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Extension operations for cuts

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Abstract

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We consider two extension operations for the cut cone $\mathscr{C}_n$: the antipodal extension and the k-extension. Their interesting feature is that they permit to express questions about the variety of realizations in $\mathscr{C}_n$ in terms of questions about the existence of a realization in $\mathscr{C}_{n+1}$. These operations are described as well as several applications.

1. Introduction and preliminaries

Set $V = \{1, 2, \ldots, n\}$. Given a subset S of V, the *cut* $\delta(S)$ determined by S is the set of all pairs (i, j), $1 \leq i < j \leq n$, such that $i \in S$, $j \in V - S$ or vice-versa. For simplicity, we also denote by $\delta(S)$ the incidence vector of the cut determined by S, so $\delta(S) \in \{0, 1\}^{\binom{n}{2}}$ with $\delta(S)_{ij} = 1$ if $i \in S$, $j \in V - S$ or vice-versa, $\delta(S)_{ij} = 0$ otherwise, for $1 \leq i < j \leq n$. The cut $\delta(S)$ is called *k-uniform* if $|S| = k$ or $n - k$.

Let $\mathscr{K}_n$ (resp. $\mathscr{K}_n^k$) denote the family of all cuts (resp. all k-uniform cuts). We use the following notation. $\mathscr{C}_n := \mathbb{R}_+(\mathscr{K}_n)$ denotes the cone generated by all cuts, i.e., $\mathscr{C}_n$ consists of the nonnegative combinations of cuts, $\mathscr{C}_n$ is called the *cut cone*. $\mathscr{L}_n := \mathbb{Z}(\mathscr{K}_n)$ denotes the *cut lattice*, i.e., consisting of all integer combinations of cuts. Similarly, let $\mathscr{C}_n^k := \mathbb{R}_+(\mathscr{K}_n^k)$, $\mathscr{L}_n^k = \mathbb{Z}(\mathscr{K}_n^k)$, denote the cone and the lattice

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generated by all k-uniform cuts, respectively. Finally, $\mathcal{P}_n$ denotes the cut polytope, defined as the convex hull of all cuts in $\mathcal{K}_n$.

Any decomposition of d as a combination of cuts, $d = \sum \lambda_S \delta(S)$, is called a *realization* of d and $\sum \lambda_S$ is called its *size*. More precisely, the realization is called a $\mathbb{R}_+$-(resp. $\mathbb{Z}$-, $\mathbb{Z}_+$-)*realization* if all $\lambda_S \in \mathbb{R}_+$ (resp. $\mathbb{Z}$, $\mathbb{Z}_+$).

A cone $C := \mathbb{R}_+(X)$ is said to be *simplicial* if the set X is linearly independent; a point $x \in C$ is said to be *simplicial* if x lies on a simplicial face of C, i.e., x admits a unique decomposition as nonnegative sum of elements of X.

Given $d = (d_{ij})_{1 \le i < j \le n} \in \mathbb{R}^{\binom{n}{2}}$, d is said to be *hypercube embeddable* or, *h-embeddable*, for short, if $d \in \mathbb{Z}_+(\mathcal{K}_n)$, i.e., d is a nonnegative integer combination of cuts. It is not difficult to see that, equivalently, d is h-embeddable if d can be embedded in some hypercube, i.e., there exist n binary vectors $x_1, \ldots, x_n \in \{0, 1\}^s$ for some s such that d_{ij} coincides with the Hamming distance between x_i, x_j for any i, j, hence justifying the terminology 'h-embeddable'. If d is h-embeddable, then d is called *rigid* if d admits a unique decomposition as a nonnegative integer sum of cuts; clearly, if d is simplicial, then d is rigid.

Given $d = (d_{ij})_{1 \le i < j \le n} \in \mathbb{R}^{\binom{n}{2}}$, d is said to be a *metric* on V if d satisfies the following triangle inequality:

$$\mathrm{Tr}(i, j; k) \cdot d := d_{ij} - d_{ik} - d_{jk} \le 0 \tag{1.1}$$

for every triple of distinct points $i, j, k \in V$. The *metric cone* $\mathcal{M}_n$ is the set of all the metrics defined on n points. Clearly every cut defines a metric on V, called *cut metric*; actually, the members of the cut cone $\mathcal{C}_n$ are precisely the metrics on n points that are *embeddable in some* ℓ_1*-space* [1].

Given a scalar $t > 0$, we denote by $t\mathbb{1}_n$ the equidistant metric on n points that takes value t on each pair of points. Given an integer $t \ge 1$, the equidistant metric $2t\mathbb{1}_n$ is always h-embeddable; for instance, $2t\mathbb{1}_n = \sum_{1 \le i \le n} t\delta(\{i\})$ is its simplest $\mathbb{Z}_+$-realization, called *star realization*. In general, $2t\mathbb{1}_n$ may have other $\mathbb{Z}_+$-realizations; let us introduce the parameter z_n^t defined as the minimum size of a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$. This parameter z_n^t and the variety of realizations of $2t\mathbb{1}_n$ have been extensively studied in [8], in particular, in connection with Design Theory.

The cut lattice is easily characterized; namely, if $d \in \mathbb{Z}^{\binom{n}{2}}$, then $d \in \mathcal{L}_n$ if and only if d satisfies the following *even condition*:

$$d_{ij} + d_{ik} + d_{jk} \equiv 0 \pmod 2 \tag{1.2}$$

for all $1 \le i < j < k \le n$.

In this paper, we consider two extension operations on cuts, namely the antipodal extension ant_α and the k-extension ext_k. Both map a given vector of $\mathbb{R}^{\binom{n}{2}}$ on some vector of $\mathbb{R}^{\binom{n'}{2}}$ where $n' > n$. The interesting feature of these extension operations is that they permit to express questions about the *variety* of realizations in $\mathcal{C}_n$ (about the minimum size of a realization, for the antipodal extension, and about the uniformity of a realization, for the k-extension) in terms of questions about the *existence* of a realization in $\mathscr{C}_{n+1}$ (or $\mathscr{C}_{n'}$, $n' > n$).

The paper is organized as follows. In Section 2, we consider the antipodal extension and give several applications, in particular, to the description of the path metric of some graphs and polytopes. In Section 3, we describe the k-extension operation and the link between the cut cone $\mathscr{C}_{n+1}$ and the k-uniform cut cone $\mathscr{C}_n^k$. Section 4 contains the characterization of the k-uniform cut lattice $\mathscr{L}_n^k$ and we group in Section 5 several results on the 1, 2-uniform cuts.

2. The antipodal extension

Given integers n, n' such that $\lceil n/2 \rceil \leq n' \leq n-1$, we consider the linear subspace $T_{n,n'}$ of $\mathbb{R}^{\binom{n}{2}}$ defined by:

$$T_{n,n'} = \{x \in \mathbb{R}^{\binom{n}{2}}: \operatorname{Tr}(i, i+n'; j) \cdot x = 0 \text{ for all } 1 \leq i \leq n-n',$$
$$1 \leq j \leq n, j \neq i, i+n'\}.$$

Let us set $V = \{1, 2, \ldots, n\}$, $V' = \{1, 2, \ldots, n-n'\}$ and $V'' = \{n'+1, \ldots, n\}$.

