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Hilbert bases of cuts

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Abstract

Let X be a set of vectors in $\mathbb{R}^m$. X is said to be a Hilbert base if every vector in $\mathbb{R}^m$ which can be written both as a linear combination of members of X with nonnegative coefficients and as a linear combination with integer coefficients can also be written as a linear combination with nonnegative integer coefficients. Denote by $\mathscr{H}$ the collection of the graphs whose family of cuts is a Hilbert base. It is known that K_5 and graphs with no K_5-minor belong to $\mathscr{H}$ and that K_6 does not belong to $\mathscr{H}$. We show that every proper subgraph of K_6 belongs to $\mathscr{H}$ and that every graph from $\mathscr{H}$ does not have K_6 as a minor. We also study how the class $\mathscr{H}$ behaves under several operations.

1. Introduction

Let X be a set of vectors in $\mathbb{R}^m$. Set

$$\mathbb{R}_+(X) := \left\{ \sum_{x \in X} \lambda_x x \mid \lambda_x \geqslant 0 \ (x \in X) \right\},$$

$$\mathbb{Z}(X) := \left\{ \sum_{x \in X} \lambda_x x \mid \lambda_x \in \mathbb{Z} \ (x \in X) \right\},$$

$$\mathbb{Z}_+(X) = \left\{ \sum_{x \in X} \lambda_x x \mid \lambda_x \in \mathbb{Z}_+ \ (x \in X) \right\}.$$

So, $\mathbb{R}_+(X)$ is the *cone* generated by X and $\mathbb{Z}(X)$ is the *lattice* generated by X. Clearly, $\mathbb{Z}_+(X) \subseteq \mathbb{R}_+(X) \cap \mathbb{Z}(X)$. The set $\mathbb{Z}_+(X)$ is sometimes called the *integer cone* generated by X. The set X is said to be a *Hilbert base* if equality holds in the above inclusion, i.e., if

$$\mathbb{Z}_+(X) = \mathbb{R}_+(X) \cap \mathbb{Z}(X).$$

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Hilbert bases were introduced in [11] to study total dual integrality. Several examples of Hilbert bases arising from combinatorial objects are described in [15].

Let $G = (V,E)$ be a graph. For each subset $S \subseteq V$, the *cut* $\delta(S)$ consists of the edges $ij \in E$ with $|S \cap \{i,j\}| = 1$. For simplicity, we also denote by $\delta(S)$ the incidence vector of the cut determined by S; so $\delta(S)_{ij} = 1$ if $|S \cap \{i,j\}| = 1$ and $\delta(S)_{ij} = 0$ otherwise, for $ij \in E$. Let $\mathscr{K}_G$ denote the set of all cuts of G. For simplicity, we let $\mathbb{R}_+(G) := \mathbb{R}_+(\mathscr{K}_G)$ denote the cone generated by the cuts of G, and $\mathbb{Z}(G) := \mathbb{Z}(\mathscr{K}_G)$ denote the lattice generated by the cuts of G. We also set $\mathbb{Z}_+(G) := \mathbb{Z}_+(\mathscr{K}_G)$.

Let $\mathscr{H}$ denote the set of graphs G whose family of cuts $\mathscr{K}_G$ is a Hilbert base, i.e., such that $\mathbb{Z}_+(G) = \mathbb{R}_+(G) \cap \mathbb{Z}(G)$.

We suppose here that the graphs are without loops and without multiple edges. This is no loss of generality since, if a graph G has multiple edges and loops, then $G \in \mathscr{H}$ if and only if the graph obtained from G by deleting the loops and replacing the multiple edges by single edges belongs to $\mathscr{H}$. We recall that a graph H is said to be a *minor* of a graph G if H can be obtained from G by deleting and/or contracting some edges.

In this paper, we show the following results.

Theorem 1.1. *Let G be a subgraph of K_6. Then, $G \in \mathscr{H}$ if and only if G is distinct from K_6.*

Proposition 1.2. *If G belongs to $\mathscr{H}$, then G does not have K_6 as a minor.*

Hence, K_6 is the smallest example of a graph which does not belong to $\mathscr{H}$. To see that $K_6 \notin \mathscr{H}$, consider the vector x defined by $x_e = 2$ for all edges of K_6 except $x_e = 4$ for one edge of K_6. Then, $x \in \mathbb{R}_+(K_6) \cap \mathbb{Z}(K_6)$ but $x \notin \mathbb{Z}_+(K_6)$ ([4]; see also Example 4.2). In fact, the proof of Proposition 1.2 is based on the fact that this counterexample for K_6 can be extended to a counterexample for any graph containing K_6. The graph K_6 provides, therefore, a counterexample to a conjecture of [12], stating that the cuts of any graph form a Hilbert basis. Actually, a major open question is to decide whether K_6 is the *only* minimal (with respect to taking minors) graph that does not belong to $\mathscr{H}$.

Let us now recall several results that we need for the paper. The lattice $\mathbb{Z}(G)$ can be easily characterized. Namely, given $x \in \mathbb{Z}^E$,

$$x \in \mathbb{Z}(G) \quad \text{if and only if } x(C) \equiv 0 \pmod 2 \tag{1}$$

for each circuit C of G. (We set $x(C) := \sum_{e \in C} x_e$ for each subset $C \subseteq E$.) On the other hand, characterizing the cone $\mathbb{R}_+(G)$ or the integer cone $\mathbb{Z}_+(G)$ are hard problems, in general. The following Theorems 1.3 and 1.4 give the characterization of $\mathbb{R}_+(G)$ and $\mathbb{Z}_+(G)$ for the class of graphs with no K_5-minor. Let $x \in \mathbb{R}_+(G)$. Then, x satisfies the following inequality:

$$x_e - x(C \setminus \{e\}) \leqslant 0 \tag{2}$$

for each circuit C of G and each $e \in C$. Inequality (2) is called a *cycle inequality*.

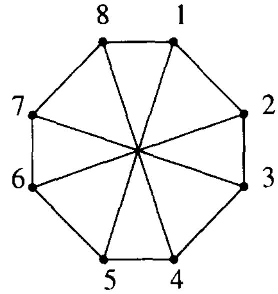

Fig. 1. V_8.

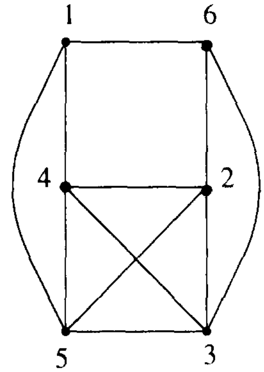

Fig. 2. H_6.

Theorem 1.3 (Seymour [18]). *Let G be a graph. Then, $\mathbb{R}_+(G)$ consists of the vectors $x \in \mathbb{R}_+^E$ satisfying the inequalities (2) for all $e \in C$, C circuit of G, if and only if G has no K_5-minor.*

Theorem 1.4 (Fu and Goddyn [10]). *Let G be a graph. Then, $\mathbb{Z}_+(G)$ consists of the vectors $x \in \mathbb{Z}_+^E$ satisfying the inequalities (2) and the condition (1) for all $e \in C$, C circuit of G, if and only if G has no K_5-minor.*

In other words, Fu and Goddyn showed that every graph with no K_5-minor belongs to $\mathscr{H}$. The proof of this result is based on the following facts:

— graphs with no K_5-minor can be obtained by means of k-sums ($k = 1,2,3$) of planar graphs and copies of the graph V_8 (shown in Fig. 1) [19],

— planar graphs belong to $\mathscr{H}$ [16],

— V_8 belongs to $\mathscr{H}$,

— $\mathscr{H}$ is closed under the k-sum operation (see Proposition 2.7).

In fact, the graph K_5, which is excluded in Theorem 1.4, also belongs to $\mathscr{H}$ [5,7]. Let H_6 denote the graph obtained by splitting evenly a node in K_5; H_6 is shown in Fig. 2. From Seymour's splitter theorem [17], every graph with no H_6-minor can be obtained by means of k-sums ($k = 1, 2$) of graphs with no K_5-minor and copies of K_5. Hence, from Theorem 1.4 and Proposition 2.7, we deduce that every graph with no H_6-minor belongs to $\mathscr{H}$. Note, however, that the graph H_6 also belongs to $\mathscr{H}$ (by Theorem 1.1; see also Example 2.8).

The proof of Theorem 1.1 relies mainly on the following Theorem 1.5. However, this result does not imply immediately that every subgraph of K_6 belongs to $\mathscr{H}$, since we do not know whether $\mathscr{H}$ is closed under deletion of edges (we have only a partial result; see Proposition 2.3).

Theorem 1.5. *The graph $K_6 \backslash e$ (obtained by deleting an edge from K_6) belongs to $\mathscr{H}$.*

The full characterization of the class $\mathscr{H}$ seems to be a hard problem. One reason for that may be that we could not prove that $\mathscr{H}$ is closed under deletion of edges. Another major difficulty for showing that a given graph G belongs to $\mathscr{H}$ is that the cone $\mathbb{R}_+(G)$ is not known in general (i.e., if G *has* a K_5-minor). For instance, for showing that $K_6 \backslash e$ belongs to $\mathscr{H}$, we need first to find the linear description of the cone $\mathbb{R}_+(K_6 \backslash e)$ (which we did using computer).

On the other hand, the dual problem, i.e., the characterization of graphs whose family of cycles is a Hilbert base, is completely solved. Namely, the family $\mathscr{C}_G$ of cycles of a graph G is a Hilbert base if and only if G does not have the Petersen graph P_{10} as a minor [1]. Note that describing the cone $\mathbb{R}_+(\mathscr{C}_G)$ is 'easy'; indeed, for any graph G, the cone $\mathbb{R}_+(\mathscr{C}_G)$ consists of the vectors $x \in \mathbb{R}_+^E$ satisfying the inequalities (2) for all $e \in C$ and all cuts C of G [16]. Hence, for a graph G with no P_{10}-minor, the integer cone $\mathbb{Z}_+(\mathscr{C}_G)$ is characterized by the inequalities (2) and the parity condition (1), for each $e \in C$ and each cut C of G.

One may ask the same questions at the more general level of binary matroids. Let $\mathscr{M}$ be a binary matroid on a set E with family of cycles $\mathscr{C}_{\mathscr{M}}$. The question of characterizing the matroids whose family of cycles forms a Hilbert basis is raised in [12].

The following result is shown in [10]: The integer cone $\mathbb{Z}_+(\mathscr{C}_{\mathscr{M}})$ consists of the vectors $x \in \mathbb{R}_+^E$ satisfying the inequalities (2) and the parity condition (1), for each $e \in C$ and each cocircuit C of $\mathscr{M}$, if and only if $\mathscr{M}$ does not have F_7^* (the dual Fano matroid), R_{10}, $\mathscr{M}^*(K_5)$ (the cographic matroid of K_5), or $\mathscr{M}(P_{10})$ (the graphic matroid of P_{10}), as a minor. The proof of this result is based on Seymour's decomposition for matroids with no F_7^*, R_{10} minor, and on the fact that the result holds for graphic matroids (the above mentioned result of [1]), for cographic matroids (Theorem 1.4) and for the Fano matroid F_7. Note that the exclusion of the minors F_7^*, R_{10} and $\mathscr{M}^*(K_5)$ ensures that the cone $\mathbb{R}_+(\mathscr{C}_{\mathscr{M}})$ is 'easy', i.e., is completely determined by the inequalities (2), for C cocircuit of $\mathscr{M}$ ([18]).

On the other hand, the binary matroids $\mathscr{M}$ for which the lattice $\mathbb{Z}(\mathscr{C}_{\mathscr{M}})$ is completely determined by the parity condition (1) are characterized in [14].

