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# Collapsing and lifting for the cut cone

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*Abstract*

The cut polytope $P_C(G)$ of a graph $G$ is the convex hull of the incidence vectors of all cuts of $G$; the cut cone $C(G)$ of $G$ is the cone generated by the incidence vectors of all cuts of $G$. We introduce the operation of collapsing an inequality valid over the cut cone $C(K_n)$ of the complete graph with $n$ vertices: it consists of identifying vertices and adding the weights of the corresponding incident edges. Using collapsing and its inverse operation (lifting), we give several methods to find facets of $C(K_n)$. We also show how to construct facets of $C(K_n)$ from the difference of inequalities valid over $C(K_n)$. When $G$ is an induced subgraph of a graph $H$, we give sufficient conditions to derive inequalities defining facets of $P_C(H)$ from inequalities defining facets of $P_C(G)$. Finally, the description (up to permutation) of the cut cone $C(K_7)$ is given.

## 1. Introduction and preliminaries

We use the standard graph-theoretical terminology as in [9, 10]. An edge with endpoints $i$ and $j$ in an undirected graph will be denoted by $ij$ (or $ji$). The complete graph on $n$ vertices is denoted by $K_n$. Let $G=(V,E)$ be a graph, and let $S$ be a (possibly empty) subset of $V$. The *cut* corresponding to $S$ is the set $\delta(S)$ of edges with exactly one endpoint in $S$. (In particular, we allow $S=\emptyset$, in which case $\delta(S)$ is a zero vector.) Throughout this paper, we shall let $\delta(S)$ stand for both a cut and its incidence vector. The *cut cone* $C(G)$ of a graph $G$ is the cone generated by the incidence vectors of all

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edge sets of cuts of $G$; the *cut polytope* $P_C(G)$ of a graph $G$ is the convex hull of the incidence vectors of all edge sets of cuts of $G$. For every graph $G$, the cone $C(G)$ and the polytope $P_C(G)$ are full dimensional. As usual, we let $B^A$ denote the set of all mappings from $A$ to $B$; elements of $B^A$ can be thought of as vectors whose components are subscripted by elements of $A$ and take values in $B$.

Let $G=(V,E)$ be a graph, and let $v$ be a vector in $\mathbb{R}^E$. If the inequality $v^{\mathrm{T}}x\leqslant 0$ is satisfied by all points in $C(G)$ or, equivalently, by all cut vectors $\delta(S)$, we say that the inequality $v^{\mathrm{T}}x\leqslant 0$ is valid over $C(G)$. The face defined by the inequality $v^{\mathrm{T}}x\leqslant 0$ is the set $F_v=\{x\in C(G)\colon v^{\mathrm{T}}x=0\}$. A *root* of the vector $v$ is a nonzero cut vector which belongs to $F_v$. The dimension of a face $F_v$, denoted by $\dim(F_v)$, is the largest number of affinely independent points in $F_v$ minus one or, equivalently, the largest number of linearly independent roots of $v$ (since $F_v$ contains the zero vector). The codimension of a face $F_v$ is equal to $\binom{n}{2}-\dim(v)$. A facet of $C(G)$ is a face of dimension $|E|-1$.

For every graph $G=(V,E)$, and for every vector $v$ in $\mathbb{R}^E$, we define a graph $G(v)$ as follows: its edges are all the edges $ij$ in $G$ for which $v_{ij}\neq 0$, and its vertices are all the endpoints of these edges; to every edge $ij$, the weight $v_{ij}$ is assigned. The graph $G(v)$ is called the *supporting graph* of $v$. Let $v^{\mathrm{T}}x\leqslant 0$ be an inequality valid over $C(G)$. If all nonzero components of $v$ are $\pm 1$, then we say that the inequality $v^{\mathrm{T}}x\leqslant 0$ is *pure*. As usual, a vector with components all equal to zero will be denoted by $\underline{0}$.

When $G$ is the complete graph $K_n$ with $n$ vertices, the corresponding cut cone will be denoted by $C_n$. Points in $C_n$ can be interpreted as semi-metrics on $n$ points; in fact, $C_n$ coincides with the family of all the semi-metrics on $n$ points which are isometrically embeddable into $L^1$; in this context, the study of the cut cone $C_n$ was started in 1960 by Deza [12]. (For more informations, see for instance [2, 6, 13, 14, 24].)

We now describe two classes of inequalities valid over the cone $C_n$. The first class is the class of *hypermetric inequalities* which were introduced by Deza [12] and later, independently, by Kelly [22]. For every integer row vector, $b=(b_1,\ldots,b_n)$ such that $b_1+\cdots+b_n=1$, the hypermetric inequality specified by the vector $b$ is the inequality

$$\sum_{1\leqslant i<j\leqslant n} b_i b_j x_{ij}\leqslant 0. \tag{1}$$

We refer to each inequality (1) as Hyp($b$). Write $b(S)=\sum_{i\in S} b_i$. To see that (1) is valid, observe that

$$\sum_{ij\in\delta(S)} b_i b_j=\sum_{i\in S} b_i \sum_{j\notin S} b_j=b(S)(1-b(S))$$

and that $t(1-t)\leqslant 0$ for all integers $t$. Let $v$ be the vector defined by $v_{ij}=b_ib_j$ for all $ij$. Note that every root of the vector $v$ is a cut for which $b(S)$ is equal to zero or one.

An hypermetric inequality that will play a special role in our paper is the inequality

$$x_{ij}-x_{ik}-x_{jk}\leqslant 0;$$

we shall refer to such inequality as *triangle inequality*. It is easy to verify that, for $n\geqslant 3$, every triangle inequality defines a facet of $C_n$.

The second class of inequalities valid over $C_n$ is the class of *cycle inequalities* which were introduced by Deza and Laurent [17]. To specify these inequalities, we need one more definition. Let $f$ be an integer greater than or equal to three; a cycle $C=(1,2,\dots,f)$ is the graph with vertices $1,2,\dots,f$ and edges $12,23,\dots,f1$. For every cycle $C$, $E(C)$ denotes the set of its edges. Let $b=(b_1,\dots,b_n)$ be an integer row vector such that $b_1+\dots+b_n=3$; order the components of $b$ in such a way that $b_1,b_2,\dots,b_f>0\geqslant b_{f+1},\dots,b_n$. Then the cycle inequality specified by the vector $b$ is the inequality

$$\sum_{1\leqslant i<j\leqslant n} b_i b_j x_{ij} - \sum_{ij\in E(C)} x_{ij}\leqslant 0; \tag{2}$$

we shall refer to each inequality (2) as Cyc(b).

For every nonnegative integer $n$, we let $[1,n]$ denote the set $\{1,\dots,n\}$, and we let $N$ stand for $\binom{n}{2}$. In the following, we describe two operations on an inequality valid over the cone $C_n$: permutation and switching.

Let $v$ be a vector in $\mathbb{R}^N$. For every permutation $\sigma$ of the set $[1,n]$, we define a vector $v^\sigma$ in $\mathbb{R}^N$ by

$$v^\sigma_{ij}=v_{\sigma(i)\sigma(j)} \quad \text{for every } 1\leqslant i<j\leqslant n;$$

we shall say that $v^\sigma$ has been obtained from $v$ via the *permutation* $\sigma$ or that $v^\sigma$ is permutation equivalent to $v$. Clearly, the inequality $v^{\mathrm{T}}x\leqslant 0$ is valid over $C_n$ if and only if the inequality $(v^\sigma)^{\mathrm{T}}x\leqslant 0$ is valid over $C_n$. It is easy to verify that if $\mathrm{Hyp}(b_1,\dots,b_n)$ is an hypermetric facet-defining inequality of $C_n$, then for every permutation $\sigma$ of the set $[1,n]$, the inequality $\mathrm{Hyp}(b_{\sigma(1)},\dots,b_{\sigma(n)})$ defines a facet of $C_n$. However, if $\mathrm{Cyc}(b_1,\dots,b_n)$ is a cycle facet-defining inequality of $C_n$, then the inequality $\mathrm{Cyc}(b_{\sigma(1)},\dots,b_{\sigma(n)})$ does not define a facet of $C_n$ for every permutation $\sigma$ of the set $[1,n]$ [17].

The second operation, called *switching*, relates the cut polytope of a graph $G$ with the cut cone of $G$ in the following sense. Since $P_C(G)\subset C(G)$, every inequality valid over $C(G)$ is also valid over $P_C(G)$. Moreover, every facet-defining inequality of $C(G)$ is facet-defining inequality of $P_C(G)$. In fact, the switching operation will show that looking for all facets of $P_C(G)$ is equivalent to looking for all facets of $C(G)$. To describe this operation, consider a graph $G=(V,E)$ and let $v$ be a vector in $\mathbb{R}^E$. For every subset $S$ of $V$, we define a vector $v^S$ in $\mathbb{R}^E$ by

$$v^S_{ij}=\begin{cases} v_{ij} & \text{if } ij\notin\delta(S),\\ -v_{ij} & \text{otherwise,}\end{cases}$$

we shall say that the vector $v^S$ has been obtained from $v$ by switching the cut $\delta(S)$. Write

$$d=-\sum_{ij\in\delta(S)} v_{ij}.$$

For the case $G=K_n$, Deza [12] (see also [17]) showed that for every vector $v$ in $\mathbb{R}^N$ and for every root $\delta(S)$ of $v$, the inequality $v^{\mathrm{T}}x \leqslant 0$ defines a facet of $C_n$ if and only if the inequality $(v^S)^{\mathrm{T}}x \leqslant 0$ defines a facet of $C_n$. For a general graph $G=(V,E)$, Barahona and Mahjoub [8] showed that for every vector $v$ in $\mathbb{R}^E$ and for every cut $\delta(S)$, $v^{\mathrm{T}}x \leqslant b$ defines a facet of $P_C(G)$ if and only if the inequality $(v^S)^{\mathrm{T}}x \leqslant b-d$ defines a facet of $P_C(G)$. Furthermore, they showed that every inequality defining a facet of $P_C(G)$ can be obtained for some inequality defining a facet of $C(G)$ by switching a cut [8]. In [15], it was shown that switching and permutation are the only symmetries of $P_C(K_n)$.

In Section 2, we introduce two operations on an inequality valid over $C_n$: collapsing and expansion; collapsing an inequality consists of identifying vertices and adding the weights of the corresponding incident edges; the expansion of an inequality is the inverse operation of collapsing.