Lemma 2.1. *$T_{n,n'}$ is a subspace of dimension $\binom{n'}{2}+1$. In fact, for $d \in T_{n,n'}$, all coordinates of d can be expressed in terms of $(d_{ij})_{1 \leq i < j \leq n'}$ and of $d_{1,n'+1}$, as indicated in relation* (2.2).

$$\begin{aligned} d_{i+n',j+n'} &= d_{ij} && \text{for } 1 \leq i < j \leq n-n', \\ d_{i,i+n'} &= d_{1,n'+1} && \text{for } 1 \leq i \leq n-n', \\ d_{i,j+n'} &= d_{1,n'+1} - d_{ij} && \text{for } 1 \leq i \leq n', 1 \leq j \leq n-n', i \neq j. \end{aligned} \tag{2.2}$$

Proof. Take $d \in T_{n,n'}$. We show that all coordinates of d can be expressed in terms of d_{ij} for $1 \leq i < j \leq n'$ and $d_{1,n'+1}$ as in relation (2.2). For this, we use the fact that d satisfies all triangle equalities: $\operatorname{Tr}(i, i+n'; j) \cdot d = 0$ for $1 \leq i \leq n-n'$ and $1 \leq j \leq n$. Let us fix some i, j, $1 \leq i < j \leq n-n'$. First,

$$2(d_{i+n',j+n'} - d_{ij}) = \operatorname{Tr}(i, i+n'; j) \cdot d + \operatorname{Tr}(j, j+n'; i) \cdot d$$
$$- \operatorname{Tr}(i, i+n'; j+n') \cdot d - \operatorname{Tr}(j, j+n'; i+n') \cdot d,$$

from which we deduce that $d_{i+n',j+n'} = d_{ij}$. Also,

$$2(d_{i,j+n'} - d_{j,i+n'}) = \operatorname{Tr}(i, i+n'; j) \cdot d - \operatorname{Tr}(j, j+n'; i) \cdot d$$
$$- \operatorname{Tr}(i, i+n'; j+n') \cdot d + \operatorname{Tr}(j, j+n'; i+n') \cdot d,$$

implying that $d_{i,j+n'} = d_{j,i+n'}$. Also

$$d_{i,i+n'} - d_{j,j+n'} = \operatorname{Tr}(i, i+n'; j) \cdot d - \operatorname{Tr}(j, j+n'; i) \cdot d - d_{i,j+n'} + d_{i+n',j},$$

implying that $d_{i,i+n'} = d_{1,1+n'}$. Finally, if $1 \leq j \leq n - n'$ and $1 \leq i \leq n'$, then

$$d_{i,j+n'} = -\mathrm{Tr}(j, j+n'; i) \cdot d - d_{ij} + d_{j,j+n'} = d_{1,n'+1} - d_{ij}.$$

This concludes the proof. □

Corollary 2.3. *Take $d \in T_{n,n'}$ and set $d' = (d_{ij})_{1 \leq i < j \leq n'} \in \mathbb{R}^{\binom{n'}{2}}$. Then, $d \in \mathcal{M}_n$ if and only if $d' \in \mathcal{M}_{n'}$ and*

$$d_{1,n'+1} \geq \max(\tfrac{1}{2}(d_{ij} + d_{ik} + d_{jk})\colon 1 \leq i < j < k \leq n').$$

Proof. It is an easy verification. □

Definition 2.4. Given integers n, n' such that $\lceil n/2 \rceil \leq n' \leq n-1$, $\alpha \in \mathbb{R}$ and $d \in \mathbb{R}^{\binom{n'}{2}}$, we define its *antipodal extension* $\mathrm{ant}_\alpha^n(d) \in \mathbb{R}^{\binom{n}{2}}$ as the unique vector of $T_{n,n'}$ such that $\mathrm{ant}_\alpha^n(d)_{ij} = d_{ij}$ for all $1 \leq i < j \leq n'$ and $\mathrm{ant}_\alpha^n(d)_{1,n'+1} = \alpha$. In other words,

$$\begin{aligned} &\mathrm{ant}_\alpha^n(d)_{i+n',j+n'} = d_{ij} && \text{for } 1 \leq i < j \leq n-n', \\ &\mathrm{ant}_\alpha^n(d)_{i,i+n'} = \alpha && \text{for } 1 \leq i \leq n-n', \\ &\mathrm{ant}_\alpha^n(d)_{i,j+n'} = \alpha - d_{ij} && \text{for } 1 \leq i \leq n',\ 1 \leq j \leq n-n',\ i \neq j. \end{aligned}$$

For simplicity, we shall adopt the following notation. When $\alpha = 1$, then we omit the subscript '1', i.e., we denote $\mathrm{ant}_1^n(d)$ by $\mathrm{ant}^n(d)$. When $n = n'+1$, we omit the superscript 'n', i.e., we denote $\mathrm{ant}_\alpha^n(d)$ by $\mathrm{ant}_\alpha(d)$. So, for $d \in \mathbb{R}^{\binom{n'}{2}}$, $\mathrm{ant}(d) \in \mathbb{R}^{\binom{n'+1}{2}}$ stands for $\mathrm{ant}_1^{n'+1}(d)$. Finally, when $n' = \lceil n/2 \rceil$, we also denote $\mathrm{ant}_\alpha^n(d)$ by $\mathrm{Ant}_\alpha(d)$.

Remark 2.5. (i) The map $d \to \mathrm{ant}_\alpha^n(d)$ is an affine map from $\mathbb{R}^{\binom{n'}{2}}$ to $\mathbb{R}^{\binom{n}{2}}$; for $\alpha = 0$ it is linear and, in fact, we have the following relation:

$$\mathrm{ant}_\alpha^n(d) = \mathrm{ant}_0^n(d) + \alpha\delta(V'').$$

Also,

$$\mathrm{ant}_\alpha^n(d) = \mathrm{ant}_1^n(d) + (\alpha - 1)\delta(V'').$$

(ii) Given a subset S of $\{1, 2, \ldots, n'\}$, define the subset $S^* = \{i + n' \colon i \notin S$ *and* $1 \leq i \leq n - n'\}$ of $V'' = \{n'+1, \ldots, n\}$. So S^* can be seen as the 'reflection' of S on V''. Then, $\mathrm{ant}^n(\delta(S)) = \delta(S \cup S^*)$. In particular, $\mathrm{ant}^n(\delta(\emptyset)) = \delta(V'')$. Note that the only cuts lying on $T_{n,n'}$ are precisely of the form $\mathrm{ant}^n(\delta(S))$ for some subset S of $\{1, \ldots, n'\}$. Observe also that, if $n' = \lceil n/2 \rceil$, then $\mathrm{ant}^n(\delta(S))$ is always an equicut of $\mathscr{C}_n$.

(iii) Given scalars $t, \alpha \in \mathbb{R}_+$ and $d \in \mathbb{R}^{\binom{n'}{2}}$, we have: $t(\mathrm{ant}_\alpha^n(d)) = \mathrm{ant}_{t\alpha}^n(td)$.

Let $X \subseteq \mathscr{K}_{n'}$ be a set of nonzero cuts in $\mathscr{C}_{n'}$ and define $Y = \{\mathrm{ant}^n(\delta(S)) \colon \delta(S) \in X\} \cup \{\delta(V'')\}$, the set of the antipodal extensions of the cuts in X and of the zero cut; so $Y \subseteq \mathcal{K}_n$. Hence, $\mathbb{R}_+(X)$ (resp. $\mathbb{R}_+(Y)$) is a subcone of $\mathcal{C}_{n'}$ (resp. of $\mathcal{C}_n$) with $|X|$ (resp. $|X|+1$) generators and, in fact, $\dim(\mathbb{R}_+(Y)) = \dim(\mathbb{R}_+(X)) + 1$. Observe that, given $d \in \mathbb{R}^{\binom{n'}{2}}$, if

$$d = \sum_{\delta(S) \in X} \lambda_S \delta(S),$$

then

$$\begin{aligned} \operatorname{ant}^n_\alpha(d) &= \sum_{\delta(S)\in X} \lambda_S \operatorname{ant}^n_\alpha(\delta(S)) + \alpha(1 - \sum \lambda_S)\delta(V'') \\ &= \sum_{\delta(S)\in X} \lambda_S \operatorname{ant}^n(\delta(S)) + (\alpha - \sum \lambda_S)\delta(V''). \end{aligned}$$

Also, if

$$\operatorname{ant}^n_\alpha(d) = \sum_{\delta(S)\in X} \lambda_S \operatorname{ant}^n(\delta(S)) + \lambda_0 \delta(V''),$$

then $\alpha = \sum \lambda_S + \lambda_0$ and, by taking the projection on V',

$$d = \sum_{\delta(S)\in X} \lambda_S \delta(S).$$

In particular, any $\mathbb{R}_+$-realization of $\operatorname{ant}^n_\alpha(d)$ has size equal to α; indeed, if $\operatorname{ant}^n_\alpha(d) = \sum \mu_A \delta(A)$ with $\mu_A > 0$, then $\delta(A) \in T_{n,n'}$ is of the form $\operatorname{ant}^n(\delta(S))$ and, thus, $\alpha = \sum \mu_A$ holds. The above observations permit to state the following result.

