monotonicity. For both examples, we use the domain of single-peaked preferences, which satisfies Condition R1 (this follows from Proposition 1).

Example 3. We consider Moulin's (1980) model as introduced in Example 2. Thus, for all $i \in N, A_i = [0, 1]$ and agents' preferences are single-peaked on $\mathbb{R}$. Let $c_1, c_2 \in [0, 1]$, $c_1 < c_2$, and $k \in N$. Then, for all $R \in \mathcal{R}^N$,

$$\varphi(R) \equiv \begin{cases} c_1 \mathbb{1} & \text{if } c_1 \; P_k \; c_2 \text{ or if } c_1 \; I_k \; c_2 \text{ and } p(R_k) \in \mathbb{Q}; \\ c_2 \mathbb{1} & \text{if } c_2 \; P_k \; c_1 \text{ or if } c_1 \; I_k \; c_2 \text{ and } p(R_k) \notin \mathbb{Q}. \end{cases}$$

It is easy to see that φ is strategy-proof, but not unilateral/Maskin monotonic. ◇

[10] Note that in the proof of Theorem 1 and in all results concerning single-peaked domains, we could strengthen Condition R1 by requiring $L(R_i, x) = L(R'_i, x)$ instead of $L(R_i, x) \subseteq L(R'_i, x)$.

Example 4. We consider Sprumont's (1991) model as introduced in Example 2. Thus, for all $i \in N$, $A_i = [0,1]$ and $F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$. Note that in this model, a two agents division problem corresponds to a two agents location problem in Moulin's (1980) model. Hence, by adapting the rule of Example 3, we can construct a strategy-proof rule φ' that is not unilaterally/Maskin monotonic for Sprumont's (1991) model as follows. Let $c_1, c_2 \in [0,1]$, $c_1 < c_2$, $k \in N$, and φ be the rule defined in Example 3. Let $j \in N \setminus \{k\}$. Then for all $R \in \mathcal{R}^N$, $\varphi'_k(R) = \varphi_k(R)$, $\varphi'_j(R) = 1 - \varphi_k(R)$, and for all $i \in N \setminus \{j,k\}$, $\varphi'_i(R) = 0$. ⋄

3.2 Restricted Domains: Strategy-Proofness implies Monotonicity

To establish our second result, we introduce the following domain "restriction" condition.

Condition R2: Let $i \in N$, $R_i, R'_i \in \mathcal{R}$, and $x \in A$ be such that $R'_i \in MT(R_i, x)$ and $R'_i \neq R_i$. Then, for all $y \in L(R_i, x) \cap U(R'_i, x)$, $y_i = x_i$.

Examples of restricted domains satisfying Condition R2 are the Arrovian domain (and any domain containing only strict preference relations), the symmetric single-peaked preference domain on $\mathbb{R}$, the separable quadratic preference domain on $\mathbb{R}^m$, and more generally any strictly convex norm induced preference domain (see Proposition 2). We will check if the domains introduced above satisfy Condition R2 in Section 4 and give a short survey in Table 1.

Theorem 2. *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R2 and let rule φ be defined on $\mathcal{R}^N$.*

(a) *If φ is strategy-proof, then it is unilaterally monotonic.*

(b) *Let F determine a public goods economy. If φ is strategy-proof, then it is Maskin monotonic.*

Proof. (a) Suppose φ is strategy-proof, but not unilaterally monotonic. Then, there exist $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $\varphi(R) = x$, $R'_i \in MT(R_i, x)$, and $\varphi_i(R'_i, R_{-i}) = y_i \neq x_i$. By strategy-proofness, $x\ R_i\ y$ and $y\ R'_i\ x$. Thus, $y \in L(R_i, x)$ and $y \in U(R'_i, x)$. Hence, $y \in L(R_i, x) \cap U(R'_i, x)$ and $y_i \neq x_i$; a contradiction with Condition R2.

(b) Next, assume that F determines a public goods economy and suppose φ is strategy-proof, but not Maskin monotonic. Then, there exist $R, R' \in \mathcal{R}^N$ such that $R' \in MT(R, x)$, $\varphi(R) = x$ and $\varphi(R') = y \neq x$. Assume that $R' = (R'_i, R_{-i})$ for some $i \in N$. By strategy-proofness, $x\ R_i\ y$ and $y\ R'_i\ x$. Thus, $y \in L(R_i, x)$ and $y \in U(R'_i, x)$. By the public goods assumption, $x_i \neq y_i$. Hence, $y \in L(R_i, x) \cap U(R'_i, x)$ and $y_i \neq x_i$; a contradiction with Condition R2. Hence, $\varphi(R) = \varphi(R')$. The proof that for all $R, R' \in \mathcal{R}^N$ such that $R' \in MT(R, x)$, $\varphi(R) = \varphi(R')$ follows from an iteration of the previous arguments (by switching agents one by one from R_i to R'_i). □

We demonstrate for the public as well as for the private goods case that unilateral/Maskin monotonicity does not necessarily imply strategy-proofness. For both examples, we use the domain of symmetric single-peaked preferences, which satisfies Condition R2 (this follows from Proposition 2).