In Sections 3–5, we give several results on lifting. Lifting is a commonly used technique in polyhedral combinatorics to derive inequalities defining facets of a polyhedron in $\mathbb{R}^n$ from inequalities defining facets of a polyhedron in $\mathbb{R}^{n'}$ with $n'<n$ (see for instance [23]).

Let $G=(V,E)$ and $H=(W,F)$ be two graphs where the former is an induced subgraph of the latter, and let $v$ be a vector in $\mathbb{R}^E$. *Lifting* the vector $v$ means to find a vector $v'$ in $\mathbb{R}^F$ such that the following two conditions hold:

- if $v^{\mathrm{T}}x \leqslant 0$ is valid over $C(G)$, then $(v')^{\mathrm{T}}x \leqslant 0$ is valid over $C(H)$;
- if $v^{\mathrm{T}}x \leqslant 0$ defines a facet of $C(G)$, then $(v')^{\mathrm{T}}x \leqslant 0$ defines a facet of $C(H)$.

If $v'=(v,\underline{0})$ where $\underline{0}$ is the vector in $\mathbb{R}^{F-E}$ with components all equal to zero, then we shall say that $v'$ was obtained from $v$ by *zero-lifting*.

Finally, in Section 6, we give the complete description of the cut cone $C_7$.

## 2. Collapsing and expansion

Let $n$ be an integer greater than or equal to two, and let $k$ be an integer such that $1 \leqslant k \leqslant n-1$. Recall that $N$ stands for $\binom{n}{2}$. For every partition, $\pi=\{V_1,\ldots,V_k\}$ of the set $[1,n]$ into $k$ nonempty subsets, and for every vector $v$ in $\mathbb{R}^N$, we define a vector $v^\pi$ in $\mathbb{R}^{\binom{k}{2}}$ by

$$v_{ij}^{\pi}=\sum_{s\in V_i,\, t\in V_j} v_{st} \quad \text{for all } 1 \leqslant i<j \leqslant k.$$

We call the vector $v^\pi$ the *$\pi$-collapsing* of $v$. If $k=n-1$, then precisely one of the $k$ subsets of $[1,n]$, say $V_1$, has size two, all the others have size one; in this case, if $V_1=\{i,j\}$ then we denote the vector $v^\pi$ simply by $v^{i,j}$, and we call the vector $v^{i,j}$ the *$(i,j)$-collapsing* of $v$. The $\pi$-collapsing of an inequality $v^{\mathrm{T}}x \leqslant 0$ is the inequality $(v^\pi)^{\mathrm{T}}x \leqslant 0$.

The $\pi$-collapsing of an hypermetric inequality $\mathrm{Hyp}(b_1,\ldots,b_n)$ can be easily obtained in the following way: define a vector $v$ in $\mathbb{R}^N$ by writing $b_ib_j$ for $v_{ij}$. Clearly, for distinct

$i$ and $j$ in $[1,n]$, the $(i,j)$-collapsing of the vector $v$ is the vector $v^{i,j}$ given by

$$v_{hk}^{i,j}=\begin{cases}(b_i+b_j)b_k & \text{if } h=i,\ k\in[1,n]-\{i,j\},\\ b_hb_k & \text{if } h,k\in[1,n]-\{i,j\}.\end{cases}$$

Now define a vector $b^{i,j}$ in $\mathbb{R}^{\binom{n-1}{2}}$ by

$$b_h^{i,j}=\begin{cases}b_i+b_j & \text{if } h=i,\\ b_h & \text{if } h\in[1,n]-\{i,j\}.\end{cases}$$

Since $b_1^{i,j}+\cdots+b_n^{i,j}=1$, the inequality $(v^{i,j})^{\mathrm{T}}x\leqslant 0$ is an hypermetric inequality. We call the vector $b^{i,j}$ the $(i,j)$-collapsing of the vector $b$. For instance, if $b=(1,1,1,-1,-1)$ then the $(1,2)$-collapsing of $b$ is the vector $(2,1,-1,-1)$.

**Proposition 2.1.** *Let $\pi$ be a partition of the set $[1,n]$ into $k$ nonempty subsets $(1\leqslant k\leqslant n-1)$. If the inequality $v^{\mathrm{T}}x\leqslant 0$ is valid over $C_n$, then the inequality $(v^\pi)^{\mathrm{T}}x\leqslant 0$ is valid over $C_k$.*

**Proof.** Write $\pi=(V_1,\ldots,V_k)$; let $S$ be a subset of $[1,k]$; and set $S'=\bigcup_{i\in S}V_i$. Clearly, $S'$ is a subset of $[1,n]$. Now it is easy to verify that $(v^\pi)^{\mathrm{T}}\delta(S)=v^{\mathrm{T}}\delta(S')$. □

Let $G=(V,E)$ be a graph; the concept of the $\pi$-collapsing of a vector $v$ can be extended to the case when $v$ is a vector in $\mathbb{R}^E$ in the following sense. For every partition $\pi=\{V_1,\ldots,V_k\}$ of $V$, let $E'$ be the set of edges of the graph $G'$ obtained from $G$ by identifying all the vertices in each $V_i$ into a single vertex (multiple edges are deleted). Define the $\pi$-collapsing of $v$ as the vector $v^\pi$ in $\mathbb{R}^{E'}$ given by

$$v_{ij}^\pi=\sum_{hk\in E,\,h\in V_i,\,k\in V_j} v_{hk}\quad \text{for all } ij\in E'.$$

Clearly, if $v^{\mathrm{T}}x\leqslant 0$ is valid over $C(G)$, then the inequality $(v^\pi)^{\mathrm{T}}x\leqslant 0$ is valid over $C(G')$.

Let $\Sigma$ be the set of all partitions of the set $[1,n]$, and let $v$ be a vector in $\mathbb{R}^N$. For every $\pi$ in $\Sigma$, we denote by $\bar{v}^\pi$ the vector in $\mathbb{R}^N$ defined by

$$\bar{v}^\pi=(v^\pi,\underline{0}).$$

The vector $\bar{v}^\pi$ is a zero-lifting of $v^\pi$. Let $L(v)=\{\bar{v}^\pi\colon \pi\in\Sigma\}$. It is easy to verify that $L(v)$ is a lattice isomorphic to the set of all the partitions of $[1,n]$; the order of $L(v)$ is the following: for all partitions $\pi=\{V_1,\ldots,V_k\}$ and $\pi'=\{W_1,\ldots,W_h\}$ in $\Sigma$, $\bar{v}^\pi\geqslant\bar{v}^{\pi'}$ if and only if for every $i$ in $\{1,\ldots,k\}$, $V_i\subseteq W_j$ for some $j$ in $\{1,\ldots,h\}$. Note that the greatest element of the lattice $L(v)$ is $v$, and that the smallest element of $L(v)$ is $\underline{0}$ (zero vector corresponding to the trivial partition $\pi=\{[1,n]\}$). For every vector $v$, we call the lattice $L(v)$ the *collapsing lattice* of $v$.

Let $v$ be a vector in $\mathbb{R}^{\binom{n}{2}}$, and let $v'$ be a vector in $\mathbb{R}^{\binom{n'}{2}}$, with $n'>n$. If $v$ is a $\pi$-collapsing of $v'$ for some partition $\pi$ of $[1,n']$, then we say that $v'$ is an *expansion* of $v$. Not every expansion of an inequality valid over the cut cone $C_n$ is valid over $C_{n'}$. In fact, every inequality $v^{\mathrm{T}}x\leqslant 0$ valid over $C_n$ admits an expansion which is not valid over some cut cone containing $C_n$. For instance, let $k$ be an arbitrary element in $[1,n]$; define a vector $v'$ by

$$v'_{ij}=\begin{cases}1 & \text{if } i=n+1,\ j=k,\\ 0 & \text{if } i\doteq n+1,\ j\neq k,\\ v_{ij} & \text{otherwise.}\end{cases}$$

Note that the vector $v$ is the $(k,n+1)$-collapsing of $V'$, and that the inequality $(v')^{\mathrm{T}}x\leqslant 0$ is not valid over $C_{n+1}$. On the other hand, every inequality which is valid over $C_n$ admits an expansion which is valid over $C_{n+1}$: its zero-lifting.

Let $v$ be a vector in $\mathbb{R}^N$, and let $v'$ be an expansion of $v$. If the inequality $(v')^{\mathrm{T}}x\leqslant 0$ is pure, then we say that $(v')^{\mathrm{T}}x\leqslant 0$ is a *purification* of the inequality $v^{\mathrm{T}}x\leqslant 0$. Every inequality $v^{\mathrm{T}}x\leqslant 0$ valid over $C_n$ admits a purification which is valid over some cut cone containing $C_n$. If $v$ is not pure then some coefficient $v_{hk}$ is greater than one in modulo. Without loss of generality, we can assume that $v_{hk}>1$; define a vector $v'$ in $\mathbb{R}^{\binom{n+1}{2}}$ by

$$v'_{ij}=\begin{cases}v_{hk}-1 & \text{if } i=h,\ j=k,\\ 1 & \text{if } i=n+1,\ j=h,\\ -1 & \text{if } i=n+1,\ j=k,\\ 0 & \text{if } i=n+1,\ j\neq h,k,\\ v_{ij} & \text{otherwise.}\end{cases}$$

Clearly, the vector $v$ is the $(k,n+1)$-collapsing of $v'$. Now, let $S$ be a subset of $[1,n+1]$; without loss of generalilty, we can assume that $n+1\notin S$. Clearly, if $h$ and $k$ are both in $S$ or both not in $S$ then $(v')^{\mathrm{T}}\delta(S)\leqslant 0$ (since$(v')^{\mathrm{T}}\delta(S)=v^{\mathrm{T}}\delta(S)$); if $h\in S$ and $k\notin S$ then $(v')^{\mathrm{T}}\delta(S)=v^{\mathrm{T}}\delta(S)-v_{hh}+v_{hk}-1+1$, and so $(v')^{\mathrm{T}}\delta(S)\leqslant 0$; if $h\notin S$ and $k\in S$ then

$$(v')^{\mathrm{T}}\delta(S)=v^{\mathrm{T}}\delta(S)-v_{hk}+v_{hk}-1-1,$$

and so $(v')^{\mathrm{T}}\delta(S)<0$. Hence, the inequality $(v')^{\mathrm{T}}x\leqslant 0$ is valid over $C_{n+1}$. Repeating this procedure on $v'$ will yield a purification of the vector $v$.