The paper is organized as follows. In Section 2, we study how the class $\mathscr{H}$ behaves under several operations, namely, under contraction and deletion of edges, under the

k-sum operation, and with respect to switching. In Section 3, we give the proof of Theorem 1.5, i.e., we show that the cuts of $K_6\backslash e$ form a Hilbert base; Section 3.1 contains the description of the cone $\mathbb{R}_+(K_6\backslash e)$. In Section 4.1, we present the description of the cones $\mathbb{R}_+(H_6)$ and $\mathbb{R}_+(H_6+e)$; in Section 4.2, we give the proof of Theorem 1.1 and, in Section 4.3, we prove Proposition 1.2.

2. Operations

In this section, we group several results showing that the class $\mathscr{H}$ is closed under some operations, namely, under contraction of an edge, under deletion of an edge *with some additional conditions*, and under the 1-, 2-, 3-sum operations. We also give a result on $\mathscr{H}$ related to the switching operation; see Proposition 2.9.

Let G/e (resp. $G\backslash e$) denote the graph obtained from G by contracting (resp. deleting) the edge e.

Proposition 2.1. *If $G \in \mathscr{H}$, then $G/e \in \mathscr{H}$ for each edge e of G.*

Proposition 2.2. *Assume that $G/e \in \mathscr{H}$ for some edge e of G. If $x \in \mathbb{R}_+(G) \cap \mathbb{Z}(G)$ and $x_e = 0$, then $x \in \mathbb{Z}_+(G)$.*

Propositions 2.1 and 2.2 can be easily checked directly. In fact, as was pointed to us by Grishukhin, the proof relies essentially on the fact that the cone $\mathbb{R}_+(G/e)$ can be seen as a face of the cone $\mathbb{R}_+(G)$. Namely, let e be the edge uv, where $u, v \in V$, and let $N_{u,v}$ denote the set of nodes of G that are adjacent to both u and v. Then, the cone $\mathbb{R}_+(G/e)$ is obviously in one-to-one correspondence with the cone $\mathbb{R}_+(G) \cap \{x \in \mathbb{R}^E : x_{ui} - x_{vi} - x_{uv} = 0$ and $x_{vi} - x_{ui} - x_{uv} = 0$ for all $i \in N_{u,v}\}$, which is a face of the cone $\mathbb{R}_+(G)$. Proposition 2.2 follows immediately, as well as Proposition 2.1 (indeed, if the generators of a given cone form a Hilbert basis, then the same property holds for any face of this cone).

We now turn to the case of deletion minors. We can prove an analogue of Proposition 2.1 only if we make some additional assumptions on the graph G. Consider the following properties:

$$v \in \{0, 1, -1\}^E, \tag{3}$$

$$v^{\mathrm{T}}\delta(S) \in 2\mathbb{Z} \quad \text{for all cuts } \delta(S) \text{ of } G \tag{4}$$

for each inequality $v^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G)$. Each cycle inequality (2) clearly satisfies the properties (3) and (4).

Proposition 2.3. *Let G be a graph satisfying* (3) *and* (4) *for each inequality $v^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G)$. If $G \in \mathscr{H}$, then $G\backslash e \in \mathscr{H}$ for each edge e of G.*

Proof. Let $y \in \mathbb{R}_+(G\backslash e) \cap \mathbb{Z}(G\backslash e)$. We show that $y \in \mathbb{Z}_+(G\backslash e)$. Let $x \in \mathbb{R}^E$, where $x_f = y_f$ for each edge $f \neq e$ of G and x_e remains to be determined. Clearly, $x \in \mathbb{R}_+(G)$ if and only if

$$x_{\max} \leqslant x_e \leqslant x_{\min}, \tag{5}$$

where $x_{\max} = \max(-v^{\mathrm{T}}y/v_e | v_e < 0,\ v^{\mathrm{T}}z \leqslant 0$ defining a facet of $\mathbb{R}_+(G))$ and $x_{\min} = \min(-v^{\mathrm{T}}y/v_e | v_e > 0,\ v^{\mathrm{T}}z \leqslant 0$ defining a facet of $\mathbb{R}_+(G))$. Moreover, $x \in \mathbb{Z}(G)$ if and only if

$$x_e \text{ has the same parity as } y(C \setminus \{e\}), \tag{6}$$

where C is an arbitrary circuit of G containing e. By (3), $x_{\min}, x_{\max} \in \mathbb{Z}$. Hence, if $x_{\max} < x_{\min}$, then $x_{\max} + 1 \leqslant x_{\min}$ and we can choose x_e satisfying the above conditions (5) and (6). If $x_{\min} = x_{\max}$, then we set $x_e = x_{\max} = x_{\min}$. We verify that x_e has indeed the correct parity. For instance, $x_e = v^{\mathrm{T}}y$, where $v^{\mathrm{T}}z \leqslant 0$ defines a facet of $\mathbb{R}_+(G)$ and $v_e = -1$. Define $x' \in \mathbb{R}^E$ by setting $x'_f = y_f$ if f is an edge of G distinct from e, and $x'_e = 0$ (resp. $x'_e = 1$) if $y(C \setminus \{e\})$ is even (resp. odd). Clearly, $x' \in \mathbb{Z}(G)$. Therefore, using (4), we deduce that $v^{\mathrm{T}}x'$ is an even integer, implying that x_e has the same parity as x'_e, i.e., as $y(C \setminus \{e\})$. Therefore, we can choose x_e in such a way that $x \in \mathbb{R}_+(G) \cap \mathbb{Z}(G)$. Since $G \in \mathscr{H}$, we have that $x \in \mathbb{Z}_+(G)$, implying that $y \in \mathbb{Z}_+(G\backslash e)$. $\square$

Note that the above proof shows, in fact, that the following weaker form of Proposition 2.3 holds.

Proposition 2.4. *Let e be an edge of a graph G. Suppose that, for each inequality $v^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G)$, $v_e \in \{0, 1, -1\}$, and $v^{\mathrm{T}}\delta(S) \in 2\mathbb{Z}$ for all cuts $\delta(S)$ of G. Then, $G\backslash e \in \mathscr{H}$ whenever $G \in \mathscr{H}$.*

Example 2.5. Every graph on at most 5 nodes belongs to $\mathscr{H}$. Indeed, $K_5 \in \mathscr{H}$ [5,7] and every proper subgraph of K_5 belongs to $\mathscr{H}$ (by Theorem 1.4).

Let us point out that K_5 satisfies the properties (3) and (4); indeed, its facets are defined by the triangle inequalities: $x_{ij} - x_{ik} - x_{jk} \leqslant 0$, for $i, j, k \in V(K_5)$, and the pentagonal inequality: $x_{12} + x_{23} + x_{13} + x_{45} - \sum_{\substack{i=1,2,3\\ j=4,5}} x_{ij} \leqslant 0$ for any labeling of the nodes of K_5 as $1, 2, 3, 4, 5$ [5,7].

Let $G_t = (V_t, E_t)$ be a graph, for $t = 1, 2$. When the subgraph induced by $V_1 \cap V_2$ is a complete graph on $k := |V_1 \cap V_2|$ nodes, the *k-sum* of G_1 and G_2 is defined as the graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

Proposition 2.6 (Barahona [2]). *Let G be the k-sum ($k = 1, 2, 3$) of two graphs G_1 and G_2. Then, a system of linear inequalities sufficient to describe the cone $\mathbb{R}_+(G)$ is obtained by juxtaposing the inequalities that define the cones $\mathbb{R}_+(G_1)$ and $\mathbb{R}_+(G_2)$ and identifying the variables associated with the common edges of G_1 and G_2.*

In particular, G satisfies property (3) (resp. (4)) if and only if G_1 and G_2 satisfy property (3) (resp. (4)).

Proposition 2.7. *Let G be the k-sum ($k = 1,2,3$) of two graphs G_1 and G_2. Then, $G \in \mathscr{H}$ if and only if $G_1 \in \mathscr{H}$ and $G_2 \in \mathscr{H}$.*

Proof. We give the proof in the case $k = 3$; the cases $k = 1,2$ are similar but easier. Set $V_1 \cap V_2 := \{u, v, w\}$. We first suppose that $G_1, G_2 \in \mathscr{H}$ and we show that $G \in \mathscr{H}$. Let $x \in \mathbb{R}_+(G) \cap \mathbb{Z}(G)$. The projection x_t of x on $\mathbb{R}^{E_t}$ belongs to $\mathbb{R}_+(G_t) \cap \mathbb{Z}(G_t)$, for $t = 1,2$. Since $G_t \in \mathscr{H}$, then $x_t \in \mathbb{Z}_+(G_t)$, for $t = 1,2$. Say, $x_1 = \sum_{A \in \mathscr{A}} \delta(A)$, $x_2 = \sum_{B \in \mathscr{B}} \delta(B)$, where $\mathscr{A}$ is a multiset of cuts of G_1, i.e., repetition is allowed in $\mathscr{A}$, and $\mathscr{B}$ is a multiset of cuts of G_2. We can suppose, without loss of generality, that $w \notin A, B$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$. Let $\mathscr{A}_0$ (resp. $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$) denote the multiset consisting of all members $\delta(A)$ of $\mathscr{A}$ such that $u, v \notin A$ (resp. ($u \in A$, $v \notin A$), ($u \notin A$, $v \in A$), ($u, v \in A$)). Define similarly $\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$. Hence,

$$x(uv) = x_1(uv) = x_2(uv) = |\mathscr{A}_1| + |\mathscr{A}_2| = |\mathscr{B}_1| + |\mathscr{B}_2|,$$

$$x(uw) = x_1(uw) = x_2(uw) = |\mathscr{A}_1| + |\mathscr{A}_3| = |\mathscr{B}_1| + |\mathscr{B}_3|,$$

$$x(vw) = x_1(vw) = x_2(vw) = |\mathscr{A}_2| + |\mathscr{A}_3| = |\mathscr{B}_2| + |\mathscr{B}_3|,$$

yielding that $|\mathscr{A}_1| = |\mathscr{B}_1| = (x(uv) + x(uw) - x(vw))/2$, $|\mathscr{A}_2| = |\mathscr{B}_2| = (x(uv) + x(vw) - x(uw))/2$ and $|\mathscr{A}_3| = |\mathscr{B}_3| = (x(uw) + x(vw) - x(uv))/2$. Since $|\mathscr{A}_k| = |\mathscr{B}_k|$, we can order the members of $\mathscr{A}_k$ as $A_1, \ldots, A_{|\mathscr{A}_k|}$, and those of $\mathscr{B}_k$ as $B_1, \ldots, B_{|\mathscr{A}_k|}$, for each $k = 1,2,3$. Then, $x = \sum_{A \in \mathscr{A}_0} \delta(A) + \sum_{B \in \mathscr{B}_0} \delta(B) + \sum_{k=1,2,3} \left(\sum_{1 \leqslant i \leqslant |\mathscr{A}_k|} \delta(A_i \cup B_i) \right)$. This shows that $x \in \mathbb{Z}_+(G)$. Hence, $G \in \mathscr{H}$.