Example 5. We consider Moulin's (1980) model as described in Examples 2 and 3, but with symmetric single-peaked preferences. Let $c_1, c_2 \in [0,1]$, $c_1 < c_2$, and $k \in N$. Then, for all $R \in \mathcal{R}^N$,

$$\varphi(R) \equiv \begin{cases} p(R_k)\mathbb{1} & \text{if } p(R_k) \leq c_1; \\ c_2\mathbb{1} & \text{otherwise.} \end{cases}$$

It is easy to see that φ is unilaterally/Maskin monotonic, but not strategy-proof. ◇

Example 6. We consider Sprumont's (1991) model discussed in Examples 2 and 4, but with symmetric single-peaked preferences. Similarly as in Example 4, we can adapt the rule of Example 5 to construct a unilaterally/Maskin monotonic rule φ' that is not strategy-proof. Let $c_1, c_2 \in [0,1]$, $c_1 < c_2$, $k \in N$, and φ be the rule defined in Example 5. Let $j \in N \setminus \{k\}$. Then for all $R \in \mathcal{R}^N$, $\varphi'_k(R) = \varphi_k(R)$, $\varphi'_j(R) = 1 - \varphi_k(R)$, and for all $i \in N \setminus \{j,k\}$, $\varphi'_i(R) = 0$. ◇

The following example demonstrates that for private goods economies Condition R2 and strategy-proofness do not necessarily imply Maskin monotonicity (hence, Theorem 2 (b) cannot be extended to private goods economies). We use the domain of separable quadratic single-peaked preferences, which satisfies Condition R2 (this follows from Proposition 2).

Example 7. We consider a two-dimensional extension of Sprumont's (1991) model with separable quadratic preferences. Then, for all $i \in N$, $A_i = [0,1]^2$ and $F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0\mathbb{1} \text{ and } \sum_{i\in N} x_i = 1\mathbb{1}\}$. Without loss of generality let $N = \{1,2,3\}$. Let $c \in [0,1)^2$. We define φ as follows. First, for all $R \in \mathcal{R}^N$, $\varphi_1(R) = c$. Second, if R_1 is symmetric, then $\varphi_2(R) = 1\mathbb{1} - c$ and $\varphi_3(R) = 0$, and otherwise, $\varphi_2(R) = 0$ and $\varphi_3(R) = 1\mathbb{1} - c$. It is easy to see that φ is strategy-proof, unilaterally monotonic, but not Maskin monotonic. ◇

Theorem 2 as well as Examples 6 and 7 show an important difference between public goods and private goods models. For the former, and for almost all the domains we cover[11], the class of Maskin monotonic rules is either a subset, a superset, or coincides with the class of strategy-proof rules (see Table 1). In the private goods case, this "set-inclusion connection" between the class of Maskin monotonic rules and the class of strategy-proof rules is lost for some domains, e.g., the symmetric single-peaked domain for which there exist rules that are Maskin monotonic but not strategy-proof, as well as rules that are strategy-proof but not Maskin monotonic.

A key feature of Example 7 is that φ violates non-bossiness. With the next lemma we can show easily that Theorem 2 (b) can be extended to private goods economies if non-bossiness is added.

Lemma 1. *Let A and F be given. Let rule φ be defined on $\mathcal{R}^N$. If φ is unilaterally monotonic and non-bossy, then it is Maskin monotonic.*

[11] Exception made of the star-shaped domain.

Proof. Suppose that φ is unilaterally monotonic and non-bossy. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $\varphi(R) = x$ and $R'_i \in MT(R_i, x)$. Then, by unilateral monotonicity, $\varphi_i(R'_i, R_{-i}) = x_i$. Hence, by non-bossiness, $\varphi(R'_i, R_{-i}) = x$. The proof that for all $R, R' \in \mathcal{R}^N$ such that $R' \in MT(R, x)$, $\varphi(R) = \varphi(R') = x$ follows from an iteration of the previous arguments (by switching agents one by one from R_i to R'_i). Hence, φ is Maskin monotonic. □

Corollary 2. *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R2 and let rule φ be defined on $\mathcal{R}^N$. If φ is strategy-proof and non-bossy, then it is Maskin monotonic.*

4 Rich Domains, Restricted Domains, and the Muller-Satterthwaite Theorem

We now analyze which of our domains are rich and which are restricted.

