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Evaluating Discrete Dynamic Strategies in Affine Models

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Abstract

We consider the problem of measuring the performance of a dynamic strategy, re-balanced at a discrete set of dates, whose objective is that of replicating a claim in an incomplete market driven by a general multi-dimensional affine process. The main purpose of the paper is to propose a method to efficiently compute the expected value and variance of the hedging error of the strategy. Representing the pay-off of the claim as an inverse Laplace transform, we are able to get semi-explicit formulas for strategies satisfying a certain property. The result is quite general and can be applied to a very rich class of models and strategies, including Delta hedging. We provide illustrations for the cases of interest rate models and Heston's stochastic volatility model.

1 Introduction

One of the most discussed assumptions of financial models, especially criticized in periods of financial turmoils, is that of market completeness, that is the perfect replication of any contingent claim by a suitable dynamic trading strategy. Theoretically, this is often achieved by ruling out any market imperfection, like illiquidity, credit risks, transactions costs, taxes, etc and by assuming the possibility of continuous time trading. Of course, real markets usually fail to satisfy most, if not all of such assumptions. One of the main challenges for financial economics is therefore to address such issue, by proposing models with less stringent hypotheses or by studying what happens when they do not hold. In this paper we focus on the impossibility of trading continuously in time. Even if all other assumptions of the model are satisfied, the inherent discreteness of trading times is a source of market incompleteness in the real world. The aim of the paper is to efficiently evaluate the impact of trading in discrete time on the final goal of the strategy.

The object of our investigation is the ex-ante assessment of the performances of dynamic trading strategies. Probably, the most notable instance of such problem is measuring the hedging error of a strategy, based on a liquid asset, that tries to replicate a future liability. Problems of such kind arise when replicating either a claim using futures contracts, or a payoff of a derivative security with a delta hedging strategy based on the underlying asset, and in any case when a dynamic strategy is adopted. Ex-ante, a possible way to measure the performance of a strategy is by evaluating expected value and variance of its hedging error. This is usually done by approximations or by Monte Carlo simulations. The approach proposed in the paper, based on Laplace transforms, allows to efficiently perform such computations for a very general class of models.

Our methodology can be applied to many important models for financial markets for equities, interest rates and credit products where the stochastic dynamics of risk factors are driven by affine processes. Affine processes are popular in modeling financial markets, because of their analytical tractability and flexibility. Their defining property, which we will often exploit in this work, is that their characteristic function is exponentially affine. Affine processes in finance were introduced first for interest rate models by Vasicek (1977), Cox et al. (1985) and by Duffie and Kan (1996) who gave a general formulation in a multivariate setting. The pioneering work in the case of equities is due to Heston (1993) who proposed a stochastic volatility model and opened the way to many other models of the kind. Duffie, Filipovic and Schachermayer (2003) formulated a general theory for affine processes for equities and interest rates. Further application to credit risk modeling can be found in Duffie and Singleton (2003). We present the general model framework of our approach in Section 2, where we provide the necessary details of affine processes as well as some examples.

The problem of measuring the hedging error in discrete time was first addressed by Toft (1996) who proposed an approximation for the variance of the Delta hedging strategy in the Black-Scholes model. Hayashi and Mykland (2005) use a weak convergence argument to derive the asymptotic distribution of the hedging error as the number of trades goes to infinity. Their approach was generalized by Tankov and Voltchkova (2009) to Levy processes with jumps. A very important problem, related to this, is that of determining a strategy that minimizes the variance of the hedging error. An extremely rich branch of the financial literature flourished after the seminal papers of Föllmer and Sondermann (1986). Schweizer (1999) contains a review of the main results and contributions in a discrete time setting. In continuous time, but in a context very close to that of the present paper, Černý and Kallsen (2006) solve the problem of computing the optimal strategy and the optimal variance for the Heston's model, and Kallsen and Vierthauer (2009) extended the results to general affine stochastic volatility models.

An important ingredient of our method is that of representing the payoff of the claim as an inverse Laplace transform. This idea was introduced by Černý (2007) and Hubalek et al. (2006) in the context of variance-optimal hedging. From this, the key idea, proposed by Angelini and Herzel (2009) in the case of Levy processes, is to express the Delta-based strategies as an inverse Laplace transform, so that one can directly compute the Laplace transform of the hedging error and, from it, its expected value and variance. In Section 3 we show how to exploit the integral representation of a dynamic strategy and the nice features of affine processes to compute such values. We conclude in Section 4 with two examples, the first is related to the hedging of an option on a bond, the second is an application to the stochastic volatility model by Heston (1993).

2 The general framework

We consider a general framework for market models with a risky asset S and a short term interest rate r that may be stochastic, depending on the specification of the model in the framework. The dynamics of the market are driven by a multi-dimensional affine process X, whose components may include $y = \ln(S)$ and/or r whenever it is stochastic as well as stochastic volatility, dividend yields, etc. For instance, in short rate models, like Cox, Ingersoll and Ross (1985) or Vasicek (1977), X_t is a one-dimensional process representing the short-term interest rate. In this case, if the risky asset is a zero coupon bond, y is an affine function of r. Another relevant case is the model proposed by Heston (1993), where X_t is a two-dimensional process of the logarithm of the asset price and its instantaneous volatility. We will examine these two cases in Section 4. Pan (2002) studied a four dimensional affine model combining stochastic volatility, interest rates and dividend yield. A very general study of affine processes in a financial setting is contained in Duffie, Pan and Singleton (2000). Another important area of application is to model the intensity of defaults in evaluating the credit risk (Duffie and Singleton (2003)). Duffie, Filipovic and Schachermayer (2003) provide a characterization of affine processes.

As the theory of affine processes is well established, we only recall those concepts that are necessary for our purpose, referring to Duffie, Filipovic and Schachermayer (2003) for a more complete exposition and technical details.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le \infty}, P)$ be a filtered probability space satisfying the usual technical conditions. We interpret P as the physical or objective probability measure. We consider a strong Markov process X defined in a state space $D \subset \mathbb{R}^d$ and the moment generating function of X_T defined in $\mathbb{C}^d \times D \times \mathbb{R}_+ \times \mathbb{R}_+$

$$\phi(u, X_t, t, T) \quad = \quad E_t\left[e^{u \cdot X_T}\right], \tag{2.1}$$

where E_t is the expected value conditional on $\mathcal{F}_t$ and $\cdot$ is the scalar product. When X is affine the moment generating function is

$$\phi(u, X_t, t, T) \quad = \quad e^{\alpha(u,t,T) + \beta(u,t,T) \cdot X_t}, \tag{2.2}$$

where $\alpha(u, t, T)$ and $\beta(u, t, T)$ are functions from $\mathbb{C}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{C}$ and to $\mathbb{C}^d$ respectively, satisfying a system of Riccati equations that depend on the process X with boundary conditions $\beta(u, T, T) = u$ and $\alpha(u, T, T) = 0$. Such equations have an explicit solution in some particular cases, otherwise can be numerically integrated. For example, in the Black-Scholes model, where X is a one-dimensional Brownian motion with coefficients μ and σ

$$\alpha^{bs}(u,t,T) = \left(\mu - \frac{u\sigma^2}{2} + \frac{u^2\sigma^2}{2}\right)(T-t), \tag{2.3}$$

$$\beta^{bs}(u,t,T) = u. \tag{2.4}$$

The functions $\alpha(u,t,T)$ and $\beta(u,t,T)$ can be computed in closed form in other important cases, like in the models by Cox, Ingersoll and Ross (1985), by Vasicek (1977) and by Heston (1993).

