Graphs and Combinatorics 8, 125–142 (1992)

Graphs and Combinatorics

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The Cut Cone III: On the Role of Triangle Facets

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Abstract. The cut polytope P_n is the convex hull of the incidence vectors of the cuts (i.e. complete bipartite subgraphs) of the complete graph on n nodes. A well known class of facets of P_n arises from the triangle inequalities: $x_{ij} + x_{ik} + x_{jk} \leq 2$ and $x_{ij} - x_{ik} - x_{jk} \leq 0$ for $1 \leq i, j, k \leq n$. Hence, the metric polytope M_n, defined as the solution set of the triangle inequalities, is a relaxation of P_n. We consider several properties of geometric type for P_n, in particular, concerning its position within M_n. Strengthening the known fact ([3]) that P_n has diameter 1, we show that any set of k cuts, $k \leq \log_2 n$, satisfying some additional assumption, determines a simplicial face of M_n and thus, also, of P_n. In particular, the collection of low dimension faces of P_n is contained in that of M_n. Among a large subclass of the facets of P_n, the triangle facets are the closest ones to the barycentrum of P_n and we conjecture that this result holds in general. The lattice generated by all even cuts (corresponding to bipartitions of the nodes into sets of even cardinality) is characterized and some additional questions on the links between general facets of P_n and its triangle facets are mentioned.

1. Introduction

In this paper, we prove several results of geometric type on the cut polytope P_n of the complete graph on n nodes. They are motivated by the study of the geometric shape of P_n, in particular, the position of the facets of P_n with respect to its barycentrum, the contribution of the important subclass of the triangle facets to the global shape of P_n, and also the study of some lattices generated by families of cuts.

We set $[1,n] = \{1,2,\ldots,n\}$. Given a subset S of $[1,n]$, the cut determined by S is the set $\delta(S)$ of all pairs (i,j) of distinct points of $[1,n]$ such that exactly one of i and j belongs to the set S. The incidence vector of the cut $\delta(S)$, also called its cut vector, is the vector $X^{\delta(S)}$ of $R^{n(n-1)/2}$ defined by $X_{ij}^{\delta(S)} = 1$ if $(i,j) \in \delta(S)$ and $X_{ij}^{\delta(S)} = 0$ otherwise, for $1 \leq i < j \leq n$. The *cut polytope* P_n is the convex hull of the incidence vectors of the cuts $\delta(S)$ for all subsets S of $[1,n]$; it is a full dimensional polytope in $R^{n(n-1)/2}$. Given $v \in R^{n(n-1)/2}$ and $v_0 \in R$, the inequality $v.x \leq v_0$ is said to be valid for P_n if it is satisfied by all cut vectors and, then, to be facet inducing if there exist $n(n-1)/2$ affinely independent cut vectors satisfying the equality $v.x = v_0$. A well known class of facets of P_n arises from the following triangle inequalities:

(1.1) $x_{ij} - x_{ik} - x_{jk} \leq 0$ for $1 \leq i, j, k \leq n$

as well as the inequalities:

(1.2) $x_{ij} + x_{ik} + x_{jk} \leq 2$ for $1 \leq i, j, k \leq n$

There are $3\binom{n}{3}$ facets of type (1.1) (homogeneous triangle facets) and $\binom{n}{3}$ facets of type (1.2) and so $4\binom{n}{3}$ triangle facets in total. Each triangle facet contains 3.2^{n-3} cut vectors, i.e. 3/4th of the total number of vertices of P_n. Although P_n has surely a lot of quite complicated (and still yet undiscovered for the greatest majority) facets, its most simple ones, the triangle facets, seem to gather already quite a lot of the properties enjoyed by P_n.

Let M_n denote the polytope in $R^{n(n-1)/2}$ defined as the solution set of the $4\binom{n}{3}$ inequalities (1.1) and (1.2), M_n is called the *metric polytope*. So M_n contains the polytope P_n and M_n is contained in the cube $[0, 1]^{n(n-1)/2}$. The cut vectors are also vertices of M_n; in fact, they are the integral vertices of M_n. The problematics of describing vertices of M_n and facets of P_n are in some sense "dual". Namely, while the vertices of P_n are easy (they are the cut vectors), it is probably very hard to find explicitly all its facets; on the other hand, the facets of M_n are easy (they are the triangle inequalities) while it is also probably very hard to find all vertices of M_n. We refer e.g. to [3], [5], [7], [8] for information on the facets of P_n and to [2], [11] for information on the vertices of M_n. Actually, [2] and [11] study the extreme rays of the metric cone $MC_n = \{x \in R^{n(n-1)/2}: x_{ij} - x_{ik} - x_{jk} \leq 0 \text{ for all } 1 \leq i, j, k \leq n\}$. But, one sees easily that, if d defines an extreme ray of MC_n and if $\alpha = \min(2/(d_{ij} + d_{ik} + d_{jk})$: $d_{ij} + d_{ik} + d_{jk} \neq 0$ for $1 \leq i, j, k \leq n)$, then αd is a vertex of the metric polytope M_n.

Since the metric polytope M_n contains the cut polytope P_n, it is natural to ask how well M_n approximates P_n, i.e. how well the triangle facets wrap P_n. In section 2, we give some elements of answer toward this question. Barahona and Mahjoub ([3]) proved that P_n has diameter one, i.e. that any two cut vectors are adjacent on P_n. It follows from a result of Padberg that any two cut vectors are also adjacent on M_n (see Remark 2.11). Therefore, the 1-skeleton of P_n (its collection of vertices and edges) is contained in the 1-skeleton of M_n; in other words, M_n has the Trubin property (see [19]) with respect to P_n. So, for $d = 0, 1$, all d-faces (faces of dimension d) of P_n are also faces of M_n; this property holds for some higher dimension faces. Namely, we show that any three cut vectors determine a simplicial face of M_n and, thus, also of P_n and, therefore, all 2-faces of P_n are faces of M_n. Generally, we prove that any k cut vectors, $k \leq \log_2 n$, which are in general position (see section 2 for the definition) determine a simplicial face of M_n and, thus, also, of P_n. We conjecture that, for $k \leq \log_2 n$, all k-faces of P_n are also faces of M_n. We show that the minimum integer k for which there exist k cuts that do not lie on any triangle facet is in $O(\log_2 n)$. This indicates that $\log_2 n$ might be the limit value for validity of our conjecture.

Several other geometrical facts are known on the cut polytope P_n, for instance, that it enjoys a lot of symmetries (see the precise description of its symmetry group below), also its circumscribed sphere, since it is immediate to check that all cut vectors lie on the sphere of center $b = (1/2, \ldots, 1/2)$, the barycentrum of P_n, and radius $\sqrt{r}/2$ with $r = n(n-1)/2$. However, the geometrical shape of P_n is not yet fully understood. For example, it is not quite excluded that P_n might become more and more "flat" for large n. This question is considered in section 3; unfortunately, we cannot completely settle it. However, we show that any facet of P_n having only 0, 1, -1 coefficients (in the left hand side of its defining inequality) has distance at least $(2\sqrt{3})^{-1}$ from the barycentrum b of P_n, this smallest distance being attained precisely by the triangle facets. We conjecture that this property holds generally for all facets of P_n, i.e. that the triangle facets are the closest ones to the barycentrum and so the inscribed sphere to P_n has radius greater or equal to $(2\sqrt{3})^{-1}$.

