Operations Research Letters 11 (1992) 105–110
North-Holland

A characterization of knapsacks with the max-flow–min-cut property

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Received January 1990
Revised April 1991

Using a result of Seymour we give a characterization of a class of knapsack problems for which the clutter of minimal covers has the max-flow–min-cut property with respect to all right-hand sides. This implies that adding the minimal cover cuts to the problem is sufficient to guarantee an integer optimum for the linear programming relaxation. We also give a characterization of all the minimal cover cuts for this class of knapsacks.

integer programming

1. Introduction

In this paper we consider the *knapsack polytope* $P_{a,b} = \text{conv}\{x \in \{0, 1\}^n\colon\ ax \le b\}$ where $a_1, \dots, a_n$, b are non-negative integers. The study of the facial structure of this polytope has received considerable attention in the literature (see e.g. [1,2,6,9,10]). As a result, several classes of facet-defining inequalities for $P_{a,b}$ are known to date. These inequalities are very useful, since for any 0–1 linear programming problem, each individual constraint can be regarded as a knapsack, and so, as indicated in the seminal work of Crowder, Johnson and Padberg [5], facets and valid inequalities for the knapsack polytope can be used as strong cutting planes in the solution of large-scale 0–1 linear programming problems.

The basic class of facet-defining inequalities for $P_{a,b}$ is that associated with the *minimal covers* of the inequality $ax \le b$. As shown by Balas and Jeroslow [2], the minimal cover inequalities provide an alternative integer programming formulation of the knapsack problem. A natural question is to characterize the parameters a, b for which the class of minimal cover facets is sufficient for describing the knapsack polytope $P_{a,b}$. In this paper we prove that $P_{a,b}$ is fully described by minimal cover facets for every choice of b if and only if the sequence $a_n, \dots, a_1$ is *weakly superincreasing*, i.e. satisfies: $a_n + \cdots + a_q \le a_{q-1}$, $q = n, \dots, 2$. For proving this fact, we use a result of Seymour on Mengerian clutters. We first recall the necessary preliminaries and notation.

A *clutter* $\mathscr{L}$ is a family of subsets of a set $E(\mathscr{L})$ with the property that $A_1 \not\subseteq A_2$ for distinct members A_1, A_2 of $\mathscr{L}$. The *blocker* $b(\mathscr{L})$ of $\mathscr{L}$ is the family of minimal sets (here minimal is meant with respect to inclusion) intersecting all sets if $\mathscr{L}$. Clearly, the family $b(\mathscr{L})$ is also a clutter.

For every clutter $\mathscr{L}$ and for every subset Z of $E(\mathscr{L})$, the *deletion* $\mathscr{L}\setminus Z$ and the *contraction* $\mathscr{L}/Z$ are defined as follows:

$$\mathscr{L}\setminus Z = \{A \in \mathscr{L} : A \cap Z = \emptyset\},$$

$$\mathscr{L}/Z = \text{minimal members of } \{A - Z : A \in \mathscr{L}\}.$$

Both $\mathscr{L}\setminus Z$ and $\mathscr{L}/Z$ are clutters and it is very easy to show that

$$b(L\setminus Z) = b(L)/Z, \qquad b(L/Z) = b(L)\setminus Z,$$

and that if $Z_1 \cap Z_2 = \emptyset$, then $(L \setminus Z_1)/Z_2 = (L/Z_2) \setminus Z_1$.

A *minor* of $\mathscr{L}$ is a clutter which may be obtained from $\mathscr{L}$ by a sequence of deletions and contractions.