Proposition 2.6. *Given* $\alpha \in \mathbb{R}$, $X \subseteq \mathcal{K}_{n'}$, $Y = \{\operatorname{ant}^n(\delta(S)) : \delta(S) \in X\} \cup \{\delta(V'')\}$, *and* $d \in \mathbb{R}^{\binom{n'}{2}}$, *the following assertions hold:*

(i) $\operatorname{ant}^n_\alpha(d) \in \mathbb{Z}(Y)$ *if and only if* $d \in \mathbb{Z}(X)$ *and* $\alpha \in \mathbb{Z}$.

(ii) $\operatorname{ant}^n_\alpha(d) \in \mathbb{Z}_+(Y)$ *if and only if* $\alpha \in \mathbb{Z}_+$, $d \in \mathbb{Z}_+(X)$ *and* d *admits a* $\mathbb{Z}_+$*-realization whose size is less or equal to* α.

(iii) $\operatorname{ant}^n_\alpha(d) \in \mathbb{R}_+(Y)$ *if and only if* $\alpha \in \mathbb{R}_+$, $d \in \mathbb{R}_+(X)$ *and* d *admits a* $\mathbb{R}_+$*-realization whose size is less or equal to* α.

(iv) *Given* $d \in \mathbb{R}_+(X)$, $\alpha \in \mathbb{R}_+$ *such that* d *admits a* $\mathbb{R}_+$*-realization of size less than or equal to* α, *then* $\operatorname{ant}^n_\alpha(d)$ *is a simplicial point of* $\mathbb{R}_+(Y)$ *if and only if* d *is a simplicial point of* $\mathbb{R}_+(X)$.

In particular, if $X = \mathcal{K}_{n'}$ with $n = 2n'$, then $Y = \{\operatorname{Ant}(\delta(S)) = \delta(S \cup S^*) : S \subseteq \{1, 2, \ldots, n'\}\}$; hence $\operatorname{Ant}(\mathcal{C}_{n'}) := \mathbb{R}_+(Y)$ is a subcone of $\mathcal{C}_n$ with $2^{n'-1}$ generators which are all equicuts of $\mathcal{C}_n$. Observe that $\operatorname{Ant}(\mathcal{C}_{n'}) = \mathcal{C}_n \cap T_{n,n'}$ is thus a face of $\mathcal{C}_n$ of dimension $\binom{n'}{2} + 1$.

Let us now treat some examples illustrating the power of the antipodal extension operation.

We first give an example of application of Proposition 2.6. Take $d \in \mathcal{C}^1_n$, $d = \sum_{1 \le i \le n} a_i \delta(\{i\})$ with all $a_i \in \mathbb{Z}_+$, and assume that d is rigid (for instance, assume that n is large with respect to $\max(a_1, \ldots, a_n)$ (see Proposition 5.3)).

Then,

$$\mathrm{ant}_\alpha(d) \in \mathcal{M}_{n+1} \text{ if and only if } \alpha \geq a_1 + \max(a_i + a_j: 2 \leq i < j \leq n)$$

and

$$\mathrm{ant}_\alpha(d) \text{ is } h\text{-embeddable if and only if } \alpha \in \mathbb{Z}_+ \text{ and } \alpha \geq a_1 + \cdots + a_n.$$

In particular, take $d = 2t\mathbb{1}_n$ with $n \geq t^2 + t + 3$, then d is rigid [4], and take $\alpha \in \mathbb{Z}_+$. Then,

$$\mathrm{ant}_\alpha(2t\mathbb{1}_n) \in \mathcal{M}_{n+1} \text{ if and only if } \alpha \geq 3t,$$

$$\mathrm{ant}_\alpha(2t\mathbb{1}_n) \text{ is } h\text{-embeddable if and only if } \alpha \geq nt, \text{ and}$$

$$\mathrm{ant}_\alpha(2t\mathbb{1}_n) \in \mathcal{C}_{n+1} \text{ if and only if } \alpha \geq 4t.$$

Indeed, the minimum size of a $\mathbb{R}_+$-realization of $2t\mathbb{1}_n$ is equal to

$$tn(n-1)/\lfloor n/2 \rfloor \lceil n/2 \rceil$$

and, thus, by Proposition 2.6(iii), $\mathrm{ant}^n_\alpha(d) \in \mathcal{C}_{n+1}$ if and only if

$$\alpha \geq tn(n-1)/\lfloor n/2 \rfloor \lceil n/2 \rceil,$$

i.e., $\alpha \geq 4t$ since α is an integer.

Given a graph G, let $d(G)$ denote its path metric, where, for two nodes i, j of G, $d(G)_{ij}$ denotes the length of a shortest path from i to j in G. We indicate how the path metrics of several graphs can be described using the antipodal extension operation. For example,

$$d(H(n,2)) = \mathrm{Ant}_n(d(H(n-1,2)))$$

and, for n odd,

$$d(H(n,2)) = \mathrm{Ant}_n(2d(\tfrac{1}{2}H(n,2)));$$

$$d(DO_{2n+1}) = \mathrm{Ant}_{2n+1}(2d(J(2n+1,n))); \qquad d(C_{2n}) = \mathrm{Ant}_n(d(P_n))$$

and, for n odd,

$$d(C_{2n}) = \mathrm{Ant}_n(2d(C_n)); \qquad d(K_{2\times n}) = \mathrm{Ant}_2(d(K_n)),$$

where, as usual, $H(n,q)$, $H(n,2)$, $\frac{1}{2}H(n,2)$, DO_{2n+1}, $J(n,d)$, C_n, P_n, $K_{2\times n}$, K_n, denote, respectively, the Hamming graph, the 1-skeleton of the n-dimensional cube, the n-dimensional halfcube, the double odd graph, the Johnson graph, the cycle, the path of length n, the cocktail party graph, the clique on n nodes (see e.g., [2] for definitions). Also, the path metric d_I of the icosahedron satisfies: $d_I = \mathrm{Ant}_3(d(\nabla C_5))$, where ∇C_5 denotes the graph obtained by adding a node adjacent to all nodes of C_5. In [9], it is shown that the following graphs are simplicial (i.e., that they lie on a simplicial face of the corresponding cut cone or, equivalently, that they admit a unique ℓ_1-embedding): Johnson graphs $J(n,d)$ for $d \geq 2$, halfcubes $\frac{1}{2}H(n,2)$ if and only if $n = 2$ or $n \geq 5$, ℓ_1-embeddable bipartite graphs (including even cycles, trees, . . .), Hamming graphs $H(n,q)$ if and only if $q \leq 3$, the Petersen graph, the 1-skeleton of the icosahedron or of the dodecahedron. Clearly, $d(K_n)$ is simplicial for $n \leq 3$ since the cut cone $\mathcal{C}_n$ is then simplicial, and $d(K_n)$ is not simplicial for $n \geq 4$ since it admits several decompositions as sum of cuts, e.g.,

$$d(K_n) = \tfrac{1}{2} \sum_{1 \leq i \leq n} \delta(\{i\}) = \frac{1}{2(n-2)} \sum_{1 \leq i \leq j \leq n} \delta(\{i, j\}).$$

As application of Proposition 2.6, we deduce that the cocktail party graph metric $d(K_{n \times 2})$ is simplicial if and only if $n \leq 3$, also that the double odd graph is always simplicial.

Note that the three graphs C_{2n}, $H(n, 2)$, DO_{2n+1} are bipartite and, moreover, they are the bipartite doubles of their halved graphs which are, respectively, C_n, $\frac{1}{2}H(n, 2)$, $J(2n+1, n)$. In fact, Weichsel [13] proved that C_{2n}, $H(n, 2)$, DO_{2n+1} are the only distance regular graphs that are hypercube embeddable; this was also proved by Koolen [10].

The following statements can be found in [8].