For our application it is necessary to compute the joint moment generating function of X at times $t_1,\dots,t_\nu$ conditional on the information in t,

$$\phi_\nu(u_1,\dots,u_\nu,X_t,t,t_1,\dots,t_\nu) = E_t\left[e^{\sum_{j=1}^{\nu} u_j\cdot X_{t_j}}\right]. \tag{2.5}$$

For an affine process we have

$$\phi_\nu(u_1,\dots,u_\nu,X_t,t,t_1,\dots,t_\nu) = e^{\alpha_\nu(u_1,\dots,u_\nu,t,t_1,\dots,t_\nu)+\beta_\nu(u_1,\dots,u_\nu,t,t_1,\dots,t_\nu)\cdot X_t}, \tag{2.6}$$

where the functions $\alpha_\nu(\cdot)$ and $\beta_\nu(\cdot)$ are equal to $\alpha(\cdot)$ and $\beta(\cdot)$ for $\nu = 1$ and can be computed recursively if $\nu > 1$. In fact,

$$\begin{aligned}\phi(u_1,\dots,u_\nu,X_t,t,t_1,\dots,t_\nu) &= E_t\left[e^{u_1\cdot X_{t_1}} E_{t_1}\left[e^{\sum_{j=2}^{\nu} u_j\cdot X_{t_j}}\right]\right] \\ &= E_t\left[e^{u_1\cdot X_{t_1}}\phi(u_2,\dots,u_\nu,X_{t_1},t_1,t_2,\dots,t_\nu)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_\nu(u_1,\dots,u_\nu,t,t_1,\dots,t_\nu) &= \alpha_{\nu-1}(u_2,\dots,u_\nu,t_1,t_2,\dots,t_\nu) \\ &+ \alpha(u_1+\beta_{\nu-1}(u_2,\dots,u_\nu,t_1,t_2,\dots,t_\nu),t,t_1) \\ \beta_\nu(u_1,\dots,u_\nu,t,t_1,\dots,t_\nu) &= \beta(u_1+\beta_{\nu-1}(u_2,\dots,u_\nu,t_1,t_2,\dots,t_\nu),t,t_1).\end{aligned}$$

We also assume that X is affine under a pricing measure Q. Conditions for a process to be affine under both measures P and Q are given by Duffie, Pan and Singleton (2000). Since for pricing purposes it is often necessary to consider discounting, in this case we state a more general version of (2.2), that is

$$\begin{aligned}\psi(u,X_t,t,T) &= E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)e^{u\cdot X_T}\right] \\ &= e^{\bar{\alpha}(u,t,T)+\bar{\beta}(u,t,T)\cdot X_t},\end{aligned} \tag{2.7}$$

where the functions $\bar{\alpha}(u,t,T)$ and $\bar{\beta}(u,t,T)$ solve a system of Riccati equations depending on the risk-neutral dynamics of X. Setting $u = 0$ in (2.7) we get the the discount factor between time t and T

$$\begin{aligned} P(t,T) &= E_t^Q \left[\exp\left(-\int_t^T r_s ds \right) \right] \\ &= e^{\bar{\alpha}(0,t,T)+\bar{\beta}(0,t,T)\cdot X_t}. \end{aligned}$$

3 Dynamic hedging strategies

In this section we give the main definitions and properties of dynamic hedging strategies in our setting and we measure the hedging error of a class of strategies in terms of expected value and variance. Let us consider a square-integrable contingent claim written on S with payoff H at time T that can be expressed as

$$H = \int_{\mathcal{C}} e^{zy_T} \Pi(dz), \tag{3.8}$$

where $\mathcal{C}$ is a contour in the complex plane, Π is a finite complex measure on $\mathcal{C}$ and $y_T = \ln(S_T)$. In other words, the payoff function is represented as an inverse Laplace transform. For instance, the payoff of a European call option with strike price $K > 0$ may be written as

$$(e^x - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{zx} \frac{K^{1-z}}{z(z-1)} dz,$$

for an arbitrary $R > 1$. Other examples are the put, the power call and the digital option, see Hubalek et al. (2006).

Let us denote by 1_y the d-dimensional vector of zeros except for the entry corresponding to y that is equal to one. From (2.7) and using Fubini, we get an expression for the value at time t of a European claim with payoff expressed as in (3.8)

$$\begin{aligned} H_t &= E_t^Q\left[\exp\left(-\int_t^T r_s ds\right) H\right] \\ &= E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)\int_{\mathcal{C}} e^{zy_T}\Pi(dz)\right] \\ &= \int_{\mathcal{C}} E_t^Q\left[\exp\left(-\int_t^T r_s ds\right) e^{zy_T}\right]\Pi(dz) \\ &= \int_{\mathcal{C}} E_t^Q\left[\exp\left(-\int_t^T r_s ds\right) e^{z1_y\cdot X_T}\right]\Pi(dz) \\ &= \int_{\mathcal{C}} e^{\bar{\alpha}(z1_y,t,T)+\bar{\beta}(z1_y,t,T)\cdot X_t}\Pi(dz). \end{aligned} \tag{3.9}$$

Note that H_t depends on all the components of X_t and not only on those involved in the definition of the payoff and in the discount factor. That is the value at time t of a claim on S_T may also depend on factors, like the stochastic volatility of S, that do not appear in the payoff function.

By differentiating (3.9), we can compute the sensitivities of the pricing formula with respect to the factor of the model. In particular, the Delta of the claim at time t is given by

$$\begin{aligned} \Delta_t^H &= \frac{\partial H_t}{\partial S_t} \\ &= \frac{1}{S_t}\frac{\partial H_t}{\partial y_t} \\ &= e^{-y_t}\frac{\partial}{\partial y_t}\int_{\mathcal{C}} e^{\bar{\alpha}(z1_y,t,T)+\bar{\beta}(z1_y,t,T)\cdot X_t}\Pi(dz) \\ &= e^{-y_t}\int_{\mathcal{C}}\frac{\partial}{\partial y_t} e^{\bar{\alpha}(z1_y,t,T)+\bar{\beta}(z1_y,t,T)\cdot X_t}\Pi(dz) \\ &= \int_{\mathcal{C}} \bar{\beta}(z1_y,t,T)\cdot 1_y e^{\bar{\alpha}(z1_y,t,T)+(\bar{\beta}(z1_y,t,T)-1_y)\cdot X_t}\Pi(dz). \end{aligned} \tag{3.10}$$

We note that the Delta (as well as any other of the so called *Greeks*) of a claim with payoff as in (3.8) is an integral of an exponential of an affine function of X.