4.1 Condition R1: Rich Domains

It is clear from Examples 5 and 6 that symmetric single-peaked preferences violate Condition R1. We show below that the convex star-shaped domain is rich. Since the single-peaked preference domain is the one-dimensional equivalent of (convex) star-shaped preferences, our result implies that any domain larger than the single-peaked preference domain is also rich – provided that all preferences in the domain are convex (see Example 8).

Proposition 1. *The domain of convex star-shaped preferences satisfies Condition R1.*

The following notation for star-shaped preferences is useful in the proof of Proposition 1. Let R_i be a star-shaped preference relation and assume that $x_i \in A_i \setminus \{p(R_i)\}$. Then, for all $z_i \in A_i$ such that $z_i \, R_i \, x_i$ there exists $x'_i \in A_i$, $x'_i \, I_i \, x_i$ and $\lambda(R_i; x_i, z_i) \in [0, 1]$ such that $z_i = \lambda(R_i; x_i, z_i)p(R_i) + (1 - \lambda(R_i; x_i, z_i))x'_i$. Note that if $\lambda(R_i; x_i, z_i) = 0$, then $z_i \; I_i \; x_i$ and if $\lambda(R_i; x_i, z_i) = 1$, then $z_i = p(R_i) \; P_i \; x_i$.

Proof. Let R_i be a convex star-shaped preference relation and assume that $x, y \in A$ such that $y \; P_i \; x$. In order to verify Condition R1 we construct convex star-shaped preferences R'_i such that $y \in b(R'_i)$ and $L(R_i, x) \subseteq L(R'_i, x)$. If $y_i = p(R_i)$ then we are done by choosing $R'_i = R_i$. Thus, we assume that $y_i \neq p(R_i)$.

Loosely speaking, we construct R'_i by "lifting y_i up" to become the peak of a new preference relation R'_i such that preferences over $L(R_i, x)$ do not change. To be more precise, we construct preferences R'_i as follows:

(i) $y_i = p(R'_i)$, i.e., y_i is the peak of R'_i;

(ii) for all $z, z' \in L(R_i, x)$, $z \, R_i \, z'$ if and only if $z \, R'_i \, z'$, i.e., preferences on $L(R_i, x)$ do not change;

(iii) for all $z \in U(R_i, x)$ and $z' \in SL(R_i, x)$, $z P'_i z'$, i.e., preferences between $U(R_i, x)$ and $SL(R_i, x)$ do not change;

(iv) for all $z, z' \in U(R_i, x)$, $z\, R'_i\, z'$ if and only if $\lambda(R'_i; x_i, z_i) \geq \lambda(R'_i; x_i, z'_i)$, i.e., we parameterize all $z, z' \in U(R_i, x)$ using line segments from the indifference set $I_i(R_i, x_i) = \{x'_i \in A_i : x'_i I_i x_i\}$ to the peak $p(R'_i) = y_i$ and $\lambda(R'_i; x_i, \cdot)$.

Note that by (i) and (iv), $b(R'_i) = \{y\}$ and by (ii), $L(R_i, x) = L(R'_i, x)$ (in particular, $I_i(R_i, x_i) = I_i(R'_i, x_i)$). Next, we prove that convex star-shapedness is preserved by our construction of R'_i from R_i.

First, we show that star-shapedness is preserved when going from R_i to R'_i. Let $w, z \in A$ and $\lambda \in (0, 1)$ be such that $w_i, z_i \in A_i \setminus \{p(R'_i)\}$ and $z_i = \lambda y_i + (1 - \lambda) w_i$. We prove that $z_i\, P'_i\, w_i$. We have two cases to consider:

Case 1. $w \in SL(R_i, x)$

Hence, (a) $w, z \in SL(R_i, x)$ or (b) $[w \in SL(R_i, x)$ and $z \in U(R_i, x)]$. For (a), since $y_i\, P_i\, w_i$, by convexity, $z_i\, R_i\, w_i$. Suppose, by contradiction, that $z_i\, I_i\, w_i$. Since, $y_i\, P_i\, w_i$ there exists $w' \in A$ with $w'_i\, I_i\, w_i$ and such that $y_i = \bar{\lambda} p(R_i) + (1 - \bar{\lambda}) w'_i$ for some $\bar{\lambda} \in (0, 1)$. If $w'_i = w_i$ or $w'_i = z_i$, then we are done.[12] So suppose that w'_i is distinct from w_i and z_i. Since $z_i = \lambda w_i + (1 - \lambda) y_i$ and $y_i = \bar{\lambda} p(R_i) + (1 - \bar{\lambda}) w'_i$, we obtain

$$z_i = \lambda w_i + (1 - \lambda)[\bar{\lambda} p(R_i) + (1 - \bar{\lambda}) w'_i] = \lambda w_i + (1 - \lambda)(1 - \bar{\lambda}) w'_i + (1 - \lambda)\bar{\lambda} p(R_i).$$