Let $\mathscr{L}$ be a clutter and let $M_{\mathscr{L}}$ be the 0–1 matrix whose rows are the incidence vectors of the subsets $A \in \mathscr{L}$. Consider the following dual pair of linear programs associated with $M_{\mathscr{L}}$:

$$(\mathrm{P}_1) \quad \max \quad 1_n y \quad \text{s.t.} \quad yM_{\mathscr{L}} \le w, \quad y \ge 0,$$

$$(\mathrm{P}_2) \quad \min \quad wx \quad \text{s.t.} \quad M_{\mathscr{L}} x \ge 1_n, \quad x \ge 0.$$

A clutter $\mathscr{L} \neq \{\emptyset\}$ has the *weak max-flow–min-cut property* (weak MFMC) [12] if the program (P_2) has an integer optimizing vector for all integral w with $w \ge 0$. The clutter $\mathscr{L}$ is called *Mengerian* [12] if both programs (P_1) and (P_2) have an integer optimizing vector for all integral w with $w \ge 0$. The weak MFMC property is implied by, but is strictly weaker than, the property of being Mengerian.

Seymour [12] also proved that both the weak MFMC property and the property of being Mengerian are hereditary with respect to taking minors, thus suggesting the possibility of characterizing both classes in terms of forbidden minors. Unfortunately, Seymour provided convincing evidences that the problem of describing all minimal (with respect to taking minors) non-Mengerian clutters is very hard. Nevertheless, he was able to solve this problem for the interesting class of clutters having no minor isomorphic to $P_4 = \{\{1, 2\}, \{2, 3\}, \{3\ 4\}\}$.

Theorem 1.1. [12]. *If a clutter $\mathscr{L}$ has no P_4 minor, then $\mathscr{L}$ is Mengerian if and only if $\mathscr{L}$ has no minor isomorphic to one of the following clutters*:

$$Q_6 = \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\},$$

$$J_q = \{\{1, \ldots, q\}, \{0, i\} \text{ for } i = 1, 2, \ldots, q\},$$

$$q \ge 2.$$

Observe that the clutter J_q is a minimal (with respect to taking minors) non-MFMC clutter. Interesting progresses in the study of the properties of minimal non-MFMC clutters have been recently made by Cornuéjols and Novick [4], Padberg [11] and Seymour [13].

A clutter $\mathscr{L}$ is a *shift clutter* if there exists an ordering σ of the elements of $E(\mathscr{L})$ such that for each member A of $\mathscr{L}$ and each pair of elements e_i, e_j of $E(\mathscr{L})$ such that $e_i \in A$, $e_j \notin A$ and $\sigma(e_j) < \sigma(e_i)$, there exists a member A' of $\mathscr{L}$ with the property that $A' \subseteq A - \{e_i\} \cup \{e_j\}$. If $\mathscr{L}$ is a shift clutter then the matrix $M_{\mathscr{L}}$ is a (proper) *regular matrix* [7] and the problem (P_2) is also called *regular set-covering problem*. In [3] it was shown that shift clutters are hereditary with respect to taking minors and an $O(mn)$ algorithm to solve problem (P_2) was given. Examples of clutters which are not shift clutters are P_4 and Q_6.

Remark 1.2. A consequence of Theorem 1.1 is that shift clutters that are Mengerian or have the MFMC property can be characterized in terms of forbidden minors; namely, a shift clutter $\mathscr{L}$ is Mengerian if and only if it has the weak MFMC property if and only if it has no J_q minor, $q \ge 2$.

A significant example of shift clutter is the clutter of minimal covers of a linear inequality $\sum_{i=1}^n a_i x_i \le b$. Let $N = \{1, \ldots, n\}$, a subset C of N is called a *minimal cover* if $\sum_{i \in C} a_i > b$ and every proper subset S of C satisfies $\sum_{i \in S} a_i \le b$.

Let $K_{a,b}$ denote the family of minimal covers associated with the inequality $ax \le b$. Then $K_{a,b}$ is a shift clutter. To see this, it suffices to order the elements of N in such a way that: $a_1 \ge a_2 \ge \cdots \ge a_n$.

Let $M_{a,b}$ denote the matrix whose rows are the incidence vectors of the members of $K_{a,b}$. In [2] Balas and Jeroslow proved that the knapsack problem $\max\{ws : ax \le b,\ x \in \{0, 1\}^n\}$ is equivalent to the set covering problem $\min\{wx : M_{a,b}x \ge 1_n,\ x \in \{0, 1\}^n\}$.