Conversely, let us assume that $G \in \mathscr{H}$. We show that $G_1 \in \mathscr{H}$. Let $y \in \mathbb{R}_+(G_1) \cap \mathbb{Z}(G_1)$. So, $y = \sum_S \lambda_S \delta(S)$ for some scalars $\lambda_S \geqslant 0$, where the cuts $\delta(S)$ are taken in G_1 with $w \notin S$. Set $x = \sum_S \lambda_S \delta(S)$, where the cuts $\delta(S)$ are now taken in the graph G. Hence, $x_{iw} = 0$, $x_{iv} = y_{vw}$, $x_{iu} = y_{uw}$ for each node $i \in V_2 - V_1$, and $x_{ij} = 0$ for all nodes $i, j \in V_2 - V_1$. This observation permits to check that $x(C) \in 2\mathbb{Z}$ for each circuit of G, i.e., $x \in \mathbb{Z}(G)$. Therefore, $x \in \mathbb{Z}_+(G)$ since $G \in \mathscr{H}$. This implies that $y \in \mathbb{Z}_+(G_1)$. Hence, $G_1 \in \mathscr{H}$. □

Example 2.8. .As an application of Proposition 2.7, we obtain that the graph $K_6 - P_3$ (i.e., K_6 with a path on three nodes deleted) belongs to $\mathscr{H}$ (since it is the 3-sum of K_4 and K_5 and $K_4, K_5 \in \mathscr{H}$, as mentioned in Example 2.5). By Propositions 2.3 and 2.6, we deduce that the graph obtained by deleting an edge from $K_6 - P_3$ still belongs to $\mathscr{H}$. In particular, the graph $H_6 + e$ (i.e., H_6 with one more edge among its nodes) belongs to $\mathscr{H}$. (H_6 is shown in Fig. 2 and $H_6 + e$ in Fig. 7.) Then, H_6 too belongs to $\mathscr{H}$ since all the inequalities defining facets of $H_6 + e$ satisfy (3) and (4) (see Section 4.1).

We conclude this section with a result related to the switching operation. Given a cut $\delta(A)$ in G and $v \in \mathbb{R}^E$, define $v^{\delta(A)} \in \mathbb{R}^E$ by $(v^{\delta(A)})_e = -v_e$ if $\delta(A)_e = 1$ and $(v^{\delta(A)})_e = v_e$ if $\delta(A)_e = 0$, for all edges $e \in E$. Then, the mapping $r_{\delta(A)} : \mathbb{R}^E \longrightarrow \mathbb{R}^E$ defined by $r_{\delta(A)}(v) = v^{\delta(A)} + \delta(A)$, for all $v \in \mathbb{R}^E$, is called a *switching mapping*. It is well known that any switching mapping $r_{\delta(A)}$ preserves the cut polytope [3].

Switching also preserves the cone $\mathbb{R}_+(G)$ in the following sense [5]. Suppose that the inequality $v^{\mathrm{T}}x \leqslant 0$ is valid for $\mathbb{R}_+(G)$ and that $v^{\mathrm{T}}\delta(A) = 0$; then, the inequality $(v^{\delta(A)})^{\mathrm{T}}x \leqslant 0$, obtained by *switching* $w^{\mathrm{T}}x \leqslant 0$ *by the cut* $\delta(A)$, is valid for $\mathbb{R}_+(G)$. Moreover, $(v^{\delta(A)})^{\mathrm{T}}x \leqslant 0$ defines a facet of $\mathbb{R}_+(G)$ if and only if $v^{\mathrm{T}}x \leqslant 0$ defines a facet of $\mathbb{R}_+(G)$.

In other words, if $\mathscr{F}$ is a face of $\mathbb{R}_+(G)$ with $\mathscr{R} := \{\delta(A_1),\ldots,\delta(A_t)\}$ denoting the set of nonzero cuts lying on $\mathscr{F}$, then the set $\mathscr{F}^{\delta(A_1)} := \{\lambda_1\delta(A_1)+\sum_{2\leqslant i\leqslant t}\lambda_i\delta(A_i\triangle A_1) \mid \lambda_1,\lambda_2,\ldots,\lambda_t \geqslant 0\}$ is also a face of $\mathbb{R}_+(G)$, obtained by switching the face $\mathscr{F}$ by the cut $\delta(A_1)$.

We now give a result which will be very useful for showing that some given graph G belongs to $\mathscr{H}$. For this, we need two more definitions. Given $x \in \mathbb{R}_+(G)$, we define its *minimum* $\mathbb{R}_+$*-size* $s(x)$ by

$$s(x) := \min\left(\sum_{S\subseteq V}\alpha_S \mid x = \sum_{S\subseteq V}\alpha_S\delta(S) \text{ with all } \alpha_S \geqslant 0\right)$$

and, given $x \in \mathbb{Z}_+(G)$, we define its *minimum* $\mathbb{Z}_+$*-size* $h(x)$ by

$$h(x) := \min\left(\sum_{S\subseteq V}\alpha_S \mid x = \sum_{S\subseteq V}\alpha_S\delta(S) \text{ with all } \alpha_S \in \mathbb{Z}_+\right).$$

As above, let $\mathscr{F}$ be a face of $\mathbb{R}_+(G)$ and let $\mathscr{R} = \{\delta(A_1),\ldots,\delta(A_t)\}$ denote the set of nonzero cuts lying on $\mathscr{F}$. We consider the following two properties:

$$\text{If } x \in \mathbb{R}_+(G)\cap\mathbb{Z}(G) \text{ and } x \in \mathscr{F}, \text{ then } x \in \mathbb{Z}_+(G) \tag{7}$$

$$\begin{aligned}&\text{If } x \in \mathbb{R}_+(G)\cap\mathbb{Z}(G) \text{ and } x \in \mathscr{F}, \text{ then } s(x) \in \mathbb{Z}; \text{ moreover,}\\ &\Sigma_{1\leqslant i\leqslant t}\lambda_i = s(x) \text{ for each decomposition } x = \Sigma_{1\leqslant i\leqslant t}\lambda_i\delta(A_i) \text{ with}\\ &\qquad\qquad \lambda_i \geqslant 0 \quad \text{for } 1\leqslant i\leqslant t.\end{aligned} \tag{8}$$

Proposition 2.9. *Assume that the face* $\mathscr{F}$ *has the property* (7) *and that both faces* $\mathscr{F}$ *and* $\mathscr{F}^{\delta(A_1)}$ *have the property* (8). *Then, the face* $\mathscr{F}^{\delta(A_1)}$ *has the property* (7).

Proof. Let $z \in \mathbb{R}_+(G)\cap\mathbb{Z}(G)$ such that $z \in \mathscr{F}^{\delta(A_1)}$. We show that $z \in \mathbb{Z}_+(G)$. By assumption, we have that $z = \lambda_1\delta(A_1)+\sum_{2\leqslant i\leqslant t}\lambda_i\delta(A_i\triangle A_1)$ for some scalars $\lambda_1,\ldots,\lambda_t \geqslant 0$. Since $\mathscr{F}^{\delta(A_1)}$ has the property (8), we have that $\sum_{1\leqslant i\leqslant t}\lambda_i = s(z) \in \mathbb{Z}$.

Set $y := \sum_{2\leqslant i\leqslant t}\lambda_i\delta(A_i)$. Hence, $y \in \mathscr{F}$. Since $\mathscr{F}$ has the property (8), we deduce that $\sum_{2\leqslant i\leqslant t}\lambda_i = s(y) \in \mathbb{Z}$. Note also that $y = r_{\delta(A_1)}(z) + \delta(A_1)(s(z)-1)$. Moreover, $y \in \mathbb{Z}(G)$; indeed, $z \in \mathbb{Z}(G)$ which implies obviously that $r_{\delta(A_1)}(z) \in \mathbb{Z}(G)$.

Therefore, from the property (7) applied to $\mathscr{F}$, we deduce that $y \in \mathbb{Z}_+(G)$, i.e., $y = \sum_{1 \leqslant i \leqslant t} \alpha_i \delta(A_i)$ for some nonnegative integers α_i. Moreover, $\sum_{1 \leqslant i \leqslant t} \alpha_i = s(y)$. Then, from $z = r_{\delta(A_1)}(y) + \delta(A_1)(s(z) - 1)$, we obtain that $z = \sum_{2 \leqslant i \leqslant t} \alpha_i \delta(A_i) + \delta(A_1)(s(z) - s(y))$. This shows that $z \in \mathbb{Z}_+(G)$, since $s(z) - s(y) = \lambda_1 \in \mathbb{Z}_+$. □

3. The cuts of $K_6 \backslash e$ form a Hilbert base

In this section, we show that the cuts of $K_6 \backslash e$ form a Hilbert base. Let G_6 denote the graph on the nodes $1,2,3,4,5,6$ whose edges are all pairs except the pair $(5,6)$, i.e., $G_6 = K_6 \backslash e$ for $e = 56$. We present the description of the facets of the cone $\mathbb{R}_+(G_6)$ in Section 3.1 and we show that $G_6 \in \mathscr{H}$ in Section 3.2. We use the following notation throughout: For $b \in \mathbb{R}^n$, the vector $Q(b) \in \mathbb{R}^{\binom{n}{2}}$ has ijth entry $b_i b_j$, for $1 \leqslant i < j \leqslant n$.

3.1. *Description of the cone* $\mathbb{R}_+(G_6)$

The facets of $\mathbb{R}_+(G_6)$ are grouped into three classes.

(a) The first class is composed of 48 *triangle facets*; they are induced by the cycle inequalities (2), where C is one of the 16 triangles of G_6, namely, $C = (i,j,k)$ for $1 \leqslant i < j < k \leqslant 4$, $C = (i,j,5)$ and $C = (i,j,6)$ for $1 \leqslant i < j \leqslant 4$. There are 23 nonzero cuts lying on each triangle facet.

(b) The second class consists of 20 *pentagonal facets*. They are induced by the inequalities

$$Q(b_1, b_2, b_3, b_4, b_5, b_6)^{\mathrm{T}} x := \sum_{1 \leqslant i < j \leqslant 6} b_i b_j x_{ij} \leqslant 0,$$

where $b = (b_1, \ldots, b_6)$ is one of the sequences ($b_i = b_j = -1$, $b_k = 1$ for $k \in \{1,2,3,4,5\} \setminus \{i,j\}$, $b_6 = 0$) for $1 \leqslant i < j \leqslant 5$, or ($b_i = b_j = -1$, $b_k = 1$ for $k \in \{1,2,3,4,6\} \setminus \{i,j\}$, $b_5 = 0$) for $i < j$, $i,j \in \{1,2,3,4,6\}$. There are 19 nonzero cuts lying on each pentagonal facet. For instance, the vector $Q(1,1,1,-1,-1,0)$ is shown in Fig. 3. We use the following notation: a plain edge ij represents a component $+1$ for the ij-coordinate and a dotted edge represents a component -1, while no edge means a component 0.

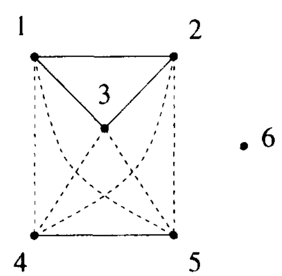

Fig. 3. $Q(1,1,1,-1,-1,0)$.

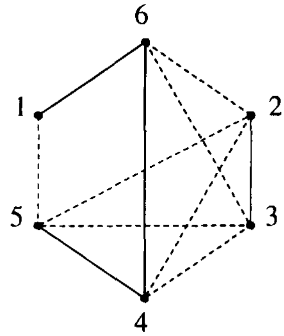

Fig. 4. w_1.

