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ON THE FACIAL STRUCTURE OF INDEPENDENCE SYSTEM POLYHEDRA*†

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A polyhedron P whose extreme points are the incidence vectors of the sets of an independence system $\mathscr{I}$ is called an independence system polyhedron. In this paper, we consider the problem of describing the facial structure of independence system polyhedra.

Using the fact that this structure is known when the independence system is a matroid, we study the polyhedron P by decomposing the independence system $\mathscr{I}$ as a union of matroidal families. We provide a system of valid inequalities for P that contains all its facets. A consequence of this result is the following: the maximum number of distinct coefficients of a facet is bounded by the minimum number of matroidal families whose union gives $\mathscr{I}$. When the independence system $\mathscr{I}$ is the union of two matroids, we give necessary and sufficient conditions for an inequality of the above system to define a facet of P.

0. Introduction. Given a finite set E, $|E| = n$, an *independence system* (IS for short) on E is a family $\mathscr{I}$ of subsets of E closed under inclusion, i.e.,

$$\text{(I1)} \qquad \text{if } J \in \mathscr{I} \text{ and } I \subseteq J, \text{ then } I \in \mathscr{I}.$$

A set belonging to the family $\mathscr{I}$ is called *independent*. When an independence system $\mathscr{I}$ satisfies the following additional property:

$$\text{(I2)} \qquad \begin{array}{c}\text{for all } I,\ J \in \mathscr{I}, \text{ if } |J| > |I|, \text{ then there exists an element}\\ x \in J - I \text{ such that } I \cup x \in \mathscr{I}\end{array}$$

then $\mathscr{I}$ is called a *matroid*.

Given an IS $\mathscr{I}$ on E, we define the polyhedron $P = P(\mathscr{I}) = \mathrm{Conv}(x^I\colon I \in \mathscr{I})$ of R^E where, for a set $S \subseteq E$, x^S denotes the *incidence vector* of S defined by $x_e^S = 1$ if $e \in S$ and $x_e^S = 0$ otherwise. In many applications, a weight is associated to each element of E and one is interested in finding an independent set whose weight (i.e., sum of the weights of its elements) is maximum. Equivalently, one is interested in maximizing a linear function over the polyhedron $P(\mathscr{I})$. This problem could be solved, at least in theory, by linear programming techniques if the minimum system of linear inequalities describing $P(\mathscr{I})$ would be available. In practice, efficient solutions of the above problem can be found even if only a partial description of $P(\mathscr{I})$ is available. Therefore, the following question is of fundamental interest:

Find a (minimum) system of linear inequalities defining $P(\mathscr{I})$, i.e., determine the faces (facets) of $P(\mathscr{I})$.

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This problem of describing the facial structure of the polytope $P(\mathcal{I})$ has been the object of many efforts for some instances of independence systems. In particular, for the node packing problem, i.e., when $\mathcal{I}$ is the family of stable sets of a graph G, some classes of facets for $P(\mathcal{I})$ are derived from the cliques, the odd holes, odd antiholes and other configurations of the graph G (see [2] for a survey). Similarly, in the case of a general IS $\mathcal{I}$, some classes of facets for $P(\mathcal{I})$ can be obtained from special configurations of the family of circuits (i.e., minimally dependent sets) of $\mathcal{I}$; for instance, from generalized cliques, odd holes or antiholes (see [7], [12], [13]). In this paper, we propose a global approach to the question of describing the polytope $P(\mathcal{I})$ based on the structural properties of $\mathcal{I}$ and, more precisely, on the fact that any IS can be decomposed as a union of matroidal families. Although there exist well structured independence systems that are the intersection of 2 or 3 matroids, we are not aware of any general example of independence systems arising as the union of a fixed number of matroids. However, graphs whose family of independent sets can be expressed as the union of two matroids, have been characterized in [3] and in a more general setting in [5].

Before presenting our results in more detail, we summarize the basic notions from linear algebra and polyhedral theory which are relevant for our treatment (for references on the theory of polyhedra, see [1], [9], [11], for instance).

Given a set of vectors $x_1, \ldots, x_k \in \mathbf{R}^n$ and $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$, the vector $\sum_{i=1}^k \lambda_i x_i$ is called an *affine* (*convex*) *combination* of the x_i's if $\sum_{i=1}^k \lambda_i = 1$ ($\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geqslant 0$ for all $i \in [1, k]$). The family $\{x_1, \ldots, x_k\}$ is called *linearly* (*affinely*) *independent* if, for all $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$, the equations $\sum_{i=1}^k \lambda_i x_i = 0$ ($\sum_{i=1}^k \lambda_i x_i = 0$ and $\sum_{i=1}^k \lambda_i = 0$) imply $\lambda_1 = \cdots = \lambda_k = 0$; notice that for a family $\{x_1, \ldots, x_k\}$ of linearly (affinely) independent vectors, we have $k \leqslant n$ ($k \leqslant n+1$). The *rank* (*affine rank*) of a set $S \subseteq \mathbf{R}^n$ is the maximum cardinality of a linearly (affinely) independent subset of S and the *dimension* of S, denoted by $\dim S$, is the affine rank of S minus 1. For a set $S \subseteq \mathbf{R}^n$, $S \neq \varnothing$, the set of all affine (convex) combinations of finitely many vectors in S is called the *affine* (*convex*) *hull* of S and is denoted by $\operatorname{Aff} S$ ($\operatorname{Conv} S$). Hence, S and $\operatorname{Aff} S$ have clearly the same affine rank. Notice that, if $0 \notin \operatorname{Aff} S$, then the rank and the affine rank of S coincide.

A *polyhedron* P is the intersection of a finite number of closed halfspaces; hence P can be represented in the form: $P = \{x \in \mathbf{R}^n\colon\ Ax \leqslant b\}$ where $b \in \mathbf{R}^n$ and A is a matrix with n columns. If P also is bounded, then P is a *polytope*. A fundamental result of polyhedral theory (see [1]) states in particular that polytopes are precisely those sets of $\mathbf{R}^n$ which are the convex hull of finitely many points. In our case, we have a polyhedron given as the convex hull of points ($P = \operatorname{Conv} \mathcal{I}$) and we want to find an inequality system defining P with as few inequalities as possible. A polyhedron P is called *full-dimensional* if $\dim P = n$. In the following, we will assume w.l.o.g. that $\{e\} \in \mathcal{I}$ for all $e \in E$ and therefore that $P(\mathcal{I}) = \operatorname{Conv} \mathcal{I}$ is full dimensional.