It is known that the integer points x belonging to the lattice generated by the cut vectors are characterized by the fact that their perimeter on any triangle must be even, i.e. $x_{ij} + x_{ik} + x_{jk}$ is even for all $1 \le i < j < k \le n$ ([1]). Here, in section 4, we characterize a sublattice of it, namely the lattice of all even cuts, i.e. all cuts $\delta(S)$ with both sets S and $[1,n]-S$ of even cardinality. Subfamilies of cuts obtained by introducing some parity conditions are well studied and classical objects in Combinatorial Optimization (see e.g. [13]).

We state in section 5 several questions concerning the links between arbitrary facets of P_n and its triangle facets, in particular, whether any facet of P_n can be decomposed as linear combination of triangles, also whether any facet collapses to some triangle inequality? Both these properties can be observed on the classes of facets of P_n known so far. Finally, we show in section 6 how the structure of the 3-hypercut polytope $HP(3)_n$ can be derived from that of the cut polytope P_n. Given a subset S of $[1,n]$, the 3-hypercut $\delta_3(S)$ is the set of triples (i,j,k) of distinct points of $[1,n]$ that intersect both S and its complement $[1,n]$-S and the polytope $HP(3)_n$ is the convex hull in $R^{n(n-1)(n-2)/6}$ of the incidence vectors of the 3-hypercuts. So, 3-hypercuts are a direct generalization of cuts (i.e. 2-hypercuts). In fact, $HP(3)_n$ is a linear bijective image of P_n.

We conclude the introduction by recalling the description of the symmetries of the cut polytope P_n. Given a cut $\delta(S)$, set $r_{\delta(S)} = \prod_{(i,j)\in\delta(S)} r_{ij}$ where r_{ij} denotes the reflection around the hyperplane $x_{ij} = 1/2$ for $1 \le i < j \le n$. Hence, $y = r_{\delta(S)}(x)$ is defined by $y_{ij} = 1 - x_{ij}$ if $(i,j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise; $r_{\delta(S)}$ is an affine map and, if we denote by $R_{\delta(S)}$ its linear part, then $r_{\delta(S)}(x) = R_{\delta(S)}(x) + X^{\delta(S)}$. For $v \in R^{n(n-1)/2}$, let v^S denote the vector of $R^{n(n-1)/2}$ defined by $v^S_{ij} = -v_{ij}$ if $(i,j) \in \delta(S)$ and $v^S_{ij} = v_{ij}$ otherwise for $1 \le i < j \le n$. If the inequality $v.x \le v_0$ is valid for P_n and defines the face F of P_n, then the inequality $v^S.x \le v_0 - v.\delta(S)$ is also valid for P_n ([3]) and, in fact, defines the face $r_{\delta(S)}(F)$ of P_n ([6]). Any permutation σ of $[1,n]$ clearly induces an isometry of $R^{n(n-1)/2}$ and, in fact, a symmetry of P_n. For $n \ne 4$, the only symmetries of P_n are the reflections $r_{\delta(S)}$ for S subset of $[1,n]$ and the permutations of $[1,n]$; in fact, the symmetry group of P_n coincides then with the central quotient of the symmetry group of the n-dimensional cube ([6]).

2. How Well Do the Triangle Facets Wrap the Cut Polytope?

The *metric polytope* M_n is the set of vectors satisfying all triangle inequalities (1.1) and (1.2), i.e. $M_n = \{x \in R^{n(n-1)/2}\colon x_{ij} - x_{ik} - x_{jk} \le 0,\ x_{ij} + x_{ik} + x_{jk} \le 2$ for $1 \le i, j,$

$k \le n\}$. Therefore, $P_n \subseteq M_n \subseteq [0,1]^{n(n-1)/2}$. We are interested in how tight this relaxation of P_n by M_n is. In fact, for $n = 3, 4$, both polytopes coincide but, for $n \ge 5$, the inclusion $P_n \subseteq M_n$ is strict. We show that some properties of P_n, in particular, concerning the structure of its low dimension faces, are retained by M_n. We shall use the following criterion for characterizing faces. Given some cuts $\delta(S_1), \ldots, \delta(S_k)$, they determine a face of M_n, namely the face $F = \{\sum_{1 \le i \le k} \alpha_i X^{\delta(Si)}: \alpha_i \ge 0$ and $\sum_{1 \le i \le k} \alpha_i = 1\}$, if one can find a vector w in $R^{n(n-1)/2}$ such that $\mathrm{Max}(w.x: x \in M_n)$ is attained precisely at the points $x \in F$. Clearly, if F is a face of M_n, then, F is also a face of P_n. The dimension of a face F is the largest number of affinely independent points in F minus one.

A first useful observation is that all the symmetries of P_n are also symmetries of M_n. Indeed, any permutation of $[1, n]$ trivially preserves M_n and the following lemma can be easily checked.

Lemma 2.1. *For any subset S of $[1, n]$, the reflection $r_{\delta(S)}$ preserves M_n.* □

Corollary 2.2. *Let $\delta(S_1), \ldots, \delta(S_k)$ be k distinct non empty cuts. Then, the set $F = \mathrm{Conv}(X^{\delta(S_i)}: 1 \le i \le k)$ is a face of M_n (resp. P_n) if and only if the set $F' = \mathrm{Conv}(X^{\delta(\varnothing)}, X^{\delta(S_i \triangle S_k)}: 1 \le i \le k-1)$ is a face of M_n (resp. P_n).*

Proof. It suffices to prove that, if F is a face, then F' too is a face. We do the proof e.g. for the case of the polytope M_n, the proof being identical for the case of P_n. Since F is a face of M_n, there exists a vector w such that $w_0 := \mathrm{Max}(w.x: x \in M_n)$ is attained precisely at the points $x \in F$. Define the vector w' by $w'_{ij} = -w_{ij}$ if $(i,j) \in \delta(S_k)$ and $w'_{ij} = w_{ij}$ otherwise. For $x \in M_n$, if $y = r_{\delta(S_k)}(x)$, then $w'.x = w.y - w.X^{\delta(S_k)} \le w_0 - w.X^{\delta(S_k)}$, since, from Lemma 2.1, $y \in M_n$. Moreover, equality holds if and only if $w.y = w_0$, i.e. $y \in F$, i.e. $y = \sum_{1 \le i \le k} \alpha_i X^{\delta(S_i)}$ for some $\alpha_i \ge 0$ with $\sum_{1 \le i \le k} \alpha_i = 1$, or equivalently, $x = r_{\delta(S_k)}(y) = \sum_{1 \le i \le k-1} \alpha_i X^{\delta(S_i \triangle S_k)}$, that is, $x \in F'$. This shows that F' is a face of M_n. □

Lemma 2.3. *Any set of four distinct cut vectors is affinely independent.*

Proof. (i) Any two non zero cut vectors $X^{\delta(S)}$, $X^{\delta(T)}$ are linearly independent. Indeed, if $\alpha X^{\delta(S)} + \beta X^{\delta(T)} = 0$, then, computing the value of the left hand side at coordinate $(i,j) \in \delta(S) - \delta(T)$ yields $\alpha = 0$ and thus $\beta = 0$ too.

(ii) Any three non zero cut vectors are linearly independent. Indeed, assume that $v := \alpha X^{\delta(S)} + \beta X^{\delta(T)} + \gamma X^{\delta(U)} = 0$. If $\delta(S) \nsubseteq \delta(T) \cup \delta(U)$, computing the value of v at coordinate $(i,j) \in \delta(S) - (\delta(T) \cup \delta(U))$ yields that $\alpha = 0$ and thus we deduce $\beta = \gamma = 0$ from case (i) above. So we can suppose that $\delta(S) \subseteq \delta(T) \cup \delta(U)$, $\delta(T) \subseteq \delta(S) \cup \delta(U)$ and $\delta(U) \subseteq \delta(S) \cup \delta(T)$. Take (i,j) in $\delta(S) - \delta(T)$, so $(i,j) \in \delta(U)$; by computing v_{ij}, we deduce that $\alpha + \gamma = 0$. Similarly, we obtain that $\alpha + \beta = 0$ and $\beta + \gamma = 0$, implying that $\alpha = \beta = \gamma = 0$.