Let $\pi=\{V_1,\dots,V_k\}$ be a partition of the set $[1,n]$, and let $S$ be a subset of $[1,n]$. We say that the cut $\delta(S)$ is *compatible* with the partition $\pi$ if, for every $i=1,\dots,k$, each $V_i$ is a subset of $S$ whenever $S\cap V_i\neq 0$. Recall that, for every vector $v$ in $\mathbb{R}^N$, the vector $v^\pi$ denotes the $\pi$-collapsing of $v$ and $v^S$ denotes the vector obtained from $v$ by switching the cut $\delta(S)$.

**Proposition 2.2.** *Let $\pi=\{V_1,\dots,V_k\}$ be a partition of the set $[1,n]$, and let $S$ be a subset of $[1,n]$. If the cut $\delta(S)$ is compatible with $\pi$ then $(v^\pi)^S=(v^S)^\pi$.*

**Proof.** Let $i$ and $j$ be two distinct elements in $[1,n]$. Clearly, it is sufficient to show that the vector obtained from the $(i,j)$-collapsing of $v$ by switching the cut $\delta(S)$ is the

$(i,j)$-collapsing of the vector obtained from $v$ by switching the cut $\delta(S)$, i.e. $(v^{i,j})^S=(v^S)^{i,j}$. Let $\pi$ be the corresponding partition of $[1,n]$. Set $x=v^S$, $y=x^{i,j}$, $z=v^{i,j}$, and $w=z^S$. Since the cut $\delta(S)$ is compatible with the partition $\pi$, $ij\notin\delta(S)$, and so we may assume that both $i$ and $j$ are in $S$. Let $h$ and $k$ be distinct elements in $[1,n]-\{i,j\}$; we have

$$y_{ik}=x_{ik}+x_{jk}=\begin{cases}-v_{ik}-v_{jk} & \text{if } k\notin S,\\ v_{ik}+v_{jk} & \text{if } k\in S,\end{cases}$$

$$y_{hk}=x_{hk}=\begin{cases}-v_{hk} & \text{if } hk\in\delta(S),\\ v_{hk} & \text{if } hk\notin\delta(S),\end{cases}$$

$$w_{ik}=\begin{cases}-z_{ik}=-v_{ik}-v_{jk} & \text{if } k\notin S,\\ z_{ik}=v_{ik}+v_{jk} & \text{if } k\in S,\end{cases}$$

$$w_{hk}=\begin{cases}-z_{hk}=-v_{hk} & \text{if } hk\in\delta(S),\\ z_{hk}=v_{hk} & \text{if } hk\notin\delta(S).\end{cases}\qquad\square$$

We refer to [16] for an extension of the notion of collapsing for the multicut polytope.

## 3. Zero-lifting

In this section, we consider two graphs $G=(V,E)$ and $H=(W,F)$ where the former is an induced subgraph of the latter. Let $v$ be a vector in $\mathbb{R}^E$, and let $\underline{0}$ be a vector in $\mathbb{R}^{F-E}$ with components all equal to zero. It is easy to see that if the inequality $v^{\mathrm{T}}x\leqslant d$ is valid over $P_C(G)$, then the inequality $(v,\underline{0})^{\mathrm{T}}x\leqslant d$ is valid over $P_C(H)$. Conversely, if the inequality $(v,\underline{0})^{\mathrm{T}}x\leqslant d$ is valid over $P_C(H)$, then the inequality $v^{\mathrm{T}}x\leqslant d$ is valid over $P_C(G)$. This is a special case of the following observation.

**Theorem 3.1** (De Simone [11]). *If $(v,\underline{0})^{\mathrm{T}}x\leqslant d$ defines a facet of $P_C(H)$ then $v^{\mathrm{T}}x\leqslant d$ defines a facet of $P_C(G)$.*

Consider the following problem:
*given an inequality $v^{\mathrm{T}}x\leqslant d$ defining a facet of $P_C(G)$,*
*is the inequality $(v,\underline{0})^{\mathrm{T}}x\leqslant d$ defining a facet of $P_C(H)$ ?* (3)

Barahona and Mahjoub [8] showed that for the inequalities

$$x_{ij}\geqslant 0,\qquad x_{ij}\leqslant 1, \tag{4}$$

the answer to (3) is 'yes' if and only if $ij$ does not belong to any triangle of $H$. In addition, they showed that the answer to (3) is again 'yes' for every other inequality they studied in [8].

We say that an inequality $v^{\mathrm{T}}x \leqslant d$ is nontrivial if the supporting graph $G(v)$ of $v$ has more than two vertices. Note that the supporting graphs of the inequalities (4) have precisely two vertices. De Simone [11] gave a sufficient condition on the graphs $G$ and $H$ under which problem (3) has a positive answer for all the inequalities of the linear description of $P_C(G)$, with the exception of (4).

**Theorem 3.2** (De Simone [11]). *Let $G=(V,E)$ be a graph with n vertices; let $n \geqslant 3$, and let $H=(V \cup \{r\}, F)$. If $N(r)-\{v\} \subseteq N(v)$ for some vertex $v$ in $G$ then problem* (3) *has a positive answer for every nontrivial inequality defining a facet of $P_C(G)$.*

**Corollary 3.3.** (De Simone [11]; Deza and Laurent [17]). *Let $G$ be a complete graph with $n$ vertices and let $n \geqslant 3$. Then $v^{\mathrm{T}}x \leqslant d$ defines a facet of $P_C(G)$ if and only if $[v, \underline{0}]^{\mathrm{T}}x \leqslant d$ defines a facet of the cut polytope of every complete graph with more than $n$ vertices.*

Now consider the graph $K_n$ with $n \geqslant 3$. Write $K_n=(V,E)$ and let $v$ be a vector in $\mathbb{R}^E$. Recall that, for every vector $v$, $G(v)$ denotes the supporting graph of $v$. Clearly, if $G(v)=(V', E')$ is a partial subgraph of $K_n$ then the vector $v$ can be written as

$$v=(v', \underline{0}), \quad \text{with } v' \in \mathbb{R}^{E'} \text{ and } \underline{0} \in \mathbb{R}^{E'-E}.$$

**Theorem 3.4.** *Let $K_n=(V,E)$ with $n \geqslant 3$; let $E'$ be a subset of $E$, and let $v$ be a vector in $\mathbb{R}^E$ such that $v=(v', \underline{0})$, with $v'$ in $\mathbb{R}^{E'}$, and such that $G(v)=(V', E')$. If $v^{\mathrm{T}}x \leqslant d$ defines a facet of $P_C(K_n)$ then, for every subgraph $H=(W,F)$ of $K_n$ containing $G(v)$, the inequality*

$$(v', \underline{0})^{\mathrm{T}}x \leqslant d,$$

*with $v' \in \mathbb{R}^{E'}$ and $\underline{0} \in \mathbb{R}^{F-E'}$, defines a facet of $P_C(H)$.*

**Proof.** Suppose the contrary: there exists a partial subgraph $H$ of $K_n$ containing $G(v)$ such that $(v', \underline{0})^{\mathrm{T}}x \leqslant d$ does not define a facet of $P_C(H)$. Then $(v', \underline{0})^{\mathrm{T}}x \leqslant d$ can be obtained as sum of two other inequalities valid over $P_C(H)$, say $v_1^{\mathrm{T}}x \leqslant d_1$ and $v_2^{\mathrm{T}}x \leqslant d_2$. But the inequalities $[v_1, \underline{0}]^{\mathrm{T}}x \leqslant d_1$ and $[v_2, \underline{0}]^{\mathrm{T}}x \leqslant d_2$, with $\underline{0} \in \mathbb{R}^{E-F}$, are valid over $P_C(K_n)$ and their sum is $[v', \underline{0}]^{\mathrm{T}}x \leqslant d$, contradicting the fact that $[v', \underline{0}]^{\mathrm{T}}x \leqslant d$ defines a facet of $P_C(K_n)$. □

An instant corollary of Theorem 3.4 is the following.

**Corollary 3.5.** *Let $G=(V,E)$ be the complete graph with n vertices; let $n \geqslant 3$; and let $H=(W,F)$ be a graph containing $G$ as induced subgraph. Then problem* (3) *has a positive answer for every facet-defining inequality of $P_C(G)$.*

**Proof.** Let $v^{\mathrm{T}}x \leqslant d$ be a facet-defining inequality of $P_C(G)$. Let $T$ denote the set of edges of the complete graph with $|W|$ vertices. Corollary 3.3 implies that the inequality $(v, \underline{0})^{\mathrm{T}}x \leqslant d$, with $\underline{0} \in \mathbb{R}^{T-E}$, defines a facet of $P_C(K_{|W|})$. Since $H$ is a partial subgraph of $K_{|W|}$, the corollary follows from Theorem 3.4. □

We end this section by considering a generalization of problem (3).

*Given an inequality $v^{\mathrm{T}}x \leqslant d$ defining a face of $P_C(G)$ of codimension $r$,*
*is the inequality $(v, \underline{0})^{\mathrm{T}}x \leqslant d$ defining a face of $P_C(H)$ of codimension $r$?*

The answer to the above problem is, in general, 'no'. For instance, consider the vector $b = (1, 1, -1, -1)$ and define a vector $v$ by $v_{ij} = b_i b_j$ $(1 \leqslant i < j \leqslant 4)$. It is easy to verify that while the inequality $v^{\mathrm{T}}x \leqslant 0$ defines a face of $P_C(K_4)$ of codimension four, the inequality $(v, \underline{0})^{\mathrm{T}}x \leqslant 0$, with $\underline{0} \in \mathbb{R}^4$, defines a face of $P_C(K_5)$ of codimension five.

## 4. Nonzero lifting

In this section, we consider a complete graph with $n$ vertices, $n \geqslant 5$. Recall that $N$ stands for $\binom{n}{2}$. Let $v$ be a vector in $\mathbb{R}^N$, and let $v^{\mathrm{T}}x \leqslant 0$ be a facet-defining inequality of $C_n$. In Section 3, we have seen that this inequality can be lifted to a facet-defining inequality of $C_{n'}$, with $n' > n$, by just adding zeroes (Corollary 3.3). In this section, we study the more general lifting problem. For this purpose, recall that for distinct $i$ and $j$ in $[1, n]$, the vector $v^{i,j}$ denotes the $(i, j)$-collapsing of $v$.