Given a polyhedron P, $c \in \mathbf{R}^n$, $c_0 \in \mathbf{R}$, the inequality $c \cdot x \leqslant c_0$ is called *valid* for P if $c \cdot x \leqslant c_0$ holds for all $x \in P$ (for all $a, b \in \mathbf{R}^n$, we use the notation $a \cdot b$ for the inner product of a, b, i.e., $a \cdot b = \sum_{i=1}^n a_i b_i$). A subset V of P is called a *face* of P if there is a valid inequality $c \cdot x \leqslant c_0$ such that $V = \{x \in P\colon\ c \cdot x = c_0\}$; one says that $c \cdot x \leqslant c_0$ *supports* (defines, induces) V. A *facet* of P is a maximal (for set inclusion) proper (i.e., distinct from P) nonempty face of P. Equivalently, a facet of P is a face of dimension $\dim(P) - 1$. Therefore, if P is full dimensional, then any facet of P is supported by a unique (up to positive multiple) valid inequality.

If we define the *rank function* $r(\cdot)$ of the IS $\mathcal{I}$ by: $r(S) = \max(|I|\colon I \subseteq S$ and $I \in \mathcal{I})$ for all $S \subseteq E$, then the inequality $\sum_{e \in S} x_e \leqslant r(S)$ is clearly a valid inequality for $P(\mathcal{I})$ called *rank inequality*; it is also said to be *boolean* since its nonzero coefficients take all the same value. A set $S \subseteq E$ is called *closed* if $r(S \cup e)$

$> r(S)$ for all $e \in E - S$ and S is called *nonseparable* if $r(S) < r(T) + r(S - T)$ for any subset T of S such that $T \neq \varnothing$ and $T \neq S$.

It is a well-known result by Edmonds that, in the particular case when $\mathscr{I}$ is a matroid, the rank inequalities are indeed sufficient for describing the polytope $P(\mathscr{I})$; more precisely, the following result holds:

THEOREM 1 ([6], see also [8]). *Let $\mathscr{I}$ be a matroid on E. Then*
(i) $P(\mathscr{I}) = \{x \in \mathbf{R}_+^E: \sum_{e \in S} x_e \leq r(S)$ *for all* $S \subseteq E\}$,
(ii) *the inequality* $\sum_{e \in S} x_e \leq r(S)$ *induces a facet of* $P(\mathscr{I})$ *if and only if* S *is closed and nonseparable.*

In the general case, it is easy to construct facet defining inequalities that are not boolean. When $\mathscr{I}$ is an arbitrary IS, we study the description of $P(\mathscr{I})$ by, using the fact that this structure is known in the matroidal case, decomposing the IS $\mathscr{I}$ as a *union of matroids*. Notice that such a decomposition always exists; for instance, $\mathscr{I} = \bigcup_{B \in \mathscr{B}} \mathscr{I}_B$ where $\mathscr{B}$ is the family of bases (i.e., maximal independent sets) of $\mathscr{I}$ and $\mathscr{I}_B = \{I \in \mathscr{I}: I \subseteq B\}$ for all $B \in \mathscr{B}$. However, this decomposition is, in some sense, the "worst" one since it involves the maximum number of matroids. We refer to [3] for the study of the smallest decomposition of an IS as a union of matroids when the IS is the family of stable sets of a graph. Notice however that, for some IS, we need to take the union of exponentially (with respect to the size n of the groundset E) many matroids for obtaining the IS; we refer to [4] for an example of such IS. In the following, we consider a decomposition of $\mathscr{I}$ as a union of m matroids: $\mathscr{I} = \mathscr{I}_1 \cup \cdots \cup \mathscr{I}_m$, $\mathscr{I}_i$ being a matroid on the set $E_i = \{x \in E: \{x\} \in \mathscr{I}_i\}$ with $r_i(\cdot)$ as a rank function, for all $i \in [1, m]$. We can furthermore choose the parameter m with the smallest possible value. We can define, for all $i \in [1, m]$, the matroid polyhedron $P_i = \text{Conv } \mathscr{I}_i$ in $\mathbf{R}^E$; note that, for every vector $x \in P_i$, we have $x_e = 0$ for all $e \in E - E_i$. It is easy to verify that the following relationships hold between the polytopes P and P_i.

REMARK 2. (i) $P = \text{Conv}(P_1 \cup \cdots \cup P_m)$.
(ii) if V is a face of P, then $V_i = V \cap P_i$ is a face of P_i and, furthermore, $V = \text{Conv}(V_1 \cup \cdots \cup V_m)$.

Notice also that the rank function $r(\cdot)$ of the IS and the matroidal rank functions $r_i(\cdot)$ are related by the following property:

$$r(S) = \max\big(r_i(S \cap E_i): i \in [1, m]\big) \text{ for all } S \subseteq E.$$

In this paper, after recalling some generalities in the first section, we state in the second part our main result which consists of an analytic description of the polytope $P(\mathscr{I})$. It is furthermore proved that if the IS $\mathscr{I}$ can be decomposed as the union of m matroids, then the nonzero coefficients of any facet of $P(\mathscr{I})$ take at most m distinct values. In the last paragraph, we give a characterization of the facets of $P(\mathscr{I})$ when $\mathscr{I}$ can be decomposed as the union of two matroids.

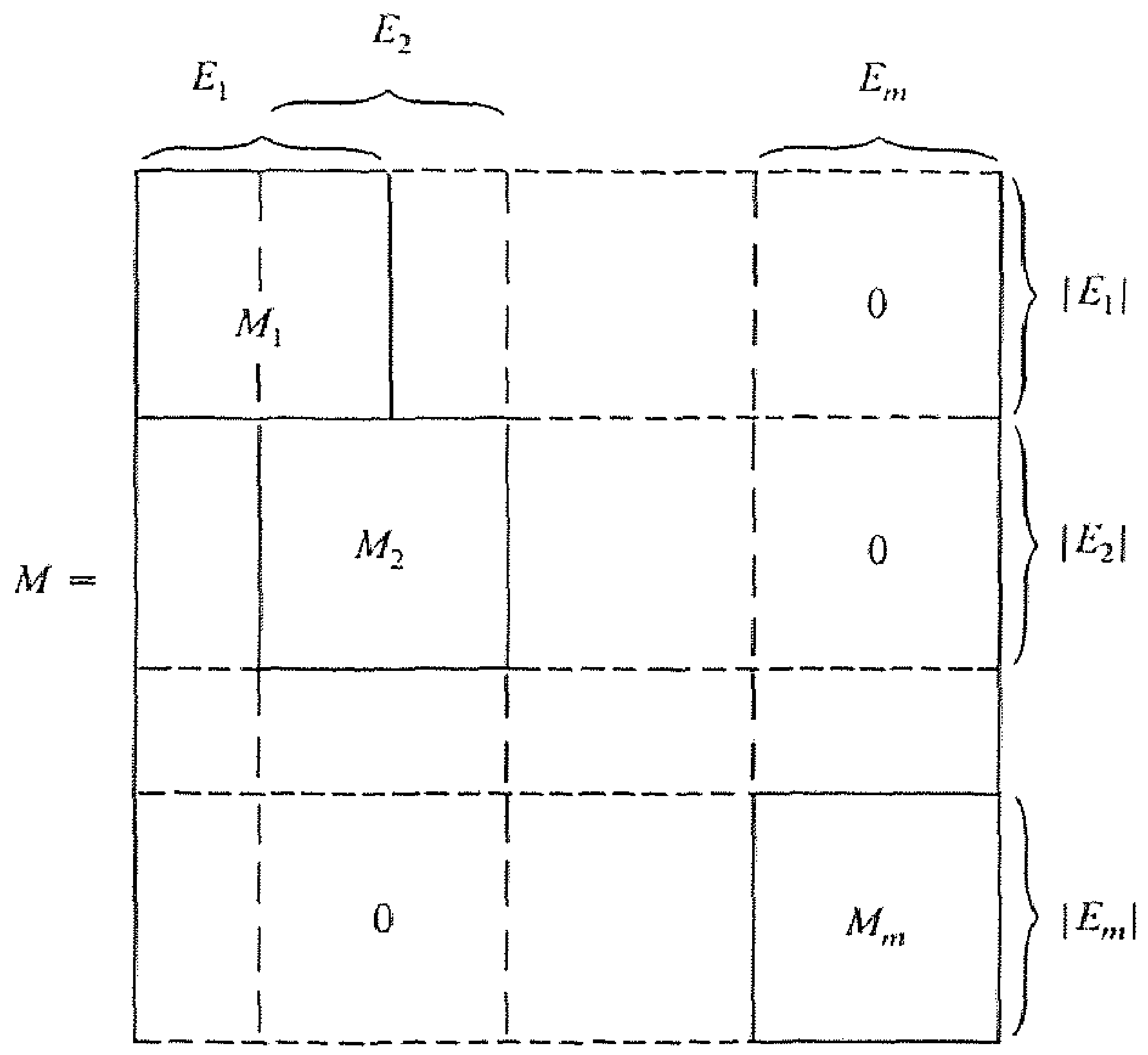
1. Some generalities on the facial structure of $P(\mathscr{I})$. We first recall some generalities on the valid inequalities supporting facets of the polyhedron $P(\mathscr{I})$, $\mathscr{I}$ being an IS on E. The following is a standard result for IS polyhedra (see [10]).