(iii) Take now four distinct cuts $\delta(S)$, $\delta(T)$, $\delta(U)$ and $\delta(V)$ and scalars α, β, γ, λ such that $\alpha + \beta + \gamma + \lambda = 0$ and $\alpha X^{\delta(S)} + \beta X^{\delta(T)} + \gamma X^{\delta(U)} + \lambda X^{\delta(V)} = 0$. If e.g. $\delta(V) = \varnothing$, then we can conclude by applying case (ii). Otherwise, by applying the reflection $r_{\delta(V)}$ to the above relation, we obtain that $\alpha X^{\delta(S \triangle V)} + \beta X^{\delta(T \triangle V)} + \gamma X^{\delta(U \triangle V)} = 0$, which, using again (ii), yields that $\alpha = \beta = \gamma = 0$. □

A polytope P is said to be *k-neighborly* (see [12]) if, for any subset X of k vertices, the set $F = \mathrm{Conv}(X)$ is a face of P with vertex set X, i.e. X determines a simplicial face of P. Let $\phi_d(P)$ denote the set of faces of dimension d of P.

Theorem 2.4. *Any set of d distinct cut vectors determines a simplicial face of M_n and, thus, also of P_n, for $1 \le d \le 3$.*

Corollary 2.5. *P_n is 3-neighborly.* □

Corollary 2.6. *$\phi_d(P_n) \subseteq \phi_d(M_n)$ for $0 \le d \le 2$.* □

Proof of Theorem 2.4. In view of Corollary 2.2, it is enough to prove the result for a set of cuts containing the empty cut. The case $d = 2$ follows from [14], but it can be checked directly as follows. Given a non empty cut $\delta(S)$, define the vector w by $w_{ij} = 0$ if $(i, j) \in \delta(S)$ and $w_{ij} = -1$ otherwise. Then, for $x \in M_n$, $w \,.\, x \le 0$ with equality if and only if $x_{ij} = 0$ for $(i, j) \notin \delta(S)$. Since x satisfies (1.1), then, for any $i, j \in S$ and $h \notin S$, $x_{ih} \le x_{ij} + x_{jh} = x_{jh}$ and $x_{jh} \le x_{ih} + x_{ij} = x_{ih}$ and thus $x_{ih} = x_{jh}$. Therefore, $x_{ij} = \alpha$ for all $(i, j) \in \delta(S)$, for some $0 \le \alpha \le 1$, i.e. $x = \alpha X^{\delta(S)}$. Hence, $\mathrm{Conv}(0, X^{\delta(S)})$ is a face of M_n.

We now turn to the case $d = 3$. We prove the result for the three cuts $\delta(S)$, $\delta(T)$ and $\delta(\varnothing)$. Set $A = S \cap T$, $B = ([1, n] - S) \cap T$, $C = S \cap ([1, n] - T)$ and $D = ([1, n] - S) \cap ([1, n] - T)$. We suppose first that the four sets A, B, C, D are non empty. Take some points $a \in A$, $b \in B$, $c \in C$ and $d \in D$. We define the vector w by $w_{ab} = w_{ac} = w_{bd} = w_{cd} = -1$, $w_{ad} = w_{bc} = 1$, $w_{ij} = -1$ if $(i, j) \in E := A^2 \cup B^2 \cup C^2 \cup D^2$ and $w_{ij} = 0$ otherwise. Thus, $w \,.\, X^{\delta(S)} = w \,.\, X^{\delta(T)} = 0$. Take $x \in M_n$; then, $w \,.\, x = -\sum_{(i,j) \in E} x_{ij} + \sigma_2$ where $\sigma_2 = x_{ad} + x_{bc} - x_{ab} - x_{ac} - x_{bd} - x_{cd}$ verifies the following relations.

(i) $\sigma_2 = (x_{ad} - x_{ac} - x_{cd}) + (x_{bc} - x_{cd} - x_{bd}) + x_{cd} - x_{ab} \le x_{cd} - x_{ab}$
(ii) $\sigma_2 = (x_{ad} - x_{ab} - x_{bd}) + (x_{bc} - x_{ab} - x_{ac}) + x_{ab} - x_{cd} \le x_{ab} - x_{cd}$
(iii) $\sigma_2 = (x_{ad} - x_{ac} - x_{cd}) + (x_{bc} - x_{ab} - x_{ac}) + x_{ac} - x_{bd} \le x_{ac} - x_{bd}$
(iv) $\sigma_2 = (x_{ad} - x_{ab} - x_{bd}) + (x_{bc} - x_{bd} - x_{cd}) + x_{bd} - x_{ac} \le x_{bd} - x_{ac}$

From (i)–(iv), we deduce that $\sigma_2 \le 0$ and thus $w \,.\, x \le 0$. Moreover, if $w \,.\, x = 0$, then $\sum_{(i,j) \in E} x_{ij} = 0$ and $\sigma_2 = 0$. Since $\sigma_2 = 0$, we deduce from (i)–(iv) that $x_{ab} = x_{cd} := \alpha$, $x_{ac} = x_{bd} := \beta$, $x_{ad} = x_{bc} = \alpha + \beta$. Since $\sum_{(i,j) \in E} x_{ij} = 0$, we have that $x_{ij} = 0$ for all $(i, j) \in E$. Next, using again the inequalities (1.1), we obtain that $x_{ij} = \alpha$ for all $(i, j) \in A \times B \cup C \times D$, $x_{ij} = \beta$ for all $(i, j) \in A \times C \cup B \times D$, $x_{ij} = \alpha + \beta$ for all $(i, j) \in A \times D \cup B \times C$. Hence, $x = \alpha X^{\delta(S)} + \beta X^{\delta(T)}$ holds with $0 \le \alpha, \beta$ and $\alpha + \beta \le 1$. We suppose now that some of the sets A, B, C, D is empty. Since $\delta(S)$, $\delta(T)$ are distinct non empty cuts, at most one of the sets A, B, C, D can be empty. Suppose, for instance, that D is empty. Then, w is defined by $w_{ab} = w_{ac} = -1$, $w_{ij} = -1$ for $(i, j) \in A^2 \cup B^2 \cup C^2$, $w_{bc} = 1$ and $w_{ij} = 0$ otherwise. The proof is then identical (but simpler). □

We conjecture that Corollary 2.6 can be generalized to low dimension faces.

Conjecture 2.7. *For $d < \log_2(n)$, $\phi_d(M_n) \subseteq \phi_d(P_n)$.*

Given k cuts $\delta(S_1), \ldots, \delta(S_k)$, we say that they are in *general position* if each of the 2^k intersection classes $C(A) := (\bigcap_{i \in A} S_i) \cap (\bigcap_{i \notin A} ([1, n] - S_i))$ is non empty for any subset A of $[1, k]$. Then, $k \leq \log_2 n$ and it is easy to see that the associated cut vectors are linearly independent. Note that, if the cuts $\delta(S_1), \ldots, \delta(S_k)$ are in general position, then the cuts $\delta(S_1 \triangle S_k), \ldots, \delta(S_{k-1} \triangle S_k)$ are also in general position. The next Theorem 2.8 is a partial contribution to Conjecture 2.7. In view of the preceeding remark and of Corollary 2.2, Theorem 2.8 implies that any k cuts in general position together with the zero cut also determine a simplicial face of M_n.