**Theorem 4.1.** *Let $v$ be a vector in $\mathbb{R}^N$ satisfying the following three conditions:*

(i) *there exists $p$ in $[1, n]$ such that $\sum_{j \in [1,n]-\{p\}} v_{pj} = 0$ ($\delta(\{p\})$ is a root of $v$);*

(ii) *there exist distinct $h$ and $k$ in $[1, n] - \{p\}$ such that both inequalities $(v^{p,h})^{\mathrm{T}}x \leqslant 0$ and $(v^{p,k})^{\mathrm{T}}x \leqslant 0$ define facets of $C_{n-1}$;*

(iii) *there exist distinct $i$ and $j$ in $[1, n] - \{p, h, k\}$ such that $v_{ij} \neq 0$.*

*If the inequality $v^{\mathrm{T}}x \leqslant 0$ is valid over $C_n$ then it defines a facet of $C_n$.*

**Proof.** Suppose the contrary: $v^{\mathrm{T}}x \leqslant 0$ does not define a facet of $C_n$. Then $v^{\mathrm{T}}x \leqslant 0$ is the sum of two inequalities, say $u^{\mathrm{T}}x \leqslant 0$ and $w^{\mathrm{T}}x \leqslant 0$ (with $u \neq \underline{0}$ and $w \neq \underline{0}$), valid over $C_n$. Let $u^{p,h}$ and $u^{p,k}$ be the $(p, h)$-collapsing and the $(p, k)$-collapsing of the vector $u$, respectively; similarly, let $w^{p,h}$ and $w^{p,k}$ be the $(p, h)$-collapsing and the $(p, k)$-collapsing of the vector $w$, respectively. Proposition 2.1 implies that the four inequalities

$$(u^{p,h})^{\mathrm{T}}x \leqslant 0, \qquad (u^{p,k})^{\mathrm{T}}x \leqslant 0,$$

$$(w^{p,h})^{\mathrm{T}}x \leqslant 0, \qquad (w^{p,k})^{\mathrm{T}}x \leqslant 0$$

are valid over $C_{n-1}$. It is easy to verify that $v^{p,h} = u^{p,h} + w^{p,h}$, and that $v^{p,k} = u^{p,k} + w^{p,k}$. Now, (ii) implies that either $u^{p,h} = \underline{0}$ or $w^{p,h} = \underline{0}$, and that either $u^{p,k} = \underline{0}$ or $w^{p,k} = \underline{0}$. Without loss of generality, we can assume that $u^{p,h} = \underline{0}$, and so

$$u_{pj} + u_{hj} = 0, \quad u_{ij} = 0 \quad \text{for all } i, j \in [1, n] - \{p, h\}. \tag{5}$$

If $w^{p,k}=\underline{0}$ then $v_{ij}=0$, for all $i,j$ in $[1,n]-\{p,h,k\}$, contradicting (iii). Hence, $u^{p,k}=\underline{0}$, and so

$$u_{pj}+u_{kj}=0, \quad u_{ij}=0 \quad \text{for all } i,j\in[1,n]-\{p,k\}. \tag{6}$$

Now, (5) and (6) imply

$$u_{ij}=\begin{cases} u_{ph} & \text{if } i=p,\ j=h \text{ or } k,\\ -u_{ph} & \text{if } i=k,\ j=h,\\ 0 & \text{otherwise.}\end{cases}$$

Since $u^{\mathrm{T}}x\leqslant 0$ is valid over $C_n$, it follows that $u_{ph}\leqslant 0$ (because $u^{\mathrm{T}}\delta(\{p\})=2u_{ph}$), and so

$$u_{ph}<0 \tag{7}$$

(for otherwise $u=\underline{0}$). Since $w^{\mathrm{T}}x\leqslant 0$ is valid over $C_n$, it follows that

$$\sum_{j\in[1,n]-\{p\}} w_{pj}\leqslant 0,$$

and so

$$\sum_{j\in[1,n]-\{p\}} v_{pj}\leqslant 2u_{ph}.$$

But, (i) implies that $\sum_{j\in[1,n]-\{p\}} v_{pj}=0$, and so $u_{ph}\geqslant 0$, contradicting (7). □

From the proof of Theorem 4.1, we get the following observation.

**Remark 1.** Let $v$ be a vector in $\mathbb{R}^N$ satisfying conditions (ii) and (iii) of Theorem 4.1. If the inequality $v^{\mathrm{T}}x\leqslant 0$ is valid over $C_n$, then either it defines a facet of $C_n$ or it is the sum of two inequalities valid over $C_n$, one of which is a positive multiple of the triangle facet-defining inequality $x_{hk}-x_{ph}-x_{pk}\leqslant 0$.

In the following, we show some applications of Theorem 4.1 on hypermetric and cycle inequalities.

**Corollary 4.2.** *Let $b=(b_1,\dots,b_{n-1})$ be an integer vector satisfying the following conditions:*
- $b_1+\cdots+b_{n-1}=1$;
- *there exist distinct $h$ and $k$ in $[1,n-1]$ such that $b_h=b_k-1$;*
- *there exist distinct $i$ and $j$ in $[1,n-1]-\{h,k\}$ such that $b_ib_j\neq 0$.*

*If the hypermetric inequality* Hyp($b$) *defines a facet of $C_{n-1}$, then the hypermetric inequality* Hyp($d$) *specified by the vector $d=(d_1,\dots,d_n)$, with*

$$d_i=\begin{cases} b_h & \text{if } i=k,\\ 1 & \text{if } i=n,\\ b_i & \text{otherwise,}\end{cases}$$

*defines a facet of* $C_n$.

**Proof.** Without loss of generality, we may assume that $h=1$ and $k=2$, and so $d=(b_1,b_1,b_3,\dots,b_{n-1},1)$. Let $d^{1,n}$ and $d^{2,n}$ be the $(1,n)$-collapsing and the $(2,n)$-collapsing of the vector $d$, respectively. We have

$$d^{1,n}=(b_1+1,b_1,b_3,\dots,b_{n-1}),$$

$$d^{2,n}=(b_1,b_1+1,b_3,\dots,b_{n-1}),$$

Since $d^{1,n}$ can be obtained from $d^{2,n}$ by a permutation of the set $[1,n-1]$, it follows that the hypermetric inequality Hyp($d^{1,n}$) is permutation equivalent to the hypermetric inequality Hyp($d^{2,n}$). Note that $d^{2,n}=b$, and so Hyp($d^{2,n}$) defines a facet of $C_{n-1}$. Define a vector $v$ in $\mathbb{R}^N$ by $v_{ij}=d_id_j$ for all $ij$. Now the vector $v$ satisfies conditions (i), (ii) and (iii) of Theorem 4.1 with $p=n$. □

For instance, consider the vector $b=(3,2,2,-1,-1,-1,-1,-2)$. Since the hypermetric inequality Hyp($b$) defines a facet of $C_8$, Corollary 4.2 implies that the hypermetric inequality Hyp$(3,2,2,-1,-1,-1,-2,-2,1)$ defines a facet of $C_9$. (Here, $h=7$ and $k=8$.)

**Corollary 4.3.** *Let* $c=(c_1,\dots,c_{n-2})$ *be an integer vector satisfying the following three conditions:*
- $c_1+\cdots+c_{n-2}=1$;
- *there exists* $h$ *in* $[1,n-2]$ *such that* $c_h=-1$;
- *there exist distinct* $i$ *and* $j$ *in* $[1,n-2]-\{h\}$ *such that* $c_ic_j\neq 0$.

*If the hypermetric inequality* Hyp($c$) *defines a facet of* $C_{n-2}$, *then the hypermetric inequality* Hyp($d$) *specified by the vector* $d=(d_1,\dots,d_n)$, *with*

$$d_i=\begin{cases} c_i & \text{if } i=1,2,\dots,n-2\\ 1 & \text{if } i=n-1,\\ -1 & \text{if } i=\text{n},\end{cases}$$

*defines a facet of* $C_n$.

**Proof.** Corollary 2.1 guarantees that the hypermetric inequality Hyp($b$) specified by the vector $b=(c_1,c_2,\dots,c_{n-2},0)$ defines a facet of $C_{n-1}$. Now, observe that the vector $b$ satisfies the assumptions of Corollary 4.2 (with $k=n-1$). □

**Corollary 4.4.** *Let* $f$ *be an integer greater than or equal to three, and let* $b=(b_1,\dots,b_{n-1})$ *be an integer vector satisfying the following four conditions:*
- $b_1+\cdots+b_{n-1}=3$;
- $b_1,\dots,b_f>0\geqslant b_{f+1},\dots,b_n$;

– *there exists distinct $h$ and $k$ in $[1,f]$ such that $b_h = b_k - 1$ with $k = h+1 \pmod f$;*
– *there exists distinct $i$ and $j$ in $[1, n-1] - \{h,k\}$ such that $b_i b_j \neq 0$.*

*Let $b'$ be the vector obtained from $b$ by permuting $h$ and $k$; let $d$ be the vector obtained from $b$ by inserting 1 between $b_h$ and $b_k$ and by replacing $b_k$ with $b_h$. If the cycle inequality* Cyc($b$) *defines a facet of $C_{n-1}$, and if the cycle inequality* Cyc($b'$) *defines a facet of $C_{n-1}$, then the cycle inequality* Cyc($d$) *defines a facet of $C_n$.*

**Proof.** Without loss of generality, we may assume that $h=1$ and $k=2$, and so $d=(b_1, 1, b_1, b_3, \ldots, b_{n-1})$. Let $v^{\mathrm{T}}x \leqslant 0$ denote the cycle inequality Cyc($d$); write

$$d^{1,2} = (b_1 + 1, b_1, b_3, \ldots, b_{n-1}),$$

$$d^{2,3} = (b_1, b_1 + 1, b_3, \ldots, b_{n-1}).$$

It is easy to verify that the (1, 2)-collapsing and (1, 3)-collapsing of the vector $v$ yield the two cycle inequalities Cyc($d^{1,2}$) and Cyc($d^{1,3}$), respectively. Since $d^{1,2} = b'$ and since $d^{2,3} = b$, both inequalities Cyc($d^{1,2}$) and Cyc($d^{2,3}$) define facets of $C_{n-1}$. Now the vector $v$ satisfies conditions (*i*), (*ii*) and (*iii*) of Theorem 4.1 with $p=2$. □

For instance, consider the vector $b=(3,2,2,-1,-1,-1,-1)$. Since the cycle inequality Cyc($b$) defines a facet of $C_7$, Corollary 4.4 implies that the cycle inequality Cyc$(2,1,2,2,-1,-1,-1,-1)$ defines a facet of $C_8$. (Here, $h=1$ and $k=2$.)

