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ON THE GEOMETRIC STRUCTURE OF INDEPENDENCE SYSTEMS

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A bouquet of matroids is a combinatorial structure that generalizes the properties of matroids. Given an independence system $\mathcal{I}$, there exist several bouquets of matroids having the same family $\mathcal{I}$ of independent sets. We show that the collection of these geometries forms in general a meet semi-lattice and, in some cases, a lattice (for instance, when $\mathcal{I}$ is the family of the stable sets in a graph). Moreover, one of the bouquets that correspond to the highest elements in the meet semi-lattice provides the smallest decomposition of $\mathcal{I}$ into matroidal families, such that the rank functions of the different matroids have the same values for common sets. In the last section, we give sharp bounds on the performance of the greedy algorithm, using parameters of some special bouquets in the semi-lattice.

Key words: Independence system, bouquet of matroids, matroid, the greedy algorithm.

1. Independence systems and bouquets of matroids

An *independence system* $\mathcal{I}$ on X is a nonempty family of subsets of a ground set X having the following property:

$$S \subseteq T \in \mathcal{I} \Rightarrow S \in \mathcal{I}.$$

An independence system (IS, for short) is a *matroid* if it satisfies the following axiom:

$$S, T \in \mathcal{I}, \quad |T| = |S|+1 \Rightarrow \exists e \in T\backslash S \quad \text{such that } S \cup e \in \mathcal{I}.$$

A set belonging to the family $\mathcal{I}$ of an IS is called *independent*, otherwise it is *dependent* and minimally dependent sets are *circuits* of the IS. The *rank* of a set $A \subseteq X$ is the maximum cardinality of an independent subset of A, and the *rank function* $r(\cdot)$ of an IS on X is the set function associating to every subset of X its rank. A subset A of S is a *flat* (or *closed* set) if $r(A \cup x) > r(A)$ for all $x \in X - A$. The *closure operator* σ associates with subset $A \subseteq X$ the set: $\sigma(A) = \{x \in X: r(A \cup x) = r(A)\}$, and if $\mathcal{I}$ is a matroid, then $\sigma(A)$ is the smallest flat containing A.

A whole wealth of combinatorial optimization problems can be formulated as the problem of maximizing a set function over the family of independent sets of a particular IS: Consider for instance all the combinatorial packing problems on graphs, such as the spanning tree, matching, and vertex packing problems. Hence the study of structural properties of independence systems and matroids has been a subject of conspicuous research efforts, and our paper can be seen as a further attempt to study the relationships between independence systems and matroids. We assume a basic knowledge of definitions and properties of matroids and independence systems; however, our paper is self-contained, and we refer to [9] as a reference for the subjects treated here.

Let $\mathcal{I}$ be an independence system on X. There are two "dual" ways for interpreting $\mathcal{I}$:

(i) As an *intersection* of matroids. For instance, if $\mathcal{D}$ denotes the set of circuits of $\mathcal{I}$, i.e., the set of minimal dependent subsets of X, define for every $D \in \mathcal{D}$: $\mathcal{I}^D = \{I \subseteq X, D \not\subset I\}$. Then $\mathcal{I} = \bigcap_{D \in \mathcal{D}} \mathcal{I}^D$ holds clearly. Let p denote the minimum number of matroids whose intersection is equal to $\mathcal{I}$, then $p \leqslant |\mathcal{D}|$. However, this bound is far from being sharp.

(ii) As a *union* of matroids. For instance, if $\mathcal{B}$ denotes the set of bases of $\mathcal{I}$, define for every $B \in \mathcal{B}$: $\mathcal{I}_B = \{I \subseteq X, I \subseteq B\}$. Then $\mathcal{I} = \bigcup_{B \in \mathcal{B}} \mathcal{I}_B$ holds clearly. In [2], a Boolean procedure is proposed for determining the different maximal matroids contained in $\mathcal{I}$. However, there is in general no "compatibility" between the different matroids whose union gives $\mathcal{I}$. For instance, as the following example shows, the different rank functions defined in each matroid do not coincide on every subset of X common to the groundsets of the matroids.

Example. Let $X = \{1, 2, 3, 4, 5\}$ and $\mathcal{I}$ be the IS on X whose bases are: $\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}$. Then $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ where $\mathcal{I}_1$ is the IS with bases: $\{1, 2, 3\}, \{1, 2, 4\}$ and $\mathcal{I}_2$ is the IS with basis: $\{3, 4, 5\}$. $\mathcal{I}_1$ is a matroid on $X_1 = \{1, 2, 3, 4\}$ with rank function r_1 and $\mathcal{I}_2$ is a matroid on $X_2 = \{3, 4, 5\}$ with rank function r_2. Since $r_1(\{3, 4\}) = 1$ and $r_2(\{3, 4\}) = 2$, r_1 and r_2 do not coincide on $\{3, 4\} = X_1 \cap X_2$.

The concept of *bouquet of matroids* provides a particular union of matroids, called *squashed union*, in which the compatibility between the different matroids is preserved. So, for example, it will be possible to define in this structure a rank function which coincides in each matroid with its own rank function. Let us mention that bouquets of matroids are in fact a particular case of *$\mathcal{F}$-squashed geometries*, this latter concept having been introduced by Deza and Frankl in [6] (see also [5]). Consider a clutter $\mathcal{F}$; then $\mathcal{F}$-squashed geometries are a generalization of the matroidal structure in which the flats, in addition to satisfying some axioms similar to the matroidal axioms, have to be contained in some element of $\mathcal{F}$. Injection geometries (see [5]) and permutation geometries (see [3]) are also particular instances of $\mathcal{F}$-squashed geometries.

In the next paragraphs, we define bouquets of matroids and give their axiomatizations through the *flats*, the *independent sets*, the *rank function*, and the *circuits*. Equivalence between these different sets of axioms is proved for the sake of clarity and completeness. We refer to [8] for an extensive treatment of axiomatizations of squashed geometries and bouquets of matroids.

1.1. On matroid axioms

It is a well-known fact that a matroid can be equivalently defined through the axioms of its independent sets, circuits, rank function, closure operator and flats (or closed sets). Equivalence between the first four of them is proved in [9]. We could not find the axiomatization for the family of flats; hence we introduce it here and prove its equivalence with the axiomatization for the closure operator, since flat axioms for bouquets of matroids depend on this result and are extensively used in our treatment.

Closure axioms [9]. A function σ: $\mathscr{Q}^X \to \mathscr{Q}^X$ is the closure operator of a matroid on X if and only if for all $A, B \subseteq X$; $x, y \in X$:

(c1) $A \subseteq \sigma(A)$;

(c2) $A \subseteq B \to \sigma(A) \subseteq \sigma(B)$;

(c3) $\sigma(A) = \sigma(\sigma(A))$;

(c4) if $y \notin \sigma(A)$, $y \in \sigma(A \cup x)$, then $x \in \sigma(A \cup y)$.

Flat axioms. A family $\mathscr{G}$ of subsets of X is the family of flats of a matroid of rank s on X if and only if $\mathscr{G}$ can be partitioned into subfamilies: $\mathscr{G}^0, \mathscr{G}^1, \ldots, \mathscr{G}^s$ satisfying:

(f1) $F \cap F' \in \mathscr{G}$ for all $F, F' \in \mathscr{G}$;

(f2) if $F \in \mathscr{G}^i$, $F' \in \mathscr{G}^j$ and $F \subsetneq F'$ (i.e., F is properly contained in F'), then $i < j$;

(f3) if $F \in \mathscr{G}^i$ $(i < s)$ and $x \in X - F$, then there exists (a unique flat) $F' \in \mathscr{G}^{i+1}$ such that $F \cup x \subseteq F'$.

Remark 1.1. The set $\mathscr{G}^i$ is exactly the family of flats having rank i for $i \in [0, s]$. Axioms (f1), (f2) imply easily that $|\mathscr{G}^0| = 1$. In fact, the unique flat F_0 in $\mathscr{G}^0$ is the (possible empty) set of elements of rank 0. Also, axioms (f1), (f2) imply the uniqueness property of the flat F' satisfying (f3).

Remark 1.2. Given a flat $F \in \mathscr{G}$, a chain of length k is a sequence of flats: $F_0, F_1, \ldots, F_k = F$ such that $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$. It is easy to show that $F \in \mathscr{G}^i$ if and only if i is the length of any maximal chain of flats between F_0 and F.

Equivalence between the axioms for flats and closure operator is now proved as follows:

Suppose first that σ is the closure operator of a matroid $\mathscr{I}$ on X with rank function $r(\cdot)$. Define the families: $\mathscr{G} = \{F \subseteq X: \sigma(F) = F\}$ and $\mathscr{G}^i = \{F \in \mathscr{G}: r(F) = i\}$ for $i \in [0, s]$. Axiom (f1) can be easily deduced from (c1). Suppose $F \in \mathscr{G}^i$, $F' \in \mathscr{G}^j$ and $F \subsetneq F'$; take $x \in F' - F$, then $x \notin \sigma(F) = F$, which implies $r(F \cup x) > r(F)$ and therefore $i < j$, and (f2) is verified. If $F \in \mathscr{G}^i$ $(i < s)$ and $x \in X - F$, then $F' = \sigma(F \cup x)$ has rank $i+1$, hence it belongs to $\mathscr{G}^{i+1}$, and its uniqueness follows from (f1), (f2).

Suppose now that $\mathcal{G} = \mathcal{G}^0 \cup \cdots \cup \mathcal{G}^s$ satisfies (f1), (f2), (f3). For $A \subseteq X$, define: $\sigma(A) = \bigcap (F: F \in \mathcal{G}, F \supseteq A)$. Then, from (f1), $\sigma(A)$ is indeed the smallest flat of $\mathcal{G}$ containing A. Thus, (c1), (c2), (c3) are clearly satisfied. Let us verify (c4). Suppose $y \in \sigma(A \cup x)$, $y \notin \sigma(A)$ and $F = \sigma(A) \in \mathcal{G}^i$ $(i < s)$. Then, $\sigma(A \cup x)$, $\sigma(A \cup y)$ are two flats of $\mathcal{G}^{i+1}$ containing $F \cup y$ and, by the uniqueness property in (f3), we deduce that: $\sigma(A \cup x) = \sigma(A \cup y)$, which achieves the proof of (c4).

1.2. *Definition and axiomatizations of bouquets of matroids*

Let us first define a bouquet of matroids through its flats.

Axiomatization through flats. A family $\mathscr{g}$ of subsets of X is the set of *flats* of a bouquet of matroids on X if and only if there exists a clutter $X_1, \ldots, X_m$ of subsets of X (i.e., $X_i \not\subset X_j$ $\forall i \neq j \in [1, m]$) such that:

(F1) $\mathscr{g} \subseteq \bigcup_{i=1}^m \mathscr{2}^{X_i}$;

(F2) $\mathscr{g} \cap \mathscr{2}^{X_i}$ is the family of flats of a matroid on X_i for all $i \in [1, m]$;

(F3) $G \cap G' \in \mathscr{g}$ for all $G, G' \in \mathscr{g}$.