(c) The third class consists of 56 facets, which are grouped into 4 switching classes. Set

$$w_1^{\mathrm{T}}x := x_{16} + x_{46} + x_{45} - x_{15} + x_{23} - \sum_{\substack{i=2,3\\ j=4,5,6}} x_{ij}.$$

The vector w_1 is shown in Fig. 4. The inequality $w_1^{\mathrm{T}}x \leqslant 0$ is valid for the cone $\mathbb{R}_+(G_6)$. There are exactly 13 nonzero cuts satisfying the equality $w_1^{\mathrm{T}}x = 0$, namely, the cuts of the set

$$\mathscr{A}_1 := \{\delta(A) \mid A = 1,4,6,14,15,24,26,34,36,124,125,134,135\}.$$

(For simplicity, we use the following notation throughout this section: we denote the set $\{1,4\}$ by the string 14, similarly for the other sets.) The set $\mathscr{A}_1$ is linearly independent. Hence, the inequality $w_1^{\mathrm{T}}x \leqslant 0$ defines a simplex facet of $\mathbb{R}_+(G_6)$. Observe that the inequality $w_1^{\mathrm{T}}x \leqslant 0$ arises as the sum of the pentagonal inequality: $Q(0,-1,-1,1,1,1)^{\mathrm{T}}x \leqslant 0$ and of the triangle inequality: $x_{16} - x_{15} - x_{56} \leqslant 0$, which both define facets of the cone $\mathbb{R}_+(K_6)$.

For each $\delta(A) \in \mathscr{A}_1$, the inequality $(w_1^{\delta(A)})^{\mathrm{T}}x \leqslant 0$, obtained by switching the inequality $w_1^{\mathrm{T}}x \leqslant 0$ by the cut $\delta(A)$, defines another (simplex) facet of $\mathbb{R}_+(G_6)$. We show in Fig. 5 the vector $w_1^{\delta(4)}$. In fact, Figs. 4 and 5 show the two possible patterns for the coefficients of the switchings of w_1.

By permuting cyclically the nodes of $(1,2,3,4)$, we obtain three more inequalities $w_2^{\mathrm{T}}x \leqslant 0$, $w_3^{\mathrm{T}}x \leqslant 0$, $w_4^{\mathrm{T}}x \leqslant 0$, defined by

$$w_2^{\mathrm{T}}x := x_{26} + x_{16} + x_{15} - x_{25} + x_{34} - \sum_{\substack{i=3,4\\ j=1,5,6}} x_{ij},$$

$$w_3^{\mathrm{T}}x := x_{36} + x_{26} + x_{25} - x_{35} + x_{14} - \sum_{\substack{i=1,4\\ j=2,5,6}} x_{ij},$$

$$w_4^{\mathrm{T}}x := x_{46} + x_{36} + x_{35} - x_{45} + x_{12} - \sum_{\substack{i=1,2\\ j=3,5,6}} x_{ij}.$$

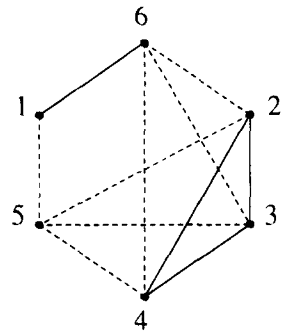

Fig. 5. $w_1^{\delta(4)}$.

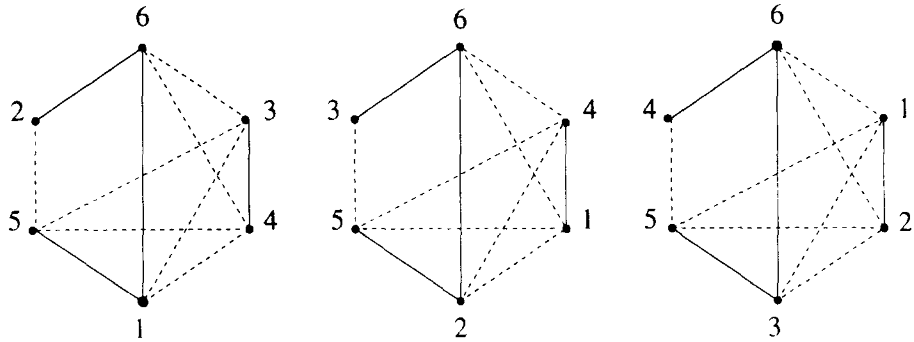

Fig. 6. w_2, w_3, w_4.

Each of them yields, via switching, 14 other facets of $\mathbb{R}_+(G_6)$. We show in Fig. 6 the vectors w_2, w_3 and w_4. Let $\mathscr{A}_i$ denote the set of nonzero cuts satisfying the equality $w_i^{\mathrm{T}}x = 0$, for $i = 2,3,4$; they are easily obtained from $\mathscr{A}_1$.

We refer to the facets of $\mathbb{R}_+(G_6)$ induced by the inequalities $w_i^{\mathrm{T}}x \leqslant 0$ and their switchings $(w_i^{\delta(A)})^{\mathrm{T}}x \leqslant 0$, for $A \in \mathscr{A}_i$, $i = 1,2,3,4$, as the *special facets* of $\mathbb{R}_+(G_6)$. We call the facet induced by $w_1^{\mathrm{T}}x \leqslant 0$ the *main special facet* of $\mathbb{R}_+(G_6)$.

We checked, using computer, that the above triangle facets, pentagonal facets and special facets constitute all the facets of $\mathbb{R}_+(G_6)$. Hence, $\mathbb{R}_+(G_6)$ has $48+20+56 = 124$ facets in total. We conclude with an observation.

Remark 3.1. (i) If $v^{\mathrm{T}}x \leqslant 0$ defines a triangle facet, then $v^{\mathrm{T}}\delta(A) \in \{0,-2\}$ for all cuts.

(ii) If $v^{\mathrm{T}}x \leqslant 0$ defines a pentagonal facet, then $v^{\mathrm{T}}\delta(S) \in \{0,-2\}$ for all cuts except two cuts for which $v^{\mathrm{T}}\delta(S) = -6$. Namely, $v^{\mathrm{T}}\delta(ij) = v^{\mathrm{T}}\delta(hkl) = -6$ for the pentagonal inequality $Q(b)^{\mathrm{T}}x \leqslant 0$ with $b_i = b_j = -1$ and $b_h = b_k = b_l = 1$.

(iii) If $v^{\mathrm{T}}x \leqslant 0$ defines a special facet, then $v^{\mathrm{T}}\delta(S) \in \{0,-2\}$ for all cuts except four cuts for which $v^{\mathrm{T}}\delta(S) = -4,-6$. Namely, for the main special facet, $w_1^{\mathrm{T}}\delta(45) = w_1^{\mathrm{T}}\delta(146) = -4$ and $w_1^{\mathrm{T}}\delta(23) = w_1^{\mathrm{T}}\delta(123) = -6$. (One deduces easily for which cuts every other special facet takes value -4 or -6 using permutation and switching; for instance, $w_2^{\mathrm{T}}\delta(15) = w_2^{\mathrm{T}}\delta(126) = -4$ and $w_2^{\mathrm{T}}\delta(34) = w_2^{\mathrm{T}}\delta(234) = -6$.)

3.2. The Proof of Theorem 1.5

In this section, we show that G_6 belongs to $\mathscr{H}$, i.e., that $\mathbb{Z}(G_6) \cap \mathbb{R}_+(G_6) \subseteq \mathbb{Z}_+(G_6)$. Our proof is by contradiction. We suppose that there exists $y \in \mathbb{Z}(G_6) \cap \mathbb{R}_+(G_6) \setminus \mathbb{Z}_+(G_6)$ and we take such y for which the sum $\sum_{e\in E(G_6)} y_e$ is minimum. Our goal is to show that no such y exists. This is done by a careful analysis of the possible locations of the point y within the cone $\mathbb{R}_+(G_6)$. Clearly, y satisfies:

$$y - \delta(A) \notin \mathbb{R}_+(G_6) \quad \text{for all cuts } \delta(A). \tag{9}$$

(For, if not, then $y - \delta(A) \in \mathbb{Z}_+(G_6)$ by the minimality of y, which would imply that $y \in \mathbb{Z}_+(G_6)$.) Let $\mathscr{F}$ denote the smallest face of $\mathbb{R}_+(G_6)$ that contains y, let $\mathscr{R}$ denote the set of nonzero cuts lying on $\mathscr{F}$, and let $\mathscr{V}$ denote the set of vectors v for which the inequality $v^{\mathrm{T}}x \leqslant 0$ defines a facet of $\mathbb{R}_+(G_6)$ such that $v^{\mathrm{T}}y = 0$. The proof can be sketched as follows: We show in Claim 3.5 and Corollary 3.7 that $\mathscr{F}$ is not contained in any pentagonal or special facet. Hence, y may lie only on some triangle facets. This fact, combined with the observation from Claim 3.3, permits to exclude many cuts from the set $\mathscr{R}$. We can show, in fact, that $\mathscr{R} \subseteq \{\delta(A) \mid A = 12, 13, 14, 23, 24, 34\}$. A direct argument permits then to conclude that $y \in \mathbb{Z}_+(G_6)$, in contradiction with our assumption.

From now on, y is a nonzero vector of $\mathbb{Z}(G_6) \cap \mathbb{R}_+(G_6) \setminus \mathbb{Z}_+(G_6)$, which satisfies (9). We shall use throughout the following notation: For distinct i, j, k, we set $y([ij]k) := y_{ij} - y_{ik} - y_{jk}$. We start with some easy observations.

Claim 3.2. $y_e \geqslant 1$ *for all* $e \in E(G_6)$.

Proof. This follows from Proposition 2.2 and the fact that every contraction minor of G_6 belongs to $\mathscr{H}$. $\square$

Claim 3.3. *For each cut* $\delta(A) \in \mathscr{R}$, *there exists an inequality* $v^{\mathrm{T}}x \leqslant 0$ *defining a facet of* $\mathbb{R}_+(G_6)$ *such that* $(v^{\mathrm{T}}y = -2, v^{\mathrm{T}}\delta(A) \in \{-4, -6\})$ *or* $(v^{\mathrm{T}}y = -4, v^{\mathrm{T}}\delta(A) = -6)$.

Proof. As $y - \delta(A) \notin \mathbb{R}_+(G_6)$, there exists an inequality $v^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G_6)$ such that $v^{\mathrm{T}}(y - \delta(A)) > 0$. Hence, $0 \geqslant v^{\mathrm{T}}y > v^{\mathrm{T}}\delta(A)$ with $v^{\mathrm{T}}y \leqslant -2$ (else, $v^{\mathrm{T}}y = v^{\mathrm{T}}\delta(A) = 0$ as $\delta(A) \in \mathscr{R}$). One can conclude using Remark 3.1. $\square$

Corollary 3.4. *Every cut of* $\mathscr{R}$ *is of the form* $\delta(A)$, *where* A *belongs to the set* $\{5, 6\} \cup \{12, 13, 14, 23, 24, 34\} \cup \{15, 16, 25, 26, 35, 36, 45, 46\} \cup \{56\} \cup \{123, 124, 134, 156\} \cup \{125, 126, 135, 136, 145, 146\}$. (*We have grouped together the sets according to the symmetries of* G_6.)