We ended Section 3 by pointing out that, in general, the zero-lifting of a face does not preserve the codimension. In the following, we show that a similar result holds for the general nonzero-lifting. For this purpose, let $n$ be an integer greater than or equal to eight; let $b^n$ be the vector in $\mathbb{R}^n$ defined by $b^n = (n-6, 2, 2, 1, 1, -1, \ldots, -1)$; and let $w$ be the vector in $\mathbb{R}^{\binom{5}{2}}$ with components $w_{12} = w_{23} = 3$, $w_{15} = w_{34} = 2$, $w_{14} = w_{35} = w_{45} = 1$, and $w_{ij} = 0$ otherwise. Consider the inequality

$$\sum_{1 \leqslant i < j \leqslant n} b_i^n b_j^n x_{ij} - \sum_{1 \leqslant i < j \leqslant 5} w_{ij} x_{ij} \leqslant 0. \tag{8}$$

The inequality (8) belongs to the class of *clique-web* inequalities valid over $C_n$ introduced by Deza and Laurent in [18]: (8) is the clique-web inequality $CW_n^2(b^n)$ with corresponding antiweb $AW_5^2(n-6, 2, 2, 1, 1)$.

**Proposition 4.5.** *Let $n \geqslant 8$. Then the inequality* (8) *defines a face of $C_n$ of dimension $\binom{n}{2} - (n-4)$.*

**Proof.** For every $n \geqslant 8$, define a vector $v^n$ in $\mathbb{R}^N$ by

$$(v^n)_{ij} = \begin{cases} b_i^n b_j^n - w_{ij} & \text{if } 1 \leqslant i < j \leqslant 5, \\ b_i^n b_j^n & \text{otherwise.} \end{cases}$$

To prove that the inequality, $(v^n)^{\mathrm{T}} x \leqslant 0$ defines a face of $C_n$ of dimension $\binom{n}{2} - (n-4)$, we use induction on $n$. A computer check guarantees that $(v^8)^{\mathrm{T}} x \leqslant 0$ defines a face of

$C_8$ of dimension 24. Now, suppose that the inequality $(v^n)^T x \leqslant 0$ defines a face of $C_n$ of dimension $\binom{n}{2}-(n-4)$. We want to show that the inequality $(v^{n+1})^T x \leqslant 0$ defines a face of $C_{n+1}$ of dimension $\binom{n+1}{2}-(n-3)$. For this purpose, let $S$ be a subset of $[2,n]$. Since $\sum_{i=1}^n b_i^n = 5$, the cut $\delta(S)$ is a root of $v^n$ if and only if

$$b^n(S)(5-b^n(S)) = \sum_{ij\in\delta(S)} w_{ij},$$

and so every root $\delta(S)$ of $v^n$, with $1 \notin S$, yields a root of $v^{n+1}$. By the inductive hypothesis, $\dim(v^n)=\binom{n}{2}-(n-4)$, and so we can find a set $R_1$ that contains $\dim(v^n)$ linearly independent roots of $v^{n+1}$. Since $\binom{n+1}{2}-(n-3)=\binom{n}{2}-(n-4)+(n-1)$, we only need find $n-1$ additional roots. Consider the following $n-1$ sets:

$$S'=\{2,3,n+1\}, \qquad S'=\{3,4,n+1\}, \qquad S'=\{3,4,5,n+1\},$$
$$S'=\{2,3,4,6,n+1\}, \qquad S'=\{2,3,k,n+1\}, \quad \text{for every } k=6,\dots,n.$$

Clearly, every set $S'$ listed above yields a root of $v^{n+1}$. Let $R_2$ denote the set of these $n-1$ new roots of $v^{n+1}$. Now it is easy to verify that all the vectors in $R_1 \cup R_2$ are linearly independent. □

## 5. Difference of inequalities

In this section, we show how to construct, from a given face of the cut cone $C_n$, a face of $C_n$ of higher dimension. Recall that $N$ stands for $\binom{n}{2}$. Let $v$ be a vector in $\mathbb{R}^N$ such that $v^T x \leqslant 0$ is valid over $C_n$; we want to find a vector $w$ in $\mathbb{R}^N$ and two nonzero real numbers $\alpha$ and $\beta$ such that both inequalities $w^T x \leqslant 0$ and $(\alpha v - \beta w)^T x \leqslant 0$ are valid over $C_n$. Clearly, if the inequality $(v-w)^T x \leqslant 0$ is valid over $C_n$, then the face $F_v$ defined by the inequality $v^T x \leqslant 0$ is contained in the face $F_w$ defined by the inequality $w^T x \leqslant 0$. For every vector $v$ in $\mathbb{R}^N$, let $m_v$ and $M_v$ denote the minimum and maximum nonzero value assumed by $|v^T \delta(S)|$ over all subsets $S$ of $[1,n]$, respectively. In this section, we let $n$ stand for an integer greater than or equal to seven.

**Proposition 5.1.** *Let $v^T x \leqslant 0$ and $w^T x \leqslant 0$ be two inequalities valid over $C_n$. If $F_v \subseteq F_w$ then the inequality*

$$(M_w v - m_v w)^T x \leqslant 0$$

*is valid over $C_n$.*

**Proof.** Let $S$ be a subset of $[1,n]$. If $\delta(S)$ is a root of $w$ then

$$(M_w v - m_v w)^T \delta(S) = M_w v^T \delta(S),$$

which is $\leqslant 0$. If $\delta(S)$ is not a root of $w$ then it is neither a root of $v$ (since $F_v \subseteq F_w$). Hence

$$(M_w v - m_v w)^{\mathrm{T}} \delta(S) = M_w v^{\mathrm{T}} \delta(S) - m_v w^{\mathrm{T}} \delta(S)$$
$$\leqslant -M_w m_v + m_v M_w = 0. \quad \square$$

Let $v^{\mathrm{T}} x \leqslant 0$ be an hypermetric inequality specified by a vector $b$ in $\mathbb{R}^N$. Clearly, for every cut vector $\delta(S)$,

$$v^{\mathrm{T}} \delta(S) = b(S)(1 - b(S)),$$

which is an even integer. Hence, $m_v \geqslant 2$. Furthermore, if $v^{\mathrm{T}} x \leqslant 0$ is a triangle inequality, then $m_v = M_v = 2$. Similarly, if $v^{\mathrm{T}} x \leqslant 0$ is a cycle inequality specified by a vector $b$ in $\mathbb{R}^N$ and a cycle $C$, then, for every cut vector $\delta(S)$,

$$v^{\mathrm{T}} \delta(S) = b(S)(3 - b(S)) - |E(C) \cap \delta(S)|,$$

which, again, is an even integer, and so $m_v \geqslant 2$.

**Corollary 5.2.** *Let $v^{\mathrm{T}} x \leqslant 0$ be an hypermetric or cycle inequality over $C_n$, and let $w^{\mathrm{T}} x \leqslant 0$ be a triangle inequality. If $F_v \subseteq F_w$ then the inequality*

$$(v - w)^{\mathrm{T}} x \leqslant 0$$

*is valid over $C_n$.*

The proof follows directly from Proposition 5.1 and from the fact that $m_v \geqslant M_w = 2$.

**Proposition 5.3.** *Let $b = (b_1, \ldots, b_n)$ be an integer vector in $\mathbb{R}^n$ such that $b_i \neq 0$ $(i = 1, \ldots, n)$ and such that $b_1 + \cdots + b_n = 1$. Let the components of $b$ be ordered in such a way that $b_1, \ldots, b_p > 0 > b_{p+1}, \ldots, b_n$, for some $p \geqslant 2$. If there exist distinct $i$ and $j$ in $[1, p]$, and if there exists a $k$ in $\{p+1, \ldots, n\}$ such that*

$$b_i + b_j - b_k \geqslant \sum_{i=1}^{p} b_i + 1, \tag{9}$$

*then the face of $C_n$ defined by the inequality* Hyp$(b)$ *is strictly contained in the face of $C_n$ defined by the triangle inequality $x_{ij} - x_{ik} - x_{jk} \leqslant 0$.*

**Proof.** Let $S$ be a subset of $[1, n]$ such that $\delta(S)$ is a root of Hyp$(b)$. Without loss of generality, we may assume that $k \notin S$. Note that $b(S)$ is equal to zero or one. If both $i$ and $j$ are in $S$ then

$$b(S) \geqslant b_i + b_j + \sum_{i=p+1}^{n} b_i - b_k,$$

and so (9) implies that $b(S) \geqslant 2$, a contradiction. If at least one of $i$ and $j$ is not in $S$, then $\delta(S)$ is a root of $x_{ij} - x_{ik} - x_{jk} \leqslant 0$. $\square$

For instance, consider the hypermetric inequality Hyp($b$) with $b=(1,3,2,-1,-1,-1,-2)$. Since $b$ satisfies (9) (with $i=2, j=3$ and $k=7$), it follows that every root of Hyp($b$) is also a root of the triangle inequality Hyp$(0,1,1,0,0,0,-1)$. Observe that an hypermetric inequality Hyp($b$) satisfying the assumptions of Proposition 5.3 does not define a facet of $C_n$.