Define $\mathcal{G}_i = \mathscr{g} \cap \mathscr{2}^{X_i}$ for $i \in [1, m]$. Then, $\mathscr{g} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ is the *bunch* (or *bouquet*) of the matroids $\mathcal{G}_i$. Notice that $X_1, \ldots, X_m$ are indeed the maximal flats of $\mathscr{g}$. Thus, $\mathscr{g}$ is a matroid on X if and only if $m = 1$ and $X_1 = X$.

For all $i \in [1, m]$, let us denote by r_i, $\mathscr{I}_i$, $\mathscr{S}_i$, σ_i the rank function, the family of independent sets, the family of circuits (or stigmas), the closure operator, respectively, of the matroid $\mathcal{G}_i$ on X_i. Then, we are naturally led to define the rank function r, the family $\mathscr{I}$ of independent sets, the family $\mathscr{D}$ of circuits, the closure operator σ for the bouquet of matroids as follows:

- For any subset $A \in \bigcup_{i=1}^m \mathscr{2}^{X_i}$, if $A \subseteq X_i$ for some $i \in [1, m]$, then $r(A) = r_i(A)$ and $\sigma(A) = \sigma_i(A)$. Therefore, r, σ are defined only for subsets of $\bigcup_{i=1}^m \mathscr{2}^{X_i}$; however, they can be extended to $\mathscr{2}^X$ by setting: $r(A) = \infty$, $\sigma(A) = X \cup \{w\}$ (w being an arbitrary element that does not belong to X) for any subset $A \subseteq X$, $A \notin \bigcup_{i=1}^m \mathscr{2}^{X_i}$.
- The family of independent sets is: $\mathscr{I} = \mathscr{I}_1 \cup \cdots \cup \mathscr{I}_m$.
- The family of circuits is the family $\mathscr{D}$ of all minimal dependent sets, i.e., $D \notin \mathscr{I}$, but $D - x \in \mathscr{I}$ for all $x \in D$. Therefore, $\mathscr{D}$ can be partitioned into $\mathscr{D} = \mathscr{S} \cup \mathscr{C}$ where

$$\mathscr{S} = \mathscr{D} \cap \left(\bigcup_{i=1}^m \mathscr{2}^{X_i} \right) = \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_m, \quad \mathscr{S}_i = \mathscr{S} \cap \mathscr{2}^{X_i},$$

being thus the family of circuits for the matroid $\mathcal{G}_i$ on X_i; and $\mathscr{C} = \mathscr{D} - \mathscr{S} = \mathscr{D} - \bigcup_{i=1}^m \mathscr{2}^{X_i}$. Elements of $\mathscr{S}$ are called *stigmas* and elements of $\mathscr{C}$ are called *critical sets*.

We now give sets of axioms for characterizing the rank function, the family of independent sets, the family of circuits, the closure operator of a bouquet of matroids.

Axiomatization through the rank function. A set function r is the rank of a bouquet of matroids on X if and only if there exists a clutter $X_1, \ldots, X_m$ of subsets of X such that:

(R1) r is defined on $\bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$;

(R2) $r|_{\mathcal{Q}^{X_i}}$ is the rank function of a matroid on X_i for all $i \in [1, m]$;

(R3) $r((X_i \cap X_j) \cup x) = r(X_i \cap X_j) + 1$ for all $x \in X_i - X_j$ and $i, j \in [1, m]$.

Axiomatization through the independent sets. A family $\mathcal{I}$ of subsets of X is the family of independent sets of a bouquet of matroids on X if and only if there exists a clutter $X_1, \ldots, X_m$ of subsets of X such that:

(I1) $\mathcal{I} \subseteq \bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$;

(I2) $\mathcal{I} \cap \mathcal{Q}^{X_i}$ is the family of independent sets of a matroid on X_i for all $i \in [1, m]$;

(I3) If $I \in \mathcal{I} \cap \mathcal{Q}^{X_i} \cap \mathcal{Q}^{X_j}$ and $x \in X_i - X_j$, then $I \cup x \in \mathcal{I}$ for all $i \neq j \in [1, m]$.

Remark 1.3. Any independence system $\mathcal{I}$ is indeed the family of independent sets of a bouquet of matroids: Choose for $X_1, \ldots, X_m$ the bases (i.e., maximal independent sets) of $\mathcal{I}$; then its family $\mathcal{C}$ of critical sets is empty.

Remark 1.4. If $\mathcal{I}$ is a bouquet of matroids, i.e., satisfies (I1), (I2), (I3), then $\mathcal{I}$ is clearly an independence system and we recall that its rank function is defined by: $r(A) = \text{Max}(|I|: I \subseteq A, I \in \mathcal{I})$ for all $A \subseteq E$. Then r and the rank function for the bouquet of matroids coincide on any subset belonging to $\bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$.

Axiomatization through the circuits. A family $\mathcal{D}$ of subsets of X is the family of circuits of a bouquet of matroids on X if and only if $\mathcal{D}$ can be partitioned into two subfamilies $\mathcal{S}$, $\mathcal{C}$ satisfying:

(D1) $D \not\subset D'$ for all $D \neq D' \in \mathcal{D}$;

(D2) $\forall S \neq S' \in \mathcal{S}, \forall x \in S \cap S'$, there exists $D' \in \mathcal{D}$ such that $D' \subseteq S \cup S' - x$;

(D3) $\forall S \in \mathcal{S}, \forall C \in \mathcal{C}, \forall x \in S \cap C$, there exists $C' \in \mathcal{C}$ such that $C' \subseteq S \cup C - x$.

Remark 1.5. (D2) implies clearly that $\mathcal{S} \cap \mathcal{Q}^{X_i}$ is a matroidal family of circuits. Therefore, the following version of (D2) is also satisfied (see [9]):

(D′2) $\forall S \neq S' \in \mathcal{S}, \forall x \in S \cap S', \forall y \in S - S'$, if $S \cup S' \notin \mathcal{C}^*$ then there exists $S'' \in \mathcal{S}$ such that $y \in S''$ and $S'' \subseteq S \cup S' - x$, where

$$\mathcal{C}^* = \{A \subseteq X : \exists C \in \mathcal{C}, C \subseteq A\}.$$

Axiomatization through the closure operator. A set function σ is the closure operator of a bouquet of matroids on X if and only if there exists a clutter $X_1, \ldots, X_m$ of subsets of X such that:

(C1) σ is defined on $\bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$;

(C2) $\sigma|_{\mathcal{Q}^{X_i}}$ is the closure operator of a matroid on X_i, for all $i \in [1, m]$.

Remark 1.6. The conditions (F3), (R3), (I3), (D3) ensure the compatibility between the different matroids $\mathscr{G}_i$ composing the bouquet.

We now show that there is equivalence between the different axiomatizations for bouquets of matroids by proving the equivalence between the following combinatorial structures:

(i) a family $\mathscr{g}$ satisfying (F1), (F2), (F3);
(ii) a set function r satisfying (R1), (R2), (R3);
(iii) a family $\mathscr{I}$ satisfying (I1), (I2), (I3);
(iv) a family $\mathscr{D}$ satisfying (D1), (D2), (D3);
(v) a set function σ satisfying (C1), (C2).

We show the following implications: (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i).

Proof of (i)$\Rightarrow$(ii)
Suppose $\mathscr{g}$ satisfies (F1), (F2), (F3). Let r_i be the rank function of the matroid $g \cap 2^{X_i}$ on X_i, for $i \in [1, m]$.

Lemma A. *If $A \subseteq X_i \cap X_j$ for $i \neq j \in [1, m]$, then $r_i(A) = r_j(A)$.*

Proof. We first show that the lemma holds for all $A \in \mathscr{g}$. Take $k = r_i(A)$; by Remark 1.2, there exists a chain of flats of $\mathscr{g} \cap 2^{X_i}$: $F_0, F_1, \ldots, F_k = A$ such that $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$; however, $F_0, \ldots, F_k$ also form a chain of flats of $\mathscr{g} \cap 2^{X_j}$, which implies $r_i(A) = k \leq r_j(A)$ and by the same argument, $r_j(A) \leq r_i(A)$; thus, $r_i(A) = r_j(A)$. $\square$

Hence, it is legitimate to define a rank function r on $\bigcup_{i=1}^{m} 2^{X_i}$ by: $r(A) = r_i(A)$ if $A \subseteq X_i$. Thus, (R1), (R2) are clearly satisfied and (R3) follows from the fact that X_i, X_j are flats of M_i, M_j, hence, by property (F3), $X_i \cap X_j$ is also a flat.

Proof of (ii)$\Rightarrow$(iii)
Suppose r is a rank function satisfying (R1), (R2), (R3). Define the families:

$$\mathscr{I} = \left\{ I \in \bigcup_{i=1}^{m} 2^{X_i} : r(I) = |I| \right\}, \qquad \mathscr{I}_i = \mathscr{I} \cap 2^{X_i} \quad \text{for } i \in [1, m].$$

Then, (I1) is trivially satisfied and (I2) follows from (R2). Let us prove (I3): Suppose $I \in \mathscr{I}$, $I \subseteq X_i \cap X_j$ and $x \in X_i - X_j$. Choose $I_0 \in \mathscr{I}$ such that: $I \subseteq I_0 \subseteq X_i \cap X_j$ and $|I_0| = r(X_i \cap X_j)$ (which is possible in the matroid $\mathscr{I}_i$ or $\mathscr{I}_j$). We deduce from (R3) that $I_0 \cup x \in \mathscr{I}_i$ and therefore $I \cup x \in \mathscr{I}$.

Proof of (iii)$\Rightarrow$(iv)
Suppose $\mathscr{I}$ satisfies (I1), (I2), (I3); thus, $\mathscr{I}$ is an independence system. Let $\mathscr{D}$ be its family of circuits (i.e., of minimal dependent sets); then $\mathscr{D}$ satisfies clearly (D1). Define: $\mathscr{S}_i = \mathscr{D} \cap 2^{X_i}$ for $i \in [1, m]$, $\mathscr{S} = \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_m$ and $\mathscr{C} = \mathscr{D} - \mathscr{S}$. We now prove in Lemmas C and D that the collection $\mathscr{S}$, $\mathscr{C}$ satisfies (D2), (D3).