- $\mathrm{ant}_{4t-1}(2t\mathbb{1}_{4t})$ is h-embeddable if and only if there exists a Hadamard matrix of order $4t$.
- $\mathrm{ant}_{t^2+2t}(2t\mathbb{1}_{t^2+t+2})$ is h-embeddable if and only if there exists a projective plane of order t.

Recall that z_n^t denotes the minimum size of a $\mathbb{Z}_+$-realization of the equidistant metric $2t\mathbb{1}_n$. Since $d(K_{n \times 2}) = \mathrm{Ant}_2(\mathbb{1}_n)$, by Remark 2.5(iii), $2td(K_{n \times 2}) = \mathrm{Ant}_{4t}(2t\mathbb{1}_n)$ and, therefore, by Proposition 2.6(ii), $2td(K_{n \times 2})$ is h-embeddable if and only if $z_n^t \leq 4t$. Observe also that $\mathrm{ant}_4(2\mathbb{1}_n) = 2d(K_{n+1} - e)$, where $K_{n+1} - e$ denotes the complete graph on $n+1$ nodes with one deleted edge; $\mathrm{ant}_4(2\mathbb{1}_n)$ is h-embeddable for $n = 3, 4$ but not for $n \geq 5$ (though it satisfies the even condition and belongs to the cut cone). For $n = 5$, $2\,\mathrm{ant}_4(2\mathbb{1}_5)$ is h-embeddable and has exactly three $\mathbb{Z}_+$-realizations:

$$\begin{aligned} 2\,\mathrm{ant}_4(2\mathbb{1}_5) &= \sum_{2 \leq i \leq 5} \delta(\{1, i\}) + \delta(\{i, 6\}) \\ &= \sum_{2 \leq i \leq 4} \delta(\{i, 6\}) + \delta(\{i, 5, 6\}) + \delta(\{1\}) + \delta(\{1, 5\}) \\ &= \delta(\{1\}) + \delta(\{6\}) + \sum_{2 \leq i < j \leq 5} \delta(\{i, j, 6\}). \end{aligned}$$

We conclude the section with a reformulation of the condition for membership in the cut cone for the antipodal extension $\mathrm{ant}_\alpha(d)$ of $d \in \mathscr{C}_n$. By Proposition 2.6(iii), $\mathrm{ant}_\alpha(d) \in \mathscr{C}_{n+1}$ if and only if $d \in \mathscr{C}_n$ and the minimum size of a $\mathbb{R}_+$-realization of d is less than or equal to α. Given $d \in \mathscr{C}_n$, the minimum size s of a $\mathbb{R}_+$-realization of d is expressed by the following minimization program:

$$\begin{aligned} & s = \mathrm{Min} \sum \lambda_S, \\ & \text{s.t.} \sum \lambda_S \delta(S)_{ij} = d_{ij}, \quad \forall 1 \leq i < j \leq n, \\ & \text{and } \lambda_S \geq 0 \quad \forall S \subset \{2, \ldots, n\}. \end{aligned} \tag{2.7}$$

From Linear Programming Duality, the minimum size s can also be expressed as the following maximization program:

$$s = \text{Max} \sum_{1 \le i < j \le n} v_{ij} d_{ij}, \qquad \text{s.t.} \sum_{1 \le i < j \le n} v_{ij} \delta(S)_{ij} \le 1 \quad \forall S. \tag{2.8}$$

Equivalently,

$$s = \max(v \cdot d : v \cdot x \le 1 \text{ defines a facet of } \mathcal{P}_n). \tag{2.9}$$

(Note that the above computation of the minimum size remains valid for an arbitrary cone; namely, the minimum size of a $\mathbb{R}_+$-realization of $x \in \mathbb{R}_+(X)$ is equal to $\max(v \cdot x : v \cdot x \le 1$ defines a facet of $\text{Conv}(X))$.) The complete description of $\mathcal{P}_n$ is known for $n \le 7$ (see [7]). As example of application, we have that: for $d \in \mathcal{C}_4$, the minimum size of a $\mathbb{R}_+$-realization of d is equal to:

$$\max\left(\frac{d_{ij} + d_{ik} + d_{jk}}{2} : 1 \le i < j < k \le 4\right),$$

and for $d \in \mathcal{C}_5$, the minimum size is equal to:

$$\max\left(\frac{d_{ij} + d_{ik} + d_{jk}}{2} : 1 \le i < j < k \le 5, \frac{\sum_{1 \le i < j \le 5} d_{ij}}{6}, \frac{\sum_{1 \le i < j \le 5} d_{ij} - 2\sum_{1 \le j \le 5, j \ne i} d_{ij}}{2} : 1 \le i \le 5\right).$$

Therefore, $\text{ant}_\alpha(d) \in \mathcal{C}_{n+1}$ if and only if $d \in \mathcal{C}_n$ and

$$\alpha \ge \max(v \cdot d : v \cdot x \le 1 \text{ defines a facet of } \mathcal{P}_n). \tag{2.10}$$

On the other hand, $\text{ant}_\alpha(d) \in \mathcal{C}_{n+1}$ if and only if $v \cdot \text{ant}_\alpha(d) \le 0$ for every valid inequality $v \cdot x \le 0$ for $\mathcal{C}_{n+1}$. Let us compute $v \cdot \text{ant}_\alpha(d)$:

$$\begin{aligned} v \cdot \text{ant}_\alpha(d) &= \sum_{1 \le i < j \le n} v_{ij} d_{ij} + v_{1,n+1}\alpha + \sum_{2 \le i \le n} v_{i,n+1}(\alpha - d_{1i}) \\ &= \sum_{2 \le i < j \le n} v_{ij} d_{ij} + \alpha v \cdot \delta(\{n+1\}) + \sum_{2 \le i \le n} (v_{1i} - v_{i,n+1}) d_{1i}. \end{aligned}$$

Therefore, if $\text{ant}_\alpha(d) \in \mathcal{C}_{n+1}$, then we have:

$$\alpha \ge \max\left(\frac{\sum_{2 \le i \le n}(-v_{1i} + v_{i,n+1})d_{1i} - \sum_{2 \le i < j \le n} v_{ij} d_{ij}}{v \cdot \delta(\{n+1\})} : v \cdot x \le 0 \text{ is valid for } \mathcal{C}_{n+1} \text{ and } v \cdot \delta(\{n+1\}) \ne 0\right). \tag{2.11}$$

How do the two maxima in (2.10), (2.11) compare? In fact, as we now check, the maximum in (2.10) always dominates the maximum in (2.11). Indeed, take a valid inequality $v \cdot x \leqslant 0$ for $\mathscr{C}_{n+1}$ such that $v \cdot \delta(\{n+1\}) = -1$. First, let us 'switch' this inequality by the cut $\delta(\{n+1\})$, i.e., consider $v' \in \mathbb{R}^{\binom{n+1}{2}}$ defined by:

$$v'_{i,n+1} = -v_{i,n+1} \quad \text{for } 1 \leqslant i \leqslant n,$$
$$v'_{ij} = v_{ij} \quad \text{for } 1 \leqslant i < j \leqslant n.$$

Then, collapse the nodes $1, n+1$ in a single node 1, i.e., define $v'' \in \mathbb{R}^{\binom{n}{2}}$ by:

$$v''_{1i} = v'_{1i} + v'_{i,n+1} \quad \text{for } 2 \leqslant i \leqslant n,$$
$$v''_{ij} = v_{ij} \quad \text{for } 2 \leqslant i < j \leqslant n.$$

Then, the inequality $v'' \cdot x \leqslant 1$ is valid for the cut polytope $\mathscr{P}_n$. Note that the quantity to be maximimized in (2.11) takes precisely the form $v'' \cdot x$ if $v \cdot x \leqslant 0$ is valid for $\mathscr{C}_{n+1}$ and $v \cdot \delta(\{n+1\}) = -1$.

3. The k-extension operation

In this section, we introduce the k-extension operation and we shall see how it permits to link the k-uniform cut cone $\mathscr{C}_n^k$ and the cut cone $\mathscr{C}_{n+1}$.