Theorem 2.8. *Let $\delta(S_1), \ldots, \delta(S_k)$ be k distinct cuts which are in general position. Then, they determine a simplicial face of M_n and, thus, also of P_n.*

In order to prove Theorem 2.8, we introduce some notation. Given an integer $k \leq \log_2 n$, let X be a set of 2^k distinct points of $[1, n]$. Hence, the elements of X can be indexed by the subsets of $[1, k]$, i.e. we can write $X = \{i(A): A \subseteq [1, k]\}$. If $x \in R^{n(n-1)/2}$, for the sake of simplicity in the notation, we write $x_{A,B}$ for denoting $x_{i(A)i(B)}$ for A, B subsets of $[1, k]$. We set $\sigma_k(x) := \sum_{|A \triangle B| = k} x_{A,B} - \sum_{|A \triangle B| = 1} x_{A,B}$. Hence, $\sigma_k(x)$ can be seen as the sum of the components of x along the main diagonals of the k-dimensional cube minus the sum of the components of x along the edges of the cube.

Lemma 2.9. *With the above notation, if $x \in M_n$, then $\sigma_k(x) \leq 0$ and $\sigma_k(x) = 0$ if and only if there exist some scalars $\alpha_1, \ldots, \alpha_k$ such that $\alpha_i \geq 0$ for $1 \leq i \leq k$, $\alpha_1 + \cdots + \alpha_k \leq 1$ and*

(2.10) $x_{A,B} = \sum_{i \in A \triangle B} \alpha_i$ *for all subsets A, B of $[1, k]$.*

Proof. First, it is easy to check that, if condition (2.10) holds, then $\sigma_k(x) = 0$ indeed holds. We now show that, for $x \in M_n$, $\sigma_k(x) \leq 0$ and, if equality holds, then one can find scalars $\alpha_1, \ldots, \alpha_k \geq 0$, $\alpha_1 + \cdots + \alpha_k \leq 1$, such that x satisfies (2.10); let us call (H_k) this property. We prove that property (H_k) holds by induction on $k \geq 2$. The proof in the case $k = 2$ is easy and, in fact, is already contained in the proof of Theorem 2.4 (case $d = 3$). We assume that (H_{k-1}) holds for $k \geq 3$ and we prove that (H_k) holds. The idea is to partition the set $X = \{i(A): A \subseteq [1, k]\}$ of size 2^k into the two sets $X' = \{i(A): A \subseteq [1, k]$ and $k \notin A\}$ and $X'' = \{i(A): A \subseteq [1, k]$ and $k \in A\}$, each of size 2^{k-1}; so this partition is done by distinguishing the point k. Correspondingly to the sets X', X'', we set:

$$\sigma'_{k-1}(x) = \sum_{k \notin A, B, |A \triangle B| = k-1} x_{A,B} - \sum_{k \notin A, B, |A \triangle B| = 1} x_{A,B}$$

and

$$\sigma''_{k-1}(x) = \sum_{k \in A, B, |A \triangle B| = k-1} x_{A,B} - \sum_{k \in A, B, |A \triangle B| = 1} x_{A,B}.$$

Then,

$$\sigma_k(x) = \sigma'_{k-1}(x) + \sigma''_{k-1}(x) + W_1(x) - W_2(x) - W_3(x) - W_4(x),$$

where W_1, W_2, W_3, W_4 are defined as follows.

$$W_1(x) = \sum_{k \notin A} x_{A,[1,k]-A} = \sum_{k \notin A} x_{A \cup \{k\},[1,k-1]-A},$$

$$W_2(x) = \sum_{k \notin A} x_{A,A \cup \{k\}},$$

$$W_3(x) = \sum_{k,k-1 \notin A} x_{A,[1,k-1]-A} = \sum_{k \notin A, k-1 \in A} x_{A,[1,k-1]-A}$$

and

$$W_4(x) = \sum_{k,k-1 \notin A} x_{A \cup \{k\},[1,k]-A} = \sum_{k \notin A, k-1 \in A} x_{A \cup \{k\},[1,k]-A}.$$

Then, one can check that $\sigma_k(x)$ can be written in the following two ways:

$$\sigma_k(x) = \sigma'_{k-1}(x) + \sigma''_{k-1}(x) + \sum_{k \notin A} (x_{A,[1,k]-A} - x_{A,A \cup \{k\}} - x_{A \cup \{k\},[1,k]-A})$$
$$+ \sum_{k \notin A, k-1 \in A} (x_{A \cup \{k\},[1,k]-A} - x_{A,[1,k-1]-A})$$

and

$$\sigma_k(x) = \sigma'_{k-1}(x) + \sigma''_{k-1}(x) + \sum_{k \notin A} (x_{A \cup \{k\},[1,k-1]-A} - x_{A,[1,k-1]-A} - x_{A,A \cup \{k\}})$$
$$+ \sum_{k \notin A, k-1 \in A} (x_{A,[1,k-1]-A} - x_{A \cup \{k\},[1,k]-A}).$$

Since $x \in M_n$, the first sum (being a sum of homogeneous triangles) in each of the above expressions of $\sigma_k(x)$ is non positive. The second sums in each of the above expressions are opposite quantities, hence, $\sigma_k(x) \leq 0$ indeed holds. Furthermore, if $\sigma_k(x) = 0$, then $\sigma'_{k-1}(x) = \sigma''_{k-1}(x) = 0$ and
(i) $x_{A,[1,k]-A} - x_{A,A \cup \{k\}} = x_{A \cup \{k\},[1,k]-A} = x_{A,[1,k-1]-A}$ for any $A \subseteq [1, k-1]$
From the induction assumption (H_{k-1}) applied to $\sigma'_{k-1}(x)$ and $\sigma''_{k-1}(x)$, we deduce, respectively, that there exist $k-1$ scalars $\alpha'_1(k), \ldots, \alpha'_{k-1}(k)$ such that $x_{A,B} = \sum_{i \in A \triangle B} \alpha'_i(k)$ for $A, B \subseteq [1, k-1]$, and there exist $k-1$ scalars $\alpha''_1(k), \ldots, \alpha''_{k-1}(k)$ such that $x_{A,B} = \sum_{i \in A \triangle B} \alpha''_i(k)$ for $A, B \subseteq [1,k]$ with $k \in A, B$. In particular, $x_{\varnothing,\{i\}} = \alpha'_i(k)$ for all $i \neq k$; also, $x_{[1,k],[1,k]-\{i\}} = \alpha''_i(k)$ for all $i \neq k$. In what preceeds, we have distinguished the point k of $[1,k]$, but any point h of $[1,k]$ could have been distinguished as well and, hence, we can define similarly the scalars $\alpha'_i(h)$ and $\alpha''_i(h)$ for any $i \neq h$ in $[1,k]$. In other words, we have $x_{\varnothing,\{i\}} = \alpha'_i(1) = \alpha'_i(2) = \cdots = \alpha'_i(i-1) = \alpha'_i(i+1) = \cdots = \alpha'_i(k) := \alpha'_i \geq 0$, for any $1 \leq i \leq k$. Also, $x_{[1,k],[1,k]-\{i\}} = \alpha''_i(1) = \cdots = \alpha''_i(i-1) = \alpha''_i(i+1) = \cdots = \alpha''_i(k) := \alpha''_i \geq 0$, for any $1 \leq i \leq k$. Using relation (i), we deduce that $\sum_{i \in [1,k-1]} \alpha'_i = \sum_{i \in [1,k-1]} \alpha''_i$, i.e. $\alpha'_k - \alpha''_k = \sum_{1 \leq i \leq k} (\alpha'_i - \alpha''_i)$; this relation remains valid for any index h instead of k, so, by summation, one obtains that $\sum_{1 \leq i \leq k} (\alpha'_i - \alpha''_i) = 0$, and, therefore, $\alpha'_i = \alpha''_i := \alpha_i$ for all $1 \leq i \leq k$. We conclude by checking that (2.10) holds, i.e. $x_{A,B} = \sum_{i \in A \triangle B} \alpha_i$ for any subsets A, B of $[1,k]$. Indeed, this follows from the induction assumption if there exists a point h in $A \cap B$ or a point h in $[1,k] - A \cup B$. Otherwise, $B = [1,k] - A$, e.g. $k \notin A$ and, using (i), we obtain that $x_{A,[1,k]-A} = x_{A,A \cup \{k\}} + \sum_{1 \leq i \leq k-1} \alpha_i = \sum_{1 \leq i \leq k} \alpha_i = \sum_{i \in A \triangle ([1,k]-A)} \alpha_i \leq 1$. Thus, we have proved that property (H_k) indeed holds. □