Lemma B. *For any subset A of X, the following two statements are equivalent*:
(a) $A \notin \bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$;
(b) *There exists $C \in \mathcal{C}$ such that $C \subseteq A$.*

Proof. (b)$\to$(a) follows from the definition of $\mathcal{C}$. To prove (a)$\to$(b), take $A \notin \bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$. Let C be a minimal subset of A not belonging to $\bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$. In order to prove that $C \in \mathcal{C}$, we have to verify that $C - x \in \mathcal{I}$ for all $x \in C$. Suppose on the contrary that $C - x \notin \mathcal{I}$ for some $x \in C$. Let $I \in \mathcal{I}$ be a maximal independent subset of $C - x$ and $y \in C - I - x$. By construction of $\mathcal{C}$, there exists $i, j \in [1, m]$ such that $C - x \subseteq X_i$ and $C - y \subseteq X_j$. Apply (I3) to the independent set $I \subseteq X_i \cap X_j$ and the point $y \in X_i - X_j$ and deduce that $I \cup y \in \mathcal{I}$, which contradicts the maximality of I. □

Lemma C. *If $S \neq S' \in \mathcal{S}$, $x \in S \cap S'$, then there exists $D \in \mathcal{D}$ such that $D \subseteq S \cup S' - x$.*

Proof. Lemma C is clearly satisfied when $S \cup S' \subseteq X_i$ for some $i \in [1, m]$, since $\mathcal{S} \cap \mathcal{Q}^{X_i}$ is the family of circuits of the matroid on X_i. Thus, we can assume that $S \cup S' \not\subseteq X_i$ for all $i \in [1, m]$. Suppose on the contrary that $S \cup S' - x \in \mathcal{I}$. Take $i \in [1, m]$ such that $S \cup S' - x \subseteq X_i$ and $j \in [1, m]$ such that $S \subseteq X_j$. Apply (I3) to the independent set $S - x \subseteq X_i \cap X_j$ and the point $x \in X_j - X_i$ and deduce that $S \in \mathcal{I}$, yielding a contradiction. Hence, $S \cup S' - x \notin \mathcal{I}$ and therefore contains a circuit of $\mathcal{D}$. □

Lemma D. *If $S \in \mathcal{S}$, $C \in \mathcal{C}$, $x \in S \cap C$, then there exists $C' \in \mathcal{C}$ such that $C' \subseteq S \cup C - x$.*

Proof. Suppose on the contrary that $S \cup C - x \subseteq X_i$ for some $i \in [1, m]$. Take $j \in [1, m]$ such that $S \subseteq X_j$. Apply (I3) to the independent set $S - x \subseteq X_i \cap X_j$ and the point $x \in X_j - X_i$ and deduce again $S \in \mathcal{I}$, which is impossible. Therefore, $S \cup C - x \notin \bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$ and, by Lemma B, contains an element of $\mathcal{C}$. □

Proof of $(iv) \Rightarrow (v)$
Suppose $\mathcal{D}, \mathcal{S}, \mathcal{C}$ satisfy (D1), (D2), (D3). Let $X_1, \ldots, X_m$ be the maximal subsets of X that do not contain any element of $\mathcal{C}$. Then, by (D2), $\mathcal{S}_i = \mathcal{S} \cap \mathcal{Q}^{X_i}$ is a matroidal family of circuits; therefore, we can define the corresponding matroidal closure operator σ_i on $\mathcal{Q}^{X_i}$ by: for $A \subseteq X_i$,

$$\sigma_i(A) = A \cup \{x \in X_i - A \colon \exists S \in \mathcal{S}_i, x \in S \subseteq A \cup x\}.$$

Lemma E. *If $A \subseteq X_i \cap X_j$, then $\sigma_i(A) = \sigma_j(A)$.*

Proof. Suppose for contradiction that there exists $x \in \sigma_i(A) - \sigma_j(A)$. Then, there exists $S \in \mathcal{S}$ such that $x \in S \subseteq A \cup x$ and, moreover, $x \in X_i - X_j$. Let $C \in \mathcal{C}$ such that $x \in C \subseteq X_j \cup x$. Apply (D3) to $S \in \mathcal{S}$, $C \in \mathcal{C}$, $x \in S \cap C$ for obtaining the existence of $C' \in \mathcal{C}$ such that $C' \subseteq S \cup C - x \subseteq X_j$ yielding a contradiction. □

It is now legitimate to define the following operator σ on $\bigcup_{i=1}^{m} \mathcal{Q}^{X_i}$: if $A \subseteq X_i$, then $\sigma(A) = \sigma_i(A)$. Then, (C1), (C2) are trivially satisfied.

Proof of $(v) \Rightarrow (i)$
Suppose σ satisfies (C1), (C2) and define:

$$\mathcal{g} = \left\{ A \in \bigcup_{i=1}^{m} \mathcal{Q}^{X_i}: \sigma(A) = A \right\}.$$

Then, (F2) follows from (C2). Let us verify (F3). If $G, G' \in \mathcal{g}$, then $\sigma(G \cap G') \subseteq \sigma(G) \cap \sigma(G') = G \cap G'$, which yields therefore the equality: $\sigma(G \cap G') = G \cap G'$ and thus $G \cap G' \in \mathcal{g}$.

Given any IS $\mathcal{I}$ on X, there may exist several bouquets of matroids having $\mathcal{I}$ as IS. For instance, consider the IS $\mathcal{I}$ on $X = \{1, 2, 3, 4\}$ whose bases are: $\{1, 2\}$; $\{1, 3\}$; $\{2, 3\}$ and $\{1, 4\}$. Then $\mathcal{I} = \mathcal{I}_{\{1,2\}} \veebar \mathcal{I}_{\{1,3\}} \veebar \mathcal{I}_{\{2,3\}} \veebar \mathcal{I}_{\{1,4\}}$ and also: $\mathcal{I} = \mathcal{I}_{\{\{1,2\},\{1,3\},\{2,3\}\}} \veebar \mathcal{I}_{\{1,4\}}$ define two distinct bouquets of matroids with the same IS $\mathcal{I}$ (recall that $\mathcal{I}_{\{1,2\}}$ denotes the IS with base {1,2}).

Let us denote by $\mathcal{L}$ the set of all bouquets of matroids $\mathcal{g}$ having $\mathcal{I}$ as IS. If m denotes the number of maximal flats in $\mathcal{g}$, we are interested in finding some element $\mathcal{g}$ of $\mathcal{L}$ providing minimum value for this parameter m, since this particular bouquet of matroids will often be used in our treatment. In the next section, an extensive study of the structure of $\mathcal{L}$ is made which provides the minimum value of m.

2. The meet semi-lattice $\mathcal{L}$

Let $\mathcal{I}$ be an IS on X, $\mathcal{D}$ its set of circuits and $\mathcal{L}$ the set of all bouquets of matroids having $\mathcal{I}$ as independence system. Any element $\mathcal{g}$ of $\mathcal{L}$ is characterized by the partition of $\mathcal{D}$ into $\mathcal{S} \cup \mathcal{C}$ and therefore is denoted by $\mathcal{g}(\mathcal{S}, \mathcal{C})$ (or simply by $\mathcal{G}(\mathcal{S})$), $\mathcal{S}$ its set of stigmas, $\mathcal{C}$ its set of critical subsets of X and the set $(\mathcal{S}, \mathcal{C})$ must satisfy axioms (D2) and (D3). For two bouquets of matroids $\mathcal{g}_1, \mathcal{g}_2$ of $\mathcal{L}$, notice that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ is equivalent to $\mathcal{C}_2 \subseteq \mathcal{C}_1$ since $\mathcal{S}_1 \cup \mathcal{C}_1 = \mathcal{S}_2 \cup \mathcal{C}_2 = \mathcal{D}$. Let us introduce an *order* on $\mathcal{L}$ as follows:

$$\mathcal{g}_1(\mathcal{S}_1, \mathcal{C}_1) \leq \mathcal{g}_2(\mathcal{S}_2, \mathcal{C}_2) \text{ if and only if } \mathcal{S}_1 \subseteq \mathcal{S}_2.$$

Proposition 2.1. *$\mathcal{L}$ is a meet semi-lattice, that is, any two elements $\mathcal{g}_1, \mathcal{g}_2$ of $\mathcal{L}$ have a meet $\mathcal{g}_1 \wedge \mathcal{g}_2$ which is defined by: $\mathcal{g}_1 \wedge \mathcal{g}_2 = \mathcal{g}(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{C}_1 \cup \mathcal{C}_2)$. Moreover, the least element of $\mathcal{L}$ is $\mathcal{g}(\emptyset, \mathcal{D})$.*

Proof. Define $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. It is easy to verify that $\mathcal{S}$, $\mathcal{C}$ satisfy axioms (D2) and (D3). Furthermore, $\mathcal{g}(\mathcal{S}, \mathcal{C})$ is clearly the meet of $\mathcal{g}_1, \mathcal{g}_2$. It is also clear that $\mathcal{g}(\emptyset, \mathcal{D})$ belongs to $\mathcal{L}$ and is smaller than every element of $\mathcal{G}$. □

We now wish to investigate whether $\mathscr{L}$ is a lattice, i.e., any two elements $\mathcal{g}_1, \mathcal{g}_2$ of $\mathscr{L}$ have a join $\mathcal{g}_1 \vee \mathcal{g}_2$ in $\mathscr{L}$. Let us first make the following observation.

Proposition 2.2. *The following two statements are equivalent*:

(a) *$\mathscr{L}$ is a lattice*;

(b) *$\mathscr{L}$ has a greatest element.*

Proof. The implication (a)→(b) is trivially satisfied. Suppose now that $\mathscr{L}$ has a greatest element $\mathcal{g}(\mathscr{S}_0, \mathscr{C}_0)$. Let $\mathcal{g}_1(\mathscr{S}_1, \mathscr{C}_1)$ and $\mathcal{g}_2(\mathscr{S}_2, \mathscr{C}_2)$ be two distinct elements of $\mathscr{L}$. Let $\mathcal{g}(\mathscr{S}, \mathscr{C})$ be a minimal element of $\mathscr{L}$ such that $\mathcal{g} \geqslant \mathcal{g}_1$ and $\mathcal{g} \geqslant \mathcal{g}_2$. It is enough to verify that there is uniqueness of such a minimal element since then it will be the upper bound $\mathcal{g}_1 \vee \mathcal{g}_2$ of $\mathcal{g}_1, \mathcal{g}_2$. Suppose on the contrary that $\mathcal{g}'(\mathscr{S}', \mathscr{C}')$ is another minimal element of $\mathscr{L}$ such that $\mathcal{g}' \geqslant \mathcal{g}_1$ and $\mathcal{g}' \geqslant \mathcal{g}_2$. Therefore, $\mathcal{g} \wedge \mathcal{g}'$ is also an element of $\mathscr{L}$ such that: $\mathcal{g} \wedge \mathcal{g}' \geqslant \mathcal{g}_1$ and $\mathcal{g} \wedge \mathcal{g}' \geqslant \mathcal{g}_2$. Hence, by minimality of $\mathcal{g}, \mathcal{g}'$, we deduce that $\mathcal{g} \wedge \mathcal{g}' = \mathcal{g} = \mathcal{g}'$, yielding a contradiction. □

Let us now introduce the following family:

$$\bar{\mathscr{C}} = \{D \in \mathscr{D}\colon \exists D' \in \mathscr{D}, D' \neq D, \exists x \in D \cap D' \text{ with } D \cup D' - x \in \mathscr{I}\}.$$

It is easy to see, by using axioms (D2), (D3), that $\bar{\mathscr{C}} \subseteq \mathscr{C}$, i.e., $\mathscr{S} \subseteq \bar{\mathscr{S}} = \mathscr{D} - \bar{\mathscr{C}}$ hold for all bouquets $\mathcal{g}(\mathscr{S}, \mathscr{C})$ in $\mathscr{L}$. Therefore, if $\mathcal{g}(\bar{\mathscr{S}}, \bar{\mathscr{C}}) = \bar{\mathcal{g}}$ is a bouquet of matroi , then it is in fact the greatest element of $\mathscr{L}$ and $\mathscr{L}$ is indeed a lattice.