Let $v \in A$ be such that v_i is the following convex combination of w_i, w'_i, and z_i:

$$v_i = \lambda w_i + (1 - \lambda)(1 - \bar{\lambda}) w'_i + (1 - \lambda)\bar{\lambda} z_i.$$

By convexity, $v_i\, R_i\, z_i\, I_i\, w_i\, I_i\, w'_i$. Notice that $z_i - (1 - \lambda)\bar{\lambda} p(R_i) = v_i - (1 - \lambda)\bar{\lambda} z_i$. Therefore,

$$z_i = \frac{1}{(1 + (1 - \lambda)\bar{\lambda})}\, v_i + \frac{(1 - \lambda)\bar{\lambda}}{(1 + (1 - \lambda)\bar{\lambda})}\, p(R_i) = \tilde{\lambda} v_i + (1 - \tilde{\lambda}) p(R_i)$$

with $\tilde{\lambda} = \frac{1}{(1 + (1 - \lambda)\bar{\lambda})} \in (0, 1)$. Hence, by star-shapedness of R_i, $z_i\, P_i\, v_i$, contradicting $v_i\, R_i\, z_i$. Therefore, $z_i\, P_i\, w_i$ and by the construction of R'_i (see (ii)), it follows that $z_i\, P'_i\, w_i$.

For (b), $w \in SL(R_i, x)$, $z \in U(R_i, x)$, and the construction of R'_i (see (iii)) imply $z_i\, P'_i\, w_i$.

Case 2. $w \in U(R_i, x_i)$

Hence, by the convex star-shapedness of R_i, $z_i\, P_i\, w_i$ and $z \in SU(R_i, x_i)$. Thus, $\lambda(R'_i; x_i, z_i) > \lambda(R'_i; x_i, w_i)$. Hence, by construction of R'_i (see (iv)), this implies $z_i\, P'_i\, w_i$.

Second, we show that convexity is preserved when going from R_i to R'_i. Instead of the standard definition of convex preferences given in Footnote 9, it is well-known that convexity of preferences can be defined via the convexity of upper contour sets. Recall that we do not change preferences on $L(R_i, x)$. An immediate implication is that for each $y' \in L(R'_i, x)$, $U(R'_i, y')$ is a convex set.

[12] If $w'_i = w_i$ or $w'_i = z_i$, then for some $\lambda^* \in (0, 1)$, $z_i = \lambda^* w_i + (1 - \lambda^*) p(R_i)$ and by star-shapedness, $z_i\, P_i\, w_i$.

Therefore, to show our claim, we only need to consider upper contour sets for points that are in $SU(R_i, x)$. Hence, let $v, w \in SU(R'_i, x)$, $v \neq w$, $v\, I'_i\, w$, and $\alpha \in (0,1)$ such that $z_i = \alpha v_i + (1-\alpha) w_i$. We have to show that $z\, R'_i\, v\, I'_i\, w$. By construction of R'_i (see (iv)), this implies that we have to prove

$$\lambda(R'_i; x_i, z_i) \geq \lambda(R'_i; x_i, v_i) = \lambda(R'_i; x_i, w_i). \tag{1}$$

Note that $v, w \in SU(R'_i, x)$ and $v \neq w$ imply that $1 > \lambda(R'_i; x_i, v_i) = \lambda(R'_i; x_i, w_i) > 0$.

Let $\hat{z}_i = \alpha v'_i + (1-\alpha) w'_i$. There exist $v', w', z' \in A$ such that $v'_i I_i w'_i I_i z'_i I_i x_i$ (recall that $I_i(R_i, x_i) = I_i(R'_i, x_i)$), $v_i = \lambda(R'_i; x_i, v_i) y_i + (1 - \lambda(R'_i; x_i, v_i)) v'_i$, $w_i = \lambda(R'_i; x_i, w_i) y_i + (1 - \lambda(R'_i; x_i, w_i)) w'_i$, and $\hat{z}_i = \lambda(R'_i; x_i, \hat{z}_i) y_i + (1 - \lambda(R'_i; x_i, \hat{z}_i)) z'_i$. By convexity, $\hat{z}\, R'_i\, v'\, I'_i\, w'$. By construction of R'_i (see (iv)), $\lambda(R'_i; x_i, \hat{z}_i) \geq \lambda(R'_i; x_i, v'_i) = \lambda(R'_i; x_i, w'_i) = 0$.