We consider now the problem of hedging the contingent claim H when trading is only allowed at a finite and prefixed set of dates from time 0 until maturity T, $0 = t_0 < t_1 < \ldots < t_N = T$. For simplicity we assume that the probabilities of jumps of the process X at the trading times are zero.

Let $\vartheta = (\vartheta_{t_k})$, for $k = 0, \ldots, N-1$, be a stochastic process representing a trading strategy. The random variable ϑ_{t_k} is the number of shares of S held from time t_k up to time t_{k+1}. We assume that it depends only on the information available at time t_k, i.e. that it is $\mathcal{F}_{t_k}$-measurable. Let

$$M(t,T) = 1/P(t,T) = e^{-\bar{\alpha}(0,t,T)-\bar{\beta}(0,t,T)\cdot r_t}$$

be the capitalization factor given by investing in the risk-less asset from t to T and

$$\bar{\Delta} S_k = S_{t_k} M(t_k,T) - S_{t_{k-1}} M(t_{k-1},T) \tag{3.11}$$

the increment in value of the risky asset. We also suppose that the local gain $\vartheta_{t_{k-1}} \bar{\Delta} S_k$ of the strategy at time t_k is square-integrable, for all $k = 1, \ldots, N$. The final value of strategy ϑ starting from an initial capital c is

$$G_T(\vartheta) = cM(0,T) + \sum_{k=1}^{N} \vartheta_{t_{k-1}} \bar{\Delta} S_k \tag{3.12}$$

and its hedging error is given by

$$\varepsilon(\vartheta, c) = H - G_T(\vartheta). \tag{3.13}$$

We call a trading strategy ϑ *affine* if

$$\vartheta_{t_k} = \int_{\mathcal{C}} e^{a(z,t_k)+b(z,t_k)\cdot X_{t_k}} \Pi(dz), \tag{3.14}$$

for all $k = 0, \ldots, N-1$, where $a(z,t_k)$ and $b(z,t_k)$ are functions from $\mathbb{C} \times \mathbb{R}_+$ to $\mathbb{C}$ and to $\mathbb{C}^d$ respectively.

In principle the contour $\mathcal{C}$ and the measure Π appearing in the definition of affine strategies could be different from those related to the claim H in Formula (3.8). This may be the case when the hedging strategy is performed by aiming at a different claim, or we may also study trading strategies that are not intended to hedge any claim. However, to fix ideas, we will consider only hedging strategies for the claim H.

We call *Delta strategy* the hedging strategy that is obtained by setting ϑ_{t_k} equal to the Delta of H given in (3.10). It is an important example of an affine strategy, with

$$\begin{aligned} a(z,t) &= \ln\left(\bar{\beta}(z1_y,t,T)\cdot 1_y\right) + \bar{\alpha}(z1_y,t,T) \\ b(z,t) &= \bar{\beta}(z1_y,t,T) - 1_y. \end{aligned}$$

The functions $\bar{\alpha}(\cdot)$ and $\bar{\beta}(\cdot)$ depend on the pricing model. In particular, for the Black-Scholes model (where $1_y = 1$) they are given by (2.3) and (2.4), with $\mu = r$.

A common strategy, often used in practice for hedging with futures contracts, is that obtained by regressing the value of a less liquid security H to a more liquid instrument S and for this reason is usually called *Beta strategy*. More precisely

$$B_{t_k} = \frac{\mathrm{cov}_{t_k}\left(H_{t_{k+1}}, S_{t_{k+1}}\right)}{\mathrm{var}_{t_k}[S_{t_{k+1}}]}. \tag{3.15}$$

In our setting this may be computed as

$$\begin{aligned}
B_{t_k} &= \frac{E_{t_k}\left[H_{t_{k+1}}S_{t_{k+1}}\right] - E_{t_k}\left[H_{t_{k+1}}\right]E_{t_k}\left[S_{t_{k+1}}\right]}{E_{t_k}\left[S^2_{t_{k+1}}\right] - E_{t_k}\left[S_{t_{k+1}}\right]^2} \\
&= \frac{1}{\phi(2\ 1_y, X_{t_k}, t_k, t_{k+1}) - \phi(1_y, X_{t_k}, t_k, t_{k+1})^2} \times \\
&\quad \int_{\mathcal{C}} \Big(E_{t_k}\left[e^{\bar{\alpha}(z1_y, t_{k+1}, T) + \left(\bar{\beta}(z1_y, t_{k+1}, T) + 1_y\right)\cdot X_{t_{k+1}}}\right] + \\
&\quad - E_{t_k}\left[e^{\bar{\alpha}(z1_y, t_{k+1}, T) + \bar{\beta}(z1_y, t_{k+1}, T)\cdot X_{t_{k+1}}}\right] E_{t_k}\left[e^{1_y \cdot X_{t_{k+1}}}\right]\Big)\,\Pi(dz) \\
&= \frac{1}{\phi(2\ 1_y, X_{t_k}, t_k, t_{k+1}) - \phi(1_y, X_{t_k}, t_k, t_{k+1})^2} \times \\
&\quad \int_{\mathcal{C}} e^{\bar{\alpha}(z1_y, t_{k+1}, T)} \big(\phi(\bar{\beta}(z1_y, t_{k+1}, T) + 1_y, X_{t_k}, t_k, t_{k+1}) + \\
&\quad - \phi(\bar{\beta}(z1_y, t_{k+1}, T), X_{t_k}, t_k, t_{k+1})\phi(1_y, X_{t_k}, t_k, t_{k+1})\big)\,\Pi(dz)
\end{aligned}$$

We see that the Beta strategy may still be represented as an integral, but in general it is not affine. However, it is an affine strategy when X is a Levy processes (see Theorem 2.1 in Hubalek et al. (2006)).

Notice that the Beta strategy has a similar structure as the local optimal strategy, that is the strategy minimizing the variance of costs over the next period, and that is obtained by backward regressions. In the case $Q = P$, the two strategies coincide and are also globally optimal, i.e. they minimize the variance of the hedging error (3.13) (see Schweizer (1995) for a complete exposition on variance-optimal hedging in discrete time). When allowing continuous time portfolio rebalancing, Černý and Kallsen (2006) prove that the globally optimal strategy in Heston's model may be written as in (3.14), hence it is affine.

The hedging error (3.13) of an affine strategy for a contingent claim whose payoff can be written as (3.8) has the following integral representation

$$\varepsilon(\vartheta, c) = -cM(0,T) + \qquad (3.16)$$
$$+ \int_{\mathcal{C}} \left(e^{zy_T} - \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}} \bar{\Delta} S_k \right) \Pi(dz).$$

The above representation and the convenient form of the characteristic function of affine processes can be exploited to compute expected value and the variance of the hedging error as shown in the following result.