Proof of Theorem 2.8. Since the k cut $\delta(S_1), \ldots, \delta(S_k)$ are in general position, each intersection class $C(A) = (\bigcap_{i \in A} S_i) \cap (\bigcap_{i \notin A} ([1,k] - S_i))$ is non empty, for any subset A of $[1,k]$. We can choose a point $i(A)$ belonging to $C(A)$ and, thus, construct a subset $X = \{i(A): A \subseteq [1,k]\}$ of 2^k points of $[1,n]$. We now define a vector w as follows: $w_{ij} = -1$ if i, j belong to a common intersection class $C(A)$, $w_{ij} = -1$ if $(i,j) = (i(A), i(B))$ for some subsets A, B of $[1,k]$ with $|A \triangle B| = 1$, $w_{ij} = 1$ if $(i,j) = (i(A), i(B))$ for some subsets A, B of $[1,k]$ with $|A \triangle B| = k$, and $w_{ij} = 0$ otherwise. Then, for $x \in M_n$, using Lemma 2.9, $w.x = \sigma_k(x) - \sum_{A \subseteq [1,k]} \sum_{i<j, i,j \in C(A)} x_{ij} \leq 0$. Furthermore, if equality holds, then $x_{ij} = 0$ whenever i, j belong to the same intersection class and, thus, since x satisfies the triangle inequalities, $x_{ij} = x_{A,B}$ for all $i \in C(A)$, $j \in C(B)$. Also, from Lemma 2.9, there exist scalars $\alpha_1, \ldots, \alpha_k \geq 0$ with $\sum_{1 \leq i \leq k} \alpha_i \leq 1$ such that $x_{A,B} = \sum_{i \in A \triangle B} \alpha_i$ for A, B subsets of $[1,k]$, or, equivalently, $x = \sum_{1 \leq i \leq k} \alpha_i X^{\delta(S_i)}$. This shows that the cuts $\delta(S_1), \ldots, \delta(S_k)$ together with the zero cut determine a face of M_n. □

Remark 2.11. Let M'_n denote the solution set of all the triangle inequalities (1.1), (1.2) passing through a given fixed node, say node 1; then, $P_n \subseteq M_n \subseteq M'_n \subseteq [0,1]^{n(n-1)/2}$. Padberg ([14]) proved that any two cut vectors are also adjacent on the polytope M'_n. In fact, Padberg proved this result in the context of the boolean quadric polytope which is a linear bijective image of the cut polytope. Therefore, the 1-skeleton of P_n (its collection of vertices and of edges) is contained in the 1-skeleton of M'_n and thus, also, in the 1-skeleton of M_n. In other words, both M_n, M'_n have the Trubin property (see [19]) with respect to P_n.

Let us consider the following question. What is the minimum number $k = k(n)$ such that there exist k cuts that do not lie on any triangle facet? Clearly, $3 \leq k(n) \leq n-1$, because the n cuts $\delta(\{1\}), \ldots, \delta(\{n\})$ do not lie on any triangle facet. We can restrict our attention to homogeneous triangle facets, because, if $k_0(n)$ is the smallest integer such that there exist $k_0(n)$ cuts that do not lie on any homogeneous triangle facet, then $k_0(n) \leq k(n) \leq k_0(n) + 1$ holds clearly. The number $k_0(n)$ admits the following alternative interpretation. A family of cuts $\{\delta(S_1), \ldots, \delta(S_k)\}$ does not lie on any homogeneous triangle facet if and only if the family $\{(S_1, [1,n] - S_1), \ldots, (S_k, [1,n] - S_k)\}$ of 2-partitions of $[1,n]$ satisfies the property $(*)$ below.

$(*)$ for all distinct h, i, j in $[1,n]$, there exists a partition $(S_r, [1,n] - S_r)$ such that $h \in S_r$ and $i, j \in [1,n] - S_r$.

In fact, in these terms, the quantity $k_0(n)$ has been investigated in ([15], Proposition 2.6, Remark 2.8, where it is denoted by $M_0(n;3,2)$). It is shown there that, for n large, $k_0(n)$ is of the order of $\log_2 n$. Therefore, for n large, $k(n)$ is in $O(\log_2 n)$. This might be an indication that $\log_2 n$ is indeed the limit value for validity of Conjecture 2.7.

We conclude the section with a few remarks. Let p denote the largest integer such that any set of p cut vectors is affinely independent. Then, from Lemma 2.3, $p \geq 4$, and $p \leq 7$, because there exist 7 cuts whose incidence vectors are linearly dependent. Indeed, $X^{\delta(\{1,2\})} + X^{\delta(\{1,3\})} + X^{\delta(\{2,3\})} = X^{\delta(\{1\})} + X^{\delta(\{2\})} + X^{\delta(\{3\})} + X^{\delta(\{1,2,3\})}$ holds. One can observe also that the set of cuts $X = \{\delta(\{1\}), \delta(\{2\}), \delta(\{3\}), \delta(\{1,2\}), \delta(\{1,3\}), \delta(\{2,3\}), \delta(\varnothing)\}$ does not determine a face of M_n, neither of P_n. Indeed, if $\mathrm{Conv}(X)$ is a face of P_n, then there exists a vector w such that $0 = \mathrm{Max}(w.x: x \in P_n)$ is attained precisely at the vectors $x \in \mathrm{Conv}(X)$; thus, $0 =$

$w.X^{\delta(\{1,2,3\})}$, implying that $X^{\delta(\{1,2,3\})} \in \mathrm{Conv}(X)$, a contradiction. Furthermore, the smallest face of M_n (or P_n) containing X has dimension 6 and is not simplicial, since it contains also $X^{\delta(\{1,2,3\})}$. Therefore, P_n is not 7-neighborly and it has some non simplicial faces already for dimension 6. Note however that P_n has some simplicial facets (e.g. Example 5.6 below, see [7]).

3. How "Flat" Is the Cut Polytope?

A certain parameter of the shape of a polytope is the radius of the largest inscribed ball. Let r_n denote the radius of the largest ball that can be inscribed in the cut polytope P_n. How does r_n change when n groes? Is it increasing, constant or decreasing? The first alternative can be easily excluded, but we are not able to decide between the latter two. However, we conjecture that r_n remains, in fact, constant and is equal to $(2\sqrt{3})^{-1}$.