We denote by $P_{a,b}$ the polytope $\mathrm{conv}\{x \in \{0, 1\} : ax \le b\}$ (*knapsack polytope*) and, finally, we say that the sequence $a_n, \ldots, a_1$ is *weakly superincreasing* if

$$a_n + \cdots + a_q \le a_{q-1} \quad \text{for all } q = 2, \ldots, n$$

and *superincreasing* if the inequality holds strictly for all $q = 2, \ldots, n$.

In this paper we show that the knapsack clutter $K_{a,b}$ associated with the inequality $ax \le b$, $a \in \mathbf{Z}_+^n$, has the weak MFMC property for every

$b \in \mathbf{Z}_+$ if and only if the components of a define a *weakly superincreasing sequence.* Moreover, in this case we provide a complete description of the clutter $K_{a,b}$ and of its blocker.

2. Knapsack clutters with the weak MFMC property

Let $a_1, a_2, \ldots, a_n, b$ be non negative integers and consider the following linear inequality

$$\sum_{i=1}^{n} a_i x_i \leq b. \tag{2.1}$$

Fact 2.1. *Let $K_{a,b}$ be the clutter of minimal covers associated with the inequality* (2.1). *Then, for each pair C, D of disjoint subsets of $N = \{1, \ldots, n\}$, the minor $K_{a,b} \backslash D/C$ is the clutter of minimal covers associated with the inequality*:

$$\sum_{i \in N-(C \cup D)} a_i x_i \leq b - \sum_{i \in C} a_i.$$

Proof. Let k be an element of N, let a' be the vector $(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$ and let $b' = b - a_k$. The result follows from the following assertions:

$$K_{a,b} \backslash \{k\} = K_{a',b}, \qquad K_{a,b}/\{k\} = K_{a',b'}.$$

Both are easy to verify. Namely, a subset A of N is a member of $K_{a,b} \backslash \{k\}$ if and only if it is a minimal cover of (2.1) which does not contain the element k, and hence if and only if it is a minimal cover of $\sum_{i \in N-\{k\}} a_i x_i \leq b$.

On the other hand, a subset A of N is a member of $K_{a,b}/\{k\}$ if and only if either A or $A \cup \{k\}$ is a minimal cover of (2.1). It follows that $A \in K_{a,b}/\{k\}$ if and only if $\sum_{i \in A} a_i x_i > b - a_k$ and, for each $j \in A$, $\sum_{i \in A-\{j\}} a_i x_i \leq b - a_k$, or equivalently, if and only if A is a minimal cover of $K_{a',b'}$. □

We are now ready to prove the main theorem.

Theorem 2.2. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be positive integers. Then the following assertions are equivalent*:

(i) *for each positive integer b, the clutter $K_{a,b}$ has no minor isomorphic to J_q for any $q \geq 2$*;

(ii) *the sequence*: $a_n, a_{n-1}, \ldots, a_2, a_1$ *is weakly superincreasing*;

(iii) *for each positive integer b, the clutter $K_{a,b}$ has the weak MFMC property*;

(iv) *for each positive integer b, the clutter $K_{a,b}$ is Mengerian.*

Proof. The equivalence of (i), (iii) and (iv) follows from Remark 1.2.

(i) ⇒ (ii). Assume that the sequence $a_n, \ldots, a_1$ is not weakly superincreasing. Let p be the first index for which this property fails, i.e.

$$a_n + a_{n-1} + \cdots + a_{p+1} > a_p, \tag{2.2}$$

$$a_n + a_{n-1} + \cdots + a_{l+1} \leq a_l, \quad l = n-1, \ldots, p-1. \tag{2.3}$$

Let the set $\{k_1, \ldots, k_q\}$, with $p+1 \leq k_1 < \cdots < k_q \leq n$, be a minimal cover for the inequality: $\sum_{j=p+1}^{n} a_j x_j \leq a_p$. Consider the knapsack inequality:

$$a_p x_p + \sum_{j=1}^{q} a_{k_j} x_{k_j} \leq a_p. \tag{2.4}$$

The clutter of minimal covers of (2.4) is precisely J_q and it is a minor of K_{a,a_p}, namely it is the minor $K_{a,a_p} \backslash (N - \{p, k_1, \ldots, k_q\})$.