By construction, the family $\bar{\mathscr{S}}$ satisfies the following property:

(D0) $\quad \forall S \in \bar{\mathscr{S}}, \forall D \in \mathscr{D}, \forall x \in S \cap D, \exists D' \in \mathscr{D}, D' \subseteq S \cup D - x.$

Hence, the collection $\bar{\mathscr{S}}, \bar{\mathscr{C}}$ satisfies axiom (D2) but, in general, axiom (3) is not verified.

Let us now describe the atoms of the semi-lattice $\mathscr{L}$. Recall that ar element $\mathcal{g}$ of $\mathscr{L}$ is an *atom* if and only if, for all $g' \in \mathscr{L}$ distinct from the least elen nt of $\mathscr{L}$ such that $g' \subseteq \mathcal{g}$, we have $\mathcal{g}' = \mathcal{g}$.

Proposition 2.3. (a) *The atoms of $\mathscr{L}$ are the bouquets of matroid $\mathcal{g}(\{S\}, \mathscr{D} - \{S\})$ for all $S \in \bar{\mathscr{S}}$.*

(b) *$\mathscr{L}$ is atomic, that is, every element $\mathcal{g}(\mathscr{S}, \mathscr{C})$ of $\mathscr{L}$ with $\mathscr{S} \neq \emptyset$ s the join of atoms of $\mathscr{L}$; more precisely, we have*:

$$\mathcal{g}(\mathscr{S}, \mathscr{C}) = \bigvee_{S \in \mathscr{S}} \mathcal{g}(\{S\}, \mathscr{D} - \{S\}).$$

Proof. The proof of (a) is easy. Let us now verify (b). T e $\mathcal{g}(\mathscr{S}, \mathscr{C})$ in $\mathscr{L}$; then $\mathcal{g}(\mathscr{S}, \mathscr{C}) \geqslant \mathcal{g}(\{S\}, \mathscr{D} - \{S\})$ for all $S \in \mathscr{S}$ and, thus,

$$\mathcal{g}(\mathscr{S}, \mathscr{C}) \geqslant \bigvee_{S \in \mathscr{S}} \mathcal{g}(\{S\}, \mathscr{D} - \{S\}).$$

Define the element of $\mathscr{L}$:

$$\mathcal{g}(\mathscr{S}', \mathscr{C}') = \bigvee_{S \in \mathscr{S}} \mathcal{g}(\{S\}, \mathscr{D} - \{S\}).$$

Then, we have $\mathcal{S}' \supseteq \mathcal{S}$ which implies the inequality: $g(\mathcal{S}', \mathcal{C}') \geq g(\mathcal{S}, \mathcal{C})$ and, therefore, equality:

$$g(\mathcal{S}, \mathcal{C}) = g(\mathcal{S}', \mathcal{C}') = \bigvee_{S \in \mathcal{S}} g(\{S\}, \mathcal{D} - \{S\})$$

holds. □

We now give a class of independence systems for which $\mathcal{L}$ is a lattice.

Theorem 2.4. *Suppose $\mathcal{I}$ is the family of the stable sets of a graph $G = (V, E)$ with V as set of vertices and E as set of edges. Then, $\mathcal{L}$ is a lattice whose least element is $g(\emptyset, \mathcal{D})$ and whose greatest element is $g(\bar{\mathcal{S}}, \bar{\mathcal{C}})$.*

An example of construction of such a lattice is given in the next paragraph.

Notice that, in this case, an edge e belongs to $\bar{\mathcal{S}}$ if and only if, for each edge e' adjacent to e, there exists an edge e'' adjacent to e and e'; that is, there exists no maximal clique of G containing a unique endnode of e (Figure 1).

Proof of Theorem 2.4. It is enough to prove that $g(\bar{\mathcal{S}}, \bar{\mathcal{C}})$ is a bouquet of matroids, i.e., satisfies (D2), (D3). (D2) being trivially verified, we show that (D3) holds. Suppose by contradiction that there exists $S \in \bar{\mathcal{S}}$, $C \subset \bar{\mathcal{C}}$, $x \in S \cap C$ such that $S \cup C - x$ contains no element of $\bar{\mathcal{C}}$. Let us denote S by $\{x, y\}$, C by $\{x, z\}$; then, by (D0), $S' = \{y, z\}$ is an edge of $\bar{\mathcal{S}}$. Since $C \in \bar{\mathcal{C}}$, there exists $C' = \{u, v\} \in \bar{\mathcal{C}}$ such that $u \in C \cap C'$ and $C \cup C' - u \in \mathcal{I}$.

Let us first suppose that $u = x$. Apply (D0) to $S, C', x \in S \cap C'$ for obtaining that $\{y, v\} \in \mathcal{D}$. Apply (D0) to $S', \{y, v\}, y \in S' \cap \{y, v\}$ for obtaining that $\{z, v\} \in \mathcal{D}$, which contradicts the assumption: $C \cup C' - u = \{v, z\} \in \mathcal{I}$ (Figure 2).

We now suppose that $u = z$. Apply (D0) to $S', C', z \in S' \cap C'$ for obtaining that $\{y, v\} \in \mathcal{D}$. Apply (D0) to $S, \{y, v\}, y \in S \cap \{y, v\}$ for obtaining that $\{x, v\} \in \mathcal{D}$, which contradicts the assumption: $C \cup C' - z = \{x, v\} \in \mathcal{I}$ (Figure 3). □

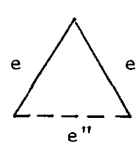

Fig. 1

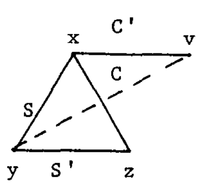

Fig. 2

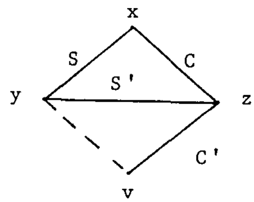

Fig. 3

In the general case when $\mathcal{I}$ is an arbitrary IS, we have the following result.

Proposition 2.5. *$\mathcal{L}$ is a lattice if and only if $\bar{g} \in \mathcal{L}$.*

Proof. If $\bar{g} \in \mathcal{L}$, then it is the greatest element of $\mathcal{L}$ and thus $\mathcal{L}$ is a lattice. Suppose now that $\mathcal{L}$ is a lattice with $g(\mathcal{S}_0, \mathcal{C}_0)$ as greatest element. If $\bar{g} \notin \mathcal{L}$, then we have: $\mathcal{S}_0 \subsetneq \bar{\mathcal{S}}$. Choose $S \in \bar{\mathcal{S}} - \mathcal{S}_0$. Then, by Proposition 2.3(a), $g(\{S\}, \mathcal{D} - \{S\}) \in \mathcal{L}$, thus $g(\{S\}, \mathcal{D} - \{S\}) \leqslant g(\mathcal{S}_0, \mathcal{C}_0)$, which implies $S \in \mathcal{S}_0$, yielding a contradiction. □

Remark 2.6. Though $\bar{g}$ is not, in general, a bouquet of matroids, we can derive from $\bar{\mathcal{S}}$, $\bar{\mathcal{C}}$ a decomposition of $\mathcal{I}$ into a union (not squashed, in general) of matroids. More precisely, let $Z_1, \ldots, Z_{\bar{m}}$ be the maximal subsets of X that do not contain any element of $\bar{\mathcal{C}}$. Then, in view of (D0), $\bar{\mathcal{S}} \cap 2^{Z_i}$ is a matroidal family of circuits defining the matroidal IS $\mathcal{I} \cap 2^{Z_i}$; therefore, we have: $\mathcal{I} = \bigcup_{i=1}^{\bar{m}} \mathcal{I} \cap 2^{Z_i}$ as a union of matroids. Also, even when $\bar{g} \notin \mathcal{L}$, $\bar{\mathcal{S}}$, $\bar{\mathcal{C}}$ will be used in the last section for obtaining sharp bounds on the performance of the greedy algorithm in the IS $\mathcal{I}$.

We now give an example of an IS $\mathcal{I}$ for which $\bar{g}$ is not a bouquet of matroids.

Example. Let $\mathcal{I}$ be the IS on $X = \{1, 2, 3, 4, 5\}$ whose set of circuits is: $\mathcal{D} = \{123, 124, 134, 234, 145\}$ ($\{1, 2, 3\}$ is denoted by 123, for short). It is easy to see that:

$$\bar{\mathcal{C}} = \{124, 145, 134\} \quad \text{and} \quad \bar{\mathcal{S}} = \{123, 234\}.$$

However, $\bar{g}$ is not a bouquet of matroids since axiom (D3) is violated (see that $123 \in \bar{\mathcal{S}}$, $124 \in \bar{\mathcal{C}}$ and $123 \cup 124 - 1 = 234$ does not contain any element of $\bar{\mathcal{C}}$). Hence, the meet semi-lattice $\mathcal{L}$ is reduced to $g(\emptyset)$, $g(123)$, $g(234)$.

This example shows also that Theorem 2.3 cannot be extended to the case when all circuits have the same size greater than 2.

2.1. *On the number of matroids which compose a bouquet of matroids*

For any bouquet of matroids g of $\mathcal{L}$, let m denote the number of its maximal flats; hence m is also the number of distinct matroids in g whose union gives $\mathcal{I}$. Our aim is to find some g for which m is minimum. We are going to see that such minimum value m is provided by some of the maximal elements of $\mathcal{L}$.