Next, we can derive $z_i = [\lambda(R'_i; x_i, v_i) + \lambda(R'_i; x_i, \hat{z}_i) - \lambda(R'_i; x_i, v_i)\lambda(R'_i; x_i, \hat{z}_i)] y_i + [(1 - \lambda(R'_i; x_i, v_i))(1 - \lambda(R'_i; x_i, \hat{z}_i))] z'_i$.[13] Since, $z_i = \lambda(R'_i; x_i, z_i) y_i + (1 - \lambda(R'_i; x_i, z_i)) z'_i$, it follows that $\lambda(R'_i; x_i, z_i) = \lambda(R'_i; x_i, v_i) + \lambda(R'_i; x_i, \hat{z}_i) - \lambda(R'_i; x_i, v_i)\lambda(R'_i; x_i, \hat{z}_i)$. Hence, $1 > \lambda(R'_i; x_i, v_i) > 0$ and $\lambda(R'_i; x_i, \hat{z}_i) \geq 0$ imply

$$\begin{array}{llll}
\Leftrightarrow & \lambda(R'_i; x_i, \hat{z}_i) & \geq & \lambda(R'_i; x_i, v_i)\lambda(R'_i; x_i, \hat{z}_i) \\
\Leftrightarrow & \lambda(R'_i; x_i, \hat{z}_i) - \lambda(R'_i; x_i, v_i)\lambda(R'_i; x_i, \hat{z}_i) & \geq & 0 \\
\Leftrightarrow & \lambda(R'_i; x_i, v_i) + \lambda(R'_i; x_i, \hat{z}_i) - \lambda(R'_i; x_i, v_i)\lambda(R'_i; x_i, \hat{z}_i) & \geq & \lambda(R'_i; x_i, v_i) \\
\Leftrightarrow & \lambda(R'_i; x_i, z_i) & \geq & \lambda(R'_i; x_i, v_i).
\end{array}$$

Hence, the desired inequality (1) holds and we have proven convexity of the preference relation R'_i. □

Corollary 3. *Let A such that for all $i \in N$, $A_i \subseteq \mathbb{R}^m$ and F be given. Let $\mathcal{R}$ be the domain of all convex star-shaped preferences and let rule φ be defined on $\mathcal{R}^N$. If φ is Maskin monotonic, then it is strategy-proof.*

The following example demonstrates that convexity of preferences is a necessary assumption for star-shaped preferences to satisfy Condition R1.

Example 8. Let $A = [0,1]^2 \times ... \times [0,1]^2$ and let $\mathcal{R}$ be the domain of star-shaped preferences. In Figure 2, we depict a preference relation R_i on $A_i = \mathbb{R}^2_+$ with peak $p(R_i)$ and with a non-convex upper contour set at $x_i \in A_i$ (marked by the indifference curve through x_i). It is easy to see that there does not exist $R'_i \in \mathcal{R}$ with $y_i = p(R'_i)$ and $L(R_i, x) \subseteq L(R'_i, x)$. Indeed, for any such R'_i, star-shapedness implies that for all $z' \in A$ with $z'_i = z_i$, $z \in SU(R'_i, x)$ while Condition R1 implies that $z \in L(R'_i, x)$; a contradiction. Thus Condition R1 is violated. ◇

[13] For completeness, $z_i = \alpha v_i + (1-\alpha) w_i = \alpha[\lambda(R'_i; x_i, v_i) y_i + (1 - \lambda(R'_i; x_i, v_i)) v'_i] + (1-\alpha)[\lambda(R'_i; x_i, v_i) y_i + (1 - \lambda(R'_i; x_i, v_i)) w'_i] = \lambda(R'_i; x_i, v_i) y_i + (1 - \lambda(R'_i; x_i, v_i))[\alpha v'_i + (1-\alpha) w'_i] = \lambda(R'_i; x_i, v_i) y_i + (1 - \lambda(R'_i; x_i, v_i)) \hat{z}_i = \lambda(R'_i; x_i, v_i) y_i + (1 - \lambda(R'_i; x_i, v_i))[\lambda(R'_i; x_i, \hat{z}_i) y_i + (1 - \lambda(R'_i; x_i, \hat{z}_i)) z'_i] = [\lambda(R'_i; x_i, v_i) + \lambda(R'_i; x_i, \hat{z}_i) - \lambda(R'_i; x_i, v_i)\lambda(R'_i; x_i, \hat{z}_i)] y_i + [(1 - \lambda(R'_i; x_i, v_i))(1 - \lambda(R'_i; x_i, \hat{z}_i))] z'_i$.