Given an integer $1 \leqslant k \leqslant n-1$ such that $k \neq n/2$ and $d \in \mathbb{R}^{\binom{n}{2}}$, we define its *k-extension* $\text{ext}_k(d) \in \mathbb{R}^{\binom{n+1}{2}}$ as follows:

$$(\text{ext}_k(d))_{ij} = d_{ij} \quad \text{for } 1 \leqslant i < j \leqslant n,$$

$$(\text{ext}_k(d))_{i,n+1} = \frac{1}{n-2k}\left(\sum_{1 \leqslant j \leqslant n, j \neq i} d_{ij} - \frac{1}{n-k}\sum_{1 \leqslant r < s \leqslant n} d_{rs}\right) \quad \text{for } 1 \leqslant i \leqslant n.$$

Let us then define $\text{int}_k(d) \in \mathbb{R}^{\binom{n}{2}}$ by

$$(\text{int}_k(d))_{ij} = \tfrac{1}{2}(\text{ext}_k(d)_{i,n+1} + \text{ext}_k(d)_{j,n+1} - \text{ext}_k(d)_{ij}) \quad \text{for } 1 \leqslant i < j \leqslant n.$$

We shall use the notion $\text{int}_k(d)$ especially in Sections 4, 5. Set

$$S_k(d) = \frac{1}{k(n-k)} \sum_{1 \leqslant i < j \leqslant n} d_{ij}.$$

We group in the following lemma several relations linking $\text{ext}_k(d)$, $\text{int}_k(d)$, $S_k(d)$ and d.

Lemma 3.1. (i) $S_k(d) = (\text{ext}_k(d))_{i,n+1} + (\text{ext}_{n-k}(d))_{i,n+1}$ *for* $1 \leqslant i \leqslant n$.

(ii) $S_k(d) = d_{ij} + (\text{int}_k(d))_{ij} + (\text{int}_{n-k}(d))_{ij}$ *for* $1 \leqslant i < j \leqslant n$.

(iii) $\sum_{1 \leqslant j \leqslant n, j \neq i} (\text{int}_k(d))_{ij} = (k-1)(\text{ext}_k(d))_{i,n+1}$ *for* $1 \leqslant i \leqslant n$.

(iv) $\sum_{1 \leqslant i \leqslant n} (\text{ext}_k(d))_{i,n+1} = kS_k(d)$.

(v) $\sum_{1 \leqslant i < j \leqslant n} (\text{int}_k(d))_{ij} = \binom{k}{2} S_k(d)$.

Take $d = \delta(S)$ for $S \subseteq \{1, \ldots, n\}$ and set $s = |S|$. Then,

$$(\mathrm{ext}_k(d))_{i,n+1} = \begin{cases} \dfrac{(n-s)(n-k-s)}{(n-k)(n-2k)} & \text{if } i \in S, \\ \dfrac{s(s-k)}{(n-k)(n-2k)} & \text{if } i \notin S. \end{cases}$$

Therefore, if $|S| = s = k$, then $\mathrm{ext}_k(\delta(S)) = \delta(S)$ (as cut in $\mathscr{C}_{n+1}$) and, if $s = n-k$, then $\mathrm{ext}_{n-k}(\delta(S)) = \delta(S \cup \{n+1\})$; also, for $s = k$, $\mathrm{int}_k(\delta(S))_{ij}$ is equal to 1 if $i, j \in S$ and to 0 otherwise. Observe that int_k is a linear injective map (use Lemma 3.1(ii), (v)). The map ext_k is a linear map from $\mathscr{C}_n^k$ to $\mathscr{C}_{n+1}^k$. The next Proposition 3.7 establishes precisely which elements of $\mathscr{C}_n$ have their k-extension in $\mathscr{C}_{n+1}$. For this, we need the following notion of quasi k-uniformity which is weaker than membership in the k-uniform cut cone $\mathscr{C}_n^k$.

Definition 3.2. Given a realisation of $d \in \mathbb{R}^{\binom{n}{2}}$, $d = \sum_S \lambda_S \delta(S)$, where the sum is taken over subsets S of $\{1, \ldots, n\}$ with $|S| \leq n/2$, we say that it is *quasi k-uniform* if the following conditions (3.3), (3.4) hold.

$$\frac{\sum_{S, i \in S} \lambda_S |S|}{\sum_{S, i \in S} \lambda_S} = k \quad \text{for any } 1 \leq i \leq n, \tag{3.3}$$

$$\frac{\sum_S \lambda_S |S|}{\sum_S \lambda_S} = k. \tag{3.4}$$

We say that d is *quasi k-uniform* if d admits a quasi k-uniform $\mathbb{R}_+$-realization.

Clearly, every k-uniform cut is quasi k-uniform and thus every member of the k-uniform cut cone $\mathscr{C}_n^k$ is quasi k-uniform. So, the quasi k-uniform points of $\mathbb{R}^{\binom{n}{2}}$ form a cone which contains $\mathscr{C}_n^k$ as a subcone. Note that in the special cases $k = 1$, $\lfloor n/2 \rfloor$, the cone of all quasi k-uniform points coincides with the k-uniform cut cone. An easy consequence of relations (3.3), (3.4) is:

$$\sum_S \lambda_S = S_k(d). \tag{3.5}$$

Indeed, by summation of (3.3) over i, we have that: $\sum_S \lambda_S |S|^2 = k \sum_S \lambda_S |S|$. Hence,

$$\begin{aligned} \sum_{1 \leq i < j \leq n} d_{ij} &= \sum_S |S| (n - |S|) = (n-k) \sum_S \lambda_S |S| \\ &= k(n-k) \sum_S \lambda_S \end{aligned}$$

(by (3.5)), implying that $S_k(d) = \sum_S \lambda_S$.

Lemma 3.6. *Given $d \in \mathbb{R}^{\binom{n}{2}}$, the following assertions are equivalent.*

(i) *$d = \sum_S \lambda_S \delta(S)$ satisfying* (3.3), (3.4).

(ii) $\mathrm{ext}_k(d) = \sum_S \lambda_S \delta(S)$ *(the cuts* $\delta(S)$ *being considered as cuts in* $\mathcal{K}_{n+1}$*) and* $\sum_S \lambda_S = S_k(d)$.

(iii) $\mathrm{ext}_{n-k}(d) = \sum_S \lambda_S \delta(S \cup \{n+1\})$ *and* $\sum_S \lambda_S = S_k(d)$.

Proof. (ii)$\Leftrightarrow$(iii) follows from Lemma 3.1(i).

(i)$\Rightarrow$(ii). Set $d' = \sum_S \lambda_S \delta(S)$. Then, $d'_{i,n+1} = \sum_{S, i \in S} \lambda_S$, implying that $\sum_{1 \leq i \leq n} d'_{i,n+1} = \sum_S \lambda_S |S|$ and thus, by (3.4), (3.5),

(a) $$\sum_{1 \leq i \leq n} d'_{i,n+1} = kS_k(d).$$

Also,

$$\sum_{1 \leq j \leq n} \frac{d'_{i,n+1} + d'_{j,n+1} - d_{ij}}{2} = \sum_{S, i \in S} \lambda_S |S| = k \sum_{S, i \in S} \lambda_S,$$

by (3.4), and thus:

(b) $$\sum_{1 \leq j \leq n} \frac{d'_{i,n+1} + d'_{j,n+1} - d_{ij}}{2} = kd'_{i,n+1} \quad \text{for } 1 \leq i \leq n.$$

From (a), (b), $d'_{i,n+1} = (\mathrm{ext}_k(d))_{i,n+1}$ holds. Therefore, $\mathrm{ext}_k(d) = d'$ indeed belongs to $\mathscr{C}_{n+1}$ and admits a realization of size $\sum \lambda_S = S_k(d)$.