Theorem 3.1 *Let H be a contingent claim satisfying condition (3.8), ϑ be an affine strategy, namely satisfying Condition (3.14), and c be the initial capital, then*

$$E[\varepsilon(\vartheta, c)] = \int_{\mathcal{C}} e(z)\Pi(dz) - cM(0,T), \qquad (3.17)$$

and

$$E[\varepsilon(\vartheta, 0)^2] = \int_{\mathcal{C}} \int_{\mathcal{C}} (v_1(w,z) - v_2(w,z) - v_3(w,z) + v_4(w,z))\Pi(dw)\Pi(dz), \qquad (3.18)$$

where $e(z)$, $v_j(w,z)$, $j = 1,2,3,4$, are sums of exponentially affine functions that depend on ϑ, on the pricing function $\psi(\cdot)$ (2.7) and on the joint moment generating function $\phi(\cdot)$ (2.6) of X conditional on the information in 0. Their explicit expression can be found in the Appendix. Therefore, the variance of the hedging error is

$$\mathrm{var}(\varepsilon(\vartheta, c)) = \mathrm{var}(\varepsilon(\vartheta, 0)) = E[\varepsilon(\vartheta, 0)^2] - E[\varepsilon(\vartheta, 0)]^2.$$

Proof. see the Appendix.

Theorem 3.1 states that the expected value and the variance of the hedging error may be represented respectively as a one-dimensional and a two-dimensional inverse Laplace transforms. Formulas (3.17) and (3.18) can be evaluated through numerical inversion of one-dimensional and two-dimensional Laplace transforms. For more details on this we refer to Angelini and Herzel (2009). Alternatively, numerical integration procedures may be adopted.

A similar argument can be applied to compute higher order moments of the hedging errors, that can be useful to get more information on the probability distribution.

The results may be used to study the effects of model mispecification or trader personal views, in terms of hedging strategies and parameters, on the performance of the hedge. This because the claim, the model and the strategy are completely independent from each other. An interesting example of how to exploit this flexibility is the case of a given underlying model generating the data (say Heston's model with a certain set of parameters) and a given strategy like the Black-Scholes Delta strategy or the model Delta strategy implemented using model parameters different from those of the data generating model.

4 Examples and Numerical illustration

In this section we illustrate two important instances of the general framework presented in Section 2, the case of an affine short rate model and the case of Heston's model. In the latter case, we also give a numerical illustration of the results in Theorem (3.1).

4.1 Affine Short Rate Models

We now show how to apply Theorem 3.1 to the computation of the Delta strategy in the case of affine short rate models, in particular we write the integral representation of a contingent claim written on a zero coupon bond and the related Delta hedging strategy. In this case the process X is the one-dimensional process of the short rate, $X_t = (r_t)$. The functions $\alpha(u,t,T)$ and $\beta(u,t,T)$ in (2.2) and $\bar{\alpha}(u,t,T)$ and $\bar{\beta}(u,t,T)$ in (2.7) may be computed explicitly in some important cases as the models by Cox, Ingersoll, Ross (1985) or Vasicek (1977), and we do not report them here. We consider a European claim H maturing at date T_1, written on the zero coupon bond with maturity $T_2 > T_1$. Hence

$$S_t = P(t,T_2) = e^{\bar{\alpha}(0,t,T_2)+\bar{\beta}(0,t,T_2)\cdot r_t}$$

and $y_t = \ln(S_t) = \bar{\alpha}(0, t, T_2) + \bar{\beta}(0, t, T_2) \cdot r_t$ is an affine function of $X_t = (r_t)$. Formula (3.8) may be written as

$$\begin{aligned} H &= \int_{\mathcal{C}} e^{zy_{T_1}} \Pi(dz) = \\ &= \int_{\mathcal{C}} e^{\bar{\alpha}(0,T_1,T_2)z + \bar{\beta}(0,T_1,T_2)z \cdot r_{T_1}} \Pi(dz). \end{aligned}$$

Hence, the price of the claim at time t can be computed as

$$\begin{aligned} H_t &= E_t^Q \left[\exp\left(-\int_t^{T_1} r_s ds \right) \int_{\mathcal{C}} e^{zy_{T_1}} \Pi(dz) \right] = \\ &= \int_{\mathcal{C}} e^{\bar{\alpha}(0,T_1,T_2)z} \psi(\bar{\beta}(0, T_1, T_2)z, r_t, t, T_1) \Pi(dz) = \\ &= \int_{\mathcal{C}} e^{\bar{\alpha}(0,T_1,T_2)z} e^{\bar{\alpha}(\bar{\beta}(0,T_1,T_2)z,t,T_1) + \bar{\beta}(\bar{\beta}(0,T_1,T_2)z,t,T_1) \cdot r_t} \Pi(dz). \end{aligned}$$

Therefore the derivative of H_t with respect to the factor r_t is

$$\begin{aligned} D_t &= \frac{\partial H_t}{\partial r_t} \\ &= \int_{\mathcal{C}} \frac{\partial}{\partial r_t} e^{\bar{\alpha}(0,T_1,T_2)z} e^{\bar{\alpha}(\bar{\beta}(0,T_1,T_2)z,t,T_1) + \bar{\beta}(\bar{\beta}(0,T_1,T_2)z,t,T_1) \cdot r_t} \Pi(dz) \\ &= \int_{\mathcal{C}} e^{\bar{\alpha}(0,T_1,T_2)z} \bar{\beta}(\bar{\beta}(0, T_1, T_2)z, t, T_1) e^{\bar{\alpha}(\bar{\beta}(0,T_1,T_2)z,t,T_1) + \bar{\beta}(\bar{\beta}(0,T_1,T_2)z,t,T_1) \cdot r_t} \Pi(dz) \\ &= \int_{\mathcal{C}} \bar{\beta}(\bar{\beta}(0, T_1, T_2)z, t, T_1) e^{\bar{\alpha}(0,T_1,T_2)z + \bar{\alpha}(\bar{\beta}(0,T_1,T_2)z,t,T_1) + \bar{\beta}(\bar{\beta}(0,T_1,T_2)z,t,T_1) \cdot r_t} \Pi(dz). \end{aligned}$$

and the Delta of the claim is

$$\begin{aligned} \Delta_t &= \frac{\partial H_t}{\partial S_t} \\ &= \frac{\partial H_t}{\partial r_t} \frac{\partial r_t}{\partial S_t} \\ &= D_t \frac{1}{\bar{\beta}(0, t, T_2) S_t} \\ &= \int_{\mathcal{C}} \frac{\bar{\beta}(\bar{\beta}(0, T_1, T_2)z, t, T_1)}{\bar{\beta}(0, t, T_2)} \times \\ &\quad e^{\bar{\alpha}(0,T_1,T_2)z + \bar{\alpha}(\bar{\beta}(0,T_1,T_2)z,t,T_1) - \bar{\alpha}(0,t,T_2) + (\bar{\beta}(\bar{\beta}(0,T_1,T_2)z,t,T_1) - \bar{\beta}(0,t,T_2)) \cdot r_t} \Pi(dz) \end{aligned}$$