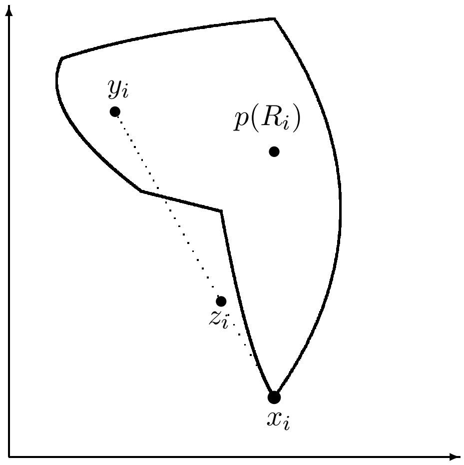

Figure 2: Star-shaped non-convex preferences that do not satisfy Condition R1

4.2 Condition R2: Restricted Domains

It is clear from Examples 3 and 4 that general single-peaked preferences violate Condition R2. We show below that all preferences that are induced by a strictly convex norm satisfy R2.

Proposition 2. *The domain of preferences that are induced by a strictly convex norm satisfies Condition R2.*[14]

Proof. Let $\|\cdot\|$ be a strictly convex norm and R_i, R_i' be preferences on $A_i \subseteq R^m$ induced by $\|\cdot\|$. Furthermore, let $x \in A$ be such that $R_i' \in MT(R_i, x)$ and $R_i' \neq R_i$. Note that then $p(R_i) \neq p(R_i')$. Let $y \in L(R_i, x) \cap U(R_i', x)$. Since $R_i' \in MT(R_i, x)$, $y \in L(R_i', x) \cap U(R_i', x)$. Hence, $y_i \, I_i' \, x_i$ and

$$\|p(R_i') - y_i\| = \|p(R_i') - x_i\|. \tag{2}$$

Furthermore, $y \in L(R_i, x)$ implies

$$\|p(R_i) - y_i\| \geq \|p(R_i) - x_i\|. \tag{3}$$

Consider $\text{line}(p(R_i), p(R_i')) = \{z_i \in \mathbb{R}^m : \text{there exists } t \in \mathbb{R} \text{ such that } z_i = tx_i + (1-t)y_i\}$. Then, there exist two distinct points $\hat{z}_i, \tilde{z}_i \in \text{line}(p(R_i), p(R_i'))$ such that $\hat{z}_i \, I_i \, x_i$ and $\tilde{z}_i \, I_i \, x_i$ (possibly $\hat{z}_i = x_i$ or $\tilde{z}_i = x_i$). Note that we can give an orientation to the line such that one of these points is to the left of $p(R_i)$ and the other is to the right of $p(R_i)$. Without loss of generality, assume that $p(R_i')$ and $\tilde{z}_i$ are to the right of $p(R_i)$. Since $R_i' \in MT(R_i, x)$, $\tilde{z}_i \, I_i \, x_i$ implies $\tilde{z}_i \in L(R_i', x)$ and

$$\|p(R_i') - \tilde{z}_i\| \geq \|p(R_i') - x_i\|. \tag{4}$$

Case 1. $p(R_i') \notin \text{conv}(p(R_i), \tilde{z}_i)$

Then, $\|p(R_i) - p(R_i')\| > \|p(R_i) - \tilde{z}_i\| = \|p(R_i) - x_i\|$. Hence, $p(R_i') \in L(R_i, x)$ and by $R_i' \in MT(R_i, x)$, $p(R_i') \in L(R_i', x)$. Hence, $x_i = p(R_i')$ and by (2), $x_i = y_i$.

[14]Note that then any subdomain satisfies Condition R2 as well.

Case 2. $p(R'_i) \in \text{conv}(p(R_i), \tilde{z}_i)$

Then, by strict convexity, $\|p(R_i) - \tilde{z}_i\| = \|p(R_i) - p(R'_i)\| + \|p(R'_i) - \tilde{z}_i\| \overset{(4)}{\geq} \|p(R_i) - p(R'_i)\| + \|p(R'_i) - x_i\| \overset{(*)}{\geq} \|p(R_i) - x_i\|$, where $(*)$ follows from the triangular inequality. However, since $\|p(R_i) - \tilde{z}_i\| = \|p(R_i) - x_i\|$, $(*)$ is an equality and by strict convexity, $p(R'_i) \in \text{conv}(p(R_i), x_i)$. Hence, $x_i = \tilde{z}_i$.

If $p(R'_i) \notin \text{conv}(p(R_i), y_i)$, then, by strict convexity, $\|p(R_i) - y_i\| < \|p(R_i) - p(R'_i)\| + \|p(R'_i) - y_i\| \overset{(2)}{=} \|p(R_i) - p(R'_i)\| + \|p(R'_i) - x_i\| = \|p(R_i) - x_i\|$. Thus, $\|p(R_i) - y_i\| < \|p(R_i) - x_i\|$; contradicting (3). Hence, $p(R'_i) \in \text{conv}(p(R_i), y_i)$. But then, (2) and (3) together imply, $x_i = y_i$.