Proposition 2.7. *Let $\mathcal{g}_1, \mathcal{g}_2$ be two bouquets of matroids of $\mathcal{L}$ whose respective numbers of maximal flats are m_1, m_2. If $\mathcal{g}_1 \leqslant \mathcal{g}_2$, then $m_2 \leqslant m_1$.*

Proof. Let σ_i denote the closure operator in $\mathcal{g}_i$, $i=1,2$. Recall that we have only defined $\sigma_i(A)$ for all subsets A of X not belonging to $\mathscr{C}_i^*$, that is, which do not contain any critical subset of $\mathscr{C}_i$, for $i=1,2$.

We define the mapping $\theta: \mathcal{g}_1 \to \mathcal{g}_2$,

$$F \mapsto \theta(F) = \sigma_2(F).$$

θ is well defined because no flat of $\mathcal{g}_1$ contains a critical subset of $\mathscr{C}_2$ since $\mathscr{C}_2 \subseteq \mathscr{C}_1$.

In Claim 1, we prove that θ is a surjective mapping from $\mathcal{g}_1$ onto $\mathcal{g}_2$. Then we use this result for showing in Claim 2 that θ induces a surjective mapping from $(\mathcal{g}_1)_{\max}$ onto $(\mathcal{g}_2)_{\max}$ from which we infer clearly that $m_1 = |(\mathcal{g}_1)_{\max}| \geqslant m_2 = |(\mathcal{g}_2)_{\max}|$.

Claim 1. θ is a surjective mapping from $\mathcal{g}_1$ onto $\mathcal{g}_2$.

Proof. We prove the following statement:

$$\text{For every } I \in \mathcal{I}, \quad \sigma_2(I) = \sigma_2(\sigma_1(I)),$$

which yields easily Claim 1, since if G is any flat of $\mathcal{g}_2$ and I is a basis of G, then we have: $G = \sigma_2(I) = \sigma_2(\sigma_1(I)) = \theta(\sigma_1(I))$ with $\sigma_1(I) \in \mathcal{g}_1$. Let I be an independent subset of $\mathcal{I}$. It is enough to show that $\sigma_2(\sigma_1(I)) \subseteq \sigma_2(I)$. Take $x \in \sigma_2(\sigma_1(I))$. Then there exists $S_2 \in \mathcal{S}_2$ such that $x \in S_2 \subseteq \sigma_1(I) \cup x$. Let $a_1, \ldots, a_p$ be the distinct elements of $\sigma_1(I) \backslash I \cup x$ that belong to S_2, so $S_2 \subseteq I \cup x \cup \{a_1, \ldots, a_p\}$. Consider a_p: since $a_p \in \sigma_1(I) \backslash I$, there exists $S_1 \in \mathcal{S}_1$ such that $a_p \in S_1 \subseteq I \cup a_p$.

Suppose first that $S_1 \cup S_2 \in \mathscr{C}_2^*$. Hence there exists $C \in \mathscr{C}_2$ which is contained in $S_1 \cup S_2$. Since $\mathscr{C}_2 \subseteq \mathscr{C}_1$, $C \in \mathscr{C}_1$ and thus $x \in C$. Consider now $S_2 \in \mathcal{S}_2$, $C \in \mathscr{C}_2$ with $x \in S_2 \cap C$. Thus there exists $C' \in \mathscr{C}_2$ such that $C' \subseteq S_2 \cup C \backslash x \subseteq \sigma_1(I)$, which contradicts $\sigma_1(I) \notin \mathscr{C}_1^*$.

Suppose now that $S_1 \cup S_2 \notin \mathscr{C}_2^*$. We can apply axiom (D'2) to $S_1 \in \mathcal{S}_2$, $S_2 \in \mathcal{S}_2$ with $x \in S_2 \backslash S_1$ and $a_p \in S_1 \cap S_2$. Hence there exists $S_2' \in \mathcal{S}_2$ such that $x \in S_2' \subseteq S_1 \cup S_2 \backslash a_p$. Therefore we have obtained a stigma $S_2' \in \mathcal{S}_2$ satisfying: $x \in S_2' \subseteq I \cup x \cup \{a_1, \ldots, a_{p-1}\}$. Thus we succeeded in deleting one element a_p of $\{a_1, \ldots, a_p\}$. We can repeat the same operation until getting the existence of a stigma $S' \in \mathcal{S}_2$ satisfying: $x \in S' \subseteq I \cup x$, which proves, therefore, that $x \in \sigma_2(I)$.

Claim 2. θ induces a surjective mapping from $(\mathcal{g}_1)_{\max}$ onto $(\mathcal{g}_2)_{\max}$.

Proof. We first show that, if $F \in (\mathcal{g}_1)_{\max}$, then $\theta(F) \in (\mathcal{g}_2)_{\max}$. Let I be a basis of F, hence $F = \sigma_1(I)$ and therefore $\theta(F) = \sigma_2(F) = \sigma_2(I)$. Suppose on the contrary that $\theta(F)$ is not a maximal flat of $\mathcal{g}_2$, thus there exists $G \in \mathcal{G}_2$ such that $G \supsetneq \theta(F)$. Choose an element $x \in X$ in $G \backslash \sigma_2(I)$. Then $I \cup x \in \mathcal{I}$, otherwise $I \cup x$ would contain some circuit $D \in \mathcal{D}$. Either $D \in \mathcal{S}_2$, which contradicts $x \notin \sigma_2(I)$, or $D \in \mathscr{C}_2$, which contradicts $G \notin \mathscr{C}_2^*$. Define $F_1 = \sigma_1(I \cup x)$, hence F_1 is a flat of $\mathcal{g}_1$ such that $F_1 \supsetneq F$, which contradicts the maximality of F.

We now verify that, for every $G \in (\mathcal{g}_2)_{\max}$, there exists $F \in (\mathcal{g}_1)_{\max}$ such that $G = \theta(F)$. Let $G \in (\mathcal{g}_2)_{\max}$, I be a basis of G and $F = \sigma_1(I)$. Thus $G = \theta(F)$. If $F \in (\mathcal{g}_1)_{\max}$, then our statement is proved. Otherwise let $F_1 \in (\mathcal{g}_1)_{\max}$ containing F. Therefore $G = \sigma_2(F) \subseteq \sigma_2(F_1)$, which implies, by maximality of G, that: $G = \sigma_2(F_1) = \theta(F_1)$. $\square$

Proposition 2.8. *The decomposition of $\mathcal{I}$ into a squashed union of matroids with minimal number m of matroids is provided by one of the maximal elements of $\mathcal{L}$.*

Proof. It follows clearly from Proposition 2.7. $\square$

Remark 2.9. For any bouquet of matroids $\mathcal{g}(\mathcal{S}, \mathcal{C})$ of $\mathcal{L}$, define the new IS $\mathcal{I}(\mathcal{C})$ whose set of circuits is $\mathcal{C}$. It is easy to see that the bases of $\mathcal{I}(\mathcal{C})$ are exactly the maximal flats of $\mathcal{g}(\mathcal{S}, \mathcal{C})$. Hence, the number of matroids composing $\mathcal{g}(\mathcal{S}, \mathcal{C})$ is equal to the number of bases of $\mathcal{I}(\mathcal{C})$.

Let us give an *example of construction of $\mathcal{L}$ when $\mathcal{L}$ is a lattice.* Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{I}$ be the IS on X whose set of circuits is: $\mathcal{D} = \{12, 13, 23, 45, 46, 56, 57\}$ (we write 12 instead of $\{1, 2\}$ for sake of brevity).

It is easy to see that $\bar{\mathcal{C}} = \{45, 56, 57\}$ and that the IS $\mathcal{I}(\bar{\mathcal{C}})$ has the bases: 1235, 123467. Hence the minimum number of matroids whose squashed union gives $\mathcal{I}$ is $m = 2$ (since $\bar{\mathcal{g}} = \mathcal{g}(\bar{\mathcal{S}}, \bar{\mathcal{C}}) \in \mathcal{L}$ by Theorem 2.4).

Any element $\mathcal{g} \in \mathcal{L}$ is characterized for instance by its family $\mathcal{S}$ of stigmas and therefore denoted by $\mathcal{g}(\mathcal{S})$. The lattice $\mathcal{L}$ has the configuration as shown in Figure 4. Every element of $\mathcal{L}$ provides a different decomposition of $\mathcal{I}$ into a squashed union of matroids. Let us first list the bases of $\mathcal{I}$: 147, 167, 247, 267, 347, 367, 15, 25, 35. For instance, the bouquet $\mathcal{g}(12, 46)$ provides a decomposition of $\mathcal{I}$ into the union of four matroids, more precisely $\mathcal{I} = \{347, 367\} \uplus \{147, 167, 247, 267\} \uplus \{15, 25\} \uplus \{35\}$. The best possible decomposition of $\mathcal{I}$ which is provided by the bouquet $\mathcal{g}(\mathcal{D} \backslash \bar{\mathcal{C}})$ is the following: $\mathcal{I} = \{147, 167, 247, 267, 347, 367\} \uplus \{15, 25, 35\}$.

Figure 5 shows the configuration of the set of flats of the bouquet of matroids $\mathcal{g}(\mathcal{D} \backslash \bar{\mathcal{C}})$.

We finally give another *example of construction of $\mathcal{L}$ when $\mathcal{L}$ is not a lattice.* Let $X = \{1, 2, 3, 4, 5\}$ and $\mathcal{I}$ be the IS on X whose set of circuits is: $\mathcal{D} = \{123, 125, 135, 145, 235, 24, 34\}$. It is easy to see that $\bar{\mathcal{C}} = \{24, 34\}$ and that $\mathcal{g}(\bar{\mathcal{S}}, \bar{\mathcal{C}}) = \bar{\mathcal{g}} \notin \mathcal{L}$ (see that $145 \in \bar{\mathcal{S}}$, $24 \in \bar{\mathcal{C}}$ and $145 \cup 24 - 4 = 125 \in \bar{\mathcal{S}}$, also $145 \in \bar{\mathcal{S}}$, $34 \in \bar{\mathcal{C}}$ and $145 \cup 34 - 4 = 135 \in \bar{\mathcal{S}}$). In fact, the meet semi-lattice $\mathcal{L}$ has the configuration as shown in Figure 6.

The bases of $\mathcal{I}$ are: 12, 13, 14, 15, 23, 25, 35, 45. It can be seen that:

- The bouquet $\mathcal{g}(123, 145)$ provides a decomposition of $\mathcal{I}$ into four matroids: $\mathcal{I} = \{12, 13, 23\} \uplus \{14, 15, 45\} \uplus \{25\} \uplus \{35\}$.
- The bouquet $\mathcal{g}(235, 145)$ too provides a decomposition of $\mathcal{I}$ into four matroids: $\mathcal{I} = \{12\} \uplus \{13\} \uplus \{14, 15, 45\} \uplus \{23, 25, 35\}$.