Proof. By Claim 3.3, $\delta(\mathrm{i}) \notin \mathscr{R}$ since no inequality $v^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G_6)$ satisfies: $v^{\mathrm{T}}\delta(i) = -4, -6$, for $i = 1,2,3,4$ (see Remark 3.1). $\square$

Claim 3.5. *The vector y does not lie on any of the special facets.*

Proof. Let $\mathscr{F}_0$ denote the main special facet of $\mathbb{R}_+(G_6)$, defined by the inequality: $w_1^{\mathrm{T}}x \leqslant 0$. We show that $\mathscr{F}_0$ has property (7). For this, let $z \in \mathbb{R}_+(G_6) \cap \mathbb{Z}(G_6) \cap \mathscr{F}_0$; so, $z = \sum_{\delta(A) \in \mathscr{A}_1} \alpha_A \delta(A)$, for some scalars $\alpha_A \geqslant 0$ and z satisfies (1). We show that all α_A's are integers:

- Since $z([16]2) = -2\alpha_{24} \in 2\mathbb{Z}$, we deduce that $\alpha_{24} \in \mathbb{Z}$. Similarly, $\alpha_{34}, \alpha_{124} \in \mathbb{Z}$, from $z([16]3), z([12]5) \in 2\mathbb{Z}$.
- From $z([16]4) \in 2\mathbb{Z}$, $\alpha_4 + \alpha_{24} + \alpha_{34} \in \mathbb{Z}$, implying that $\alpha_4 \in \mathbb{Z}$.
- From $z([12]3) - z([12]4) \in 2\mathbb{Z}$, $\alpha_{36} + \alpha_{124} - \alpha_4 \in \mathbb{Z}$ and, thus, $\alpha_{36} \in \mathbb{Z}$.
- From $z(2[36]) \in 2\mathbb{Z}$, $\alpha_{24} + \alpha_{36} + \alpha_{124} + \alpha_{125} \in \mathbb{Z}$, implying that $\alpha_{125} \in \mathbb{Z}$.
- From $z([12]6) \in 2\mathbb{Z}$, $\alpha_6 + \alpha_{36} + \alpha_{124} + \alpha_{125} \in \mathbb{Z}$, implying that $\alpha_6 \in \mathbb{Z}$.
- From $z(1[23]) - z(1[34]) \in 2\mathbb{Z}$, $\alpha_{14} - \alpha_{34} - \alpha_{125} \in \mathbb{Z}$, implying that $\alpha_{14} \in \mathbb{Z}$.
- From $z(1[35]) \in 2\mathbb{Z}$, $\alpha_1 + \alpha_{14} + \alpha_{124} \in \mathbb{Z}$, i.e., $\alpha_1 \in \mathbb{Z}$.
- From $z([14]2) - z(1[25]) \in 2\mathbb{Z}$, $\alpha_{26} - \alpha_1 \in \mathbb{Z}$, i.e., $\alpha_{26} \in \mathbb{Z}$.
- From $z(1[34]) \in 2\mathbb{Z}$, $\alpha_1 + \alpha_{15} + \alpha_{34} + \alpha_{125} \in \mathbb{Z}$, i.e., $\alpha_{15} \in \mathbb{Z}$.
- From $z(2[14]) \in 2\mathbb{Z}$, $\alpha_{14} + \alpha_{26} + \alpha_{134} \in \mathbb{Z}$, i.e., $\alpha_{134} \in \mathbb{Z}$.
- Finally, $z(1[24]) \in 2\mathbb{Z}$, i.e., $\alpha_1 + \alpha_{15} + \alpha_{24} + \alpha_{135} \in \mathbb{Z}$, i.e., $\alpha_{135} \in \mathbb{Z}$.

Hence, the main special facet $\mathscr{F}_0$ has property (7). Moreover, $\mathscr{F}_0$ has property (8); indeed, $s(x) = (x_{14}+x_{16}+x_{46})/2$ for any $x \in \mathscr{F}_0$ (since the triangle $(1,4,6)$ cuts all the cuts of $\mathscr{A}_1$). It is easy to see that every switching $\mathscr{F}_0^{\delta(B)}$ of $\mathscr{F}_0$ by a cut $\delta(B) \in \mathscr{A}_1$ also has property (8). Therefore, by Proposition 2.9, the face $\mathscr{F}_0^{\delta(B)}$ has property (7). Hence, by symmetry, every special facet has property (7). This shows that the vector y cannot lie on any special facet. $\square$

Let $\mathscr{G}$ denote the face of $\mathbb{R}_+(G_6)$ which consists of the vectors $x \in \mathbb{R}_+(G_6)$ satisfying the pentagonal equality: $Q(1,1,1,-1,-1,0)^{\mathrm{T}}x = 0$ and the three triangle equalities: $x(1[45]) = 0$, $x(2[45]) = 0$ and $x(3[45]) = 0$. The set of nonzero cuts lying on $\mathscr{G}$ is

$$\mathscr{R}_{\mathscr{G}} := \{\delta(A) \mid A = 6, 14, 146, 15, 156, 24, 246, 25, 256, 34, 346, 35, 356\}.$$

Note that the only cuts lying on the pentagonal facet defined by $Q(1,1,1,-1,-1,0)^{\mathrm{T}}x \leqslant 0$ but not on $\mathscr{G}$ are $\delta(A)$ for $A \in \{1,2,3,16,26,36\}$.

Claim 3.6. *The vector y does not lie on the face $\mathscr{G}$.*

Proof. Suppose, for contradiction, that $y \in \mathscr{G}$. Then, $y = \sum_{\delta(A) \in \mathscr{R}_{\mathscr{G}}} \alpha_A \delta(A)$ for some scalars $\alpha_A \geqslant 0$. We can assume that $0 \leqslant \alpha_A < 1$ for all $\delta(A) \in \mathscr{R}_{\mathscr{G}}$. (Else, if $\alpha_A \geqslant 1$ for some $A \in \mathscr{R}$, then $y - \delta(A)$ would still belong to the cone $\mathbb{R}_+(G_6)$, contradicting (9).)

Set

$$\varepsilon_S := \alpha_S + \alpha_{S\cup\{6\}} \quad \text{for } S \in \mathscr{S} := \{14, 15, 24, 25, 34, 35\}.$$

Let K_5 denote the subgraph of G_6 induced by the nodes: $1,2,3,4,5$ and let y_{K_5} denote the projection of y on the edge set of K_5. Then,

$$y_{K_5} = \sum_{S\in\mathscr{S}} \varepsilon_S \delta(S) \quad \text{and} \quad y = \alpha_6\delta(\{6\}) + \sum_{S\in\mathscr{S}} \alpha_S\delta(S) + (\varepsilon_S - \alpha_S)\delta(S\cup\{6\}).$$

As a consequence,

$$\varepsilon_S \in \{0,1\} \quad \text{for all } S \in \mathscr{S}.$$

(This follows from the fact that y_{K_5} lies on a simplex facet of $\mathbb{R}_+(K_5)$, namely, the one defined by the pentagonal inequality: $Q(1,1,1,-1,-1)^{\mathrm{T}}x \leqslant 0$, and the fact that $K_5 \in \mathscr{H}$. Alternatively, this follows from the fact that $y(4[ij]), y(5[ij]) \in 2\mathbb{Z}$ for $1 \leqslant i < j \leqslant 3$.) The following conditions on the α_S's and ε_S's can be derived from the parity condition (1):

- As $y(6[45]) = 2\alpha_6 \in 2\mathbb{Z}$, we deduce that

$$\alpha_6 = 0, \qquad y_{46} + y_{56} = \sum_{S\in\mathscr{S}} \varepsilon_S. \tag{10}$$

- From $y(6[ij]) \in 2\mathbb{Z}$, for $1 \leqslant i < j \leqslant 3$, we obtain that

$$\alpha_{14} + \alpha_{15}, \alpha_{24} + \alpha_{25}, \alpha_{34} + \alpha_{35} \in \{0,1\}. \tag{11}$$

- From $y(i[46]) \in 2\mathbb{Z}$, for $1 \leqslant i \leqslant 3$, we obtain that

$$\begin{aligned} &\alpha_{15} + \varepsilon_{24} - \alpha_{24} + \varepsilon_{34} - \alpha_{34} \in \mathbb{Z}, \\ &\alpha_{25} + \varepsilon_{14} - \alpha_{14} + \varepsilon_{34} - \alpha_{34}, \alpha_{35} + \varepsilon_{14} - \alpha_{14} + \varepsilon_{24} - \alpha_{24} \in \mathbb{Z}. \end{aligned} \tag{12}$$

- From $y(6[i4]) \in 2\mathbb{Z}$, for $1 \leqslant i \leqslant 3$, we obtain that

$$\begin{aligned} &\alpha_{14} + \varepsilon_{25} - \alpha_{25} + \varepsilon_{35} - \alpha_{35} \in \mathbb{Z}, \\ &\alpha_{24} + \varepsilon_{15} - \alpha_{15} + \varepsilon_{35} - \alpha_{35}, \alpha_{34} + \varepsilon_{15} - \alpha_{15} + \varepsilon_{25} - \alpha_{25} \in \mathbb{Z}. \end{aligned} \tag{13}$$

We now distinguish two cases depending whether some ε_S is equal to 0 or not. In both cases, we find that y belongs to $\mathbb{Z}_+(G_6)$, which contradicts (9).

Case A: $\varepsilon_S = 1$ for all $S \in \mathscr{S}$. Then, $y_{46} + y_{56} = 6$ and $y_{12} = y_{13} = y_{23} = 4$, $y_{14} = y_{15} = y_{24} = y_{25} = y_{34} = y_{35} = 3$, $y_{45} = 6$. Moreover,

$$\begin{aligned} y_{16} &= 4 + \alpha_{14} + \alpha_{15} - \alpha_{24} - \alpha_{25} - \alpha_{34} - \alpha_{35}, \\ y_{26} &= 4 - \alpha_{14} - \alpha_{15} + \alpha_{24} + \alpha_{25} - \alpha_{34} - \alpha_{35}, \\ y_{36} &= 4 - \alpha_{14} - \alpha_{15} - \alpha_{24} - \alpha_{25} + \alpha_{34} + \alpha_{35}, \\ y_{46} &= 3 + \alpha_{14} - \alpha_{15} + \alpha_{24} - \alpha_{25} + \alpha_{34} - \alpha_{35}. \\ y_{56} &= 3 - \alpha_{14} + \alpha_{15} - \alpha_{24} + \alpha_{25} - \alpha_{34} + \alpha_{35}. \end{aligned}$$

From (11), we know that the three sums: $\alpha_{14}+\alpha_{15}, \alpha_{24}+\alpha_{25}, \alpha_{34}+\alpha_{35}$ belong to $\{0,1\}$. This gives (up to symmetry) the following four possibilities:

Case A1: $\alpha_{14}+\alpha_{15} = \alpha_{24}+\alpha_{25} = \alpha_{34}+\alpha_{35} = 0$. Then, $y = \sum_{S\in\mathscr{S}} \varepsilon_S\delta(S) \in \mathbb{Z}_+(G_6)$.

Case A2: $\alpha_{14}+\alpha_{15} = 1$ and $\alpha_{24}+\alpha_{25} = \alpha_{34}+\alpha_{35} = 0$. Then, $y_{16} = 5$, $y_{26} = y_{36} = 3$ and $y_{46} \in \{2,4\}$. By symmetry between the nodes 4 and 5, we can suppose that $y_{46} = 2$. Then, $y = \delta(146)+\delta(15)+\delta(246)+\delta(256)+\delta(346)+\delta(356) \in \mathbb{Z}_+(G_6)$

Case A3: $\alpha_{14}+\alpha_{15} = \alpha_{24}+\alpha_{25} = 1$ and $\alpha_{34}+\alpha_{35} = 0$. Then, $y_{16} = y_{26} = 4$, $y_{36} = 2$ and $y_{46} \in \{1,3,5\}$. By symmetry, we can suppose that $y_{46} \in \{1,3\}$.

— If $y_{46} = 1$, then $y = \delta(146)+\delta(15)+\delta(246)+\delta(25)+\delta(346)+\delta(356) \in \mathbb{Z}_+(G_6)$.

— If $y_{46} = 3$, then $y = \delta(14)+\delta(156)+\delta(246)+\delta(25)+\delta(346)+\delta(356) \in \mathbb{Z}_+(G_6)$.