**Proposition 5.4.** *Let $b=(b_1,\ldots,b_n)$ be an integer vector in $\mathbb{R}^n$ such that $b_1+\cdots+b_n=3$. Let the components of $b$ be ordered in such a way that $b_1,\ldots,b_f>0>b_{f+1},\ldots,b_n$ for some $f\geqslant 3$. If there exist distinct $i$ and $j$ in $[1,f]$, and if there exists a $k$ in $\{f+1,\ldots,n\}$ such that*

$$b_i+b_j-b_k\geqslant\sum_{i=1}^{f} b_i,$$

*then the face of $C_n$ defined by the inequality* Cyc($b$) *is strictly contained in the face of $C_n$ defined by the triangle inequality $x_{ij}-x_{ik}-x_{jk}\leqslant 0$.*

**Proof.** The proof is similar to the proof of Proposition 5.3 and relies on the fact that if $\delta(S)$ is a root of Cyc($b$) then $b(S)$ is equal to one or two. □

Consider again the hypermetric inequality Hyp($b$) with $b=(1,3,2,-1,-1,-1,-2)$, and let Hyp$(d_1,\ldots,d_7)$ denote the triangle inequality Hyp$(0,1,1,0,0,0,-1)$. We have seen that every root of Hyp($b$) is also a root of Hyp($d$). Hence, Corollary 5.2 implies that the inequality

$$\sum_{1\leqslant i<j\leqslant 7}(b_ib_j-d_id_j)x_{ij}\leqslant 0 \tag{10}$$

is valid over $C_7$. Now, the inequality Hyp($b$) has 19 roots, all of which are linearly independent. Since every root of Hyp($b$) is a root of (10), to show that (10) defines a facet of $C_7$, we only need find one root of (10) which is linearly independent from the other 19; our choice for such a root is $\delta(\{1,7\})$. Hence, (10) defines a facet of $C_7$. Theorem 5.5 will generalize this procedure. Incidentally, inequality (10) can be obtained from the cycle inequality Cyc$(3,2,2,-1,-1,-1,-1)$ by switching the cut $\delta(\{1,7\})$.

**Theorem 5.5.** *Let $b=(2n-13,3,2,-1,-1,-1,-2,\ldots,-2)$ and $d=(n-7,1,1,0,0,0,-1,\ldots,-1)$ be two vectors in $\mathbb{R}^n$. Then the inequality*

$$\sum_{1\leqslant i<j\leqslant n}(b_ib_j-d_id_j)x_{ij}\leqslant 0$$

*defines a facet of $C_n$.*

**Proof.** Write

$$v^{\mathrm{T}}x=\sum_{1\leqslant i<j\leqslant n}(b_ib_j-d_id_j)x_{ij}.$$

To prove validity of the inequality $v^{\mathrm{T}}x \leqslant 0$, let $S$ be a subset of the set $[1,n]$. Without loss of generality, we can assume that $1 \notin S$. We have

$$v^{\mathrm{T}}\delta(S) = b(S)(1-b(S)) - d(S)(1-d(S))$$
$$= (b(S)-d(S))(1-b(S)-d(S)).$$

Set

$$\alpha = |S \cap \{2,3\}|, \qquad \beta = |S \cap \{4,5,6\}|, \qquad \gamma = |S \cap ([1,n]-[1,6]).$$

It is easy to verify that

$$b(S) = k - \beta - 2\gamma, \qquad d(S) = \alpha - \gamma,$$

where $k \in \{0,2,3,5\}$, and so

$$v^{\mathrm{T}}\delta(S) = (k-\alpha-\beta-\gamma)(1-k-\alpha+\beta+3\gamma).$$

Now it is a routine but tedious matter to verify that $v^{\mathrm{T}}\delta(S) \leqslant 0$.

We prove that $v^{\mathrm{T}}x \leqslant 0$ defines a facet of $C_n$, for all $n \geqslant 7$, by induction on $n$. For this purpose, note that when $n=7$, $v^{\mathrm{T}}x \leqslant 0$ is inequality (10), and so it defines a facet of $C_7$. Now assume that $v^{\mathrm{T}}x \leqslant 0$ defines a facet of $C_n$, and let $b'$ and $d'$ be two vectors in $\mathbb{R}^{n+1}$ given by

$$b' = (2(n+1)-13, 3, 2, -1, -1, -1, -2, \ldots, -2),$$
$$d' = ((n+1)-7, 1, 1, 0, 0, 0, -1, \ldots, -1).$$

Write

$$(v')^{\mathrm{T}}x = \sum_{1 \leqslant i < j \leqslant n} (b'_i b'_j - d'_i d'_j)x_{ij}.$$

We want to show that the inequality

$$(v')^{\mathrm{T}}x \leqslant 0$$

defines a facet of $C_{n+1}$, i.e. we want to exhibit $\binom{n+1}{2}-1$ linearly independent roots of the vector $v'$. For this purpose, let $\delta(S)$ be a root of the vector $v$. Without loss of generality, we may assume that $1 \notin S$, and so every root of $v$ is also a root of $v'$. By the inductive hypothesis, $\dim(v) = \binom{n}{2}-1$, and so there exist $\binom{n}{2}-1$ linearly independent roots of $v'$; let $R_1$ be the set containing such roots. Since $\binom{n+1}{2} = \binom{n}{2}+n$, we only need find $n$ additional roots. For this purpose, let $S'$ be a subset of $[1,n+1]$. Since

$$(v')^{\mathrm{T}}\delta(S) = b'(S)(1-b'(S)) - d'(S)(1-d'(S))$$
$$= (b'(S)-d'(S))(1-b'(S)-d'(S)),$$

it follows that $\delta(S')$ is a root of $v'$ if and only if either $b'(S') = d'(S')$ or $b'(S')+d'(S') = 1$. Let $S'$ be a subset of $[1,n+1]$ such that $1 \notin S'$ and such that $n+1 \in S'$; set $S = S' - \{n+1\}$. We have

$$b'(S') = b(S) - 2, \qquad d'(S') = d(S) - 1,$$

and so $\delta(S')$ is a root of $v'$ if and only if

$$\text{either } b(S)=d(S)+1 \quad \text{or} \quad b(S)+d(S)=4. \tag{11}$$

Hence, to find $n$ additional roots $\delta(S')$ of $v'$, we only need find $n$ subsets $S$ of $[1,n]-\{1\}$ satisfying (11). Our choice for such sets is as follows.

$$S=\{2\}, \qquad S=\{3\}, \qquad S=\{2,4\}, \qquad S=\{2,5\},$$

$$S=\{2,6\}, \qquad S=\{2,k\}, \text{ for every } k=7,\ldots,n, \qquad S=\{2,3,n-1,n\}.$$

Clearly, every set $S$ listed above satisfies (11). Let $R_2$ denote the set of these $n$ new roots of $v'$. Now it is easy to verify that all the roots in $R_1 \cup R_2$ are linearly independent. □

We end this section by exhibiting a class of hypermetric inequalities defining faces of $C_n$, which are not contained in the face of $C_n$ defined by any triangle inequality.

**Proposition 5.6.** *Let $n$ be an odd integer greater than or equal to seven, and let $b$ be the vector in $\mathbb{R}^n$ given by $b=(c,c,c,-c,-1,\ldots,-1)$, where $c=(n-3)/2$. Then the face of $C_7$ defined by* Hyp$(b)$ *is not contained in the face of $C_n$ defined by any triangle inequality. Furthermore, let $d=(1,1,1,-1,-1,0,\ldots,0)$ be a vector in $\mathbb{R}^n$. Then the inequality*

$$\sum_{ij}(b_ib_j-d_id_j)x_{ij}\leqslant 0$$

*is valid over $C_n$.*

**Proof.** Let $i,j$, and $k$ be arbitrary distinct elements in $[1,n]$. Clearly, to show that the face of $C_n$ defined by Hyp$(b)$ is not contained in the face of $C_n$ defined by the triangle inequality $x_{jk}-x_{ij}-x_{ik}\leqslant 0$, we only need exhibit a root $\delta(S)$ of Hyp$(b)$ such that $S\cap\{i,j,k\}=\{i\}$. If $i\in\{1,2,3\}$ then it is easy to verify that the desired root is $\delta(\{i\}\cup T)$, where $T$ is any subset of $\{5,\ldots,n\}-\{j,k\}$ of size $c-1$; if $i=4$ then it is easy to verify that the desired root is $\delta(\{1,i\})$; if $i=5,\ldots,n$ then it is easy to verify that the desired root is $\delta(\{1,i\}\cup T)$, where $T$ is any subset of $\{5,\ldots,n\}-\{i,j,k\}$ of size $c-2$.

The proof of validity of $\sum_{ij}(b_ib_j-d_id_j)x_{ij}\leqslant 0$ is similar to the proof of validity in Theorem 5.5. □

## 6. The cut cone on seven points

In 1960, Deza [12, 14] proved that all the facet-defining inequalities of $C_4$ and $C_5$ are hypermetric; $C_4$ has 12 triangle facets and $C_5$ has 40 facets (30 triangle facets and 10 facets of the type Hyp$(1,1,1,-1,-1)$ called *pentagonal* facets). In 1988, Avis and Mutt [7] proved using computer that all the facet-defining inequalities of $C_6$ are hypermetric; there are precisely 210 of them (60 triangle facets, 60 pentagonal facets, 60 facets of the type Hyp$(2,1,1,-1,-1,-1)$, and 30 facets of the type

Hyp(1, 1, 1, 1, −1, −2)). This is not true for $C_7$: Avis [4, 5] and Assouad [1] were the first to prove this. In 1989, Grishukhin [21] proved using computer that all the facet-defining inequalities of $C_7$ are (up to switching by a root and permutation) of four types: hypermetric inequalities, cycle inequalities, *parachute inequalities* and *Grishukhin* inequalities; the cut cone $C_7$ has precisely 38780 facets [19]. Let $S$ be a subset of $[1, 7]$; in this section, for every vector $v$ in $\mathbb{R}^{\binom{7}{2}}$, $v^S$ denotes the vector obtained from $v$ by switching a root $\delta(S)$ of $v$.

Below we give a list of 36 facet defining inequalities of $C_7$; they are split into four groups.

(1) The first group consists of the following ten hypermetric facet-defining inequalities:

(H1): Hyp(1, 1, −1, 0, 0, 0, 0);

(H2): Hyp(1, 1, 1, −1, −1, 0, 0);

(H3): Hyp(1, 1, 1, 1, −1, −1, −1);

(H4): Hyp(2, 1, 1, −1, −1, −1, 0);

(H5): Hyp(−2, 1, 1, 1, 1, −1, 0);

(H6): Hyp(2, 2, 1, −1, −1, −1, −1);

(H7): Hyp(−2, 2, 1, 1, 1, −1, −1);

(H8): Hyp(−2, −2, 1, 1, 1, 1, 1);

(H9): Hyp(3, 1, 1, −1, −1, −1, −1);

(H10): Hyp(−3, 1, 1, 1, 1, 1, −1).