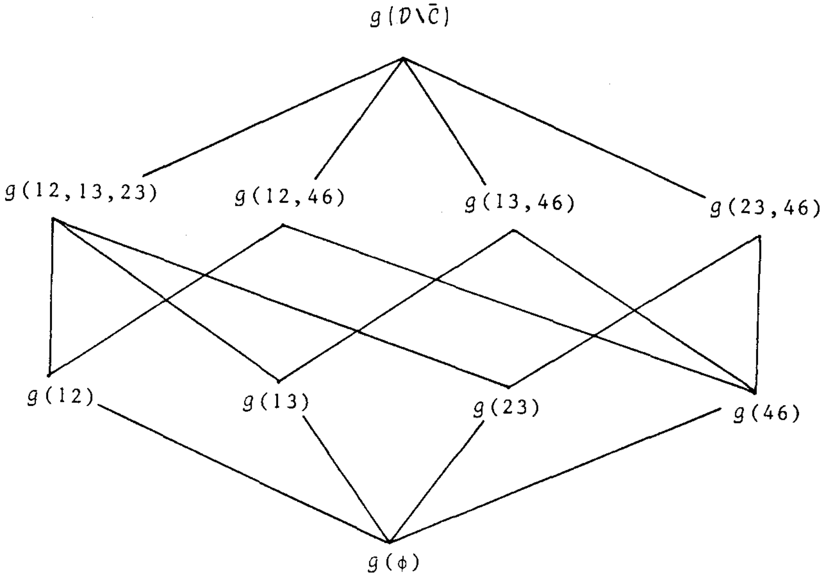

Fig. 4

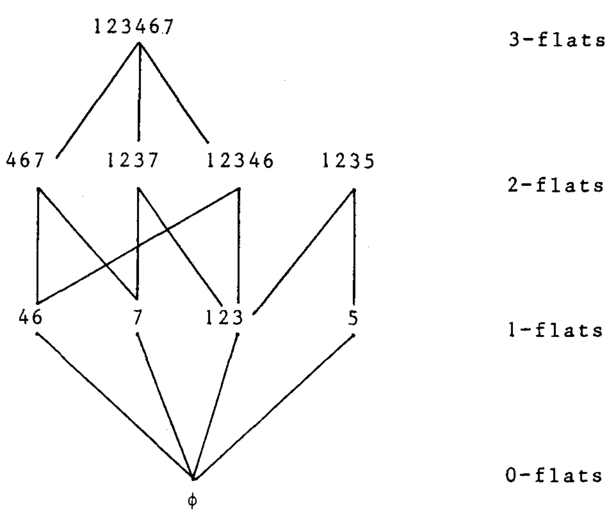

Fig. 5

- The bouquet $\mathcal{g}(123, 125, 135, 235)$ provides the best decomposition of $\mathcal{I}$ into three matroids: $\mathcal{I} = \{12, 13, 15, 23, 25, 35\} \cup \{14\} \cup \{45\}$.

This example shows therefore that not every maximal element of $\mathcal{L}$ provides a best decomposition of $\mathcal{I}$, even when all maximal elements of $\mathcal{L}$ have the same height in the meet semi-lattice $\mathcal{L}$.

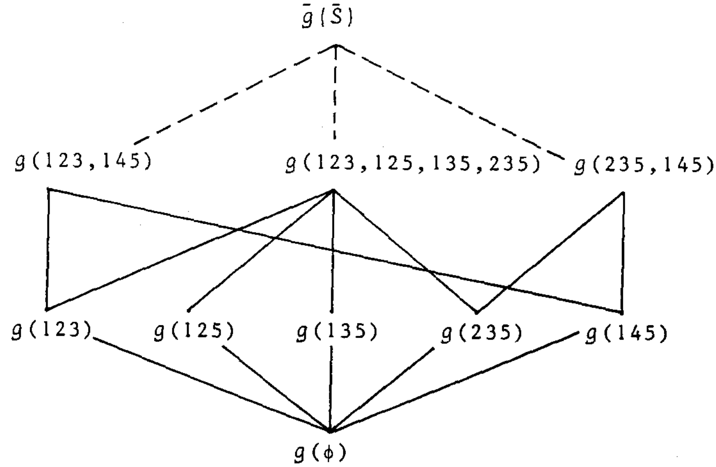

Fig. 6

2.2. *Intersection and squashed union numbers*

Let p be the minimum number of matroids whose intersection is $\mathscr{I}$ and m be the minimum number of matroids whose squashed union is $\mathscr{I}$. In this section, we wish to investigate the relationships between these two parameters.

Proposition 2.10. *The following inequality holds: $p \leq m + |\mathscr{C}|$ for all bouquets of matroids $\mathscr{g}(\mathscr{S}, \mathscr{C})$ of $\mathscr{L}$ composed of m matroids.*

Proof. For every subset $I \subseteq X$, we have:

$$I \in \mathscr{I} \Leftrightarrow \begin{cases} I \subseteq X_i & \text{for some maximal flat } X_i \text{ of } g, \\ |I \cap G| \leq r(G) & \text{for all flats } G \in \mathscr{g}. \end{cases}$$

Hence:

$$I \in \mathscr{I} \Leftrightarrow \begin{cases} I \not\supset C & \text{for all critical subsets } C \text{ of } \mathscr{C}, \\ |I \cap G| \leq r(G) & \text{for all flats } G \in \mathscr{g}. \end{cases}$$

Since $\mathscr{g} \cap [\emptyset, X_i]$ is the set of flats of a matroid on X_i, the IS

$$\mathscr{I}'_i = \{I \subseteq X, |I \cap G| \leq r(G) \text{ for all } G \in \mathscr{g} \text{ such that } G \subseteq X_i\}$$

is the set of independent subsets of a matroid on X, for all $i \in [1, m]$. Therefore $\mathscr{I}$ can be obtained as the intersection of $|\mathscr{C}| + m$ matroids, hence $p \leq m + |\mathscr{C}|$. □

Let us give an example of IS for which $p = 2$ but m may be chosen arbitrarily large. Consider the bipartite graph $G(V_1, V_2, E)$ with sets of vertices: $V_1 = \{a_1, a_2, \ldots, a_m\}$, $V_2 = \{b_1, b_2, \ldots, b_m\}$ and set of edges: $E = \{(a_i, b_i), (a_1, b_i)$ for $i \in [1, m]\}$. (See Figure 7.) Let $\mathscr{I}$ be the IS of the matchings of G. Its bases are:

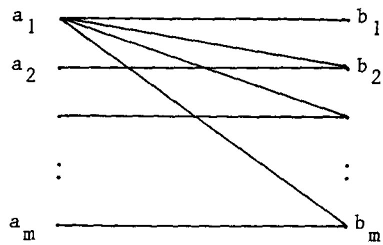

Fig. 7

$B_1 = \{(a_i, b_i) \text{ for } i \in [1, m]\}$ and $B_j = \{(a_1, b_j)(a_i, b_i) \text{ for } i \neq j \text{ and } i \neq 1\}$ for $j \in [2, m]$. It is easy to see that the only possible squashed decomposition of $\mathcal{I}$ is: $\mathcal{I} = \mathcal{I}_{B_1} \veebar \mathcal{I}_{B_2} \veebar \cdots \veebar \mathcal{I}_{B_m}$, having therefore m matroids. However, it is well known that the family of matchings in a bipartite graph is a collection of independent sets in the intersection of two matroids.

The inverse situation may also happen, that is, there exist independence systems for which $m = 2$ but p may be chosen arbitrarily large. For instance, consider the IS of the stable sets of the graph $K_{1,p}$ with vertices $0, 1, 2, \ldots, p$ and edges: $(0, i)$ for $i \in [1, p]$. (See Figure 8.) The two maximal stable sets are: $\{0\}$ and $\{1, 2, 3, \ldots, p\}$; therefore, we have $m = 2$. It is easy to see that the minimum number of matroids whose intersection is $\mathcal{I}$ is equal to p.

Another question arises: Given an IS $\mathcal{I}$ on a finite set X of size n, how big is m with respect to n, or more precisely, is m always polynomial in terms of n? The answer is no, as shown by the following example of IS for which m is exponential with respect to n.

Claim 2.11. *There exists an integer k and a collection $\mathcal{A}$ of subsets of X, $|X| = n$, satisfying:*

(i) $|A| = k \ \forall A \in \mathcal{A}$;

(ii) $|A \cap B| \leq k - 2 \ \forall A \neq B \in \mathcal{A}$;

(iii) $|\mathcal{A}|$ *is exponential with respect to n.*

Proof. For every subset $A \in \binom{n}{k}$, define:

$$\mathcal{B}(A) = \{A' \in \tbinom{n}{k}, |A \cap A'| = k - 1\}.$$

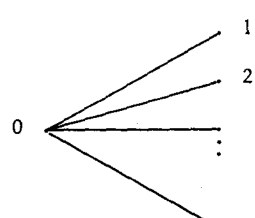

Fig. 8

Hence $|\mathscr{B}(A)|=k(n-k)$. Choose first a subset A_1 in $\binom{n}{k}$, then a subset A_2 in $\binom{n}{k}\backslash\mathscr{B}(A_1)\cup\{A_1\}$ and recursively a subset A_m in $\binom{n}{k}\backslash\mathscr{B}(A_1)\cup\cdots\cup\mathscr{B}(A_{m-1})\cup\{A_1,\ldots,A_{m-1}\}$. By construction, we have $|A_i\cap A_j|\leqslant k-2$ for all $i\neq j$. Such a construction is possible if $\mathscr{B}(A_1)\cup\cdots\cup\mathscr{B}(A_{m-1})\cup\{A_1,\ldots,A_{m-1}\}\subsetneq\binom{n}{k}$. Since the size of $\mathscr{B}(A_1)\cup\cdots\cup\mathscr{B}(A_{m-1})\cup\{A_1,\ldots,A_{m-1}\}$ is less than $m(k(n-k)+1)$, we have only to verify that it is possible to choose m exponential in n and satisfying $m<\binom{n}{k}/(k(n-k)+1)$, which can be easily obtained by choosing, for instance, $k=[\frac{1}{2}n]$. The proof of Claim 2.11 is now finished by considering $\mathscr{A}=\{A_1,\ldots,A_m\}$. $\square$

For all subsets $A\neq B$ of the preceding family $\mathscr{A}$, define

$$\mathscr{C}(A,B)=\{A-a+b,\ a\in A\backslash B,\ b\in B\backslash A\}$$

and

$$\mathscr{B}=\binom{n}{k}\supsetneq\bigcup_{\substack{A\neq B\\ A,B\in\mathscr{A}}}\mathscr{C}(A,B).$$

It is clear from Claim 2.11(ii) that $\mathscr{B}$ contains $\mathscr{A}$. Let $\mathscr{I}$ be the IS whose set of bases is $\mathscr{B}$ and m_1 denote the minimum number of matroids whose squashed union is $\mathscr{I}$. By construction, no two subsets of $\mathscr{A}$ can be included in a same matroid; therefore, $m\geqslant|\mathscr{A}|$ which infers that m is exponential in terms of n.