They are both affine strategies, in particular the Delta is affine with

$$\begin{aligned}
a(z,t) &= \ln\left(\frac{\bar{\beta}(\bar{\beta}(0,T_1,T_2)z,t,T_1)}{\bar{\beta}(0,t,T_2)}\right) + \\
&+ \bar{\alpha}(0,T_1,T_2)z + \bar{\alpha}(\bar{\beta}(0,T_1,T_2)z,t,T_1) - \bar{\alpha}(0,t,T_2) \\
b(z,t) &= \bar{\beta}(\bar{\beta}(0,T_1,T_2)z,t,T_1) - \bar{\beta}(0,t,T_2).
\end{aligned}$$

4.2 Heston's Model

In this section we apply Theorem 3.1 to compare the errors produced by different strategies to hedge a European call option in the stochastic volatility model by Heston (1993). For a simpler exposition, we assume that the pricing measure and the objective measure are equal and that the risk-free rate is zero. In this case the dynamics of the process $X_t = (y_t, v_t)$ is

$$\begin{aligned}
dy_t &= (\mu - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_t^1 \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2, \qquad (4.19)
\end{aligned}$$

where W^1 and W^2 are correlated Brownian motion, $d < W_t^1, W_t^2 >= \rho dt$, and $\mu = r = 0$. The expressions of $\alpha(\cdot)$ and $\beta(\cdot)$ can be found in Elices (2007). We report them for convenience of the reader

$$\begin{aligned}
\alpha(u,t,T) &= \mu\tau u \cdot 1_y + \\
&+ \frac{\kappa\theta}{\sigma 2}\left((\kappa - \rho\sigma u \cdot 1_y - d)\tau - 2\ln\left(\frac{1-\tilde{g}e^{-d\tau}}{1-\tilde{g}}\right)\right) \qquad (4.20) \\
\beta(u,t,T) &= \left(u \cdot 1_y, \frac{\kappa - \rho\sigma u \cdot 1_y + d}{\sigma 2}\left(\frac{g - \tilde{g}e^{-d\tau}}{1-\tilde{g}e^{-d\tau}}\right)\right), \qquad (4.21)
\end{aligned}$$

where $\tau = T - t$,

$$\begin{aligned}
g &= \frac{\kappa - \rho\sigma u \cdot 1_y - d}{\kappa - \rho\sigma u \cdot 1_y + d}, \\
\tilde{g} &= \frac{\kappa - \rho\sigma u \cdot 1_y - d - \sigma 2\sigma u \cdot 1_v}{\kappa - \rho\sigma u \cdot 1_y + d - \sigma 2\sigma u \cdot 1_v}, \\
d &= \sqrt{(\kappa - \rho\sigma u \cdot 1_y)2 + \sigma 2(u \cdot 1_y - (u \cdot 1_y)2)},
\end{aligned}$$

where 1_v is the analogous of 1_y.

We consider the hedging errors of the following strategies:

1. Heston Delta. It is the correct Delta strategy for the model, given by (3.10) and (4.20) and (4.21).
2. Black-Scholes Delta. This is the case of a trader who wishes to adopt the standard Black-Scholes strategy, neglecting some elements of the true dynamics of the underlying. The only parameter the trader has to choose is the volatility parameter σ_t^{bs} at time t. If σ_t^{bs} does not depend on y_t, the strategy is given by (3.10)

$$\Delta_{t_k}^{bs} = \int_{\mathcal{C}} e^{\ln(z)+\bar{\alpha}^{bs}(z,t,T)+(z-1)y_{t_k}} \Pi(dz), \tag{4.22}$$

with the function $\bar{\alpha}^{bs}(\cdot)$ of the Black-Scholes model as in (2.3), where μ is set to zero and $\sigma = \sigma_t^{bs}$. For some choices of σ_t^{bs} the strategy is affine. The strategy is obviously affine when σ_t^{bs} is deterministic, with

$$\begin{aligned} a(z,t) &= \ln(z) + \bar{\alpha}^{bs}(z,t,T) \\ b(z,t) &= (z-1)1_y. \end{aligned}$$

Otherwise it has to be checked case by case. We consider three choices for the volatility parameter σ_t^{bs}, the first two are deterministic, the last one stochastic:

   (a) The implied volatility of the option at time 0.

   (b) The *optimal hedging volatility*, that is the volatility parameter that minimizes the variance of the error produced by the Black-Scholes Delta hedging strategy, i.e. the solution to the problem

   $$\min_{\sigma\in\mathbb{R}} \operatorname{var}\left(H - \sum_{k=1}^{N} \Delta_{t_{k-1}}^{bs}(\sigma)\bar{\Delta}S_k\right),$$

   where $\Delta_{t_{k-1}}^{bs}(\sigma)$ is the Black-Scholes Delta at time t_{k-1} as a function of the volatility parameter σ. The optimal value can be determined numerically from Theorem 3.1.

   (c) The expected volatility over the life to maturity of the option. That is we consider a dynamic σ_t that is computed as

   $$\sigma_{t_k}^2 = \frac{1}{T-t_k} E_{t_k}\left[\int_{t_k}^{T} v_u du\right].$$

In Heston model we have

$$\sigma_{t_k}^2 = \alpha_0(t_k, T) + \alpha_1(t_k, T) v_{t_k},$$

where

$$\begin{aligned}
\alpha_0(t_k, T) &= \theta \left(1 - \frac{1 - e^{-\kappa(T - t_k)}}{\kappa(T - t_k)} \right), \\
\alpha_1(t_k, T) &= \frac{1 - e^{-\kappa(T - t_k)}}{\kappa(T - t_k)}.
\end{aligned}$$

Then, plugging into (4.22), we get

$$\begin{aligned}
a(z, t_k) &= \ln(z) + \left(-\frac{1}{2} z + \frac{1}{2} z^2 \right) (T - t_k) \alpha_0(t_k, T); \\
b(z, t_k) &= \left(-\frac{1}{2} z + \frac{1}{2} z^2 \right) (T - t_k) \alpha_1(t_k, T) 1_v + (z - 1) 1_y.
\end{aligned}$$

3. Variance-optimal strategy in case trading may be done in continuous time. This is computed in Černý and Kallsen (2006) and, in the present case, with $P = Q$, is

$$\xi_t = \int_{\mathcal{C}} (z + \rho\sigma\beta(z1_y, t, T) \cdot 1_v) e^{\alpha(z1_y, t, T) + (\beta(z1_y, t, T) - 1_y) \cdot X_t} \Pi(dz).$$

   It is then affine. Observe that, when $\rho = 0$, this is equal to the Heston Delta.

Notice that the common practice of employing the implied volatility as σ_t^{bs}, re-computed at each trading date, does not fall in the picture above, because the implied volatility is a function of y. Hence such a strategy is not representable as in (4.22). It would still have an integral representation, but in general not with an exponentially affine integrand, so it is not affine as defined in (3.14).

Since we supposed that $P = Q$, the Beta strategy defined in (3.15) is also the globally optimal hedging strategy for the discrete time case. We can compute it at each trading date t_k, but, since it is not affine, we cannot use Theorem 3.1 to compute the expected value and the variance of the related hedging error. As far as we know, there is no result in literature that allows to make such a computation.