(ii) ⇒ (i). Assume that the sequence $a_n, \ldots, a_1$ is weakly superincreasing and that, for some positive integer b, the clutter $K_{a,b}$ has a minor K' isomorphic to J_q for some $q \geq 2$. Then, there exist disjoint subsets C and D of N such that $K' = K_{a,b} \backslash D/C$. Set $N' = N - C \cup D = \{i_0, i_1, \ldots, i_q\}$ and suppose that $K' = \{\{i_1, \ldots, i_q\}, \{i_0, i_1\}, \ldots, \{i_0, i_q\}\}$. By Fact 2.1, we have that K' is the clutter of minimal covers of the knapsack:

$$\sum_{j \in N'} a_j x_j \leq b' = b - \sum_{k \in C} a_k.$$

Hence, we have the following relations:

$$a_{i_1} + \cdots + a_{i_q} > b', \qquad a_{i_0} \leq b'.$$

The subsequence $a_{i_q}, \ldots, a_{i_1}, a_{i_0}$ is also weakly superincreasing, implying that $a_{i_q} + \cdots + a_{i_1} \leq a_{i_0}$, in contradiction with the above relations. □

Let $M_{a,b}$ be the matrix whose rows are the incidence vectors of the minimal covers in $K_{a,b}$. The above theorem asserts that, if $a_1, \ldots, a_n$ is a weakly superincreasing sequence, then the polyhedron $P = \{x \in \mathbb{R}_+^n : M_{a,b} x \geq 1_n\}$ has only integral vertices. The following proposition establishes a relationship between the polyhedron P and the knapsack polytope $P_{a,b}$.

Proposition 2.3. *Given positive integers* $a_1, \dots, a_n$, b, *the following assertions are equivalent*:

(i) $K_{a,b}$ *has the weak* MFMC *property*.

(ii) *The polyhedron* $P = \{x \in \mathbb{R}^n_+ : M_{a,b}x \geq 1_n\}$ *has only integral vertices*.

(iii) *The polytope* $Q = \{x \in \mathbb{R}^n_+ : M_{a,b}x \geq 1_n,\ x \geq 1_n\}$ *has only integral vertices*.

(iv) $P_{a,b}$ *has only trivial and minimal cover facets, i.e.*

$$P_{a,b} = \left\{x \in \mathbb{R}^n : \sum_{i \in C} x_i \leq |C| - 1,\ C \in K_{a,b}\right\} \cap \{x \in \mathbb{R}^n : 0_n \leq x \leq 1_n\}.$$

Proof. (i) $\Leftrightarrow$ (ii). (ii) $\Rightarrow$ (i) follows from the definition. To prove that (i) $\Rightarrow$ (ii) observe that all the vertices of the polyhedron P are integral if the problem $\min\{wx : M_{a,b}x \geq 1,\ x \geq 0\}$ has an integral optimizing vector for every integral vector w for which the minimum is finite. Observe also that the latter problem is unbounded if and only if the vector w has some negative component. It follows that all the vertices of P are integral if the problem $\min\{wx : M_{a,b}x \geq 1,\ x \geq 0\}$ has an integral optimizing vector for every integral and *nonnegative* vector w. But the latter condition is exactly the weak MFMC property.

(iii) $\Leftrightarrow$ (iv). Obvious, using transformation $y = 1_n - x$.

(ii) $\Rightarrow$ (iii). Set $P_I = \text{conv}(x \in \mathbf{Z}^n_+ : M_{a,b}x \geq 1_n\}$ and $Q_I = \text{conv}(x \in \{0, 1\}^n : M_{a,b}x \geq 1_n)$. Condition (ii) implies that $P_I = P$; to prove (iii) we have to show that $Q_I = Q = P \cap \{x \in \mathbb{R}^n : 0_n \leq x \leq 1_n\}$. This can be done by showing that every non trivial inequality which induces a facet of Q_I also induces a facet of P_I.