PROPOSITION 3. (i) *The only facets, called the trivial facets, of P containing the vector* $0_n = (0, \ldots, 0)$ *are induced by the inequalities* $x_e \geq 0$, *for* $e \in E$.
(ii) *If V is a facet of P not containing* 0_n, *then there exists a unique valid inequality* $c \cdot x \leq 1$ *supporting V. Moreover, its coefficients are nonnegative.*

Suppose that $\mathcal{I}$ can be decomposed as the union of m matroids: $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_m$, $\mathcal{I}_i$ being a matroid on $E_i \subseteq E$ for $i \in [1, m]$. As we mentioned in Remark 2, for any face V of $P(\mathcal{I})$, $V_i = V \cap P_i$ is a face of $P_i = \mathrm{Conv}\,\mathcal{I}_i$ for all $i \in [1, m]$ and $V = \mathrm{Conv}(V_1 \cup \cdots \cup V_m)$. The following result gives a sufficient condition for V to be a facet of $P(\mathcal{I})$.

PROPOSITION 4. *Let V be a face of $P(\mathcal{I})$. If $V_i = V \cap P_i$ is a facet of P_i for all $i \in [1, m]$, then V is a facet of $P(\mathcal{I})$.*

PROOF. Since V_i is a facet of P_i, there exists an affine basis B_i of V_i of cardinality $|E_i|$, for all $i \in [1, m]$. Define the family $B = B_1 \cup \cdots \cup B_m$ of vectors of V. Let M be the incidence matrix of the family B, its first $|E_1|$ rows are the incidence vectors of the elements of $B_1, \ldots,$ its last $|E_m|$ rows are the incidence vectors of the elements of B_m. Hence, M is of the form:



where M_i is a $|E_i| \times |E_i|$ nonsingular matrix whose columns are indexed by E_i, for $i \in [1, m]$. We show that the columns $(\mathcal{U}_e\colon e \in E)$ of M are linearly independent. Suppose that we have $\sum_{e \in E} \lambda_e \mathcal{U}_e = 0$. Take the projection of the vector $\sum_{e \in E} \lambda_e \mathcal{U}_e$ on $\mathbf{R}^{E_1}$ (i.e., on the first $|E_1|$ rows of M); we obtain $\sum_{e \in E_1} \lambda_e \mathcal{U}_e^1$ where $(\mathcal{U}_e^1, e \in E_i)$ are the columns of M_1 and therefore we deduce that $\lambda_e = 0$ for all $e \in E_1$. By applying the same argument for $i \in [1, m]$, we obtain that $\lambda_e = 0$, $\forall e \in E$. Therefore, the matrix M has rank $|E|$ which implies that B has rank $|E|$ and thus that V is a facet. ∎

COROLLARY 5. *Let $E_1, \ldots, E_m$ be some pairwise disjoint sets and $E = E_1 \cup \cdots \cup E_m$. Let $\mathcal{I}_i$ be a matroid on E_i for $i \in [1, m]$ and $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_m$. Then:*

$$(6) \qquad P(\mathcal{I}) = \left\{ x \in \mathbf{R}^E\colon x_e \geqslant 0 \text{ for all } e \in E \text{ and } \sum_{i=1}^{m} \sum_{e \in S_i} \frac{x_e}{r(S_i)} \leqslant 1 \text{ for all subsets } S_i \subseteq E_i \text{ and } i \in [1, m] \right\}.$$

Moreover, (6) induces a facet of $P(\mathcal{I})$ if and only if every S_i is a closed and nonseparable set in the matroid $\mathcal{I}_i$ for $i \in [1, m]$.

PROOF. Let us first show that every nontrivial facet of P is of the form (6). Let V be a nontrivial facet of P, then $V = \mathrm{Conv}(V_1 \cup \cdots \cup V_m)$. We have the following relation on the affine ranks of the sets V, V_i for $i = 1, \ldots, m$:

$$|E| = r_a(V) \leqslant r_a(V_1) + \cdots + r_a(V_m) \leqslant |E_1| + \cdots + |E_m| = |E|.$$

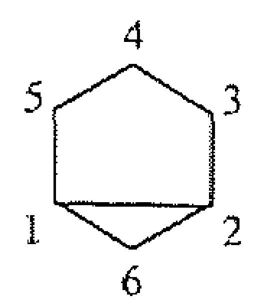
This implies therefore that $r_a(V_i) = |E_i|$ and thus V_i is a facet of P_i for all $i \in [1, m]$. By Theorem 1, there exists a closed nonseparable subset S_i of E_i such that $V_i = \{x \in P_i\colon \sum_{e \in S_i} x_e = r(S_i)\}$, for $i \in [1, m]$. Now let $c \cdot x \leqslant 1$ be a valid inequality supporting V. Then, clearly, $V_i = \{x \in P_i\colon \sum_{e \in E_i} c_e x_e = 1\}$. Using the fact that V_i is a facet, we deduce that $c_e = 1/r(S_i)$ for all $e \in S_i$ and $c_e = 0$ for all $e \in E_i - S_i$ and, therefore,

$$V = \left\{ x \in P\colon \sum_{i=1}^{m} \sum_{e \in S_i} \frac{x_e}{r(S_i)} = 1 \right\}.$$

We now prove that, if $S_i \subseteq E_i$ is a closed and nonseparable set in $\mathcal{I}_i$ for all $i \in [1, m]$, then the face V induced by the inequality (6) is indeed a facet. For all $i \in [1, m]$, we have: $V_i = V \cap P_i = \{x \in P_i\colon \sum_{e \in S_i} x_e = r(S_i)\}$ and thus, from Theorem 1, V_i is a facet of P_i. Now use Proposition 4 for deducing that V is a facet of $P(\mathcal{I})$. ■

The following example shows that the converse of Proposition 4 is not true in general.