Case A4: $\alpha_{14}+\alpha_{15} = \alpha_{24}+\alpha_{25} = \alpha_{34}+\alpha_{35} = 1$. Then, $y_{16} = y_{26} = y_{36} = 3$ and, up to symmetry, $y_{46} = 2$. Then, $y = \delta(14)+\delta(156)+\delta(246)+\delta(25)+\delta(346)+\delta(35) \in \mathbb{Z}_+(G_6)$.

Case B: Some ε_S is equal to 0. Say, $\varepsilon_{14} = 0$. Then, $\alpha_{14} = 0$ and, using (11), $\alpha_{15} = 0$. From (11) and (12), we deduce that $\alpha_{24}+\alpha_{25}, \alpha_{34}+\alpha_{35}, \alpha_{25}+\varepsilon_{34}-\alpha_{34} \in \{0,1\}$. If one of these three quantities is equal to 0, then $\alpha_{24} = \alpha_{25} = \alpha_{34} = \alpha_{35} = 0$, which implies that $y = \sum_{S\in\mathscr{S}} \varepsilon_S\delta(S) \in \mathbb{Z}_+(G_6)$. Otherwise, the above three quantities are equal to 1, which implies that $\alpha_{25} = \alpha_{34} := \alpha$ and $\alpha_{24} = \alpha_{35} = 1-\alpha$ for some $0 \leqslant \alpha < 1$. Moreover, using (12), (13), $\varepsilon_{24} = \varepsilon_{25} = \varepsilon_{34} = \varepsilon_{35} = 1$. Hence, $y = \alpha(\delta(25)+\delta(246)+\delta(34)+\delta(356)) + (1-\alpha)(\delta(24)+\delta(256)+\delta(346)+\delta(35))$. Therefore, $y_e = 2$ for all edges except $y_{23} = y_{45} = 2$, in which case $y = \delta(246)+\delta(25)+\delta(34)+\delta(356) \in \mathbb{Z}_+(G_6)$. □

Corollary 3.7. *y does not lie on any pentagonal facet.*

Proof. There are, up to symmetry, two pentagonal facets to consider, namely, those defined by the inequalities $Q(1,1,1,-1,-1,0)^{\mathrm{T}}x \leqslant 0$ and $Q(1,1,-1,1,-1,0)^{\mathrm{T}}x \leqslant 0$. Note that the second one arises by switching the first one by the cut $\delta(34)$.

Suppose first that $Q(1,1,1,-1,-1,0)^{\mathrm{T}}y = 0$. Then, $y = \sum_{\delta(A)\in\mathscr{R}} \alpha_A\delta(A)$ for some scalars $0 \leqslant \alpha_A < 1$, where $\mathscr{R} \subseteq \mathscr{R}_{\mathscr{G}} \cup \{\delta(16), \delta(26), \delta(36)\}$ (recall that $\delta(1), \delta(2), \delta(3) \notin \mathscr{R}$ by Corollary 3.4). From $y(i[45]) \in 2\mathbb{Z}$, for $i = 1,2,3$, we obtain that $\alpha_{i6} \in \mathbb{Z}$ and, thus, $\alpha_{i6} = 0$, for $i = 1,2,3$. Hence, y lies on the face $\mathscr{G}$, contradicting Claim 3.6.

Suppose now that $Q(1,1,-1,1,-1,0)^{\mathrm{T}}y = 0$. Then, $y = \sum_{\delta(A)\in\mathscr{R}} \alpha_A\delta(A)$ for some scalars $0 \leqslant \alpha_A < 1$, where $\mathscr{R} \subseteq \mathscr{R}_{\mathscr{G}^{\delta(34)}} \cup \{\delta(16), \delta(26), \delta(46)\}$ and $\mathscr{R}_{\mathscr{G}^{\delta(34)}} = \{\delta(A) \mid A = 6, 13, 136, 15, 156, 23, 145, 25, 134, 34, 125, 45, 123\}$ denotes the set of nonzero cuts lying on the switching $\mathscr{G}^{\delta(34)}$ of $\mathscr{G}$ by $\delta(34)$. Again, from $y(i[35]) \in 2\mathbb{Z}$, for $i = 1,2,4$, we obtain that $\alpha_{i6} = 0$, for $i = 1,2,4$. Hence, y lies on the face $\mathscr{G}^{\delta(34)}$. Note that the proof of Claim 3.6 shows that the face $\mathscr{G}$ has the property (7). Moreover, both faces $\mathscr{G}$ and $\mathscr{G}^{\delta(34)}$ have the property (8); indeed, $s(x) = (x_{45}+x_{46}+x_{56})/2$ if $x \in \mathscr{G}$ and $s(x) = (x_{35}+x_{36}+x_{56})/2$ if $x \in \mathscr{G}^{\delta(34)}$. Therefore, by Proposition 2.9, the face $\mathscr{G}^{\delta(34)}$ also has the property (7). Hence, $y \in \mathbb{Z}_+(G_6)$, contradicting (9). □

From now on, we can suppose that y does not lie on any pentagonal or special facet, i.e., the set $\mathscr{V}$ of the facets of $\mathbb{R}_+(G_6)$ that contain y consists only of triangle facets. We conclude the proof in the following way. In the following Claims 3.8–3.11, we show that

$$\mathscr{R} \subseteq \{\delta(A) \mid A = 12, 13, 14, 23, 24, 34\}.$$

Therefore, $y = \alpha_{12}\delta(12) + \alpha_{13}\delta(13) + \alpha_{14}\delta(14) + \alpha_{23}\delta(23) + \alpha_{24}\delta(24) + \alpha_{34}\delta(34)$ for some nonnegative α's. Using the fact that $y([ij]k) \in 2\mathbb{Z}$ for $1 \leqslant i < j \leqslant 3$ and $k = 4, 5$, we deduce that the α's are all integers, which contradicts (9). This terminates the proof of Theorem 1.5.

Claim 3.8. *The cuts $\delta(5), \delta(6), \delta(56)$ do not belong to $\mathscr{R}$.*

Proof. Suppose that $\delta(5) \in \mathscr{R}$. By Claim 3.3, there exists an inequality $u^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G_6)$ such that $u^{\mathrm{T}}\delta(5) \in \{-4, -6\}$ and $u^{\mathrm{T}}y > u^{\mathrm{T}}\delta(5)$. There are four possibilities for u, namely, $u = w_1^{\delta(4)}$, $w_2^{\delta(15)}$, $w_3^{\delta(2)}$ and $w_4^{\delta(3)}$, for which $u^{\mathrm{T}}\delta(5) = -4$. By symmetry, it suffices to consider the case $u = w_1^{\delta(4)}$. Hence, we have that $(w_1^{\delta(4)})^{\mathrm{T}}y = -2$. On the other hand, we know from Corollary 3.4 that $\delta(1) \notin \mathscr{R}$. Hence, there exists $v \in \mathscr{V}$ such that $v^{\mathrm{T}}\delta(1) < 0$; it is necessarily a triangle inequality and there are, up to symmetry, the following three triangle inequalities: $x(1[23]) \leqslant 0$, $x(1[25]) \leqslant 0$, $x(1[26]) \leqslant 0$ to consider.

(i) Suppose that the inequality $x(1[23]) \leqslant 0$ belongs to $\mathscr{V}$, i.e., $y(1[23]) = 0$. After rearranging the terms, we obtain that $y(1[23]) + (w_1^{\delta(4)})^{\mathrm{T}}y = Q(-1,1,1,1,0,-1)^{\mathrm{T}}y + y(5[14]) + y(5[23])$. But, $Q(-1,1,1,1,0,-1)^{\mathrm{T}}y \leqslant 0$, $y(5[14]) \leqslant -2$ and $y(5[23]) \leqslant -2$ (indeed, the inequalities $x(5[14]) \leqslant 0$ and $x(5[23]) \leqslant 0$ do not belong to $\mathscr{V}$ since they are not satisfied at equality by $\delta(5)$). Hence, $y(1[23]) + (w_1^{\delta(4)})^{\mathrm{T}}y \leqslant -4$, contradicting the fact that $y(1[23]) = 0$ and $(w_1^{\delta(4)})^{\mathrm{T}}y = -2$.

(ii) Suppose that $y(1[25]) = 0$. Then, $y(1[25]) + (w_1^{\delta(4)})^{\mathrm{T}}y = Q(-1,1,1,1,0,-1)^{\mathrm{T}}y + y(5[13]) + y(5[14]) \leqslant -4$, yielding again a contradiction.

(iii) Suppose that $y(1[26]) = 0$. Then, $y(1[26]) + (w_1^{\delta(4)})^{\mathrm{T}}y = y(6[34]) + y(5[24]) + y(1[23]5) \leqslant -4$, yielding a contradiction.

So, we have shown that $\delta(5) \notin \mathscr{R}$. Similarly, $\delta(6) \notin \mathscr{R}$, implying that $\delta(56) \notin \mathscr{R}$. □

Claim 3.9. *The cuts $\delta(123), \delta(124), \delta(134), \delta(156)$ do not belong to $\mathscr{R}$.*

Proof. Suppose, for instance, that $\delta(123) \in \mathscr{R}$. By Claim 3.3, there exists $u^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G_6)$ such that $u^{\mathrm{T}}\delta(123) \in \{-4, -6\}$ and $u^{\mathrm{T}}y > u^{\mathrm{T}}\delta(123)$. The possibilities for u are two pentagonal facets and four switchings for each special facet w_i, $i = 1, 2, 3, 4$. By symmetry, it suffices to consider the cases (i) $u^{\mathrm{T}}x = Q(1,1,1,-1,-1,0)^{\mathrm{T}}x \leqslant 0$, (ii) $u = w_1$, (iii) $u = w_1^{\delta(1)}$ (for which $u^{\mathrm{T}}\delta(123) = -6$), and (iv) $u = w_1^{\delta(15)}$, (v) $u = w_1^{\delta(6)}$ (for which $u^{\mathrm{T}}\delta(123) = -4$).

(i) Suppose that $Q(1,1,1,-1,-1,0)^{\mathrm{T}}y = 0$. Since $\delta(5) \notin \mathscr{R}$ (by Claim 3.8), let $v \in \mathscr{V}$ such that $v^{\mathrm{T}}\delta(5) < 0$; it is the triangle inequality $x(5[i4]) \leqslant 0$, for $i = 1, 2, 3$.

Suppose, for instance, that $y(5[14]) = 0$. Then, $y(5[14]) + Q(1,1,1,-1,-1,0)^{\mathrm{T}}y = y(4[23]) + y(5[13]) + y(5[12]) \leqslant -6$, yielding a contradiction.

(ii) Suppose that $w_1^{\mathrm{T}}y \in \{-2,-4\}$. Since $\delta(6) \notin \mathscr{R}$, there exists $v \in \mathscr{V}$ such that $v^{\mathrm{T}}\delta(6) < 0$; it is one of the triangle inequalities $x(6[14]) \leqslant 0$, $x(6[24]) \leqslant 0$ (or $x(6[34]) \leqslant 0$). But, $y(6[14]) + w_1^{\mathrm{T}}y = y(6[23]) + y(2[45]) + y([14]35) \leqslant -6$ and $y(6[24]) + w_1^{\mathrm{T}}y = y(6[23]) + y(3[45]) + y([61]52) \leqslant -6$, yielding a contradiction.

(iii) The case when $(w_1^{\delta(1)})^{\mathrm{T}}y \in \{-2,-4\}$ is identical to the case (ii), exchanging the nodes 5 and 6.