For this purpose, it is enough to check that any facet-inducing inequality $dx \geq 1$ of Q_I, with $d \geq 0$, is valid for P_I. Take any vector x in $\mathbf{Z}^n_+$ which satisfies $M_{a,b}x \geq 1$, set $x' \in \{0, 1\}^n$ with $x'_i = x_i$ if $x_i \leq 1$ and $x'_i = 1$ if $x_i > 1$. Then, $x' \in P_I$, and so $dx \geq dx' \geq 1$. Therefore, $dx \geq 1$ has the form $\sum_{i \in S} x_i \geq 1$ for $S \in K_{a,b}$.

(iii) $\Rightarrow$ (ii). Follows from the fact that every vertex of P is a vertex of Q. □

Let $\mathcal{I}_{a,b}$ be the family of feasible solutions of the knapsack problem $ax \leq b$; i.e. the family of subsets $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} a_i x_i \leq b$. The family $\mathcal{I}_{a,b}$ defines an independence system and its members are said to be independent (see [8]). The maximal feasible solutions of $ax \leq b$ are the *bases* of $\mathcal{I}_{a,b}$; the *circuits* of $\mathcal{I}_{a,b}$ are the minimal covers associated with $ax \leq b$ (minimal non feasible solutions). Observe that the circuits of $\mathcal{I}_{a,b}$ are the members of $K_{a,b}$ and that the bases of $\mathcal{I}_{a,b}$ are the complements of the members of $b(K_{a,b})$. It follows that a characterization of circuits and bases of $\mathcal{I}_{a,b}$ immediately implies a characterization of $K_{a,b}$ and of its blocker.

The following two theorems characterize both the bases and the circuits of $\mathcal{I}_{a,b}$. We denote by $[a, b]$ the set of all integers between a and b (including the extrema).

Theorem 2.4. *Let* $a_1, \dots, a_n$ *be a weakly superincreasing sequence and let* $a_1 \leq b < \sum_{i=1}^n a_i$. *There exist integers* $k_1, k_2, \dots, k_q$ *with* $q \geq 2$ *and* $1 = k_1 < k_2 < \cdots < k_q \leq n + 1$, *such that the bases of* $\mathcal{I}_{a,b}$ *are the following* q *sets*:

$$B_i = \{k_1, k_2, \dots, k_{i-1}\} \cup [k_i + 1, n] \quad \text{for } 1 \leq i \leq q - 1,$$

and $B_q = \{k_1, k_2, \dots, k_{q-1}\} \cup [k_q, n]$. *Furthermore, the integers* $k_1, \dots, k_q$ *are defined by the following relation*:

$$k_i = \min\left\{h > k_{i-1} : \sum_{j=1}^{i-1} a_{k_j} + a_h \leq b\right\} \quad \text{for } 1 \leq i \leq q, \tag{2.5}$$

with the convention that $k_q = n + 1$ *when the minimum in* (2.5) *does not exist*.

Proof. We construct recursively the integers $k_1, \dots, k_p$ and the bases $B_1, \dots, B_p$ of $\mathcal{I}_{a,b}$. The number q will then denote the number of steps after which the process stops.

Step 1. The set $B_1 = [2, n]$ is a base since $a_2 + \cdots + a_n \leq a_1 \leq b$ and $a_1 + \cdots + a_n > b$. We now examine the bases of $\mathcal{I}_{a,b}$ containing the element 1.

Step 2. Set $k_2 = \min\{i > 1 : a_1 + a_i \leq b\}$ if the minimum exists, else set $k_2 = n + 1$. If $k_2 = n + 1$ then the set $B_2 = \{1\}$ is a base, and $\mathcal{I}_{a,b}$ has exactly two bases: namely B_1 and B_2, so the theorem holds with $q = 2$.

If $k_2 \leq n$ then we deduce, from the definition of k_2, that $a_1 + a_{k_2} \leq b$; moreover, if $k_2 \leq n - 1$, the hypothesis that $a_n, \dots, a_1$ is weakly superincreasing implies that $a_1 + a_{k_2+1} + \cdots + a_n \le a_1 + a_{k_2} \le b$. As a consequence, the set $\{1\} \cup [k_2 + 1, n]$ is independent. We have two cases:

Either $a_1 + a_{k_2} + \cdots + a_n \le b$, i.e. the set $B_2 = \{1\} \cup [k_2, n]$ is a base, then $ax \le b$ has two bases: namely B_1 and B_2 and the theorem holds with $q = 2$.