The barycentrum b of the cut polytope P_n is the point defined by $b = (\sum_{S \subseteq [2,n]} X^{\delta(S)})/2^{n-1}$, hence $b = (1/2, \ldots, 1/2)$.

Lemma 3.1. *The distance of any triangle facet from the barycentrum of P_n is equal to $(2\sqrt{3})^{-1}$.*

Proof. The distance from a point $(y_1, \ldots, y_n)$ to a hyperplane $a_1x_1 + \cdots + a_nx_n \le b$ is given by the formula

(3.2) $|a_1y_1 + \cdots + a_ny_n - b|/\|a\|$

where $\|a\|^2 = a_1^2 + \cdots + a_n^2$.

Hence, the distance of the triangle facet $x_{ij} - x_{ik} - x_{jk} \le 0$ or $x_{ij} + x_{ik} + x_{jk} \le 2$ from the barycentrum b is equal to $(2\sqrt{3})^{-1}$. □

We conjecture that this is the smallest possible distance of a facet from the barycentrum.

Conjecture 3.3. The distance of any facet of the cut polytope P_n from its barycentrum is at least $(2\sqrt{3})^{-1}$, independently of n, this smallest distance being attained precisely by the triangle facets.

It is enough to prove the validity of Conjecture 3.3 for the homogeneous facets of P_n. Indeed, the two facets defined by $v.x \le v_0$ and its switching by the cut $\delta(S)$, $v^S.x \le v_0 - v.X^{\delta(S)}$, are at the same distance from the barycentrum b. We can only prove that the above conjecture is valid for all *pure* facets, i.e. the facets defined by an inequality $v.x \le 0$, where all components of v are 0, 1 or -1.

Theorem 3.4. *Let $v.x \le 0$ be an inequality which defines a facet of P_n such that the components of v belong to $\{0, 1, -1\}$. Then, the distance of this facet from the barycentrum $(1/2, \ldots, 1/2)$ is at least $(2\sqrt{3})^{-1}$. Moreover, this smallest distance is realized precisely by the triangle facets.*

In order to prove Theorem 3.4, we prove a more general result, which gives a lower bound on the maximum cut in a weighted graph with weights 1, -1 on its edges.

We recall some notation. Let $G = (V, E)$ be a graph with weights $c(e)$, $e \in E$, on its edges. We set $c(E') := \sum_{e \in E'} c(e)$ for any subset E' of E and we denote by $MC(G,c)$ the maximum weight $c(\delta(S))$ of a cut, i.e. $MC(G,c) = \max(c(\delta(S)): S \subseteq V)$. Let us remark that a special case of Theorem 3.5 below, when all weights are 1, has been first proved by Edwards ([10]). An algorithmic proof has been given later by Poljak and Turzik ([17], [18]). We will use the method of the latter proof.

Theorem 3.5. *Let $G = (V, E)$ be a connected graph on n vertices with edge weights $c(e) \in \{1, -1\}$ for $e \in E$. Then,*

(3.6) $MC(G,c) \geq c(E)/2 + (n-1)/4.$

Proof. We proceed by induction on n, the number of vertices of G. The statement is trivially valid if $n = 1$ or 2. We suppose that $n \geq 3$. We distinguish two cases.

Case (i). Assume that G is not 2-connected, i.e. G has an articulation vertex. Let $G_i(V_i, E_i)$, $i = 1, 2$, be connected subgraphs of G such that $E = E_1 \cup E_2$ and $|V_1 \cap V_2| = 1$, set $n_i = |V_i|$, so $n = n_1 + n_2 - 1$. By the induction hypothesis, (3.6) is valid for both G_1 and G_2, and one easily concludes that it is valid for G as well, because $MC(G,c) = MC(G_1,c) + MC(G_2,c) \geq c(E_1)/2 + (n_1 - 1)/4 + c(E_2)/2 + (n_2 - 1)/4 = c(E)/2 + (n-1)/4$.

Case (ii). Assume that G is 2-connected. Then, one can show the existence of an edge uv of G such that the graph $G' = G \backslash \{u, v\}$ (i.e. the nodes u, v are deleted) is still connected. The proof of this statement is given in ([17], case 3 in the proof of Theorem 1). We consider two subcases, depending on the value 1 or -1 of $c(u,v)$. Suppose that $c(u,v) = 1$. Let S be a subset of the nodes of G' which realizes the max-cut of G', i.e. $c(\delta(S)) = MC(G',c)$. Note that $MC(G,c) \geq \max(c(\delta(S \cup \{u\})), c(\delta(S \cup \{v\}))) \geq (c(\delta(S \cup \{u\})) + c(\delta(S \cup \{v\})))/2 = MC(G',c) + c(u,v) + (\sum_{i \neq u,v,\, j=u,v} c_{ij})/2$. By the induction hypothesis, we have that $MC(G',c) \geq c(E')/2 + (n-3)/4$. Hence, $MC(G,c) \geq c(E)/2 + (n-3)/4 + c(u,v)/2 = c(E)/2 + (n-1)/4$.

Suppose now that $c(u,v) = -1$. Consider the pair of cuts $\delta(S \cup \{u,v\})$ and $\delta(S)$ instead of $\delta(S \cup \{u\})$ and $\delta(S \cup \{v\})$. As in the previous subcase, one can check that $MC(G,c) \geq (c(\delta(S)) + c(\delta(S \cup \{u,v\})))/2 = MC(G',c) + (\sum_{i \neq u,v,\, j=u,v} c_{ij})/2 \geq c(E)/2 + (n-3)/4 - c(u,v)/2 = c(E)/2 + (n-1)/4$. $\square$

Corollary 3.7. *Let $c = (c_{ij})_{1 \leq i < j \leq n} \in \{0, 1, -1\}^{n(n-1)/2}$. Then,*

(3.8) $MC(K_n, c) = \max(c(\delta(S)): S \subseteq [1,n]) \geq \left(\sum_{1 \leq i < j \leq n} c_{ij} \right) \Big/ 2 + \|c\| (2\sqrt{3})^{-1}.$

Moreover, if $c \neq 0$, then equality can only occur for c such that $c_{ij} \neq 0$ for $i, j \in \{h, k, l\}$ and $c_{ij} = 0$ otherwise, for some $1 \leq h < k < l \leq n$.

Proof. Let $G = (V = [1,n], E)$ denote the subgraph of K_n formed by the edges (i,j) with non zero weight. If G is connected, then, from relation (3.6), $MC(G,c) \geq c(E)/2 + (n-1)/4$. Now, $(n-1)/4 \geq (2\sqrt{3})^{-1} \binom{n}{2}^{1/2} \geq (2\sqrt{3})^{-1} \|c\|$, with equality between the first and the last term if and only if $n = 3$ and $c_{ij} \in \{1, -1\}$ for all i, j.