3. Independence systems, bouquets of matroids and the greedy algorithm: worst-case bounds

Let $\mathscr{I}$ be an IS on X and w be a nonnegative weight function that is defined on all elements of X; hence the weight of every subset A of X is defined by: $w(A)=\sum_{x\in A}w(x)$. Consider the following optimization problem:

$$\operatorname*{Max}_{I\in\mathscr{I}} w(I). \tag{3.1}$$

A natural way for finding a reasonable approximation to the solution of this problem is provided by the following *greedy algorithm*:

> Start with the empty set and recursively add to the current solution set S an element $x\in X$ with maximum weight among all $x\in X\backslash S$ such that $S\cup x\in\mathscr{I}$. Stop when no such element exists.

It is well known (see [9] for example) that the greedy algorithm provides an optimum solution to (3.1) for every nonnegative weight function if and only if $\mathscr{I}$ is a matroid. However, the performance of the greedy algorithm applied to any IS can be measured by computing a lower bound to the following ratio: $\rho=w(S)/w(B_0)$ where S is a basis selected by the greedy algorithm (greedy basis) and B_0 a basis that yields an optimum solution to (3.1). If the value of the ratio ρ is 1, it means that the greedy algorithm selects an optimum solution and very small values of ρ indicate a poor performance.

For every subset $A \subseteq X$, we define:

- The *lower rank* of A: $\mathrm{Lr}(A) = \mathrm{Min}(|I|, I$ is a maximal independent subset of $A)$.
- The *upper rank* of A: $\mathrm{Ur}(A) = \mathrm{Max}(|I|, I$ is an independent subset of $A)$.

So, the quantity $\mathrm{Min}_{A \subseteq X}(\mathrm{Lr}(A)/\mathrm{Ur}(A))$ can be interpreted as a measure of how much $\mathscr{I}$ differs from being a matroid.

Edmonds, also Baumgarten [1], proved the following inequality:

$$\rho \geq \mathop{\mathrm{Min}}_{A \subseteq X} \frac{\mathrm{Lr}(A)}{\mathrm{Ur}(A)} \tag{3.2}$$

(for a proof, see also [7]). It is shown in [7] that, if $\mathscr{I}$ is the intersection of p matroids, then:

$$\mathop{\mathrm{Min}}_{A \subseteq X} \frac{\mathrm{Lr}(A)}{\mathrm{Ur}(A)} \geq \frac{1}{p}. \tag{3.3}$$

Also the following bound is proved in [4]:

$$\rho \geq h/r, \tag{3.4}$$

where r is the rank of $\mathscr{I}$, i.e., $r = \max(|I|, I \in \mathscr{I})$ and $h+1$ is the girth of $\mathscr{I}$, i.e., $h+1 = \min(|D|, D \in \mathscr{D})$, $\mathscr{D}$ being the set of circuits of $\mathscr{I}$.

Let $\mathscr{g}$ be a bouquet of matroids whose IS is $\mathscr{I}$. We give bounds for the value $\mathrm{Min}_{A \subseteq X}(\mathrm{Lr}(A)/\mathrm{Ur}(A))$ and for ρ, in function of parameters of $\mathscr{g}$, also in function of the families $\bar{\mathscr{S}}$, $\bar{\mathscr{C}}$ introduced in Section 2. Moreover, we will see that the choice of $\mathscr{g}$ among the maximal elements of $\mathscr{L}$ provides the best possible bound.

Let $\mathscr{I}$ be an IS on X, $\mathscr{D}$ be its set of circuits. Let $\mathscr{g}$ be a bouquet of matroids with $\mathscr{I}$ as IS, $\mathscr{S}$ be its set of stigmas and $\mathscr{C}$ be its set of critical subsets satisfying: $\mathscr{D} = \mathscr{S} \cup \mathscr{C}$. In order to give other bounds for the quantity $\mathrm{Min}_{A \subseteq X}(\mathrm{Lr}(A)/\mathrm{Ur}(A))$ in terms of the parameters of $\mathscr{g}$, we need some definitions generalizing the notion of star of a graph.

Definition 3.1. Let $\mathscr{E} = \{D_1, \ldots, D_k\}$ be a family of circuits of $\mathscr{D}$. $\mathscr{E}$ is called a *star of type* (1) if the following conditions hold:

(i) There exists an element $a \in X$ belonging to $\bigcap_{i=1}^{k} D_i$.

(ii) There exist pairwise distinct elements of X: $x_1, \ldots, x_k$ such that $x_i \in D_i$ for all $i \in [1, k]$.

(iii) $\bigcup_{i=1}^{k} D_i \backslash a$ is an independent subset of X.

Let k_1 denote the maximum number of circuits in a star of type (1).

Remark 3.2. Suppose $k_1 = 1$. Hence, for all distinct circuits D, D', if a is an element of X belonging to $D \cap D'$, then $D \cup D' \backslash a$ is not an independent subset of X and therefore contains a circuit. Thus $\mathscr{D}$ is the family of circuits of a matroid.

Proposition 3.3. *If $\mathscr{E}$ is a star of type* (1) *of size* $k \geqslant 2$, *then all members of $\mathscr{E}$ belong to $\mathscr{C}$ and therefore are critical subsets of $\mathcal{I}$.*

Proof. Suppose, for instance, that a member D_1 of $\mathscr{E}$ belongs to $\bar{\mathscr{I}}$. Let D_2 be another member of $\mathscr{E}$. Since $a \in D_1 \cap D_2$, we deduce from axiom (D0) the existence of a circuit D such that $D \subseteq D_1 \cup D_2 \backslash a$. Thus we contradict assumption (iii) of Definition 3.1. □

Definition 3.4. Let $\mathscr{E} = \{D_1, \ldots, D_k\}$ be a family of circuits of $\mathscr{D}$. $\mathscr{E}$ is called a *star of type* (2) if and only if:

(i) There exists an element $a \in X$ belonging to $\bigcap_{i=1}^{k} D_i$.

(ii) There exist some elements $x_1, \ldots, x_k$ of X such that $x_i \in D_i \backslash \bigcup_{j=1, j \neq i}^{k} D_j$ for all $i \in [1, k]$.

(iii) $\{x_1, \ldots, x_k\}$ and $\bigcup_{i=1}^{k} D_i \backslash x_i$ are independent subsets of X.

Let k_2 denote the maximum number of circuits in a star of type (2). We also define k_2^c as the maximum number of circuits in a star of type (2) formed only by critical subsets of $\mathscr{C}$.

Remark 3.5. If all circuits have size 2, that is, if $\mathscr{I}$ is the set of stable sets in a graph, then both Definitions 3.1 and 3.4 coincide with the definition of a star of a graph and also $k_2 = k_2^c = k_1$.

Theorem 3.6. *The following inequalities are valid*:

$$\frac{1}{k_2} \leqslant \operatorname*{Min}_{A \subseteq X} \frac{\operatorname{Lr}(A)}{\operatorname{Ur}(A)} \leqslant \frac{k_1(\bar{H}_c - 1) + 1}{k_1 \bar{H}_c},$$

where $\bar{H}_c + 1 = \operatorname{Max}(|C|, C \in \mathscr{C})$.

Proof. Let us first prove the upper bound. Let $\mathscr{E} = \{D_1, \ldots, D_{k_1}\}$ be a star of type (1) of size k_1. Consider the subset $A = \bigcup_{i=1}^{k_1} D_i$ of X. The set $\bigcup_{i=1}^{k_1} D_i \backslash a = A \backslash a$ is an independent subset of A of maximum size, hence $\operatorname{Ur}(A) = |A| - 1$. Let us show that $\bigcup_{i=1}^{k_1} D_i \backslash x_i$ is an independent subset of X. Suppose by contradiction that there exists a circuit D which is contained in $\bigcup_{i=1}^{k_1} D_i \backslash x_i$. Since $\bigcup_{i=1}^{k_1} D_i \backslash a \in \mathscr{I}$, a belongs to D. Choose an element $x \in D - a$; thus $x_1, x_2, \ldots, x_{k_1}, x$ are distinct elements of X. Hence $\{D_1, \ldots, D_{k_1}, D\}$ is a star of type (1) with $k_1 + 1$ circuits which yields a contradiction. Therefore, $\bigcup_{i=1}^{k_1} D_i \backslash x_i$ is a maximal independent subset of A; thus $\operatorname{Lr}(A) \leqslant |A| - k_1$. Since $|A| \leqslant 1 + k_1 \bar{H}_c$, we have:

$$\frac{\operatorname{Lr}(A)}{\operatorname{Ur}(A)} \leqslant \frac{|A| - k_1}{|A| - 1} \leqslant \frac{k_1(\bar{H}_c - 1) + 1}{k_1 \bar{H}_c}.$$

(See that $x \rightarrow (x - k_1)/(x - 1)$ is increasing since $k_1 \geqslant 1$.)

We now prove the lower bound. Let A be a subset of X and I, U be two maximal independent subsets of A such that: $|I| = \operatorname{Lr}(A)$ and $|U| = \operatorname{Ur}(A)$. If $|U| = |I|$, then

$\mathrm{Lr}(A)=\mathrm{Ur}(A)$. Otherwise, for every element $x\in U\backslash I$, since $I\cup x\notin\mathcal{I}$, there exists a circuit D such that: $x\in D\subseteq I\cup x$.

We now define a bipartite graph $G(V_1, V_2, E)$ where: $V_1=I\backslash U$, $V_2=U\backslash I$ and E is defined as follows: For any element $x\in U\backslash I$ and $a\in I\backslash U$, $(a,x)\in E$ if and only if there exists $D\in\mathcal{D}$ such that $\{a,x\}\subseteq D\subseteq I\cup x$. We count in two ways the total number of edges, which is obviously equal to:

$$\sum_{x\in U\backslash I}\deg x=\sum_{a\in I\backslash U}\deg a.$$

Since U contains no circuit, every element $x\in U\backslash I$ is connected to at least one element of $I\backslash U$ and therefore:

$$\sum_{x\in U\backslash I}\deg x\geqslant|U\backslash I|.$$

Consider now an element $a\in I\backslash U$ and $x_1, x_2,\ldots,x_t$ the elements of $U\backslash I$ that are connected to a. Thus, there exist some circuits $D_1, D_2,\ldots,D_t$ such that: $\{a,x_i\}\subseteq D_i\subseteq I\cup x_i$ for all $i\in[1,t]$. Hence $\{D_1,\ldots,D_t\}$ is a star of type (2) and thus $\deg a\leqslant k_2$. Therefore, we have:

$$\sum_{a\in I\backslash U}\deg a\leqslant k_2|I\backslash U|.$$

Consequently, we obtain:

$$|I\backslash U|k_2\geqslant|U\backslash I|,\quad\text{which yields }\frac{|I|}{|U|}\geqslant\frac{|I\backslash U|}{|U\backslash I|}\geqslant\frac{1}{k_2}.\qquad\Box$$

A slight change in the proof of Theorem 3.6 enables us to improve the lower bound for the value $\mathrm{Min}_{A\subseteq X}(\mathrm{Lr}(A)/\mathrm{Ur}(A))$ in the sense that the new bound does not involve all circuits of $\mathcal{D}$ but only the critical subsets of $\mathcal{C}$.