Or $a_1 + a_{k_2} + \cdots + a_n > b$, i.e., the set $B_2 = \{1\} \cup [k_2 + 1, n]$ is a base and one must examine the maximal feasible solutions which contain both elements 1 and k_2.

We iterate the process and it eventually stops after q steps. Then we have constructed q bases $B_1, \dots, B_q$ with $B_p = \{1, k_2, \dots, k_{p-1}\} \cup [k_p + 1, n]$ for $1 \le p \le q - 1$ (else the process would stop earlier) and $B_q = \{1, k_2, \dots, k_{q-1}\} \cup [k_q, n]$ (else the process would have one more step) with the integers $k_1, \dots, k_q$ being given by (2.5). □

Let $k_1, \dots, k_q$ be the integers given by (2.5); define the sets $A_u = [k_u + 1, k_{u+1} - 1]$ for $u \in \{1, \dots, q - 1\}$.

Theorem 2.5. *The circuits of the independence system* $\mathcal{I}_{a,b}$ *are the sets*

$$S_{u,i} = \{k_1, \dots, k_u, i\} \quad \textit{for } i \in A_u,\ 1 \le u \le q - 1.$$

Proof. Every set $S = S_{u,i}$ is indeed a circuit of $\mathcal{I}_{a,b}$ since, by definition of k_{u+1}, S is not independent, the set $S - \{i\}$ is contained in the base B_q and the set $S - \{k_t\}$ is contained in the base B_t for $1 \le t \le u$.

Conversely, let S be a circuit of $\mathcal{I}_{a,b}$; hence $1 \in S$. Let p be the first index in $\{1, \dots, q\}$ such that $k_{p+1} \notin S$, i.e., for $p = q$, $k_1, \dots, k_q \in S$. Then, $S \cap A_u = \emptyset$ for $u = 1, 2, \dots, p - 1$; else if $i \in S \cap A_u$, then S would properly contain the circuit $S_{u,i}$. It follows that $p \le q - 1$; else S would be contained in the base B_q.

Furthermore, we must have that $S \cap A_p \ne \emptyset$; else S would be contained in the independent set $\{k_1, \dots, k_p\} \cup [k_{p+1} + 1, n]$. If $|S \cap A_p| \ge 2$, then S would properly contain a circuit $S_{p,i}$ for $i \in S \cap A_p$. Therefore, $S \cap A_p = \{i\}$ which implies that $S \supseteq S_{p,i}$ and thus $S = S_{p,i}$. □

A corollary of Theorems 2.2 and 2.5 is the following:

Corollary 2.6. *Let $a_1, \dots, a_n$ be a weakly superincreasing sequence and let $a_1 \le b < \sum_{i=1}^n a_i$. The non trivial facets of the knapsack polytope $P_{a,b}$ are defined by the following inequalities:*

$$x_{k_1} + \cdots + k_{k_u} + x_i \le u \quad \textit{for } i \in A_u,\ 1 \le u \le q - 1. \tag{2.6}$$

As a consequence of Corollary 2.6 one can easily deduce, by using the characterization of the facets of $P_{a,b}$ with coefficients 0 and 1 and right hand side $k \ge 1$ given in [1], that every inequality (2.6) induces a facet of the polytope $P_{a,b}$.

Remark 2.7. Proposition 2.6 implies that the optimal solution to the knapsack problem $\max\{cx : ax \le b,\ x \in \{0, 1\}^n\}$ can be found in polynomial time by solving a linear programming problem with at most n constraints. In fact, once the integers $k_1, \dots, k_q$ have been computed, the problem is solved by the trivial (linear) algorithm which consists of computing the total weight of each base and choosing the largest value.

Acknowledgments

The authors thank the Associate Editor and the anonymous referee for several suggestions which have greatly improved the paper.

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