Hence, (3.8) follows. If G is not connected, let $G_1(V_1, E_1)$, ..., $G_k(V_k, E_k)$ be the connected components of G and let c_i denote the restriction of the vector c to the pairs (i, j) of V_i, $i = 1, \ldots, k$. It is easy to see that $\|c\| \leq \|c_1\| + \cdots + \|c_k\|$ and hence (3.8) is valid for G since it is valid for each connected component. Moreover, if equality holds in (3.8), then $\|c\| = \|c_1\| + \cdots + \|c_k\|$, implying that all c_i except one are zero (i.e. G_i is isolated vertex), say $c_1 \neq 0$, and hence $|V_1| = 3$ and $c_{ij} \in \{1, -1\}$ for $i, j \in V_1$. □

Proof of Theorem 3.4. Let $v.x \leq 0$ be an inequality that defines a facet of P_n with $v_{ij} \in \{0, 1, -1\}$ for all i, j. Consider the max-cut problem on K_n with edge weights c_{ij} on the edges. Since $v.x \leq 0$ is valid and facet inducing, we have that $MC(K_n, c) = 0$ and, from (3.8), $MC(K_n, c) \geq (\sum_{1 \leq i < j \leq n} c_{ij})/2 + (2\sqrt{3})^{-1} \|c\|$. Note that $v.b = (\sum_{1 \leq i < j \leq n} c_{ij})/2 \leq 0$. Therefore, we deduce that $|\sum_{1 \leq i < j \leq n} c_{ij}|/2 \geq (2\sqrt{3})^{-1} \|c\|$ and, hence, using formula (3.2), the distance of the facet $v.x \leq 0$ from the barycentrum b is at least $(2\sqrt{3})^{-1}$. From Corollary 3.7, equality can only occur if $v.x \leq 0$ is a triangle facet. □

Let us remark that Conjecture 3.3 would follow if one could prove Relation (3.8) for arbitrary edge weights (not necessarily 1, -1, 0), i.e. the following Conjecture 3.9 implies Conjecture 3.3.

Conjecture 3.9. Let $c = (c_{ij})_{1 \leq i < j \leq n}$. Then, $MC(K_n, c) \geq (\sum_{1 \leq i < j \leq n} c_{ij})/2 + (2\sqrt{3})^{-1} \|c\|$.

Remark 3.10. We checked, by direct computation, that the following class of hypermetric inequalities satisfies Conjecture 3.3. Hypermetric inequalities $Hyp_n(b_1, \ldots, b_n)$ are of the form $\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0$, where b_1, ..., b_n are integers whose sum $b_1 + \cdots + b_n$ is equal to 1. They are valid for the cut polytope P_n and facet defining for large classes of parameters b (see [5], [7], [8]).

4. The Lattice of Even Cuts

A cut $\delta(S)$ is called *even* (resp. *odd*) if both sets S and $[1, n] - S$ are of even (resp. odd) cardinality, so n must be even. The *even* (resp. *odd*) *cut polytope* EvP_n (resp. OdP_n), defined as the convex hull of all even (resp. odd) cut vectors, was studied in [9]; in fact, $OdP_n = r_{\delta(A)}(EvP_n)$ for any odd cut $\delta(A)$. Those polytopes share some of the properties of P_n. In particular, for $n \neq 6$, their only symmetries are the permutations of $[1, n]$ together with the reflections $r_{\delta(S)}$, but now only for the even cuts $\delta(S)$. Let L_n denote the lattice generated by the cut vectors, i.e. $L_n = \{\sum_{S \subseteq [1,n]} a_S \delta(S)$: a_S integer for $S \subseteq [1, n]\}$ and let LE_n denote the lattice generated by all even cut vectors; L_n is called the *cut lattice*, LE_n the *even cut lattice*. Thus, LE_n is a sublattice of L_n. The cut lattice L_n admits the following simple characterization.

Proposition 4.1 ([1]). *Given $d \in R^{n(n-1)/2}$, then d belongs to the cut lattice L_n if and only if d has integer components and satisfies the following condition*:

(4.2) $d_{ij} + d_{ik} + d_{jk}$ *is even for all* $1 \leq i < j < k \leq n$.

Given a partition of $[1, n]$ into k non empty disjoint subsets $S_1, \ldots, S_k$, the *k-cut* $\delta(S_1, \ldots, S_k)$ is the set of pairs (i, j) such that $i \in S_a$, $j \in S_b$ for distinct a, b in $[1, k]$. So, the 2-cut $\delta(S, [1, n] - S)$ is the usual cut $\delta(S)$. Note that the lattice generated by the incidence vectors of all k-cuts for $k \geq 2$ is simply the ring of integers $Z^{n(n-1)/2}$, because $X^{\delta(\{i\})} + X^{\delta(\{j\})} - X^{\delta(\{i\}, \{j\}, [1,n]-\{i,j\})} = e_{ij}$ (the coordinate vector with all zero components except one component equal to 1 in position (i, j)) for any i, j in $[1, n]$.

The dual lattice L_n^* of L_n too is well known, L_n^* coincides with the lattice generated by the half triangles $(e_{ij} + e_{ik} + e_{jk})/2$ for $1 \leq i < j < k \leq n$, and the coordinate vectors e_{ij} for $1 \leq i < j \leq n$. The less trivial inclusion is easily checked as follows. If $d \in L_n^*$, then $2d_{ij} = d \,.\, X^{\delta(\{i\})} + d \,.\, X^{\delta(\{j\})} - d \,.\, X^{\delta(\{i,j\})}$ is integer and so can be written as $2d_{ij} = y_{ij} + 2z_{ij}$ with $y_{ij} \in \{0, 1\}$ and z_{ij} integer for any i, j. Hence $y \,.\, X^{\delta(S)}$ is even for any cut yielding that y is integer combination of triangles and double unit vectors $2e_{ij}$ and thus d is integer combination of half triangles and unit vectors. In other words, given a vector d, $d \,.\, X^{\delta(S)}$ is an even integer for any cut $\delta(S)$ if and only if d is linear combination of triangles $(e_{ij} + e_{ik} + e_{jk})$ and double edges $2e_{ij}$. Note that the lattice generated by the triangles and the double edges coincides with the lattice generated by the incidence vectors of all cycles of the complete graph on n nodes. As application, the separation problem for the lattice L_n can be solved in polynomial time. Given a vector d, it consists of deciding whether or not $d \in L_n$ and, if not, of finding a vector $c \in L_n^*$ such that $c \,.\, d$. is not an integer.

Given a subset A of $[1, n]$, we define the following linear form $Q_A \,.\, x := \sum_{1 \leq i < j \leq n, (i,j) \notin \delta(A)} x_{ij} - \sum_{1 \leq i < j \leq n, (i,j) \in \delta(A)} x_{ij}$. For i, j in $[1, n]$, we also set $Q_i := Q_{\{i\}}$ and $Q_{i,j} := Q_{\{i,j\}}$. For any even cut $\delta(S)$, $Q_A \,.\, X^{\delta(S)} = z(2a - n - z)$ where $a = |A|$ and $z = 2|A \cap S| - |S|$ is an even integer, say $z = 2y$, and thus, $Q_A \,.\, X^{\delta(S)} = 4y(a - n/2 - y)$. Therefore, the following relations hold.

(4.3) if $A = N = [1, n]$, then $Q_N \,.\, X^{\delta(S)} \equiv 0 \pmod 4$ for all even cuts $\delta(S)$

(4.4) if $a - n/2$ is odd, then $Q_A \,.\, X^{\delta(S)} \equiv 0 \pmod 8$ for all even cuts $\delta(S)$

Hence, from relations (4.3), (4.4), one can derive easy necessary conditions for membership in the even cut lattice LE_n. In fact, these conditions, together with the condition (4.2) on the perimeter of triangles, are sufficient for characterizing lattice points in LE_n and, even more, it is sufficient to consider the condition derived from (4.4) by taking $a = 1$ if $n \equiv 0 \pmod 4$ and $a = 2$ if $n \equiv 2 \pmod 4$.