To summarize, we have proven that for any $y \in L(R_i, x) \cap U(R'_i, x)$, it follows that $y_i = x_i$. Hence, preferences that are induced by a strictly convex norm satisfy Condition R2. □

Examples of preferences induced by a strictly convex norm for $A_i = \mathbb{R}^m$ are symmetric (Euclidean) and separable quadratic preferences.

Corollary 4. *Let A such that for all $i \in N$, $A_i \subseteq \mathbb{R}^m$ and F be given. Let $\mathcal{R}$ be a domain of preferences that are induced by a strictly convex norm and let rule φ be defined on $\mathcal{R}^N$.*

(a) *If φ is strategy-proof, then it is unilaterally monotonic.*

(b) *Let F determine a public goods economy. If φ is strategy-proof, then it is Maskin monotonic.*

We provide in Table 1 a summary of the results obtained so far. We now turn our attention to the Muller-Satterthwaite Theorem and its extensions.

Preference Domain	Condition R1	Condition R2
Arrovian preferences	**Yes**	**Yes**
strict single-peaked preferences on $\mathbb{R}$	**Yes**	**Yes**
left-right single-peaked preferences on $\mathbb{R}$	**Yes**	**Yes**
right-left single-peaked preferences on $\mathbb{R}$	**Yes**	**Yes**
symmetric single-peaked (Euclidean) preferences on $\mathbb{R}^m$	**No**	**Yes**
separable quadratic preferences on $\mathbb{R}^m$	**No**	**Yes**
convex norms induced preferences on $\mathbb{R}^m$	**No**	**Yes**
single-peaked preferences on $\mathbb{R}$	**Yes**	**No**
single-peaked preferences on $\mathbb{R}$ with an outside option	**Yes**	**No**
single-plateaued preferences on $\mathbb{R}$	**Yes**	**No**
DHM single-plateaued preferences on $\mathbb{R}$	**Yes**	**No**
convex star-shaped preferences on $\mathbb{R}^m$	**Yes**	**No**
star-shaped preferences on $\mathbb{R}^m$	**No**	**No**

Table 1: Preference Domains and Conditions R1 and R2

4.3 An Extended Muller-Satterthwaite Theorem

To conclude the section, we now state some immediate consequences of Theorems 1 and 2, and Corollaries 1 and 2.

Theorem 3. ***An Extension of the Muller-Satterthwaite Theorem***
Let A and F be given. Let $\mathcal{R}$ satisfy Conditions R1 and R2 and let rule φ be defined on $\mathcal{R}^N$.

(a) *Then, φ is unilaterally monotonic if and only if it is strategy-proof.*
(b) *Let F determine a public goods economy. Then, φ is Maskin monotonic if and only if it is strategy-proof.*
(c) *Then, φ is Maskin monotonic if and only if it is strategy-proof and non-bossy.*

Theorem 3 states an extension of the Muller-Satterthwaite Theorem that covers both the public goods and the private goods case. Items (a) and (c) establish that the only monotonicity condition equivalent to strategy-proofness in a private goods model is the unilateral monotonicity condition. As Corollary 2 made clear, for domains satisfying both R1 and R2, only a subset of the set of strategy-proof rules coincide with the set of Maskin monotonic rules, namely the set of strategy-proof rules that satisfy non-bossiness. Because non-bossiness is vacuous in public goods models, item (c) directly implies item (b). The equivalence between Maskin monotonicity and strategy-proofness as stated in the original version of the Muller-Satterthwaite Theorem can thus be obtained only for public goods models.

Corollary 5. ***The Muller-Satterthwaite Theorem***
Let A and F be given such that F determines a public goods economy. Let rule φ be defined on the Arrovian domain $\mathcal{R}_A$. Then, φ is Maskin monotonic if and only if it is strategy-proof.

Next, we show that the conclusion of the Muller-Satterthwaite Theorem is not only limited to the Arrovian domain; Theorem 3 has bite for various single-peaked preference domains. A first example is the domain of strict single-peaked preferences on $\mathbb{R}$ or the domain of strict single-peaked preferences defined on a finite set of alternatives. Indeed, preferences being single-peaked implies Condition R1 and preferences being strict implies Condition R2.

Finally, we introduce a new "Muller-Satterthwaite domain". Suppose that a public facility, e.g., a phone booth is to be located on a street that is very safe on one end of the street and becomes more and more dangerous when moving towards the other end of the street. Then, it is natural to assume that agents' preferences are single-peaked (the phone booth in front of one's house would be best) and prefer any location in the safer part of the street to a location in the more dangerous part of the street. The following preference domain describes the situation when the street is very safe on its "left side" and becomes more dangerous towards its "right side".[15]

[15]We thank Bernardo Moreno for suggesting this type of preference domain.