EXAMPLE. Let $\mathcal{I}$ be the family of stable sets of the following graph:



Then $\mathcal{I}$ can be decomposed as the union of four matroids: $\mathcal{I} = \{13, 14\} \cup \{24, 25\} \cup \{356\} \cup \{46\}$ (we list only the bases and, for short, 13 denotes the set $\{1, 3\}$). It is easy to see that $V = \{x \in P(\mathcal{I})\colon x_1 + x_2 + x_3 + x_4 + x_5 = 2\}$ is a facet of $P(\mathcal{I})$. However, $V \cap P_i$ is never a facet of P_i and, furthermore, we have $V \cap P_4 = \varnothing$ (take $\mathcal{I}_4 = \{46\}$).

2. On the facial structure of $P(\mathcal{I})$ when $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_m$. In this section, we state the basic result of our paper. We first give a key lemma on which our theorem is based.

LEMMA 7. *Let $\mathcal{I}$ be a matroid on E with $r(\cdot)$ as rank function and $c \in \mathbf{R}^E$. For any nonnegative scalar a, we define the set $T = \{e \in E\colon c_e > a\}$. Let $I \in \mathcal{I}$ be an independent set satisfying: $c \cdot x^I = \max(c \cdot x^J\colon J \in \mathcal{I})$. Then $|I \cap T| = r(T)$ holds.*

PROOF. Suppose for contradiction that $|I \cap T| < r(T)$. Then there exists an element $e \in T - I$ such that $J = I \cap T + e \in \mathcal{I}$.

Suppose first that $|I| < |J|$ which implies that $I \subseteq T$ and $J = I + e$. Then we have that: $c \cdot x^J = c \cdot x^I + c \cdot x_e > c \cdot x^I + a$ since $e \in T$, implying that: $c \cdot x^J > c \cdot x^I$ since $a \geqslant 0$ which yields a contradiction.

We can therefore assume that $|J| \leqslant |I|$. It follows from axiom (I2) applied to matroid $\mathcal{I}$ that there exists a (possibly empty) set $K \subseteq I - J$ such that $J' = J \cup K \in \mathcal{I}$ and $|J'| = |I|$. It is now clear that $J' = I + e - e'$ for some $e' \in I - T$. Then the assumption: $c \cdot x^{J'} \leqslant c \cdot x^I$ implies that $c_e \leqslant c_{e'}$ contradicting the fact that $e \in T$, $e' \notin T$, i.e. $a < c_e$ and $c_{e'} \leqslant a$. ■

Notice that Theorem 1 part (i) follows from Lemma 7 since, if $c \cdot x \leqslant 1$ induces a facet V for the matroid polyhedron $P(\mathscr{I})$ and if $T = \{e \in E: c_e > 0\}$, then $V \subseteq \{x \in P(\mathscr{I}): \sum_{e \in T} x_e = r(T)\}$ and therefore equality holds.

Before presenting our result, we introduce some notation. In the following, $\mathscr{I}$ is an IS on E that can be decomposed as the union of m matroids: $\mathscr{I} = \mathscr{I}_1 \cup \cdots \cup \mathscr{I}_m$, $\mathscr{I}_i$ being a matroid on $E_i \subseteq E$ with $r_i(\cdot)$ as rank function.

Let $T_1, \dots, T_p$ be $p \leqslant m$ subsets of E. For each pair of indices $i \in [1, m]$ and $q \in [1, p]$, the integer $t_i^{(q)} = r_i(T_q \cap E_i)$ denotes the rank of the set $T_q \cap E_i$ in the matroid $\mathscr{I}_i$. Let $\mathscr{I}_{i_1}, \dots, \mathscr{I}_{i_p}$ be p matroids chosen out of the m matroids $I_1, \dots, I_m$. Define, for each $q \in [1, p]$, the p-vector $\nu_q = (t_{i_1}^{(q)}, \dots, t_{i_p}^{(q)})$ of the ranks of the traces of the set T_q in each of the matroids $\mathscr{I}_{i_1}, \dots, \mathscr{I}_{i_p}$. We can then introduce the following $p \times p$ determinants: $d = \det(\nu_1, \dots, \nu_p)$ and, for all $q \in [1, p]$, $d_q = \det(\nu_1, \dots, \nu_{q-1}, 1_p, \nu_{q+1}, \dots, \nu_p)$ where 1_p denotes the p-vector $(1, 1, \dots, 1)$.

DEFINITION 8. Let $T_1, \dots, T_p$ be $p \leqslant m$ subsets of E and $i_1, \dots, i_p$ be p indices in $[1, m]$. We say that the pair $((T_1, \dots, T_p), (i_1, \dots, i_p))$ forms an admissible family if the following conditions are satisfied:

(a) $dd_q > 0$ for all $q \in [1, p]$,

(b) $\sum_{q=1}^{p} t_i^{(q)} d_q/d \leqslant 1$ for all $i \in [1, m]$.

PROPOSITION 9. *If $((T_1, \dots, T_p), (i_1, \dots, i_p))$ forms an admissible family, then the inequality*:

$$\sum_{q=1}^{p} d_q/d\left(\sum_{e \in T_q} x_e\right) \leqslant 1 \tag{10}$$

is a valid inequality for $P(\mathscr{I})$.

PROOF. It follows trivially from conditions (a) and (b). ∎

The following theorem states that every nontrivial facet of $P(\mathscr{I})$ is induced by an inequality of type (10) where the sets T_q form a chain, i.e., $T_1 \supset T_2 \supset \cdots \supset T_p$.