Note that inequality (H5) arises from inequality (H4) by switching the root $\delta(\{1, 4, 5\})$; (H7) arises from (H6) by switching the root $\delta(\{1, 4, 5\})$; (H8) arises from (H6) by switching the root $\delta(\{3\})$; (H10) arises from (H9) by switching the root $\delta(\{2, 3, 4\})$.

(2) The second group consists of 16 inequalities obtained from the following three cycle facet-defining inequalities by switching:

(C1): Cyc(1, 1, 1, 1, 1, −1, −1),
(C2): Cyc(2, 2, 1, 1, −1, −1, −1),
(C3): Cyc(3, 2, 2, −1, −1, −1, −1).

Let $u^{\mathrm{T}}x \leqslant 0$, $v^{\mathrm{T}}x \leqslant 0$, and $w^{\mathrm{T}}x \leqslant 0$ denote inequalities (C1), (C2), and (C3), respectively. Switching roots of inequalities (C1), (C2), and (C3), yields the following (noncycle) inequalities:

(C4): $(u^{\{1\}})^{\mathrm{T}}x \leqslant 0$,

(C5): $(u^{\{1,2\}})^{\mathrm{T}}x \leqslant 0$,

(C6): $(u^{\{1,2,6\}})^{\mathrm{T}}x \leqslant 0$,

(C7): $(v^{\{1\}})^{\mathrm{T}}x \leqslant 0$,

(C8): $(v^{\{3\}})^{\mathrm{T}}x \leqslant 0$,

(C9): $(v^{\{1,5\}})^{\mathrm{T}}x \leqslant 0$,

(C10): $(v^{\{3,4\}})^{\mathrm{T}}x \leqslant 0$,

(C11): $(v^{\{1,4,5\}})^{\mathrm{T}}x \leqslant 0$,

(C12): $(v^{\{3,4,5\}})^{\mathrm{T}}x \leqslant 0$;

(C13): $(w^{\{2\}})^{\mathrm{T}}x \leqslant 0$,

(C14): $(w^{\{2,4\}})^{\mathrm{T}}x \leqslant 0$,

(C15): $(w^{\{1,4\}})^{\mathrm{T}}x \leqslant 0$,

(C16): $(w^{\{1,4,5\}})^{\mathrm{T}}x \leqslant 0$.

(3) The third group consists of a parachute inequalilty and its two switchings. This parachute inequality is the inequality

(P1) $p^{\mathrm{T}}x \leqslant 0$,

where the vector $p=(p_{12}, \ldots, p_{17}; \ldots; p_{67})^{\mathrm{T}}$ is given by

$$(0, -1, -1, -1, -1, 0;\ 1, 0, -1, -1, -1;\ 1, 0, 0, -1;\ 1, 0, -1;\ 1, 0;\ 1)^{\mathrm{T}}.$$

Switching roots of the inequality $(P1)$, yields the following two inequalities:

(P2): $(p^{\{3,7\}})^{\mathrm{T}}x \leqslant 0$,

(P3): $(p^{\{1,3,6\}})^{\mathrm{T}}x \leqslant 0$,

The graphs $P_1$, $P_2$ and $P_3$ in Fig. 1 are the supporting graphs of the vectors $p$, $p^{\{3,7\}}$, and $p^{\{1,3,6\}}$, respectively: a plain line $ij$ corresponds to an edge $ij$ with weight equal to 1, a dashed line $ij$ corresponds to an edge $ij$ with weight equal to $-1$.

(4) The fourth group consists of the Grishukhin inequality and its six switchings. The Grishukhin is the inequality

(G1): $g^{\mathrm{T}}x \leqslant 0$,

where the vector $g=(g_{12}, \ldots, g_{17}; \ldots; g_{67})^{\mathrm{T}}$ is given by

$$(1, 1, 1, -2, -1, 0;\ 1, 1, -2, 0, -1;\ 1, -2, -1, 0;\ -2, 0, -1;\ 1, 1;\ -1)^{\mathrm{T}}.$$

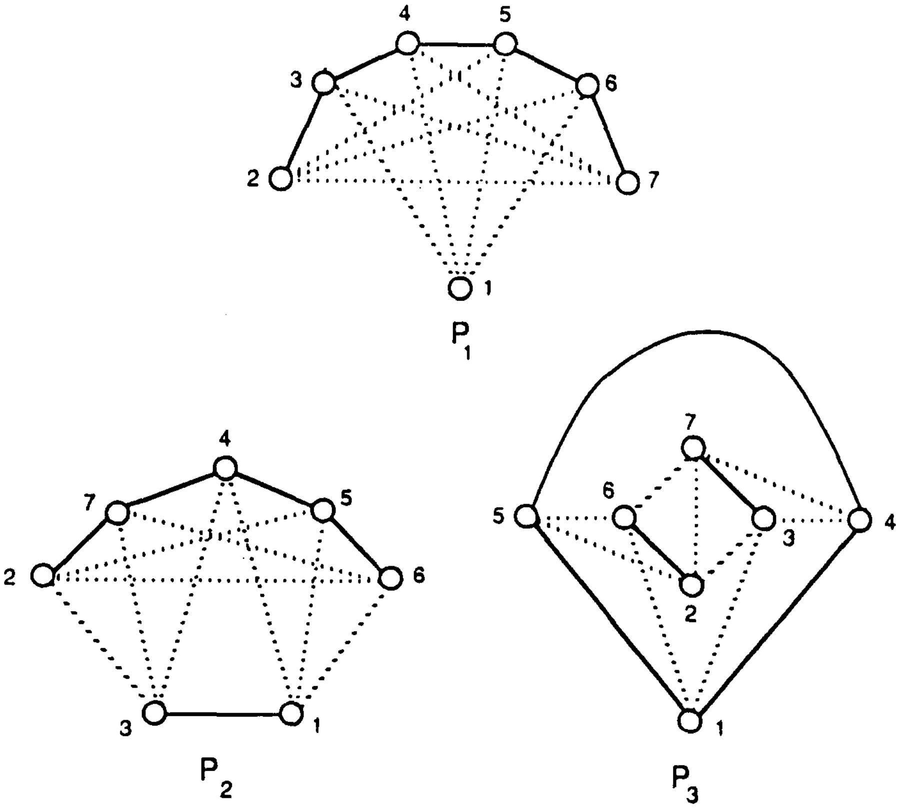


Fig. 1.

The graph in Fig. 2 is the supporting graph of the vector $g$: a plain line $ij$ corresponds to an edge $ij$ with weight equal to 1, a dashed line $ij$ corresponds to an edge $ij$ with weight equal to $-1$, a double dashed line $ij$ corresponds to an edge $ij$ with weight equal to $-2$.

Switching roots of the inequality ($G1$), yields the following six inequalities:

(G2): $(g^{\{1\}})^T x \leqslant 0$,

(G3): $(g^{\{1,6\}})^T x \leqslant 0$,

(G4): $(g^{\{1,6,7\}})^T x \leqslant 0$,

(G5): $(g^{\{1,3,5\}})^T x \leqslant 0$,

(G6): $(g^{\{1,3,6\}})^T x \leqslant 0$,

(G7): $(g^{\{1,2,5\}})^T x \leqslant 0$,

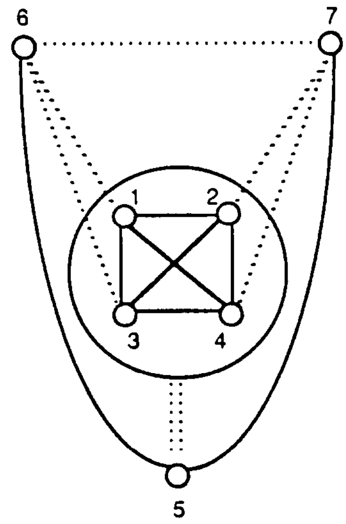


Fig. 2

Inequality (C3) was found by Avis [4, 5]; Assouad [1, 3] found the inequalities (Ci) ($i=1,\dots,6,13$) and the inequalities (Pi) ($i=1,2,3$). Inequality (P1) is called parachute inequality since it belongs to the class of parachute inequalities introduced in [17]; inequality (G1) is called Grishukhin inequality since it was found by Grishukhin along with its six switchings [20, 21].

Let $L$ denote the set of the 36 inequalities listed above. Grishukhin [21] proved that every facet-defining inequality of $C_7$ is switching or permutation equivalent to some inequality in $L$. We now show that the list $L$ is (up to permutation) complete.

**Theorem 6.1.** *Every facet-defining inequality of $C_7$ is permutation equivalent to some inequality in $L$.*

**Proof.** We only need verify that every inequality obtained by switching a root of some inequality in $L$ is also in $L$. For this purpose, let $S$ and $S'$ be two subsets of $[1,7]$; clearly, $(v^S)^{S'}=v^{S\Delta S'}$ for every vector $v$ in $\mathbb{R}^{\binom{7}{2}}$, where $S\Delta S'=(S-S')\cup(S'-S)$. It follows that every inequality obtained from some inequality $(v^S)^{\mathrm{T}}x\leqslant 0$ by switching a root of $v^S$ belongs to the family of all the inequalities $(v^S)^{\mathrm{T}}x\leqslant 0$ obtained from the inequality $v^{\mathrm{T}}x\leqslant 0$ by switching all the roots $\delta(S)$ of $v$.

First, consider an arbitrary hypermetric inequality $\mathrm{Hyp}(b_1,\dots,b_n)$, and let $\delta(S)$ be one of its roots; assume that $\sum_{i\in S}b_i=0$. It is easy to verify that switching $\mathrm{Hyp}(b_1,\dots,b_n)$ by $\delta(S)$ yields the hypermetric inequality $\mathrm{Hyp}(b'_1,\dots,b'_n)$ with $b'_i=-b_i$ if $i\in S$ and $b'_i=b_i$ if $i\notin S$. Now it is easy to verify that every switching of an $(Hi)$ ($i=1,2,3,4,6,9$) is permutation equivalent to one of $(H_i)$, $i=1,\dots,10$.