(ii)$\Rightarrow$(i). By assumption, $d' := \mathrm{ext}_k(d) = \sum \lambda_S \delta(S)$ where $\lambda_S \geq 0$, $\sum \lambda_S = S_k(d)$ and the sum is over subsets S of $\{1, \ldots, n\}$. Hence, $d'_{i,n+1} = \sum_{S, i \in S} \lambda_S$ and thus $\sum_{1 \leq i \leq n} d'_{i,n+1} = \sum \lambda_S |S|$. On the other hand, by using the definition of $\mathrm{ext}_k(d)$, one can compute that $\sum_{1 \leq i \leq n} d'_{i,n+1} = kS_k(d)$. Now using the assumption that $S_k(d) = \sum \lambda_S$, we deduce that $\sum \lambda_S |S| = k \sum \lambda_S$, i.e., (3.4) holds. Note that

$$\sum_{1 \leq j \leq n} d_{ij} = \sum_{S, i \in S} (n - |S|)\lambda_S + \sum_{S, i \notin S} |S| \lambda_S$$
$$= \sum_{S, i \in S} (n - 2|S|)\lambda_S + kS_k(d).$$

Thus, $d'_{i,n+1} = 1/(n-2k) \sum_{S, i \in S} (n - 2|S|)\lambda_S$. Comparing with the fact that $d'_{i,n+1} = \sum_{S, i \in S} \lambda_S$, we obtain that $\sum_{S, i \in S} \lambda_S |S| = k \sum_{S, i \in S} \lambda_S$, i.e., (3.3) holds. □

An instant corollary of Lemma 3.6 is as follows.

Proposition 3.7. *Given* $d \in \mathbb{R}^{\binom{n}{2}}$*, the following assertions are equivalent.*

(i) *d is quasi k-uniform.*

(ii) $\mathrm{ext}_k(d) \in \mathscr{C}_{n+1}$ *and admits a* $\mathbb{R}_+$*-realization of size* $S_k(d)$.

(iii) $\mathrm{ext}_{n-k}(d) \in \mathscr{C}_{n+1}$ *and admits a* $\mathbb{R}_+$*-realization of size* $S_k(d)$.

Note that a similar result holds at the level of the cut polytope instead of the cut cone. Namely, given $d \in \mathbb{R}^{\binom{n}{2}}$, the following assertions are equivalent:

(i) d admits a convex quasi k-uniform realization: $d = \sum \lambda_S \delta(S)$ where $\lambda_S \geq 0$ satisfy (3.3), (3.4) and $\sum \lambda_S = 1$,

(ii) $\mathrm{ext}_k(d) \in \mathscr{P}_{n+1}$ and $S_k(d) = 1$.

4. The k-uniform cut lattice $\mathcal{L}_n^k$

In this section, we give the description of the k-uniform cut lattice $\mathcal{L}_n^k$, i.e., the lattice generated by all k-uniform cuts. Actually it follows from a result of Wilson as we now explain.

Given integers $1 \leqslant t \leqslant k \leqslant n$, let W_{tk} denote the $\binom{n}{t} \times \binom{n}{k}$ binary matrix whose rows are indexed by all t-subsets A of $\{1, \ldots, n\}$, whose columns are indexed by all k-subsets B of $\{1, \ldots, n\}$ and the (A, B)th entry of W_{tk} is equal to 1 if $A \subseteq B$ and is equal to 0 otherwise. Let $L_{t,k}$ denote the lattice generated by the columns of W_{tk}.

Theorem 4.1 (Wilson [14]). *Given $x \in \mathbb{Z}^{\binom{n}{t}}$, $x \in L_{t,k}$ if and only if $W_{it} \cdot x \equiv 0 \pmod{\binom{k-i}{t-i}}$ for all $0 \leqslant i \leqslant t$.*

We shall use the case $t = 2$ in the above theorem.

Lemma 4.2. *Assume that $k \neq n/2$. Then, given $d \in \mathbb{Z}^{\binom{n}{2}}$, $d \in \mathcal{L}_n^k$ if and only if $\mathrm{int}_k(d) \in L_{2,k}$.*

Proof. We use the following property of $\mathrm{int}_k(d)$ mentioned in Section 3. If $|S| = k$, then $\mathrm{int}_k(\delta(S))_{ij}$ is equal to 1 if $i, j \in S$ and is equal to 0 otherwise. Therefore, for $|S| = k$, $\mathrm{int}_k(\delta(S))$ is precisely the corresponding column of the matrix W_{2k}. The result now follows since int_k is an injective linear map. □

Proposition 4.3. *Assume that $k \neq n/2$. Then, given $d \in \mathbb{Z}^{\binom{n}{2}}$, the following assertions are equivalent.*

(i) $d \in \mathcal{L}_n^k$.

(ii) $$\begin{cases} S_k(d) \in \mathbb{Z}, \text{ i.e., } \sum_{1 \leqslant i < j \leqslant n} d_{ij} \equiv 0 \pmod{k(n-k)}, \\ \mathrm{ext}_k(d)_{i,n+1} \in \mathbb{Z} \quad \text{for all } 1 \leqslant i \leqslant n, \\ \mathrm{ext}_k(d)_{i,n+1} + \mathrm{ext}_k(d)_{j,n+1} + \mathrm{ext}_k(d)_{ij} \equiv 0 \pmod 2 \\ \quad \text{for all } 1 \leqslant i < j \leqslant n. \end{cases}$$

(iii) $S_k(d) \in \mathbb{Z}$ *and* $\mathrm{ext}_k(d) \in \mathcal{L}_{n+1}$.

(iv) $\mathrm{ext}_{n-k}(d) \in \mathbb{Z}^{\binom{n+1}{2}}$ *and* $\mathrm{ext}_k(d) \in \mathcal{L}_{n+1}$.

(v) In the case n odd, $d \in \mathcal{L}_n$, $\mathrm{ext}_k(d)$, $\mathrm{ext}_{n-k}(d) \in \mathbb{Z}\binom{n+1}{2}$.

Proof. We first verify that (i), (ii) are equivalent. For this, we apply Theorem 4.1 and Lemma 4.2. So, $d \in \mathcal{L}_n^k$ if and only if $W_{i2} \cdot \mathrm{int}_k(d) \equiv 0 \pmod{\binom{k-i}{2-i}}$ for $i = 0, 1, 2$. But, $W_{02} \cdot \mathrm{int}_k(d) = \sum_{1 \leqslant i < j \leqslant n} \mathrm{int}_k(d)_{ij}$ which, by Lemma 3.1(v), is equal to $\binom{k}{2} S_k(d)$, thus yielding the condition: $S_k(d) \in \mathbb{Z}$. Also, $W_{12} \cdot \mathrm{int}_k(d)$ has its coordinates equal to $\sum_{1 \leqslant j \leqslant n, j \neq i} \mathrm{int}_k(d)_{ij}$ and thus, by Lemma 3.1(iii), to $(k-1)\mathrm{ext}_k(d)_{i,n+1}$; this yields the condition that $\mathrm{ext}_k(d) \in \mathbb{Z}^{\binom{n+1}{2}}$. Finally, the coordinates of $W_{22} \cdot \text{int}_k(d)$ are equal to $\text{int}_k(d)_{ij}$; from this we deduce the third condition that $\text{ext}_k(d)_{i,n+1} + \text{ext}_k(d)_{j,n+1} + \text{ext}_k(d)_{ij} \equiv 0 \pmod 2$.

The implications (iii)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) follow easily using the fact that the lattice $\mathcal{L}_n$ is described by the condition (1.2). The implications (i)$\Rightarrow$(iv) and (iv)$\Rightarrow$(ii) follow using Lemma 3.1(i). Finally, we check the equivalence with (v) in the case n odd. The implication (i)$\Rightarrow$(v) is clear. Assume that (v) holds; it suffices to check that $\text{ext}_k(d)_{i,n+1} + \text{ext}_k(d)_{j,n+1} + \text{ext}_k(d)_{ij}$ is an even integer for any i, j. This fact follows from the following identities:

$$\begin{aligned}
&\text{ext}_k(d)_{i,n+1} + \text{ext}_k(d)_{j,n+1} + \text{ext}_k(d)_{ij} \\
&\quad \equiv \text{ext}_k(d)_{i,n+1} - \text{ext}_k(d)_{j,n+1} + d_{ij} \\
&\quad = \frac{1}{n-2k}\Big(\sum_h d_{hi} - \sum_h d_{hj}\Big) + d_{ij} \\
&\quad = \frac{1}{n-2k}\Big(\sum_h d_{hi} - d_{hj} + d_{ij}\Big) + \frac{2n-2k}{n-2k} d_{ij} \pmod 2. \qquad \square
\end{aligned}$$

Observe that (i), (v) are not equivalent when n is even. For example, take $d = 2d(K_6 - e) = \text{ant}_4(2\mathbb{1}_5)$; then $d \notin \mathcal{L}_6^2$ because $\text{ext}_2(d) \notin \mathcal{L}_7$, but $d \in \mathcal{L}_6$, $\text{ext}_2(d)$, $\text{ext}_4(d) \in \mathbb{Z}^{\binom{7}{2}}$. Indeed, say $e = (1, 2)$, then $S_2(d) = 4$, $\text{ext}_2(d)_{i7} = 2, 2, 1, 1, 1, 1$ if $i = 1, 2, 3, 4, 5, 6$ and thus $\text{ext}_2(d)_{17} + \text{ext}_2(d)_{37} + \text{ext}_2(d)_{13} = 5 \not\equiv 0 \pmod 2$.