THEOREM 11. *We have*:

$$P(\mathscr{I}) = \Big\{x \in \mathbf{R}^E: x_e \geqslant 0 \textit{ for all } e \in E$$

$$\sum_{q=1}^{p} d_q/d\left(\sum_{e \in T_q} x_e\right) \leqslant 1 \textit{ for all admissible families such that} \tag{10}$$

$$T_1 \supset \cdots \supset T_p \supset \varnothing \textit{ with } 1 \leqslant p \leqslant m\Big\}.$$

PROOF. We show that every nontrivial facet of $P(\mathscr{I})$ is induced by an inequality of type (10). Let $c \cdot x \leqslant 1$ be an inequality that induces a nontrivial facet V of $P(\mathscr{I})$. From Proposition 3, we have that $c_e \geqslant 0$ for all $e \in E$. Let us assume that the coefficients c_e take exactly p distinct nonzero values $\lambda_1, \dots, \lambda_p$ such that $\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_p$. We define the sets $T_q = \{e \in E: c_e > \lambda_{q-1}\}$ for all $q \in [1, p+1]$; then $T_1 \supset T_2 \supset \cdots \supset T_p \supset T_{p+1} = \varnothing$ holds and also:

$$c \cdot x = \sum_{q=1}^{p} (\lambda_q - \lambda_{q-1})\left(\sum_{e \in T_q} x_e\right). \tag{0}$$

Our goal is to prove that there exist some indices $i_1, \dots, i_p$ in $[1, m]$ such that $((T_1, \dots, T_p), (i_1, \dots, i_p))$ is an admissible family and $c \cdot x = \sum_{q=1}^{p} d_q/d(\sum_{e \in T_q} x_e)$.

Recall that $t_i^{(q)} = r_i(T_q \cap E_i)$ is the rank of the set $T_q \cap E_i$ in the matroid $\mathscr{I}_i$, for $i \in [1, m]$ and $q \in [1, p]$. We first state some conditions relating the coefficients c_e and the parameters $t_i^{(q)}$.

Since V is a facet of $P(\mathscr{I})$, there exists a collection $I_1, \ldots, I_n$ of n independent sets in $\mathscr{I}$ whose incidence vectors are on the facet V and form a linear basis of $\mathbf{R}^E$. Notice that it may happen that, for some $i \in [1, m]$, the face $V_i = V \cap P_i$ is empty; therefore, we can assume the existence of an integer k, $1 \leqslant k \leqslant m$, for which $V_i \neq \varnothing$ for all $i \in [1, k]$ and $V_i = \varnothing$ for all $i \in [k+1, m]$. Hence the independent sets $I_1, \ldots, I_n$ belong to $\mathscr{I}_1 \cup \cdots \cup \mathscr{I}_k$. For each element $e \in E$, there exist some scalars $\alpha_1, \ldots, \alpha_n$ such that:

$$x^{\{e\}} = \sum_{j=1}^{n} \alpha_j x^{I_j}. \tag{1}$$

By taking the scalar product of both sides of (1) with the vector $c = (c_e)_{e \in E}$, we obtain:

$$c_e = \sum_{j=1}^{n} \alpha_j = \sum_{i=1}^{k} a_i \tag{2}$$

where $a_1 = \Sigma\{\alpha_j\colon I_j \in \mathscr{I}_1\}$ and $a_i = \Sigma\{\alpha_j\colon I_j \in \mathscr{I}_i - \mathscr{I}_1 \cup \cdots \cup \mathscr{I}_{i-1}\}$ for $i \in [2, k]$. We know from Lemma 7 that $|I \cap T_q| = t_i^{(q)}$ for all $I \in V_i$, $i \in [1, k]$ and $q \in [1, p]$. Hence, by taking the scalar product of both sides of (1) with the incidence vector x^{T_q} of the set T_q, we obtain:

$$\delta_q(e) = \sum_{i=1}^{k} a_i t_i^{(q)} \quad \text{for all } q \in [1, p] \tag{3}$$

where $\delta_q(e) = 1$ if $e \in T_q$ and $\delta_q(e) = 0$ otherwise. Therefore, for each $e \in E$, the set of relations (2), (3) forms a system (4) of $p + 1$ equations with the unknowns $a_1, \ldots, a_k$:

$$\begin{cases} a_1 + \cdots + a_k = c_e \\ t_1^{(1)} a_1 + \cdots + t_k^{(1)} a_k = \delta_1(e) \\ \vdots \qquad \vdots \qquad \vdots \\ t_1^{(p)} a_1 + \cdots + t_k^{(p)} a_k = \delta_p(e). \end{cases} \tag{4}$$

Notice that a different choice of element e affects only the right-hand side of the system (4) and that the unknowns $a_1, \ldots, a_k$ take distinct solution values for distinct choices of element e in E.

Let A be the $(p+1) \times k$ coefficient matrix of the system (4) and $b(e) \in \mathbf{R}^{p+1}$ be its right-hand side vector. Define also the $p \times k$ matrix A_1 as the coefficient matrix of the last p equations of (4). We now have the following two claims:

CLAIM 12. The matrix A_1 has rank p and thus $p \leqslant k \leqslant m$.

PROOF. We show that the set $\{\mathscr{W}_1, \ldots, \mathscr{W}_p\}$ of row vectors of the matrix A_1 is linearly independent in $\mathbf{R}^k$. Let $\mu_1, \ldots, \mu_p$ be scalars such that $\mu_1 \mathscr{W}_1 + \cdots + \mu_p \mathscr{W}_p = 0_k$. We deduce from (4) that:

$$\mu_1 \delta_1(e) + \cdots + \mu_p \delta_p(e) = 0 \quad \text{for all } e \in E. \tag{5}$$

We show by induction on $q \geqslant 1$ that $\mu_q = 0$. Take first an element e of $T_1 - T_2$; then $\delta_q(e) = 0$ for $q \in [2, p]$ and thus (5) implies that $\mu_1 = 0$. Suppose by induction that $\mu_1 = \cdots = \mu_q = 0$. Choose an element $e \in T_{q+1} - T_{q+2}$; then (5) yields again that $\mu_{q+1} = 0$. ■

CLAIM 13. The matrix A has rank p.

PROOF. For this, we verify that the set of rows of A is linearly dependent. For all $i \in [1, k]$, we can choose an independent set $I \in V_i$ since $V_i \neq \varnothing$. Hence we have from (0) that $1 = c \cdot x^I = \Sigma_{q=1}^{p}(\lambda_q - \lambda_{q-1})t_i^{(q)}$ for all $i \in [1, k]$ from which we deduce the relation: $1_k = \Sigma_{q=1}^{p}(\lambda_q - \lambda_{q-1})\mathscr{W}_q$ which is therefore a nontrivial linear combination between the rows $1_k, \mathscr{W}_1, \ldots, \mathscr{W}_p$ of A.