Secondly, consider an arbitrary cycle inequality $\mathrm{Cyc}(b_1, \ldots, b_f, \ldots, b_n)$ with cycle $C=(1, \ldots, f)$, and let $\delta(S)$ be one of its roots such that $\sum_{i \in S} b_i = 1$. Recall that $b_1, \ldots, b_f > 0 > b_{f+1}, \ldots, b_n$, that $f \geqslant 3$, and that $E(C)$ stands for the edge set of the cycle $C$. It is easy to show that switching $\mathrm{Cyc}(b_1, \ldots, b_n)$ by $\delta(S)$ yields the inequality

$$\sum_{i=1}^{n} b_i' b_j' x_{ij} - \left( - \sum_{ij \in \delta(S) \cap E(C)} x_{ij} + \sum_{ij \in E(C) - \delta(S)} x_{ij} \right) \leqslant 0, \tag{12}$$

with $b_i' = -b_i$ if $i \in S$ and $b_i' = b_i$ if $i \notin S$. Since $b_1', \ldots, b_n' = 1$, (12) is the sum of two inequalities one of which is the hypermetric inequality $\mathrm{Hyp}(b_1', \ldots, b_n')$ and the other one is a 'switched' cycle. We simply write (12) as

$$\mathrm{Hyp}(b_1', \ldots, b_n') - \left( - \sum_{ij \in \delta(S) \cap E(C)} x_{ij} + \sum_{ij \in E(C) - \delta(S)} x_{ij} \right) \leqslant 0.$$

Now consider the cycle inequality (C1); set

$$R_1 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\};$$

$$R_2 = \{\{3,4,5,6,7\}, \{1,4,5,6,7\}, \{1,2,5,6,7\}, \{1,2,3,6,7\}, \{2,3,4,6,7\}\};$$

$$R_3 = \{\{1,2,6\}, \{1,2,7\}, \{2,3,6\}, \{2,3,7\}, \{3,4,6\}, \{3,4,7\}, \{4,5,6\}, \{4,5,7\}, \{1,5,6\}, \{1,5,7\}\}.$$

It is easy to verify that the set $R$ of all the roots of (C1) is the union of all sets $\delta(S)$ with $S$ in $\bigcup_{i=1}^{3} R_i$, and that every switching of the inequality (C1) by a root in $R$ yields an inequality that is permutation equivalent to one of the following three inequalities:

$$\mathrm{Hyp}(-1, 1, 1, 1, 1, -1, -1) - (-x_{12} + x_{23} + x_{34} + x_{45} - x_{15}),$$

$$\mathrm{Hyp}(1, 1, -1, -1, -1, 1, 1) - (x_{12} - x_{23} + x_{34} + x_{45} - x_{15}),$$

$$\mathrm{Hyp}(-1, -1, 1, 1, 1, 1, -1) - (x_{12} - x_{23} + x_{34} + x_{45} - x_{15}),$$

which are not permutation equivalent (they are (C4), (C5), and (C6), respectively). A similar proof holds for (C2) and (C3).

Thirdly, consider the parachute inequality (P1); set

$$R_1 = \{\{3\}, \{6\}, \{3,5\}, \{4,6\}, \{2,4,6\}, \{3,5,7\}\};$$

$$R_2 = \{\{4\}, \{5\}, \{2,6\}, \{3,6\}, \{3,7\}, \{2,3,5,7\}, \{2,4,6,7\}\};$$

$$R_3 = \{\{2,5\}, \{4,7\}, \{2,5,7\}, \{2,4,7\}, \{2,5,6\}, \{3,4,7\}, \{2,4,5,7\}\}.$$

It is easy to verify that the set of all the roots of (P1) is the union of all sets $\delta(S)$ with $S$ in $\bigcup_{i=1}^{3} R_i$ [17], and that every switching of the nequality (P1) by a root $\delta(S)$ with $S$ in $R_i$ $(i=1,2,3)$ yields an inequality that is permutation equivalent to inequality (P$i$) $(i=1,2,3)$; in addition, inequalities (P1), (P2), and (P3) are not permutation equivalent.

Finally, consider the Grishukhin inequality (G1); let $R$ denote the set of all its roots. Set

$$R_1=\{\{1\},\{2\},\{3\},\{4\}\},$$

$$R_2=\{\{1,6,7\},\{2,6,7\},\{3,6,7\},\{4,6,7\}\},$$

$$R_3=\{\{1,6\},\{3,6\},\{2,7\},\{4,7\}\},$$

$$R_4=\{\{2,4,5\},\{1,3,5\}\},$$

$$R_5=\{\{1,3,6\},\{2,4,7\}\},$$

$$R_6=\{\{1,4,5\},\{3,4,5\},\{1,2,5\},\{2,3,5\}\}.$$

It is easy to verify that $R$ is the union of all sets $\delta(S)$ with $S$ in $\bigcup_{i=1}^{6} R_i$, and that every switching of the inequality (G1) by a root in $R$ yields an inequality that is permutation equivalent to one of (G$i$) ($i=2,\dots,7$). □

**Theorem 6.2.** *Every facet-defining inequality of* $C_7$ *collapses to some triangle inequality.*

**Proof.** By Theorem 6.1, we only need verify that every inequality in $L$ collapses to some triangle inequality. For this purpose, recall that for every partition $\pi$ of $[1,7]$ and for every vector $v$ in $\mathbb{R}^{\binom{7}{2}}$, the vector $v^\pi$ is the $\pi$-collapsing of $v$.

First, consider an arbitrary hypermetric inequality Hyp$(b_1,b_2,\dots,b_n)$, and observe that, for every nonnegative integer $n$ greater than or equal to three, Hyp$(b_1,b_2,\dots,b_n)$ collapses to a triangle inequality if and only if the set $[1,n]$ can be partitioned into three subsets, say $V_1, V_2$, and $V_3$, in such a way that

$$\sum_{i\in V_1} b_i=\sum_{i\in V_2} b_i=-\sum_{i\in V_3} b_i=1.$$

Now it is easy to verify that all hypermetric inequalities in $L$ collapse to some triangle inequality.

To show that every cycle inequality in $L$ and all its switchings collapse to some triangle inequality, set

$$u_1=u^{\{1\}},\quad u_2=u^{\{1,2\}},\quad u_3=u^{\{1,2,6\}},$$

$$v_1=v^{\{1\}},\quad v_2=v^{\{3\}},\quad v_3=v^{\{1,5\}},\quad v_4=v^{\{3,4\}},\quad v_5=v^{\{1,4,5\}},\quad v_6=v^{\{3,4,5\}},$$

$$w_1=w^{\{2\}},\quad w_2=w^{\{2,4\}},\quad w_3=w^{\{1,4\}},\quad w_4=w^{\{1,4,5\}}.$$

Now it is easy to verify that the inequalities

$$(u^\pi)^{\mathrm{T}}x\leqslant 0,\quad \text{with } \pi=\{\{1\},\{3\},\{2,4,5,6,7\}\},$$

$$(v^\pi)^{\mathrm{T}}x\leqslant 0,\quad \text{with } \pi=\{\{1\},\{4\},\{2,3,5,6,7\}\},$$

$$(w^\pi)^{\mathrm{T}}x\leqslant 0,\quad \text{with } \pi=\{\{2\},\{3,4,5\},\{1,6,7\}\},$$

and the inequalities

$$(u_1^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{1\},\{3\},\{2,4,5,6,7\}\},$$

$$(u_2^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{6\},\{7\},\{1,2,3,4,5\}\},$$

$$(u_3^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{6\},\{7\},\{1,2,3,4,5\}\},$$

$$(v_1^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(v_2^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(v_3^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(v_4^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(v_5^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(v_6^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(w_1^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(w_2^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(w_3^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(w_4^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\}$$

are triangle inequalities.

To show that the inequalities (P1), (P2), and (P3) collapse to some triangle inequality, set

$$p_1 = p^{\{3,7\}}, \qquad p_2 = p^{\{1,3,6\}}.$$

Now it is easy to verify that the inequality

$$(p^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

and the inequalities

$$(p_1^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\},$$

$$(p_2^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{4\},\{5\},\{1,2,3,6,7\}\}$$

are triangle inequalities.

Finally, consider the inequalities (G$i$) ($i = 1, \ldots, 7$). Set

$$g_1 = g^{\{1\}}, \qquad g_2 = g^{\{1,6\}}, \qquad g_3 = g^{\{1,6,7\}}, \qquad g_4 = g^{\{1,3,5\}}, \qquad g_5 = g^{\{1,3,6\}},$$
$$g_6 = g^{\{1,2,5\}}.$$

Now it is easy to verify that the inequality

$$(g^\pi)^{\mathrm{T}}x \leqslant 0, \quad \text{with } \pi = \{\{1\},\{3\},\{2,4,5,6,7\}\}$$

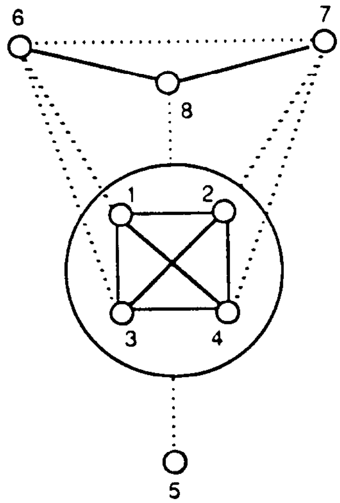


Fig.3

and the inequalities

$$(g_1^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{1\}, \{6\}, \{2, 3, 4, 5, 7\}\},$$

$$(g_2^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{6\}, \{7\}, \{1, 2, 3, 4, 5\}\},$$

$$(g_3^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{2\}, \{3\}, \{1, 4, 5, 6, 7\}\},$$

$$(g_4^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{2\}, \{4\}, \{1, 3, 5, 6, 7\}\},$$

$$(g_5^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{5\}, \{7\}, \{1, 2, 3, 4, 6\}\},$$

$$(g_6^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{3\}, \{4\}, \{1, 2, 5, 6, 7\}\}$$

are triangle inequalities. □

We do not know any facet-defining inequality of $C_n$ which does not collapse to some triangle inequality. Moreover, we do not know any facet-defining inequality of $C_n$ which does not admit a purification; in other words, every facet-defining inequality that we know has an expansion that is pure and facet-defining. In particular, the facet-defining inequality (G1) admits a purification; the graph in Fig. 3 is the supporting graph corresponding to this pure inequality: a plain line $ij$ corresponds to an edge $ij$ with weight equal to 1, a dashed line $ij$ corresponds to an edge $ij$ with weight equal to $-1$.

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