For our illustration, we consider European call options with maturity $T = 0.5$ years and we suppose that the initial price of the underlying asset is $S_0 = 100$. We assume that the underlying follows a process given in (4.19) with parameters $v_0 = 0.05$, $\theta = 0.05$, $\kappa = 3$, $\sigma = 0.5$. We analyze five different determination of the correlation coefficient $\rho = (-0.9, -0.5, 0, 0.5, 0.9)$. We fix the number of trading dates to be $N = 6$ and $t_k - t_{k-1} = \frac{T}{N}$ for all $k = 1, \dots, N$, namely the position is adjusted once a month. We also consider various strike prices from 80 to 120.

Since S is a martingale, in all cases the expected value of the hedging error does not depend on the strategy adopted, but only on the difference between the price H_0 of the option at time 0 and the initial capital c (given also that the risk-free rate is 0). For instance, if H is a liability, the expected final gain would be the extra money $c - H_0$ invested in the strategy. To evaluate the performance of the various strategies in this case, we look at the variance of the related hedging errors. In the general case, when S is not a martingale, one may look at performance indices like the Sharpe ratio.

The best performance in all cases is obtained by the variance-optimal strategy for continuous time trading. To get an idea on how this is related to the variance-optimal in our discrete time setting, we computed them at time 0, for all the strike considered. We see that the two are close, for all ρ, and we gather that the variances of their hedging error must not be far. We show their values at time 0 in Figure 1 for $\rho = (-0.9, 0, 0.5)$ (for opposite values of ρ the pictures are analogous), together with the Heston Delta and the Black-Scholes Delta with expected volatility. As noticed above, for $\rho = 0$ the Heston Delta and the continuous time variance-optimal strategy coincide. The performances of the different strategies are all quite similar in case $\rho = 0$, with slightly better results obtained with the variance-optimal in continuous time (and Heston Delta) and the Black-Scholes Delta with expected volatility, as shown in Figure 2, middle panel. The differences get more evident the higher the absolute value of the correlation, either positive or negative. When ρ is not zero, it is interesting to see that all the Black-Scholes Deltas perform better than the model Delta (except for very far out-of-the-money options where the Black-Scholes Delta with expected volatility has higher variance than the model Delta). We choose the case $\rho = -0.9$ and $\rho = 0.5$, respectively in the top and lower panels of Figure 2, to represent these results. We see that, in terms of comparative performances of the different strategies, we have analogous situations for positive and negative correlation coefficients. The difference between positive and negative ρ is in the general shape of the variance as a function of the strike: while for negative ρ all strategies tend to have a decreasing variance in the out-of-the-money part, for positive ρ this is not the case. This is due to the positively proportional relation in the model between the correlation coefficient and the skewness of the return, showing the impact of the latter on the variance of the hedging error.

5 Conclusions

We found semi-explicit formulas for the expected value and the variance of the hedging error of a dynamic strategy when trading is possible only at a discrete set of dates. The results are valid for the general class of affine processes and can then be applied to different financial markets like fixed-income, equities and credit products. The method is based on the representation of the target payoff as an inverse Laplace transform and applies to a class of strategies having an analogous representation which include important strategies like Delta hedging.

We gave a numerical illustration of our results to compare the performances of various hedging strategies for European call options in Heston's stochastic volatility model. We showed how, in this case, the best performance is obtained by adopting the strategy that minimizes the variance of the hedging error when allowing continuous time trading. Our methodology does not however allow to compute the variance-optimal variance in our discrete time setting and this is a topic of future research. We also found that, when the correlation between spot return and its variance is not negligible, Black-Scholes Delta type strategies perform better than model Delta.

6 Appendix

We provide the explicit expression for the integrands involved in the statement of Theorem (3.1).

$$\begin{aligned} e(z) &= \phi(z1_y, X_0, 0, T) - \sum_{k=1}^{N} e^{a(z,t_{k-1})} \times \\ & \quad \big(e^{-\bar{\alpha}(0,t_k,T)}\phi(b(z,t_{k-1}), -\bar{\beta}(0,t_k,T)1_r + 1_y, X_0, 0, t_{k-1}, t_k) - \\ & \quad e^{-\bar{\alpha}(0,t_{k-1},T)}\phi(b(z,t_{k-1}) - \bar{\beta}(0,t_{k-1},T)1_r + 1_y, X_0, 0, t_{k-1})\big)\,, \end{aligned}$$

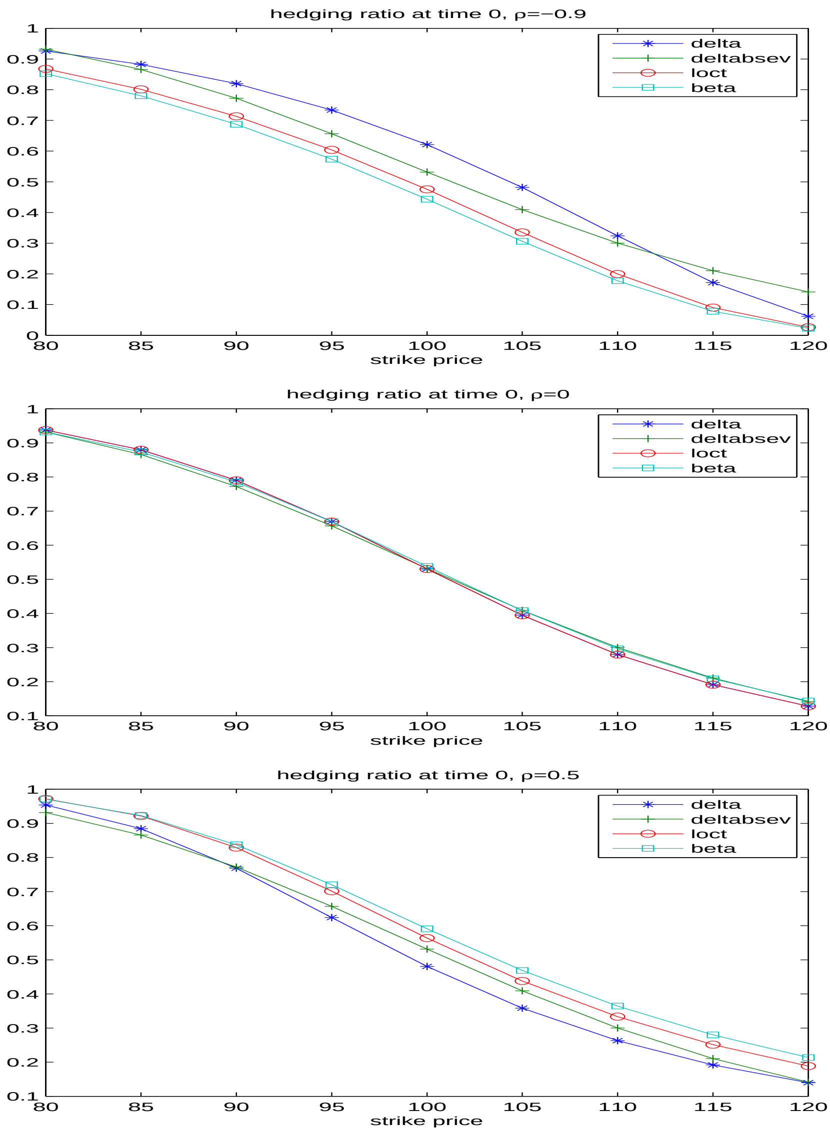

Figure 1 *Hedging ratios at time 0 for European call options with maturity* $T = 0.5$ *as a function of the strike price. The current value of the underlying is* 100*. Hedging ratio: model Delta (delta), Black-Scholes Delta with expected volatility (deltabsev), variance-optimal in continuous time (loct) and beta or variance-optimal (beta). Heston model with parameters* $v_0 = 0.05$, $\mu = 0$, $\theta = 0.05$, $\kappa = 3$, $\sigma = 0.5$*. The correlation coefficient is -0.9 (top panel), 0 (middle panel), 0.5 (lower panel).*