It follows from Claim 12 that there exist p indices in $[1, k]$, say, for simplicity, $1, 2, \ldots, p$, such that the following determinant:

$$d = \begin{vmatrix} t_1^{(1)} & \cdots & t_p^{(1)} \\ t_1^{(2)} & \cdots & t_p^{(2)} \\ \vdots & & \vdots \\ t_1^{(p)} & \cdots & t_p^{(p)} \end{vmatrix}$$

is not equal to zero. Since equations (2) and (3) show that the system (4) admits a solution for all $e \in E$, then, for each $e \in E$, the rank of the matrix $[A, b(e)]$ (obtained by adding the vector $b(e)$ as a new column to A) is equal to the rank of A which is equal to p from Claim 13. Therefore, every square $(p + 1) \times (p + 1)$ submatrix of $[A, b(e)]$ is singular and, in particular, we deduce that the determinant:

$$\begin{vmatrix} c_e & 1 \cdots 1 \\ \delta_1(e) & \\ \vdots & d \\ \delta_p(e) & \end{vmatrix}$$

is equal to zero. By developing it with respect to the first column and by defining, for each $q \in [1, p]$, the determinant d_q obtained by replacing in the determinant d its qth row by the vector 1_p, we obtain the relation: $c_e d - \Sigma_{q=1}^{p}\delta_q(e)d_q = 0$, i.e.:

$$c_e = \sum_{q=1}^{p} \delta_q(e)d_q/d. \tag{6}$$

In order to finish the proof of Theorem 11, we have to verify that the conditions (a) and (b) for the admissibility of the pair $((T_1, \ldots, T_p), (1, \ldots, p))$ are satisfied and that $c \cdot x = \Sigma_{q=1}^{p} d_q/d(\Sigma_{e \in T_q} x_e)$ holds. We use (6) for computing the values of the coefficients λ_q. Take an element $e \in T_q - T_{q+1}$ with $q \in [1, p]$; hence $c_e = \lambda_q$ and $e \in T_1 \cap \cdots \cap T_q$, $e \notin T_{q+1} \cup \cdots \cup T_p$ and thus:

$$\lambda_q = \sum_{i=1}^{q} d_i/d. \tag{7}$$

We deduce from (0) and (7) that:

$$c \cdot x = \sum_{q=1}^{p} d_q/d \sum_{e \in T_q} x_e. \tag{8}$$

Condition (a) of Definition 8 follows from (7) and the fact that $0 < \lambda_1 < \cdots < \lambda_p$. Let us now verify condition (b). For this, given $i \in [1, m]$, we can choose a basis $I_p \in \mathcal{I}_i$ of $T_p \cap E_i$ and then, recursively, for $q \in [1, p-1]$, a basis $I_q \in \mathcal{I}_i$ of $T_q \cap E_i$ obtained by extending I_{q+1}. Therefore, we have by construction that $|I_1 \cap T_q| = t_i^{(q)}$ for $q \in [1, p]$. From the validity of the inequality $c \cdot x \leqslant 1$, we deduce that $c \cdot x^{I_1} \leqslant 1$, i.e., from (8), $\sum_{q=1}^{p} d_q/dt_i^{(q)} \leqslant 1$ which completes (b). Notice that, from Claims 12 and 13, condition (b) is indeed satisfied with equality for all $i \in [1, k]$. ■

3. On the facets of $P(\mathcal{I})$ when $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. In this section, we consider an IS $\mathcal{I}$ that can be decomposed as the union of two matroids: $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$, $\mathcal{I}_i$ being a matroid on $E_i \subseteq E$ with $r_i(\cdot)$ as rank function for $i = 1, 2$. Moreover, the rank function of $\mathcal{I}$ is denoted by $r(\cdot)$ and we recall that $r(S) = \max(r_1(S \cap E_1), r_2(S \cap E_2))$ for all subsets $S \subseteq E$.

In the following, for the sake of brevity, we will use the following notations: if S, T are subsets of E, then s_i, t_i will respectively denote $r_i(S \cap E_i)$, $r_i(T \cap E_i)$, for $i = 1, 2$.

It is easy to verify that, for all sets $S, T \subseteq E$, the pair (S, T) (or, following the notations of Definition 8, the pair $((S, T), (1, 2)))$ forms an admissible family if and only if $(s_1 - s_2)(t_1 - t_2) < 0$ holds. Hence a direct rewriting of Proposition 9 and Theorem 11 yields the following results.

PROPOSITION 14. *Let S, T be two subsets of E such that $(s_1 - s_2)(t_1 - t_2) < 0$. Then the inequality: $c(S, T) \cdot x \leqslant 1$ is valid for $P(\mathcal{I})$, where*:

$$c(S,T) \cdot x = \frac{t_2 - t_1}{s_1 t_2 - s_2 t_1} \sum_{e \in S} x_e + \frac{s_1 - s_2}{s_1 t_2 - s_2 t_1} \sum_{e \in T} x_e.$$

REMARK. A necessary and sufficient condition for an independent set $I \in \mathcal{I}_i$ with $i \in \{1, 2\}$ to verify $c(S, T) \cdot x^I = 1$ is that $|I \cap S| = s_i$ and $|I \cap T| = t_i$.

THEOREM 15. *We have*:

$$P(\mathcal{I}) = \Big\{ x \in \mathbf{R}^E \colon x_e \geqslant 0 \text{ for all } e \in E$$

$$(16) \qquad \sum_{e \in S} x_e \leqslant r(S) \text{ for all subsets } S \subseteq E$$

$$(17) \qquad c(S,T) \cdot x \leqslant 1 \text{ for all subsets } S, T \text{ of } E \text{ such that } S \supset T \text{ and}$$

$$(s_1 - s_2)(t_1 - t_2) < 0 \Big\}.$$

We now want to characterize those valid inequalities (16), (17) that induce facets of $P(\mathcal{I})$.

THEOREM 19. *Let V be the face induced by the valid inequality* (16). *Then V is a facet of $P(\mathcal{I})$ if and only if the following conditions hold*:

(a) *either $s_1 > s_2$, $E = E_1$ and S is a closed, nonseparable set in the matroid $\mathcal{I}_1$.*

(b) *or $s_2 > s_1$, $E = E_2$ and S is a closed, nonseparable set in the matroid $\mathcal{I}_2$.*

(c) *or $s_1 = s_2$ and*

  (ci) *S is closed in $\mathcal{I}$,*

  (cii) *S is nonseparable in $\mathcal{I}$,*

  (ciii) *if A, B are two subsets of S that satisfy: $|I \cap A| = a_i$, $|I \cap B| = b_i$ for all I, $x^I \in V_i$, $i = 1, 2$ and $(a_1 - a_2)(b_1 - b_2) < 0$, then they are disjoint.*

PROOF. Let us first assume that $s_1 \neq s_2$; for instance, $s_1 > s_2$. Then it follows that $V_2 = \varnothing$ and thus $V = V_1$. Therefore, V is a facet of $P(\mathscr{I})$ if and only if $E = E_1$ and $V = V_1$ is a facet of $P(\mathscr{I}_1)$, i.e., according to Theorem 1, if and only if S is a closed, nonseparable set in the matroid $\mathscr{I}_1$, which completes condition (a).

We can now assume that $s_1 = s_2$. We prove that V is a facet if and only if conditions (ci), (cii), and (ciii) hold.

If condition (ci) is violated, then there exists an element $e_0 \in E - S$ such that $r(S \cup e_0) = r(S)$. Hence, we have that $V \subseteq \{x \in P: \Sigma_{e \in S \cup e_0} x_e = r(S)\}$ implying that V is not a facet.