Theorem 3.7. *The following bounds hold*:

$$\frac{1}{1+k_2^c}\leqslant\operatorname*{Min}_{A\subseteq X}\frac{\mathrm{Lr}(A)}{\mathrm{Ur}(A)}\leqslant\frac{k_1(\bar{H}_c-1)+1}{k_1\bar{H}_c},$$

where $\bar{H}_c+1=\mathrm{Max}(|C|, C\in\bar{\mathcal{C}})$.

Proof. We have only to prove the lower bound. We consider again a subset A of X, I, U two independent subsets of A such that $|I|=\mathrm{Lr}(A)$, $|U|=\mathrm{Ur}(A)$. Let X_1 be a maximal flat of $\mathcal{g}$ containing I. We partition U into $U=U_1\cup U_2$ where $U_1=U\cap X_1$ and $U_2=U\backslash X_1$. Again, for every element $x\in U\backslash I$, there exists a circuit D such that $x\in D\subseteq I\cup x$. We show that D is a stigma if and only if $x\in X_1$. If x belongs to X_1, then D is clearly a stigma since $D\subseteq X_1$. Suppose now that D is a stigma and $x\notin X_1$. Since $D\in\mathcal{S}$, there exists a maximal flat X_2 of $\mathcal{g}$, $X_2\neq X_1$, such that $D\subseteq X_2$. Since $D\backslash x\in\mathcal{I}$, $D\backslash x\subseteq X_1\cap X_2$, $x\in X_2\backslash X_1$, the independence axiom (I2) implies that $D\in\mathcal{I}$, which yields a contradiction.

We now define in the same way as before a bipartite graph $G(U_2, I\setminus U, E)$ where U_2, $I\setminus U$ are its sets of vertices and the set E of edges is defined as in the proof of Theorem 3.6. Since a circuit containing $a \in I\setminus U$, $x \in U_2$ and contained in $I \cup x$ can only be a critical subset of $\mathscr{C}$, we obtain therefore the inequality:

(i) $|I\setminus U| k_2^c \geq |U_2|$.

Let us prove:

(ii) $|I\setminus U| \geq |U_1\setminus I|$.

Suppose by contradiction that: $|U_1\setminus I| > |I\setminus U|$. Hence we have: $|U_1| > |I|$. Since I and U_1 are both contained in X_1, these are independent subsets of the matroid $\mathscr{g} \cap [\emptyset, X_1]$ on X_1 and, therefore, there exists an element $x \in U_1\setminus I$ such that $I \cup x \in \mathscr{I}$ which contradicts the maximality of I.

Thus, we infer from (i) and (ii) that:

$$|U\setminus I| = |U_2| + |U_1\setminus I| \leq |I\setminus U|(k_2^c + 1)$$

and therefore:

$$|I|/|U| \geq 1/(1+k_2^c). \qquad \square$$

Remark 3.8. Consider two bouquets of matroids of $\mathscr{L}$ satisfying: $\mathscr{g}(\mathscr{S}, \mathscr{C}) \leq \mathscr{g}(\mathscr{S}', \mathscr{C}')$, then $\mathscr{C}' \subseteq \mathscr{C}$, which implies

$$1/(1+k_2^c) \leq 1/(1+k_2^{c'});$$

therefore, the best possible value for the lower bound $1/(1+k_2^c)$ in Theorem 3.7 is provided by a maximal element of the meet semi-lattice $\mathscr{L}$ and thus, by $\bar{\mathscr{g}}(\bar{\mathscr{S}}, \bar{\mathscr{C}})$, when $\mathscr{L}$ is a lattice.

Let us now derive bounds for the greedy ratio ρ. Let $X_1, \ldots, X_m$ denote the maximal flats of a bouquet of matroids $\mathscr{g}$ having $\mathscr{I}$ as IS.

Lemma 3.9. *We have*: $H_c \leq m-1$, *where*: $H_c + 1 = \mathrm{Max}(|C|, C \in \mathscr{C})$.

Proof. Let C be a critical subset of $\mathscr{C}$. Since C is not included in any X_i and for every element $x \in C$, $C\setminus x$ is included in some X_i, the lemma follows. $\square$

Let A be the collection of subsets of $[1, m]$ defined by:

$$A = \left\{ \Lambda \subseteq [1, m]: \bigcup_{i \in \Lambda} X_i \neq X \right\}.$$

Lemma 3.10. *Suppose $H_c = 1$. Then the maximum size k_1 of a star of type* (1) *is given by*:

$$k_1 = \mathrm{Max}\left[r\left(\bigcup_{i \in \Lambda} X_i - \bigcup_{i \notin \Lambda} X_i \right), \Lambda \in A \right].$$

Proof. Let $\mathscr{C}=\{C_1,\ldots,C_{k_1}\}$ a star of type (1) of size k_1. We know that its members are all critical subsets of $\mathscr{C}$. Since $H_c=1$, every C_i has size 2 and can be written $C_i=\{a,x_i\}$ where $a,x_1,x_2,\ldots,x_{k_1}$ are distinct elements of X. Moreover, $I=\{x_1,\ldots,x_{k_1}\}$ is an independent subset of X. Define $A=\{i\in[1,m], I\cap X_i\neq\emptyset\}$. Then $a\in\bigcup_{i\notin A}X_i$ and $I\subseteq\bigcup_{i\in A}X_i\backslash\bigcup_{i\notin A}X_i$, which implies:

$$k_1=|I|\leqslant r\left(\bigcup_{i\in A}X_i\supsetneq\bigcup_{i\notin A}X_i\right)\quad\text{and}\quad\bigcup_{i\in A}X_i\neq X.$$

Equality in the preceding inequality is easy to see. □

Corollary 3.11. *Let ρ denote the greedy ratio, i.e., $\rho=w(S)/w(B_0)$, where S is a greedy basis and B_0 an optimum solution to the problem* (3.1). *Then the following bound holds:*

$$\rho\geqslant\text{Min}(1/(1+r(X_i\backslash X_j))\ \textit{for all}\ i,j\in[1,m]).$$

Proof. We first prove the corollary in the case $m=2$. Lemma 3.9 yields $H_c=1$ and therefore all critical sets have size 2. We infer from Lemma 3.10 that $k_1=\text{Max}(r(X_1\backslash X_2),r(X_2\backslash X_1))$. Since all critical subsets have size 2, we have $k_1=k_2^c$. We now obtain from Theorem 3.7 and inequality (3.2) that:

$$\rho\geqslant\underset{A\subseteq X}{\text{Min}}\frac{\text{Lr}(A)}{\text{Ur}(A)}\geqslant\text{Min}\left(\frac{1}{1+r(X_1\backslash X_2)},\frac{1}{1+r(X_2\backslash X_1)}\right).$$

We now prove the corollary in the general case. Consider a greedy solution S, an optimum solution B_0 and two maximal flats X_i,X_j of $\mathscr{G}$ such that $X_i\supseteq S$ and $X_j\supseteq B_0$. If $X_i=X_j$, then $\rho=1$. Otherwise, consider the set of flats of $\mathscr{G}$ that are contained in X_i or in X_j. It is still a bouquet of matroids and it has only two maximal flats: X_i and X_j. Therefore, we infer from the preceding case that

$$\rho\geqslant\text{Min}\left(\frac{1}{1+r(X_i\backslash X_j)},\frac{1}{1+r(X_j\backslash X_i)}\right).\qquad\square$$

Let us now treat as an application of the preceding results the case of the IS of the *stable sets of a graph.*

Proposition 3.12. *Let $G=(V,E)$ be a graph and $\mathscr{I}$ be the set of stable sets of G. Then*

$$\rho\geqslant\underset{A\subseteq V}{\text{Min}}\frac{\text{Lr}(A)}{\text{Ur}(A)}=\frac{1}{k}$$

where k is the maximum size of a star of G.

Proof. Since all circuits have size 2, we have $\bar{H}_c=1$ and $k_1=k_2=k$. Therefore, Theorem 3.6 yields: $\text{Min}_{A\subseteq V}(\text{Lr}(A)/\text{Ur}(A))=1/k$. □

Remark 3.13. It is easy to see that the same result holds when $\mathscr{I}$ is the set of matchings of G, i.e.,

$$\rho\geqslant\underset{A\subseteq E}{\text{Min}}\frac{\text{Lr}(A)}{\text{Ur}(A)}=\frac{1}{k},$$

k being equal to 2 except $k=1$ in the matroidal case (see also [7]).

Corollary 3.14. *Let $\mathcal{I}$ be the set of stable sets of the graph G, p be the minimum number of matroids whose intersection is $\mathcal{I}$ and k be the maximum size of a star of G. Then $k \leq p$.*

Proof. The inequality (3.3) and Proposition 3.12 yield trivially the result. □

Proposition 3.15. *For any* IS, *we have $p \geq k_1$ where k_1 is the maximum size of a star of type* (1).

Proof. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_p$ be matroids with respective sets of circuits $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_p$ such that: $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \cdots \cap \mathcal{I}_p$. Let $\mathscr{E} = \{D_1, \ldots, D_{k_1}\}$ be a star of type (1). For every $i \in [1, k_1]$, since $D_i \notin \mathcal{I}$, there exists $\alpha_i \in [1, p]$ such that $D_i \notin \mathcal{I}_{\alpha_i}$ which yields easily that $D_i \in \mathcal{D}_{\alpha_i}$. Suppose that there exists $i \neq j \in [1, k_1]$ such that $\alpha_i = \alpha_j$. Hence D_i, D_j are two distinct circuits of $\mathcal{D}_{\alpha_i}$. Since $a \in D_i \cap D_j$, the circuit axioms in the matroidal family $\mathcal{D}_{\alpha_i}$ imply the existence of $D \in \mathcal{D}_{\alpha_i}$ such that $D \subseteq D_i \cup D_j \backslash a$ which contradicts the assumption $\bigcup_{i=1}^{k_1} D_i \backslash a \in \mathcal{I}$. □

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