Left-right single-peaked preferences on $\mathbb{R}$: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *left-right single-peaked* if R_i is single-peaked on $\mathbb{R}$ with peak $p(R_i) \in A_i$ and such that for all $x_i, y_i \in A_i$ satisfying $x_i \leq p(R_i) < y_i$, $x_i \, P_i \, y_i$.

Note that any left-right single-peaked preference relation is uniquely defined by its peak.

Proposition 3. *Left-right single-peaked preferences satisfy Conditions R1 and R2.*

Proof. Note that the domain of left-right single-peaked preferences only contains strict preferences and therefore satisfies Condition R2. In order to verify Condition R1, let R_i be a left-right single-peaked preference relation and assume that $x, y \in A$ such that $y \, P_i \, x$. Consider the left-right single-peaked preference relation R'_i with $p(R'_i) = y_i$. By the definition of left-right single-peaked preferences:

(i) if $x_i > p(R_i)$, then $L(R_i, x) = A \cap [x, \infty) = L(R'_i, x)$;

(ii) if $x_i \leq p(R_i)$, then $x_i < y_i \leq p(R_i)$ and
$L(R_i, x) = A \cap ((-\infty, x] \cup [p(R_i)\mathbb{1}, \infty)) \subseteq A \cap ((-\infty, x] \cup [y, \infty)) = L(R'_i, x)$.

Hence, $y \in b(R'_i)$ and $L(R_i, x) \subseteq L(R'_i, x)$. Thus, the domain of left-right single-peaked preferences also satisfies Condition R1. □

Similarly, we can define the domain of right-left single-peaked preferences on $\mathbb{R}$ by assuming that the street is very safe on its "right side" and becomes more dangerous towards its "left side".

A Appendix: Richness Conditions

First, we introduce Dasgupta et al.'s (1979) richness condition. A domain is (Dasgupta, Hammond, and Maskin) rich if it satisfies the following condition.

Condition DHM: Let $R_i, R'_i \in \mathcal{R}$ and $a, b \in A$ such that (a) $a \, R_i \, b \Rightarrow a \, R'_i \, b$ and (b) $a \, P_i \, b \Rightarrow a \, P'_i \, b$. Then, there exists $R''_i \in \mathcal{R}$ such that (i) $R''_i \in MT(R_i, a)$ and (ii) $R''_i \in MT(R'_i, b)$.

Maskin (1985) called the Dasgupta et al. (1979) rich domain monotonically closed. Note that Condition DHM does not imply Condition R1. For instance, *strictly monotonic domains* satisfying Condition DHM do not satisfy Condition R1.[16] On the other hand, all the domains satisfying Condition R1 that we look at in the paper satisfy Condition DHM.

Fleurbaey and Maniquet (1997) also use a domain richness condition under the name of strict monotonic closedness. Their rich domain satisfies the following condition.

Condition FM: Let $R_i, R'_i \in \mathcal{R}$ and $a, b \in A$ such that (a) $a \, P_i \, b$. Then, there exists $R''_i \in \mathcal{R}$ such that for all $c \in A$, $c \neq a, b$, (i) $a \, R'_i \, c$ implies $a \, P''_i \, c$, (ii) $b \, R_i \, c$ implies $b \, P''_i \, c$, and (iii) [not $a \, I''_i \, b$].

[16] A domain $\mathcal{R}$ is strictly monotonic with respect to A_i, $|A_i| = \infty$, if for each $R_i \in \mathcal{R}$, and each $x_i, y_i \in A_i$ with $y_i > x_i$, $y_i \, P_i \, x_i$

Note that Conditions $R1$ and FM are logically independent. The domain of single-plateaued preferences on $\mathbb{R}$ is rich according to Condition $R1$, but not according to Condition FM (on the single-plateaued domain it might not be possible to satisfy Condition FM(iii)). On the other hand, strictly monotonic domains satisfying Condition FM do not satisfy Condition R1.

Finally we consider Le Breton and Zaporozhets's (2009) rich domain condition.

Condition LBZ: Let $R_i \in \mathcal{R}$ and $x, y \in A$ such that $y \, P_i \, x$ and $y \in b(\bar{R}_i)$ for some $\bar{R}_i \in \mathcal{R}$, there exists $R_i' \in \mathcal{R}$ such that $y \in b(R_i')$ and for all z with $z_i \neq x_i$ such that $x \, R_i \, z$, $x \, P_i' \, z$.

While Condition LBZ implies Condition R1, the converse is not true. Observe that Condition LBZ requires that $L(R_i, x) \setminus \{x\} \subseteq SL(R_i', x)$; a stronger requirement than $L(R_i, x) \subseteq L(R_i', x)$ imposed by Condition R1. Condition LBZ requires sufficient degrees of freedom to undo at R_i' the possible indifferences with respect to x present at R_i. On the other hand, all the domains satisfying Condition R1 that we look at in the paper satisfy Condition LBZ.

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