If condition (cii) is violated, then there exist two nonempty disjoint sets A, B such that $S = A \cup B$ and $r(S) = r(A) + r(B)$. Thus the inequality $\Sigma_{e \in S} x_e \leqslant r(S)$ is the sum of the two valid inequalities $\Sigma_{e \in A} x_e \leqslant r(A)$ and $\Sigma_{e \in B} x_e \leqslant r(B)$ implying again that V is not a facet.

Assume now that (ciii) is not satisfied. Hence, there exist two subsets A, B of S such that the inequality $c(A, B) \cdot x \leqslant 1$ is valid and, furthermore $V \subseteq \{x \in P: c(A, B) \cdot x = 1\}$. It is easy to see that, since A, B are not disjoint, the inequality $c(A, B) \cdot x \leqslant 1$ is not boolean and therefore the linear forms $\Sigma_{e \in S}(x_e/r(S))$ and $c(A, B) \cdot x$ are distinct, yielding that V is not a facet.

We now show that, if conditions (ci), (cii), and (ciii) are satisfied, then V is a facet. For this, we consider another valid inequality $a \cdot x \leqslant 1$ defining V and we prove that $a \cdot x = \Sigma_{e \in S}(x_e/r(S))$ holds. Take an element $e \in E - S$. Since S is closed, there exists an independent set I such that $x^I \in V$ and $e \in I$; also, $x^{I-e} \in V$ since $e \notin S$. Therefore, we have that $1 = a \cdot x^I = a \cdot x^{I-e}$ which implies that $a_e = 0$. Hence, we have that $a_e = 0$ for all $e \in E - S$ and it is easy to verify that $a_e > 0$ for all $e \in S$. We now show that $a_e = a_{e'}$ for all e, $e' \in S$ since this implies that $a \cdot x = \Sigma_{e \in S}(x_e/r(S))$. Suppose by contradiction that $\alpha = \min(a_e\colon e \in S) < \beta = \max(a_e\colon e \in S)$ and define the sets $A = \{e \in E\colon a_e > \alpha\}$, $B = \{e \in E\colon a_e = \beta\}$, $A' = S - A$ and $B' = S - B$.

CLAIM 20. Define, for any scalar $a_0 > 0$, the sets $T = \{e \in E\colon a_e > a_0\}$ and $T' = S - T$. Then, we have $|I \cap T| = t_i$, $|I \cap T'| = t_i'$ for all I, $x^I \in V_i$, $i = 1, 2$ and, furthermore, $r(S) = t_1 + t_1' = t_2 + t_2'$.

PROOF. It follows from Lemma 7 that $|I \cap T| = t_i$ for all I, $x^I \in V_i$, $i = 1, 2$. For proving Claim 20, it is enough to verify that $r(S) = t_1 + t_1' = t_2 + t_2'$. Suppose by contradiction that, for some $i \in \{1, 2\}$, $r(S) < t_i + t_i'$. Choose a basis $J \in \mathscr{I}_i$ of $T' \cap E_i$ and a basis $I \in \mathscr{I}_i$ of $S \cap E_i$ obtained by extending J; then $x^I \in V$ since $|I \cap S| = s_i = r(S)$. Hence, we have by assumption that $|I - J| < t_i$ and thus there exists an element $e \in T \cap E_i - (I - J)$ such that $I - J + e \in \mathscr{I}_i$. By applying recursively axiom (I2) in the matroid $\mathscr{I}_i$, we deduce that $I' = I + e - e'$ belongs to $\mathscr{I}_i$ for some element $e' \in I - T$. By validity of the inequality $a \cdot x \leqslant 1$, we have that $a \cdot x^{I'} \leqslant 1 = a \cdot x^I$ which implies that $a_e \leqslant a_{e'}$, contradicting the fact that $e \in T$, $e' \notin T$, i.e., $a_0 < a_e$ and $a_{e'} \leqslant a_0$. ∎

Let I be a set such that $x^I \in V_i$, it follows from Claim 20 that $|I \cap A| = a_i$, $|I \cap B| = b_i$, $|I \cap A'| = a_i'$, $|I \cap B'| = b_i'$ for $i = 1, 2$ and furthermore $r(S) = a_i + a_i' = b_i + b_i'$ for $i = 1, 2$. We finish the proof by considering three cases:

– either: $a_1 = a_2$. Then, we also have that $a_1' = a_2'$ and thus $r(S) = r(A) + r(A')$, yielding a contradiction with condition (cii).

– or: $b_1 = b_2$. Similarly, this contradicts (cii).

– or: $a_1 \neq a_2$, $b_1 \neq b_2$. Assume for instance that $a_1 > a_2$. Since the sets A, B are not disjoint, it follows from condition (ciii) that $b_1 > b_2$. Use again (ciii) for obtaining that A, B' are disjoint and therefore form a partition of S. Hence, we have $a \cdot x = \alpha \Sigma_{e \in B'} x_e + \beta \Sigma_{e \in B} x_e$. Choose a set $I \in V_i$ for obtaining that $1 = a \cdot x^I = \alpha b_i' + \beta b_i$

for $i = 1, 2$ from which we deduce that $\alpha = \beta = 1/r(S)$, contradicting our assumption $\alpha < \beta$. ■

REMARK. The following example shows that conditions (ci), (cii) are not sufficient for the validity of Theorem 19. Let $\mathscr{I}$ be the IS on $E = [1, 6]$ with bases: 123, 124, 134, 234 and 456. Then $\mathscr{I}$ is the union of the two matroids $\mathscr{I}_1$ on $E_1 = [1, 4]$ with bases 123, 124, 134, 234 and $\mathscr{I}_2$ on $E_2 = [4, 6]$ with basis 456. Then, the rank inequality $\sum_{i=1}^{6} x_i \leqslant 3$ does not induce a facet of $P(\mathscr{I})$ and, however, the set $S = \{1, 2, 3, 4, 5, 6\}$ is closed and nonseparable in $\mathscr{I}$.

THEOREM 21. *Let V be the face induced by the valid inequality* (17). *Then V is a facet of $P(\mathscr{I})$ if and only if the following condition is satisfied*:

(i) *the only nonempty subsets A of E such that $|I \cap A| = a_i$ for all I such that $x^I \in V_i$, $i = 1, 2$, are $A = S, T$ or $S - T$.*

PROOF. We can assume w.l.o.g. that $s_1 > s_2$ and $t_2 > t_1$. Notice that, since S and T are not disjoint, the inequality $c(S, T) \cdot x \leqslant 1$ is not boolean. We first prove that if V is a facet and A is a nonempty set such that $|I \cap A| = a_i$ for all I, $x^I \in V_i^i$, $i = 1, 2$, then $A = S, T$ or $S - T$. We consider three cases:

– either $a_1 = a_2$. Then we have that $V \subseteq \{x \in P\colon \sum_{e \in A} x_e = r(A)\}$, contradicting the fact that V is a facet.