(iv) Suppose that $(w_1^{\delta(15)})^{\mathrm{T}}y = -2$. As in (ii), we can suppose that $y(6[14]) = 0$ or $y(6[24]) = 0$. But, $y(6[14]) + (w_1^{\delta(15)})^{\mathrm{T}}y = Q(-1,1,1,-1,1,0)^{\mathrm{T}}y + y(6[12]) + y(6[13]) \leqslant -4$ and $y(6[24]) + (w_1^{\delta(15)})^{\mathrm{T}}y = y(4[35]) + y([23]6) + y(1[52]6) \leqslant -4$, yielding a contradiction.

(v) The case when $(w_1^{\delta(6)})^{\mathrm{T}}y = -2$ is identical to the case (iv), exchanging the nodes 5 and 6. □

Claim 3.10. *The cuts $\delta(125), \delta(126), \delta(135), \delta(136), \delta(145), \delta(146)$ do not belong to $\mathscr{R}$.*

Proof. Suppose, for instance, that $\delta(146) \in \mathscr{R}$. By Claim 3.3, let $u^{\mathrm{T}}x \leqslant 0$ define a facet of $\mathbb{R}_+(G_6)$ such that $u^{\mathrm{T}}\delta(146) \in \{-4,-6\}$ and $u^{\mathrm{T}}y > u^{\mathrm{T}}\delta(146)$. So, $u^{\mathrm{T}}x \leqslant 0$ is the pentagonal inequality $Q(1,-1,-1,1,0,1)^{\mathrm{T}}x \leqslant 0$, $u = w_1^{\delta(15)}$, for which $u^{\mathrm{T}}\delta(146) = -6$, or $u = w_1$, for which $u^{\mathrm{T}}\delta(146) = -4$. (The case when u is one of two switchings of w_2, w_3, or w_4 follows by symmetry.)

(i) Suppose that $Q(1,-1,-1,1,0,1)^{\mathrm{T}}y \in \{-2,-4\}$. Since $\delta(6) \notin \mathscr{R}$, there exists $v \in \mathscr{V}$ such that $v^{\mathrm{T}}\delta(6) < 0$; we can suppose that it is one of the inequalities $x(6[12]) \leqslant 0$ or $x(6[14]) \leqslant 0$. But, $y(6[12]) + Q(1,-1,-1,1,0,1)^{\mathrm{T}}y = y(2[46]) + y(6[23]) + y(3[14]) \leqslant -6$ and $y(6[14]) + Q(1,-1,-1,1,0,1)^{\mathrm{T}}y = y(6[23]) + y(2[14]) + y(3[14]) \leqslant -6$, yielding a contradiction.

(ii) Suppose that $(w_1^{\delta(15)})^{\mathrm{T}}y \in \{-2,-4\}$. From the fact that $\delta(5) \notin \mathscr{R}$, we know that one of the inequalities $x(5[1i]) \leqslant 0$ $(i = 2,3)$, $x(5[23]) \leqslant 0$, $x(5[i4]) \leqslant 0$ $(i = 2,3)$ belongs to $\mathscr{V}$. But, $y(5[12]) + (w_1^{\delta(15)})^{\mathrm{T}}y = Q(1,1,1,-1,0,-1)^{\mathrm{T}}y + y(1[35]) + y([14]5) \leqslant -6$, $y(5[23]) + (w_1^{\delta(15)})^{\mathrm{T}}y = y([23]6) + y([23]4) + y(15[46]) \leqslant -6$ and $y(5[24]) + (w_1^{\delta(15)})^{\mathrm{T}}y = y([23]6) + y([35]4) + y(15[46]) \leqslant -6$, yielding a contradiction.

(iii) Suppose that $w_1^{\mathrm{T}}y = -2$. From the fact that $\delta(6) \notin \mathscr{R}$, we can assume that one of the inequalities $x(6[12]) \leqslant 0$, $x(6[14]) \leqslant 0$, $x(6[24]) \leqslant 0$ belongs to $\mathscr{V}$. But, $y(6[12]) + w_1^{\mathrm{T}}y = y(2[46]) + y([23]6) + y([12]5) + y(3[45]) \leqslant -4$, $y(6[14]) + w_1^{\mathrm{T}}y = y([23]6) + y(2[45]) + y(3[41]5) \leqslant -4$ and $y(6[24]) + w_1^{\mathrm{T}}y = y(3[45]) + y([23]6) + y([16]25) \leqslant -4$, yielding a contradiction. □

Claim 3.11. *The cuts $\delta(15), \delta(16), \delta(25), \delta(26), \delta(35), \delta(36), \delta(45), \delta(46)$ do not belong to $\mathscr{R}$.*

Proof. Suppose, for instance, that $\delta(45) \in \mathscr{R}$. Then, there exists $u^{\mathrm{T}}x \leqslant 0$ defining a facet of $\mathbb{R}_+(G_6)$ such that $u^{\mathrm{T}}\delta(45) \in \{-4,-6\}$ and $u^{\mathrm{T}}y > u^{\mathrm{T}}\delta(45)$; it is (up to symmetry) $Q(1,1,1,-1,-1,0)^{\mathrm{T}}x \leqslant 0$, $(w_1^{\delta(6)})^{\mathrm{T}}x \leqslant 0$, for which $u^{\mathrm{T}}\delta(45) = -6$, or $w_1^{\mathrm{T}}x \leqslant 0$, for which $u^{\mathrm{T}}\delta(45) = -4$.

(i) Suppose that $Q(1,1,1,-1,-1,0)^{\mathrm{T}}y \in \{-2,-4\}$. We can suppose that $x([14]5) \leqslant 0$ belongs to $\mathscr{V}$ (since $\delta(5) \notin \mathscr{R}$ and using symmetries). But, $y([14]5) + Q(1,1,1,-1,-1,0)^{\mathrm{T}}y = y([12]5) + y([13]5) + y([23]5) \leqslant -6$, yielding a contradiction.

(ii) Suppose that $(w_1^{\delta(6)})^{\mathrm{T}}y \in \{-2,-4\}$. We can suppose that $x([14]5) \leqslant 0$ or $x([24]5) \leqslant 0$ belongs to $\mathscr{V}$. But, $y([14]5) + (w_1^{\delta(6)})^{\mathrm{T}}y = Q(-1,1,1,-1,0,1)^{\mathrm{T}}y + y([12]5) + y([13]5) \leqslant -6$ and $y([24]5) + (w_1^{\delta(6)})^{\mathrm{T}}y = y(4[36]) + y([23]5) + y(15[26]) \leqslant -6$, yielding a contradiction.

(iii) Suppose that $w_1^{\mathrm{T}}y = -2$. We can suppose that $x([14]5) \leqslant 0$ or $x([24]5) \leqslant 0$ belongs to $\mathscr{V}$. But, $y([24]5) + w_1^{\mathrm{T}}y = Q(-1,1,1,-1,0,1)^{\mathrm{T}}y + y([13]5) + y([12]5) \leqslant -4$ and $y([24]5) + w_1^{\mathrm{T}}y = y([23]5) + y([61]52) + y(3[46]) \leqslant -4$, yielding a contradiction. □

4. The role of K_6 in the class $\mathscr{H}$

In this section, we give the proofs of Theorem 1.1 and Proposition 1.2, i.e., we show that every proper subgraph of K_6 belongs to $\mathscr{H}$, and that every graph belonging to $\mathscr{H}$ has no K_6-minor. For the proof of Theorem 1.1, we need to know the explicit description of the facets of the cone $\mathbb{R}_+(H_6+e)$, where H_6+e is the graph H_6 with one added edge; see Fig. 7. We present this description in Section 4.1; we also give there, for information, the description of the cone $\mathbb{R}_+(H_6)$. We give the proof of Theorem 1.1 in Section 4.2 and the proof of Proposition 1.2 in Section 4.3.

4.1. Description of the cones $\mathbb{R}_+(H_6)$ and $\mathbb{R}_+(H_6+e)$

We consider the graphs H_6 and H_6+e from Figs. 2 and 7. So, H_6+e is obtained from H_6 by adding the edge $e = 46$ and $H_6+e = K_6\backslash\{12,13,56\}$. We checked, using

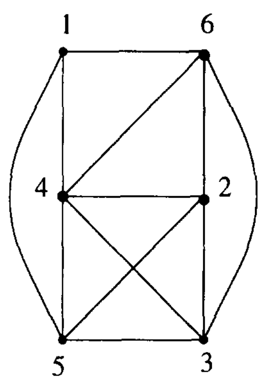

Fig. 7. H_6+e.

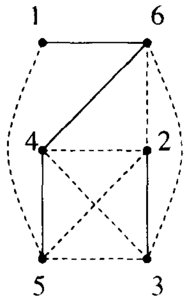

Fig. 8. u.

computer, that the cone $\mathbb{R}_+(H_6 + e)$ has 49 facets in total. They are grouped in two classes.

- The first class consists of the $9 \times 3 + 2 \times 4 = 35$ facets that are defined by the cycle inequalities (2), where C is one of the 9 triangles $(i,4,j)$ ($i = 1,2,3$; $j = 5,6$), $(2,3,i)$ ($i = 4,5,6$), or of the two circuits $(1,5,2,6)$ and $(1,5,3,6)$.
- The second class consists of 14 facets, that are all switching equivalent. Set

$$u^{\mathrm{T}}x := x_{16} - x_{15} + x_{23} + x_{45} + x_{46} - \sum_{\substack{i=2,3\\ j=4,5,6}} x_{ij}.$$

The vector u is shown in Fig. 8. The inequality $u^{\mathrm{T}}x \leqslant 0$ defines a facet of $\mathbb{R}_+(H_6 + e)$. There are exactly 13 nonzero cuts satisfying the equality $u^{\mathrm{T}}x = 0$; namely, the cuts of the set

$$\mathscr{A}_u := \{\delta(A) \mid A = 1,4,6,14,15,24,26,34,36,124,125,134,135\}.$$

Hence, for each $\delta(A) \in \mathscr{A}_u$, the inequality $(u^{\delta(A)})^{\mathrm{T}}x \leqslant 0$ defines a facet of $\mathbb{R}_+(H_6 + e)$. Observe that all the inequalities defining facets of $\mathbb{R}_+(H_6 + e)$ satisfy both conditions (3) and (4).

For information, we also give the description of the facets of $\mathbb{R}_+(H_6)$. The cone $\mathbb{R}_+(H_6)$ has 46 facets in total. Besides the facet defined by the inequality $x_{16} \geqslant 0$, they are grouped in two classes.

- The first class consists of the $6 \times 3 + 4 \times 4 = 34$ facets that are defined by the cycle inequalities (2), where C is one of the 6 triangles $(i,4,5)$ ($i = 1,2,3$), $(2,3,i)$ ($i = 4,5,6$), or one of the 4 circuits $(1,2,4,6)$, $(1,5,3,6)$, $(1,6,3,4)$ and $(1,6,2,5)$.
- The second class consists of 11 facets, that are all switching equivalent. Set

$$w^{\mathrm{T}}x := 2x_{16} + x_{23} + x_{45} - x_{26} - x_{36} - \sum_{\substack{i=1,2,3\\ j=4,5}} x_{ij}.$$

The vector w is shown in Fig. 9 (the double edge indicates the coefficient 2 for the variable x_{16}). The inequality $w^{\mathrm{T}}x \leqslant 0$ defines a simplex facet of $\mathbb{R}_+(H_6)$. There are 10 nonzero cuts satisfying $w^{\mathrm{T}}x = 0$, namely, the cuts of the set

$$\mathscr{A}_w := \{\delta(A) \mid A = 1,6,14,15,26,36,125,124,134,135\}.$$

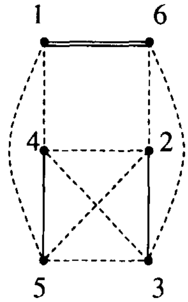

Fig. 9. w.