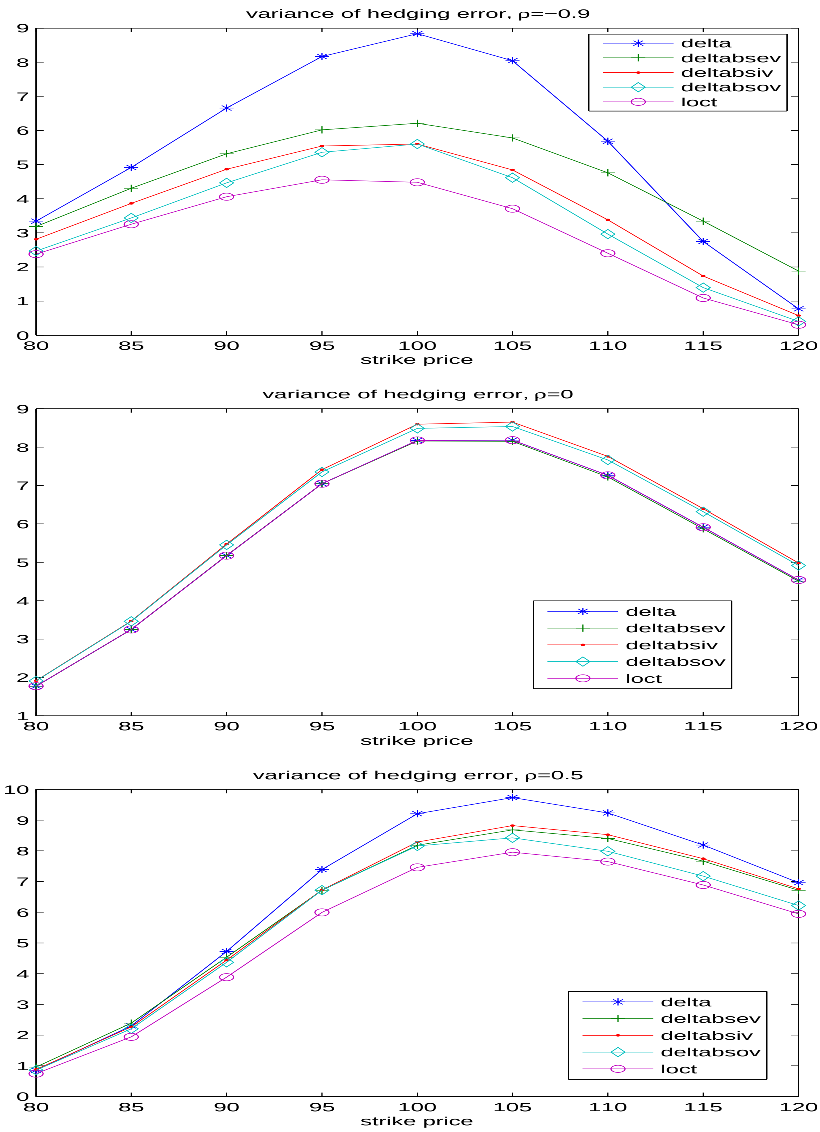

Figure 2 *Variances of hedging strategies for European call options with maturity $T = 0.5$ as a function of the strike price. The current value of the underlying is 100. Hedging ratio: model Delta (delta), Black-Scholes Delta with expected volatility (deltabsev), implied volatility (deltabsiv) and optimal volatility (deltabsov) and variance-optimal in continuous time (loct). Heston model with parameters $v_0 = 0.05$, $\mu = 0$, $\theta = 0.05$, $\kappa = 3$, $\sigma = 0.5$. The correlation coefficient are -0.9 (top panel), 0 (middle panel), 0.5 (lower panel).*

where 1_r has the analogous meaning of 1_y.

$$v_1(w,z) = \phi((w+z)1_y, X_0, 0, T),$$

$$\begin{aligned} v_2(w,z) &= \sum_{k=1}^{N} e^{a(w,t_{k-1})} \times \\ &\quad \big(e^{-\bar{\alpha}(0,t_k,T)}\phi(b(w,t_{k-1}), -\bar{\beta}(0,t_k,T)1_r + 1_y, z1_y, X_0, 0, t_{k-1}, t_k, T) - \\ &\quad e^{-\bar{\alpha}(0,t_{k-1},T)}\phi(b(w,t_{k-1}) - \bar{\beta}(0,t_{k-1},T)1_r + 1_y, z1_y, X_0, 0, t_{k-1}, T)\big)\,, \end{aligned}$$

$$v_3(w,z) = v_2(z,w).$$

To write $v_4(w,z)$, we define

$$\bar{m}(j_1, j_2, k_1, k_2) = \begin{cases} \phi(b(w,t_{j_1}), -\bar{\beta}(0,t_{j_2},T)1_r + 1_y, b(z,t_{k_1}), -\bar{\beta}(0,t_{k_2},T)1_r + 1_y, X_0, 0, t_{j_1}, t_{j_2}, t_{k_1}, t_{k_2}) \\ \text{for } j_1 \le j_2 \le k_1 \le k_2; \\ \phi((b(z,t_{k_1}), -\bar{\beta}(0,t_{k_2},T)1_r + 1_y, b(w,t_{j_1}), -\bar{\beta}(0,t_{j_2},T)1_r + 1_y, X_0, 0, t_{k_1}, t_{k_2}, t_{j_1}, t_{j_2}) \\ \text{for } k_1 \le k_2 \le j_1 \le j_2. \end{cases}$$

Then we have

$$\begin{aligned} v_4(w,z) &= \sum_{j=1}^{N}\sum_{k=1}^{N} e^{a(w,t_{j-1})} e^{a(z,t_{k-1})} \times \\ &\quad (\bar{m}(j-1,j,k-1,k) - \bar{m}(j-1,j,k-1,k-1) - \\ &\quad \bar{m}(j-1,j-1,k-1,k) + \bar{m}(j-1,j-1,k-1,k-1))\,. \end{aligned}$$