– or: $a_1 > a_2$. Then the inequality $c(A, T) \cdot x \leqslant 1$ is valid and, furthermore, $V \subseteq \{x \in P\colon\ c(A, T) \cdot x = 1\}$. Since V is a facet, this implies that $c(S, T) \cdot x$, $c(A, T) \cdot x$ are identical linear forms, i.e.:

$$\frac{t_2 - t_1}{s_1 t_2 - s_2 t_1}\left(\sum_{e \in S} x_e\right) + \frac{s_1 - s_2}{s_1 t_2 - s_2 t_1}\left(\sum_{e \in T} x_e\right) \tag{22}$$

$$= \frac{t_2 - t_1}{a_1 t_2 - a_2 t_1}\left(\sum_{e \in A} x_e\right) + \frac{a_1 - a_2}{a_1 t_2 - a_2 t_1}\left(\sum_{e \in T} x_e\right)$$

We have that $A \subseteq S$; else, if there exists an element $e \in A - S$, then, by taking the scalar product of both sides of (22) with the vector $x^{\{e\}}$, we obtain that $t_2 - t_1 = 0$, yielding a contradiction. Similarly, $S - T \subseteq A$ holds. We show that $A = S$ or $A = S - T$. Suppose by contradiction that $S - T \subset A \subset S$. Then we can choose some elements $e \in T - A$, $e' \in A \cap T$. By taking the scalar product of (22) by the vectors $x^{\{e\}}$, $x^{\{e'\}}$, we obtain, respectively:

$$\frac{t_2 - t_1 + s_1 - s_2}{s_1 t_2 - s_2 t_1} = \frac{a_1 - a_2}{a_1 t_2 - a_2 t_1}, \tag{23}$$

$$\frac{t_2 - t_1 + s_1 - s_2}{s_1 t_2 - s_2 t_1} = \frac{t_2 - t_1 + a_1 - a_2}{a_1 t_2 - a_2 t_1}, \tag{24}$$

from which we deduce that $t_2 - t_1 = 0$, yielding a contradiction.

– or: $a_2 > a_1$. Then, the inequality $c(S, A) \cdot x \leqslant 1$ is valid and $V \subseteq \{x \in P\colon c(S, A) \cdot x = 1\}$ which yields that $c(S, T) \cdot x$, $c(S, A) \cdot x$ are identical linear forms from which we deduce, by using similar arguments than before, that $A = S$ or $A = T$.

Let us now assume that (i) holds and show that V is indeed a facet. For this, consider another valid inequality $a \cdot x \leqslant 1$ defining V. We prove that $a \cdot x = c(S, T) \cdot x$.

CLAIM 22. We have $a_e = 0$ for all $e \in E - S$.

PROOF. Take $e \in E - S$. Assume first that, for some $i \in \{1,2\}$, $r_i((S \cup e) \cap E_i) > s_i$. Then, we can find an independent set I, $x^I \in V_i$ such that $I \cup e \in \mathscr{I}_i$ and, in fact, $x^{I \cup e} \in V_i$ since $e \notin S$. We deduce that $a \cdot x^I = a \cdot x^{I \cup e} = 1$ and thus $a_e = 0$. Suppose now that $r_i((S \cup e) \cap E_i) = s_i$ for $i = 1,2$. Define the set $A = S \cup e$. We have that $|I \cap A| = s_i = r_i(A \cap E_i)$ for all I, $x^I \in V_i$, $i = 1,2$, contradicting condition (i). ∎

We define the scalars $\alpha = \min(a_e\colon\ e \in S)$, $\beta = \max(a_e\colon\ e \in S)$ and the sets $A = \{e \in E\colon\ a_e > \alpha\}$, $B = \{e \in E\colon\ a_e = \beta\}$. We deduce from Lemma 7 and condition (i) that the sets A, B are equal to $\varnothing$, T, S or $S - T$. We can immediately discard the cases $A = S$ and $B = \varnothing$. Moreover, if $A = \varnothing$ or $B = S$, then, in view of Claim 22, the inequality $a \cdot x \leqslant 1$ is boolean, which implies that $s_1 = s_2$, yielding a contradiction. Since $B \subset A$, we have that $A = B = T$ or $A = B = S - T$. If $A = B = T$, then we have that $a \cdot x = \beta(\sum_{e \in T} x_e) + \alpha(\sum_{e \in S-T} x_e)$ from which it can be easily deduced that

$$\alpha = \frac{t_2 - t_1}{s_1 t_2 - s_2 t_1}, \quad \beta = \frac{s_1 - s_2 + t_2 - t_1}{s_1 t_2 - s_2 t_1}$$

and thus $a \cdot x = c(S,T) \cdot x$. If $A = B = S - T$, then we obtain that

$$\alpha = \frac{s_1 - s_2 + t_2 - t_1}{s_1 t_2 - s_2 t_1}, \quad \beta = \frac{t_2 - t_1}{s_1 t_2 - s_2 t_1}$$

which contradicts $\alpha \leqslant \beta$. ∎

Finally, we give an example for illustrating this method.

EXAMPLE. Let $\mathscr{I}$ be the independence system on $E = \{1,2,3,4,5,6\}$ whose bases are: 123, 124, 134, 456. We see that $\mathscr{I}$ is the union of the two matroids: $\mathscr{I}_1$ on $E_1 = \{1,2,3,4\}$ with bases: 123, 124, 134 and $\mathscr{I}_2$ on $E_2 = \{4,5,6\}$ with basis: 456. By using Theorem 19, we see that boolean facets for $P(\mathscr{I})$ are the rank inequalities (16) for the sets: $S = \{4\}, \{1,5\}, \{1,6\}, \{2,5\}, \{2,6\}, \{3,5\}, \{3,6\}$. By using Theorem 20, we see that the nonboolean facets for $P(\mathscr{I})$ are the inequalities: $c(S,T) \cdot x \leqslant 1$ for the sets: $S = \{1,2,3,4,5\}$, $T = \{5\}$ and $S = \{1,2,3,4,6\}$, $T = \{16\}$. Hence we have the following description of P:

$$P(\mathscr{I}) = \Big\{ x \in R^6\colon\ x_i \geqslant 0 \text{ for all } i \in [1,6]$$

$$x_4 \leqslant 1$$

$$x_1 + x_5 \leqslant 1$$

$$x_1 + x_6 \leqslant 1$$

$$x_2 + x_5 \leqslant 1$$

$$x_2 + x_6 \leqslant 1$$

$$x_3 + x_5 \leqslant 1$$

$$x_3 + x_6 \leqslant 1$$

$$\frac{x_1 + x_2 + x_3 + x_4}{3} + \frac{2}{3} x_5 \leqslant 1$$

$$\frac{x_1 + x_2 + x_3 + x_4}{3} + \frac{2}{3} x_6 \leqslant 1 \Big\}.$$

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