5. Some results on 1, 2-uniform cuts

In this section we group several results on the k-uniform cut cone $\mathcal{C}_n^k$ and other hulls of k-uniform cuts in the cases $k = 1, 2$. Note that the cone $\mathcal{C}_n^k$ is simplicial if and only if $k = 1, 2$ or ($n = 6$, $k = 3$). We first give some results for the 1-uniform case.

Lemma 5.1. *Given $d \in \mathbb{R}^{\binom{n}{2}}$, the following assertions are equivalent.*

(i) *$d \in \mathbb{R}(\delta(\{1\}), \ldots, \delta(\{n\}))$, i.e., $d = \sum_{1 \le i \le n} \lambda_i \delta(\{i\})$ for some $\lambda_i \in \mathbb{R}$.*

(ii) *There exist scalars $\lambda_i \in \mathbb{R}$ such that $(d_{ij} + d_{ik} - d_{jk})/2 = \lambda_i$ for all j, k and all i.*

(iii) *There exist some scalars $\lambda_i \in \mathbb{R}$ such that $(d_{ij} + d_{ik} - d_{jk})/2 = \lambda_i$ for all j, k and some i.*

Proof. Clearly, (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii). Let us check the implication (iii)$\Rightarrow$(ii). Assume that, for instance, $(d_{1j} + d_{1k} - d_{jk})/2 = \lambda_1$ for all $2 \le j < k \le n$. Then,

$$\begin{aligned}
d_{ij} + d_{ik} - d_{jk} &= (d_{ij} - d_{1i} - d_{1j}) + (d_{ik} - d_{1i} - d_{1k}) + (d_{1j} + d_{1k} - d_{jk}) + 2d_{1i} \\
&= 2(d_{1i} - \lambda_1).
\end{aligned}$$

Therefore, (ii) holds with $\lambda_i = d_{1i} - \lambda_1$. $\square$

Corollary 5.2. *Take* $d \in \mathbb{R}(\mathcal{K}_n^1) = \mathbb{R}(\delta(\{1\}), \ldots, \delta(\{n\}))$. *Then,*

(i) $d \in \mathscr{C}_n^1 = \mathbb{R}_+(\mathcal{K}_n^1)$ *if and only if* $\text{ext}_1(d)_{i,n+1} \in \mathbb{R}_+$ *for* $1 \leq i \leq n$ *or, equivalently,* $d \in \mathcal{M}_n$.

(ii) $d \in \mathcal{L}_n^1 = \mathbb{Z}(\mathcal{K}_n^1)$ *if and only if* $\text{ext}_1(d)_{i,n+1} \in \mathbb{Z}$ *for* $1 \leq i \leq n$ *or, equivalently,* $d \in \mathcal{L}_n$.

(iii) $d \in \mathbb{Z}_+(\mathcal{K}_n^1)$ *if and only if* $\text{ext}_1(d)_{i,n+1} \in \mathbb{Z}_+$ *for* $1 \leq i \leq n$ *or, equivalently,* $d \in \mathcal{M}_n \cap \mathcal{L}_n$.

Proof. Take $d = \sum_{1 \leq i \leq n} \lambda_i \delta(\{i\}) \in \mathbb{R}(\mathcal{K}_n^1)$. Then, since the cuts $\delta(\{1\}), \ldots, \delta(\{n\})$ are linarly independent, $d \in \mathbb{R}_+(\mathcal{K}_n^1)$ (resp. $\mathbb{Z}(\mathcal{K}_n^1)$, $\mathbb{Z}_+(\mathcal{K}_n^1)$) if and only if $\lambda_i \in \mathbb{R}_+$ (resp. $\mathbb{Z}$, $\mathbb{Z}_+$) for all $1 \leq i \leq n$. Observe that $\text{ext}_1(d)_{i,n+1} = \lambda_i$ for every i and $d_{ij} + d_{ik} - d_{jk} = 2\lambda_i$ for any i, j, k. Therefore, $d \in \mathbb{R}_+(\mathcal{K}_n^1)$ if and only if $\lambda_i = \text{ext}_1(d)_{i,n+1} \geq 0$ for all i or, equivalently, $d_{ij} + d_{ik} - d_{jk} \geq 0$ for all i, j, k, i.e., $d \in \mathcal{M}_n$. Also, $d \in \mathbb{Z}(\mathcal{K}_n^1)$ if and only if $\lambda_i = \text{ext}_1(d)_{i,n+1} \in \mathbb{Z}$ for all i or, equivalently, $d_{ij} + d_{ik} - d_{jk} \equiv 0 \pmod 2$ for all i, j, k, i.e., $d \in \mathcal{L}_n$. Finally, $d \in \mathbb{Z}_+(\mathcal{K}_n^1)$ if and only if $\lambda_i = \text{ext}_1(d)_{i,n+1} \in \mathbb{Z}_+$ for all i or, equivalently, by the above argument, $d \in \mathcal{M}_n \cap \mathcal{L}_n$. □

Let $\mathcal{B}$ be a family of subsets of a groundset X and $\lambda \geq 1$ be an integer. Then, $\mathcal{B}$ is called a *λ-design* (or pairwise balanced design) if any two distinct elements of X belong to exactly λ common blocks of $\mathcal{B}$.

Take $d = \sum_{1 \leq i \leq n} \lambda_i \delta(\{i\})$ with $\lambda_i \in \mathbb{Z}_+$; take an arbitrary $\mathbb{Z}_+$-realization of d, $d = \sum \alpha_S \delta(S)$ where the sum is over the subsets of $\{2, 3, \ldots, n\}$. Let $\mathcal{B}$ denote the family consisting of all subsets S each repeated α_S times; it is easy to see that $\mathcal{B}$ is a λ_1-design on the groundset $\{2, 3, \ldots, n\}$. In other words, there is a one-to-one correspondence between

- the $\mathbb{Z}_+$-realizations of d, and
- the λ_1-designs on $n - 1$ varieties.

Given $d \in \mathscr{C}_n^1$, $d = \sum_{1 \leq i \leq n} \lambda_i \delta(\{i\})$, $\lambda_i \geq 0$, assume that d is h-embeddable, i.e., that $\lambda_i \in \mathbb{Z}$. In general, d may have several other $\mathbb{Z}_+$-realizations. The following result, which is a reformulation of Theorem 7(i) in [5], states that, if n is large enough, then d is rigid, i.e., admits a unique $\mathbb{Z}_+$-realization.

Proposition 5.3. *Assume* $d \in \mathscr{C}_n^1 \cap \mathcal{L}_n$ *and* $d_{ij} \neq 0$ *for all* $1 \leq i < j \leq n$. *If n is large with respect to* $\max(d_{ij}: 1 \leq i < j \leq n)$, *then d is rigid.*

Proposition 5.4. *Assume* $d \in \mathscr{C}_n^1 \cap \mathcal{L}_n$ *and* $d_{ij} \neq 0$ *for all* $1 \leq i < j \leq n$. *Let* $d = \sum \alpha_S \delta(S)$ *be a $\mathbb{Z}_+$-realization of d, where the sum is over the subsets S of* $\{2, \ldots, n\}$. *Then,*

(i) (Majumdar [11]) $s := \sum \alpha_S \geq n - 1$.