Proof of Theorem 3.1. Given (3.17), we have, by Fubini's Theorem,

$$E\left[H-\sum_{k=1}^{N}\vartheta_{t_{k-1}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right]=$$

$$=\int_{\mathcal{C}}\left\{E\left[e^{zy_T}\right]-\sum_{k=1}^{N}E\left[e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\times\right.\right.$$
$$\left.\left.\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right]\right\}\Pi(dz)=$$

$$=\int_{\mathcal{C}}\left\{E\left[e^{zy_T}\right]-\sum_{k=1}^{N}e^{a(z,t_{k-1})}\times\right.$$
$$\left.E\left[e^{b(z,t_{k-1})\cdot X_{t_{k-1}}}\left(e^{y_{t_k}-\bar{\alpha}(0,t_k,T)-\bar{\beta}(0,t_k,T)r_{t_k}}-e^{y_{t_{k-1}}-\bar{\alpha}(0,t_{k-1},T)-\bar{\beta}(0,t_{k-1},T)r_{t_{k-1}}}\right)\right]\right\}\Pi(dz)=$$

$$=\int_{\mathcal{C}}\left\{E\left[e^{zy_T}\right]-\sum_{k=1}^{N}e^{a(z,t_{k-1})}\times\left(e^{-\bar{\alpha}(0,t_k,T)}E\left[e^{b(t_{k-1},z)\cdot X_{t_{k-1}}+(-\bar{\beta}(0,t_k,T)1_r+1_y)\cdot X_{t_k}}\right]\right.\right.$$
$$\left.\left.-\ e^{-\bar{\alpha}(0,t_{k-1},T)}E\left[e^{(b(z,t_{k-1})-\bar{\beta}(0,t_{k-1},T)1_r+1_y)\cdot X_{t_{k-1}}}\right]\right)\right\}\Pi(dz)=$$

$$=\int_{\mathcal{C}}\left\{\phi(z1_y,X_0,0,T)-\sum_{k=1}^{N}e^{a(z,t_{k-1})}\times\right.$$
$$\left(e^{-\bar{\alpha}(0,t_k,T)}\phi(b(z,t_{k-1}),-\bar{\beta}(0,t_k,T)1_r+1_y,X_0,0,t_{k-1},t_k)\ -\right.$$
$$\left.\left.e^{-\bar{\alpha}(0,t_{k-1},T)}\phi(b(z,t_{k-1})-\bar{\beta}(0,t_{k-1},T)1_r+1_y,X_0,0,t_{k-1})\right)\right\}\Pi(dz)$$

which is (3.17). To prove (3.18) we need to compute

$$
\begin{aligned}
&E\left[(H-\sum_{k=1}^{N}\vartheta_{t_k}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right))^2\right]=\\
&=E\left[\int_{\mathcal{C}}\left(H(z)-\sum_{k=1}^{N}e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right)\Pi(dz)\right.\\
&\left.\int_{\mathcal{C}}\left(H(w)-\sum_{k=1}^{N}e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right)\Pi(dw)\right]=\\
&=E\left[\int_{\mathcal{C}}\int_{\mathcal{C}}\left(H(z)-\sum_{k=1}^{N}e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right)\times\right.\\
&\left.\left(H(w)-\sum_{k=1}^{N}e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right)\Pi(dz)\Pi(dw)\right]=\\
&=\int_{\mathcal{C}}\int_{\mathcal{C}}E\left[\left(H(z)-\sum_{k=1}^{N}e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right)\times\right.\\
&\left.\left(H(w)-\sum_{k=1}^{N}e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right)\right]\Pi(dz)\Pi(dw).
\end{aligned}
$$

Let us compute all the expectations needed:

$$
E\left[H(z)H(w)\right]=E\left[e^{(z+w)y_T}\right]=E\left[e^{(z+w)1_y\cdot X_T}\right]=\phi((z+w)1_y,X_0,0,T).
$$

$$
\begin{aligned}
&E\left[H(z)\sum_{k=1}^{N} e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right]=\\
&=\sum_{k=1}^{N} e^{a(w,t_{k-1})}E\left[e^{b(w,t_{k-1})\cdot X_{t_{k-1}}}e^{zy_T}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right]=\\
&=\sum_{k=1}^{N} e^{a(w,t_{k-1})}\times\\
&\Big(e^{-\bar{\alpha}(0,t_k,T)}E\left[e^{b(w,t_{k-1})\cdot X_{t_{k-1}}-\bar{\beta}(0,t_k,T)r_{t_k}+y_{t_k}+zy_T}\right]-\\
&e^{-\bar{\alpha}(0,t_{k-1},T)}E\left[e^{(b(w,t_{k-1})+\cdot X_{t_{k-1}}-\bar{\beta}(0,t_{k-1},T)r_{t_{k-1}}+y_{t_{k-1}}+zy_T}\right]\Big)=\\
&=\sum_{k=1}^{N} e^{a(w,t_{k-1})}\times\\
&\Big(e^{-\bar{\alpha}(0,t_k,T)}E\left[e^{b(w,t_{k-1})\cdot X_{t_{k-1}}+(-\bar{\beta}(0,t_k,T)1_r+1_y)\cdot X_{t_k}+z1_y\cdot X_T}\right]-\\
&e^{-\bar{\alpha}(0,t_{k-1},T)}E\left[e^{(b(w,t_{k-1})-\bar{\beta}(0,t_{k-1},T)1_r+1_y)\cdot X_{t_{k-1}}+z1_y\cdot X_T}\right]\Big)=\\
&=\sum_{k=1}^{N} e^{a(w,t_{k-1})}\times\\
&\big(\phi(b(w,t_{k-1}),-\bar{\beta}(0,t_k,T)1_r+1_y,z1_y,X_0,0,t_{k-1},t_k,T)-\\
&\phi(b(w,t_{k-1})-\bar{\beta}(0,t_{k-1},T)1_r+1_y,z1_y,X_0,0,t_{k-1},T)\big)=\\
&=v_2(w,z).
\end{aligned}
$$

The expectation

$$
E[H(w)\sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)]
$$

is obtained as above after interchanging w with z.

The last term is

$$E\left[\sum_{j=1}^{N}\sum_{k=1}^{N} e^{a(w,t_{j-1})+b(w,t_{j-1})\cdot X_{t_{j-1}}}\left(S_{t_j}M(t_j,T)-S_{t_{j-1}}M(t_{j-1},T)\right)\times\right.$$
$$\left. e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right]$$
$$=\sum_{j=1}^{N}\sum_{k=1}^{N} e^{a(w,t_{j-1})}e^{a(z,t_{k-1})}\times$$
$$E\left[e^{b(w,t_{j-1})\cdot X_{t_{j-1}}+b(z,t_{k-1})\cdot X_{t_{k-1}}}\times\right.$$
$$\left.\left(S_{t_j}M(t_j,T)-S_{t_{j-1}}M(t_{j-1},T)\right)\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)\right].$$

Expanding the products

$$\left(S_{t_j}M(t_j,T)-S_{t_{j-1}}M(t_{j-1},T)\right)\left(S_{t_k}M(t_k,T)-S_{t_{k-1}}M(t_{k-1},T)\right)$$

one gets $v_4(w,z)$. □

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