For each $\delta(A) \in \mathscr{A}_w$, the inequality $(w^{\delta(A)})^T x \leqslant 0$ defines a facet of $\mathbb{R}_+(H_6)$. (Note that the inequality $w^T x \leqslant 0$ arises by summing the inequality $u^T x \leqslant 0$ and the triangle inequality $x_{16} - x_{14} - x_{46} \leqslant 0$, both defining facets of the cone $\mathbb{R}_+(H_6 + e)$.)

Remark that the property (4) is closed under deleting edges (since the facets of $\mathbb{R}_+(G \backslash e)$ arise from those of $\mathbb{R}_+(G)$ by projecting out the variable x_e). However, this is not the case for the property (3). For instance, the facets of $\mathbb{R}_+(H_6 + e)$, or of $\mathbb{R}_+(K_6 \backslash e)$, have the property (3), but not those of $\mathbb{R}_+(H_6)$.

4.2. Proof of Theorem 1.1

Let D be a nonempty subset of edges of K_6 and let $G = K_6 \backslash D$ denote the graph obtained by deleting D from K_6. We show that $G \in \mathscr{H}$. This is the case if $|D| = 1$ by Theorem 1.5.

- If $|D| = 2$, then $G \in \mathscr{H}$; this follows from Theorem 1.5 since all the facets of $K_6 \backslash e$ satisfy (3) and (4).
- If $|D| = 3$, then we are in one of the following cases:
  (i) $D = K_{1,3}$ (e.g. $D = \{12, 13, 14\}$),
  (ii) $D = P_2 \cup P_3$ (e.g. $D = \{12, 13, 56\}$),
  (iii) $D = P_4$ (e.g. $D = \{12, 23, 34\}$),
  (iv) $D = C_3$ (e.g. $D = \{12, 23, 13\}$),
  (v) $D = P_2 \cup P_2 \cup P_2$ (e.g. $D = \{12, 34, 56\}$).

  In the cases (iii)–(v), $G \in \mathscr{H}$ since G has no K_5-minor.In the case (i), $G \in \mathscr{H}$ since G is the 2-sum of K_3 and K_5 (recall Example 2.8). In case (ii), $G \in \mathscr{H}$ since G arises by deleting an edge from $K_6 - P_3$ which is the 3-sum of K_4 and K_5.
- Suppose that $|D| = 4$. If G is a subgraph of $K_6 - P_4$, then $G \in \mathscr{H}$ since G has no K_5-minor. Else, we are in one of the following cases:
  (i) $D = K_{1,4}$ (e.g. $D = \{12, 13, 14, 15\}$),
  (ii) $D = K_{1,3} \cup P_2$ (e.g. $D = \{12, 13, 14, 56\}$),
  (iii) $D = P_3 \cup P_3$ (e.g. $D = \{12, 13, 46, 56\}$).

  In the case (i), $G \in \mathscr{H}$ since G is the 1-sum of K_5 and K_2. In the cases (ii) and (iii), $G \in \mathscr{H}$ since G arises by deleting an edge from the graph $H_6 + e$ (see Fig. 7) whose facets all satisfy (3) and (4) (see Section 4.1) and $H_6 + e$ belongs to $\mathscr{H}$ (see Example 2.8).

- Suppose that $|D| \geqslant 5$. Then, G is a subgraph of K_5 or of $K_6 - P_4$, implying that $G \in \mathscr{H}$.

This concludes the proof of Theorem 1.1. □

4.3. Proof of Proposition 1.2

We start by recalling some facts on the antipodal extension operation (see, e.g., [9]). Given $x \in \mathbb{R}^{\binom{n}{2}}$ and $\alpha \in \mathbb{R}$, define the *antipodal extension* $y = ant_\alpha(x) \in \mathbb{R}^{\binom{n}{2}}$ of x by

$$\begin{aligned} y_{ij} &= x_{ij} \quad \text{if } 1 \leqslant i < j \leqslant n, \\ y_{1,n+1} &= \alpha, \\ y_{i,n+1} &= \alpha - x_{1i} \quad \text{if } 2 \leqslant i \leqslant n. \end{aligned}$$

It is easy to see that, if $x \in \mathbb{R}_+(K_n)$ and $x = \sum_{S \subseteq \{1,\dots,n\}} \alpha_S \delta(S)$ with $\alpha_S \geqslant 0$, then $ant_\alpha(x) = \sum_{S | 1 \in S} \alpha_S \delta(S) + \sum_{S | 1 \notin S} \delta(S \cup \{n+1\}) + (\alpha - \sum_S \alpha_S)\delta(\{n+1\})$ and, conversely, if $ant_\alpha(x) \in \mathbb{R}_+(K_{n+1})$, then every decomposition of $ant_\alpha(x)$ as a nonnegative combination of cuts has the above form. Hence, we have the following result.

Proposition 4.1 (Deza and Laurent [9]). (i) *$ant_\alpha(x) \in \mathbb{R}_+(K_{n+1})$ if and only if $x \in \mathbb{R}_+(K_n)$, $\alpha \in \mathbb{R}_+$ and $\alpha \geqslant s(x)$.*

(ii) *$ant_\alpha(x) \in \mathbb{Z}_+(K_{n+1})$ if and only if $x \in \mathbb{Z}_+(K_n)$, $\alpha \in \mathbb{Z}_+$ and $\alpha \geqslant h(x)$.*

(iii) *$ant_\alpha(x) \in \mathbb{Z}(K_{n+1})$ if and only if $x \in \mathbb{Z}(K_n)$ and $\alpha \in \mathbb{Z}$.*

Note that Proposition 4.1 remains valid under the following conditions: G is a graph with node set $\{1,\dots,n\}$ and with the node 1 being adjacent to all other nodes of G, G' is the graph obtained from G by adding a new node $n+1$ adjacent to all nodes of G, $x \in \mathbb{R}^{E(G)}$, and $y = ant_\alpha(x) \in \mathbb{R}^{E(G')}$ is defined by $y_e = x_e$ for $e \in E(G)$ and $y_{i,n+1} = \alpha - x_{1i}$ for all nodes i of G.

Proposition 4.1 provides a useful tool for constructing counterexamples for the Hilbert base property. Indeed, if we can find $x \in \mathbb{R}_+(K_n) \cap \mathbb{Z}(K_n)$ and $\alpha \in \mathbb{Z}$ such that $s(x) \leqslant \alpha < h(x)$, then $ant_\alpha(x) \in \mathbb{R}_+(K_{n+1}) \cap \mathbb{Z}(K_{n+1}) \setminus \mathbb{Z}_+(K_{n+1})$. We now present such an example.

Example 4.2. Consider the vector $x_n \in \mathbb{R}^{\binom{n}{2}}$ defined by $(x_n)_{ij} = 2$ for all $1 \leqslant i < j \leqslant n$ and set $a_{n+1} = ant_4(x_n)$. So, all components of a_{n+1} are equal to 2 except $(a_{n+1})_{1,n+1} = 4$. Clearly, $s(x_n) = n(n-1)/\lfloor n/2 \rfloor \lceil n/2 \rceil$ since x_n can be written as a nonnegative combination of cuts using only equicuts, i.e., cuts with $\lfloor n/2 \rfloor \lceil n/2 \rceil$ edges. Moreover, for $n \geqslant 5$, $h(x_n) = n$ since $x_n = \sum_{1 \leqslant i \leqslant n} \delta(i)$ is the only way of writing x_n as an integer nonnegative sum of cuts [6]. Hence, for $n \geqslant 5$, $s(x_n) \leqslant 4 < h(x_n)$, and we deduce from Proposition 4.1 that $a_{n+1} \in \mathbb{Z}(K_{n+1}) \cap \mathbb{R}_+(K_{n+1}) \setminus \mathbb{Z}_+(K_{n+1})$.

One can also show directly that $a_{n+1} \notin \mathbb{Z}_+(K_{n+1})$ by checking that $a_{n+1} - \delta(A) \notin \mathbb{R}_+(K_{n+1})$ for all cuts $\delta(A)$. Indeed, $a_{n+1} - \delta(A)$ violates either the pentagonal inequality

$Q(1,1,1,-1,-1,0,\ldots,0)^{\mathrm{T}}x \leqslant 0$, or the inequality $Q(2,1,1,-1,-1,-1,0,\ldots,0)^{\mathrm{T}}x \leqslant 0$ (for a suitable labeling of the nodes), which both define facets of $\mathbb{R}_+(K_{n+1})$ if $n \geqslant 5$.

Explicit decompositions of x_n and a_{n+1} are as follows. Let $\mathscr{E}_n$ denote the set of the equicuts of K_n. Then,

$$x_n = \frac{2}{c_n} \sum_{\delta(S) \in \mathscr{E}_n} \delta(S),$$

$$a_{n+1} = \frac{2}{c_n}\left(\sum_{\delta(S) \in \mathscr{E}_n,\ 1 \in S} \delta(S) + \sum_{\delta(S) \in \mathscr{E}_n,\ 1 \notin S} \delta(S \cup \{n+1\}) \right) + (4 - s(x_n))\delta(\{n+1\}),$$

where $c_n = \binom{n-2}{n/2-1}$ if n is even and $c_n = 2\binom{n-2}{(n-3)/2}$ if n is odd.

Therefore, the cuts of K_6 do not form a Hibert basis, as the point a_6 from Example 4.2 belongs to $\mathbb{Z}(K_6) \cap \mathbb{R}_+(K_6) \setminus \mathbb{Z}_+(K_6)$. It is shown in [13] that the cuts of K_6 together with the 15 permutations of a_6 form a Hilbert basis. Moreover, all elements of $\mathbb{Z}(K_6) \cap \mathbb{R}_+(K_6) \setminus \mathbb{Z}_+(K_6)$ are described; up to permutation, they are of the form $a_6 + \alpha\delta(\{i\})$, where $\alpha \in \mathbb{Z}$ and $i \in \{2,3,4,5\}$. For $n \geqslant 7$, several other classes of vectors belonging to $\mathbb{R}_+(K_n) \cap \mathbb{Z}(K_n) \setminus \mathbb{Z}_+(K_n)$ are constructed in [8], in particular, using other extension operations.

Claim 4.3. *Let G be a graph which contains K_6 as a subgraph. Then, G does not belong to $\mathscr{H}$.*

Proof. By assumption, the edge set E of G contains the edge set $E(K_6)$ of a K_6 subgraph. Define $a \in \mathbb{R}^E$ by $a_e = 2$ for all edges $e \in E$ except $a_e = 4$ for one edge $e \in E(K_6)$. Then, $a \in \mathbb{Z}(G) \cap \mathbb{R}_+(G)$, but $a \notin \mathbb{Z}_+(G)$. Indeed, $a \in \mathbb{R}_+(G)$ since a is the projection of $a_n \in \mathbb{R}_+(K_n)$ (n is the number of nodes of G); $a \notin \mathbb{Z}_+(G)$ since its projection a_6 on $\mathbb{R}^{E(K_6)}$ does not belong to $\mathbb{Z}_+(K_6)$. This shows that $G \notin \mathscr{H}$. □

Proposition 1.2 now follows easily. Indeed, suppose G has a K_6-minor, i.e., $G \backslash D / C = K_6$ for some disjoint subsets C and D of the edge set of G. Then, G/C does not belong to $\mathscr{H}$ since it contains K_6 as a subgraph (by Claim 4.3) which implies that $G \notin \mathscr{H}$ (by Proposition 2.1).

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