(ii) (Ryser [12]). *If* $s = n - 1$, *then there exists an integer* k, $1 \leq k \leq n - 1$, *such that* $|S| = k$ *or* $n - k$ *for any S for which* $\alpha_S \neq 0$.

Observe that in case (ii), when all subsets S with $\alpha_S \neq 0$ have the same size k, i.e., the realization is k-uniform, then the associated set family $\mathcal{B}$ is a symmetric $(k, \lambda_1 = (d_{1i} + d_{1j} - d_{ij})/2, n-1)$-design. Actually this is the most frequent situation. Indeed it follows from a result of Woodall [15] that there is only a finite number of h-embeddable points $d \in \mathcal{C}_n^1$ with $\lambda_1 > 0$ admitting a non-uniform $\mathbb{Z}_+$-realization of size $s = n-1$. For more details and a conjecture about the classification of the above objects, see [15, 12].

We now turn to the 2-uniform case.

Proposition 5.5. *Take* $d \in \mathbb{R}^{\binom{n}{2}}$. *Then,*

(i) $d \in \mathcal{C}_n^2 = \mathbb{R}_+(\mathcal{K}_n^2)$ *if and only if* $\text{ext}_2(d) \in \mathcal{M}_{n+1}$,

(ii) $d \in \mathcal{L}_n^2 = \mathbb{Z}(\mathcal{K}_n^2)$ *if and only if* $\text{ext}_2(d) \in \mathcal{L}_{n+1}$,

(iii) $d \in \mathbb{Z}_+(\mathcal{K}_n^2)$ *if and only if* $\text{ext}_2(d) \in \mathcal{M}_{n+1} \cap \mathcal{L}_{n+1}$.

Proof. In all three cases, the conditions are clearly necessary (using Proposition 3.7 for (i) and Proposition 4.3 for (ii)). We check the sufficiency.

(i) Assume that $\text{ext}_2(d) \in \mathcal{M}_{n+1}$. Hence, by definition,

$$\text{int}_2(d)_{ij} = \frac{\text{ext}_2(d)_{in+1} + \text{ext}_2(d)_{jn+1} - \text{ext}_2(d)_{ij}}{2} \geq 0.$$

We verify that $d = x := \sum_{1 \leq i < j \leq n} \text{int}_2(d)_{ij} \delta(\{i, j\})$ holds. Indeed, by Lemma 3.1(iii),

$$\begin{aligned} x_{ij} &= \sum_{s \neq i,j} \text{int}_2(d)_{is} + \sum_{s \neq i,j} \text{int}_2(d)_{js} = \text{ext}_2(d)_{i,n+1} + \text{ext}_2(d)_{j,n+1} - 2\,\text{int}_2(d)_{ij} \\ &= \text{ext}_2(d)_{ij} = d_{ij}. \end{aligned}$$

(ii) Assume that $\text{ext}_2(d) \in \mathcal{L}_{n+1}$. For showing that $d \in \mathcal{L}_n^2$, by Proposition 4.3, we need only to check that $S_2(d) \in \mathbb{Z}$. By assumption, $\text{ext}_2(d)_{i,n+1} + \text{ext}_2(d)_{j,n+1} + \text{ext}_2(d)_{ij} \equiv 0 \pmod 2$, i.e., $\text{int}_2(d)_{ij} \in \mathbb{Z}$ for all i, j. Using Lemma 3.1(v), we deduce that $S_2(d) \in \mathbb{Z}$.

(iii) If $\text{ext}_2(d) \in \mathcal{M}_{n+1} \cap \mathcal{L}_{n+1}$, then d admits a $\mathbb{R}_+$-realization and a $\mathbb{Z}_+$-realization both using only cuts of the form $\delta(\{i, j\})$, but then these two realizations must coincide since the cuts $\delta(\{i, j\})$, $1 \leq i < j \leq n$, are linearly independent. □

We can deduce all facets of the cone $\mathcal{C}_n^2$ very easily from the triangle inequalities, as we now explain.

Generally, a valid inequality $v^* \cdot x \leq 0$ for the k-uniform cut cone $\mathcal{C}_n^k$ can be deduced from any valid inequality $v \cdot x \leq 0$ for the cut cone $\mathcal{C}_{n+1}$. Indeed, take $v \in \mathbb{R}^{\binom{n+1}{2}}$ and assume that the inequality $v \cdot x \leq 0$ is valid for the cut cone $\mathcal{C}_{n+1}$.

Take $d \in \mathbb{R}^{\binom{n}{2}}$ and its k-extension $\text{ext}_k(d) \in \mathbb{R}^{\binom{n+1}{2}}$. Then,

$$\begin{aligned}
&v \cdot \text{ext}_k(d)\\
&= \sum_{1\le i<j\le n} v_{ij} d_{ij} + \sum_{1\le i\le n} v_{i,n+1}\left(\frac{\sum_{1\le j\le n, j\ne i} d_{ij}}{n-2k} - \frac{\sum_{1\le i<j\le n} d_{ij})}{(n-2k)(n-k)}\right)\\
&= \sum_{1\le i<j\le n} v_{ij} d_{ij} + \frac{1}{n-2k}\sum_{1\le i<j\le n} d_{ij}(v_{i,n+1} + v_{j,n+1})\\
&\quad - \frac{\sum_{1\le i<j\le n} d_{ij}}{(n-k)(n-2k)} v \cdot \delta(\{n+1\})\\
&= \sum_{1\le i<j\le n} v^*_{ij} d_{ij}\\
&= v^* \cdot d
\end{aligned}$$

where we have set

$$v^*_{ij} = v_{ij} + \frac{v_{i,n+1} + v_{j,n+1}}{n-2k} - \frac{v \cdot \delta(\{n+1\})}{(n-k)(n-2k)}$$

for all $1 \le i < j \le n$.

Proposition 5.6. *Given $v \in \mathbb{R}^{\binom{n+1}{2}}$, if the inequality $v \cdot x \le 0$ is valid for the cut cone $\mathscr{C}_{n+1}$, then the inequality $v^* \cdot x \le 0$ is valid for the k-uniform cut cone $\mathscr{C}^k_n$.*

Proof. Take $d \in \mathscr{C}^k_n$, then $\text{ext}_k(d) \in \mathscr{C}_{n+1}$; hence, $v^* \cdot d = v \cdot \text{ext}_k(d) \le 0$. □

For example, let us compute the inequalities $v^* \cdot x \le 0$ if $v \cdot x \le 0$ is a triangle inequality. For $v \cdot x := x_{i,n+1} - x_{j,n+1} - x_{ij}$ with $1 \le i < j \le n$, we have

$$v^* \cdot x = \sum_{1\le h\le n, h\ne i,j} (x_{ih} - x_{jh}) - (n-2k)x_{ij}.$$

For $v \cdot x := x_{ij} - x_{i,n+1} - x_{j,n+1}$ with $1 \le i < j \le n$, we have

$$v^* \cdot x = 2\sum_{1\le h<k\le n} x_{hk} + (n-k)(n-2k-2)x_{ij} - (n-k)\sum_{1\le h\le n, h\ne i,j}(x_{ih} + x_{jh}) \le 0.$$

Therefore, from Proposition 5.5(i), the cone $\mathscr{C}^2_n$ is fully described as follows: $d \in \mathscr{C}^2_n$ if and only if d satisfies the following inequalities:

$$\begin{aligned}
&d_{ij} - d_{ij} - d_{jk} \le 0 \quad \text{for all } i, j, k \text{ in } \{1, \ldots, n\},\\
&\sum_{1\le h\le n, h\ne i,j} (d_{ih} - d_{jh}) - (n-4)d_{ij} \le 0 \quad \text{for all } 1 \le i < j \le n, \text{ and}\\
&2\sum_{1\le h<k\le n} d_{hk} + (n-2)(n-6)d_{ij} - (n-2)\sum_{1\le h\le n, h\ne i,j} (d_{ih} + d_{jh}) \le 0
\end{aligned}$$

for all $1 \le i < j \le n$. In fact, it is shown in [6] that the latter class of $\binom{n}{2}$ inequalities suffices to describe the cone $\mathscr{C}^2_n$ which is a simplicial cone of dimension $\binom{n}{2}$.

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