Theorem 4.5. *Let n be an even integer, $n \geq 6$. Given $d \in R^{n(n-1)/2}$, then d belongs to the even cut lattice LE_n if and only if d has integer components and satisfies the conditions* (4.2) *and* (4.6), (4.7) *below.*

(4.6) $Q_N \,.\, d = \sum_{1 \leq i < j \leq n} d_{ij} \equiv 0 \pmod 4$

(4.7) $Q_i \,.\, d \equiv 0 \pmod 8$ *for all* $1 \leq i \leq n$, *if* $n \equiv 0 \pmod 4$, *and* $Q_{i,j} \,.\, d \equiv 0 \pmod 8$ *for all* $1 \leq i < j \leq n$, *if* $n \equiv 2 \pmod 4$.

Consequently, membership in LE_n can be tested in polynomial time. The remaining of the section is devoted to the proof of Theorem 4.5. Given c, $d \in R^{n(n-1)/2}$, we set $c \approx d$ if $c - d \in LE_n$, i.e. $c \in LE_n$ if and only if $d \in LE_n$.

Lemma 4.8. (i) $2(e_{ij} + e_{jh} + e_{hk} + e_{ki}) \in LE_n$ *for all distinct* i, j, h, k *in* $[1,n]$
(ii) $4(e_{ij} + e_{ik}), 4(e_{ij} + e_{hk}) \in LE_n$ *for all distinct* i, j, h, k *in* $[1,n]$
(iii) $8e_{ij} \in LE_n$ *for all* i, j *in* $[1,n]$.

Proof. Note first that $\delta(\{1,2\}) + \delta(\{3,4\}) - \delta(\{1,2,3,4\}) = 2(e_{13} + e_{23} + e_{24} + e_{14}) \in LE_n$, hence implying assertion (i). Similarly, $2(e_{12} + e_{14} + e_{23} + e_{34}) \in LE_n$ and $2(e_{12} + e_{13} + e_{24} + e_{34}) \in LE_n$. By combination of these three relations, we obtain that $4(e_{13} + e_{24}) \in LE_n$. Similarly, $4(e_{13} + e_{56}) \in LE_n$ and $4(e_{24} + e_{56}) \in LE_n$, yielding that $8e_{56} \in LE_n$ and thus stating (iii). Finally, $4(e_{12} + e_{56}) \in LE_n$ and $4(e_{13} + e_{56}) \in LE_n$, implying that $4(e_{12} + e_{13}) \in LE_n$, thus concluding the proof. □

Proof of Theorem 4.5. Take $d \in R^{n(n-1)/2}$ with integer components and assume that d satisfies the conditions (4.2), (4.6), (4.7). We show below that d indeed belongs to the even cut lattice LE_n.

We first remark that we can assume that d has only even components. Indeed, set $F = \{(i,j): d_{ij} \text{ is odd}\}$. From assumption (4.2), F is a complete bipartite graph and thus, if its node partition is A and $[1,n] - A$, then $d' = d + \delta(A)$ has only even components. From assumption (4.6), we deduce that $\delta(A)$ is an even cut and, thus, $d \approx d'$.

From now on, we suppose that $d_{ij} \equiv 0 \pmod 2$ for all i, j. The basic idea is now to apply some reductions on d using Lemma 4.8. Set $E = \{(i,j): d_{ij} \neq 0\}$. In view of Lemma 4.8 (iii), we can assume that $d_{ij} \equiv 2, 4$ or $6 \pmod 8$ for all $(i,j) \in E$ and, in view of Lemma 4.8 (ii), we can assume that $d_{ij} \equiv 4 \pmod 8$ for at most one pair $(i,j) \in E$.

Claim 4.9. *We can assume that* E *is contained in the set* $E' = \{(2,3)\} \cup \{(1,i); 2 \leq i \leq n\}$.

Proof. It is based on the reduction of d by repeated applications of Lemma 4.8 (i). First, we can assume that $d_{ij} = 0$ for all $3 \leq i < j \leq n$. Indeed, this can be achieved by doing the following reductions on d. If $d_{ij} \equiv 2 \pmod 8$, then replace d by $d - 2(e_{1i} + e_{ij} + e_{2j} + e_{12})$; if $d_{ij} \equiv 6 \pmod 8$, then replace d by $d + 2(e_{1i} + e_{ij} + e_{2j} + e_{12})$ and, if $d_{ij} \equiv 4 \pmod 8$, then $d \approx d + 4(e_{1i} + e_{ij} + e_{2j} + e_{12})$. We can also assume that $d_{2i} = 0$ for $4 \leq i \leq n$. For this, it suffices to replace d by $d + a(e_{1i} + e_{2i} + e_{23} + e_{13})$ with $a = -2$ if $d_{2i} \equiv 2 \pmod 8$, $a = 4$ if $d_{2i} \equiv 4 \pmod 8$ and $a = 2$ if $d_{2i} \equiv 6 \pmod 8$. Similarly, we can assume that $d_{3i} = 0$ for $4 \leq i \leq n$. □

Claim 4.10. *We can assume that* $d_{12} \equiv d_{13} \equiv 0 \pmod 4$, $d_{14} \equiv d_{15} \equiv \cdots \equiv d_{1n} := a \pmod 4$ *and* $d_{23} \equiv a(n-3) \pmod 4$.

Proof. We now use assumption (4.7). We first show that $d_{12} \equiv 0 \pmod 4$. Indeed, if $n \equiv 0 \pmod 4$, then $Q_2 . d + Q_1 . d = -2d_{12} \equiv 0 \pmod 8$ and, if $n \equiv 2 \pmod 4$, then $Q_{2,n} . d + Q_{1,n} . d = -2d_{12} \equiv 0 \pmod 8$. Then, $d_{12} = d_{13} \pmod 4$, because, for $n \equiv 0 \pmod 4$, $Q_2 . d - Q_3 . d = 2(d_{13} - d_{12}) \equiv 0 \pmod 8$ and, for $n \equiv 2 \pmod 4$, $Q_{2,n} . d - Q_{3,n} . d = 2(d_{13} - d_{12}) \equiv 0 \pmod 8$. Finally, for $5 \leq i \leq n$, for $n \equiv 0 \pmod 4$, $Q_4 . d - Q_i . d = 2(d_{1i} - d_{14}) \equiv 0 \pmod 8$ and, for $n \equiv 2 \pmod 4$, $Q_{2,4} . d - Q_{2,i} . d = 2(d_{1i} - d_{14}) \equiv 0 \pmod 8$. The last statement follows from assumption (4.6). □

Claim 4.11. *If* $a \equiv 0 \pmod 4$ (*a being defined in Claim* 4.10), *then* $d \in LE_n$.

Proof. From Claim 4.10, we have that $d_{ij} \equiv 0$ or $4 \pmod 8$ for all i, j. In order to show that $d \in LE_n$, it suffices to verify that the set $E = \{(i,j)\colon d_{ij} \equiv 4 \pmod 8\}$ is of even cardinality. To see it, note that, for $n \equiv 0 \pmod 4$, $Q_1 . d \equiv 4|E| - \sum_{2 \le i \le n} 2d_{1i} \equiv 0 \pmod 8$ and, for $n \equiv 2 \pmod 4$, $Q_{1,2} . d \equiv 4|E| - \sum_{3 \le i \le n} 2(d_{1i} + d_{2i}) \equiv 0 \pmod 8$, which, in both cases, implies that $|E|$ is even. □

Let us make the following observation. Set $c = 2(e_{23} + \sum_{4 \le i \le n} e_{1i})$, then $c \in LE_n$ because $c = \delta(\{1,3)\} - \delta(\{2,3\}) + \delta(\{1,2\})$.

Claim 4.12. *If* $a \equiv 2 \pmod 4$, *then* $d \in LE_n$.

Proof. Using Lemma 4.8 (ii), we can assume that $d_{1i} \equiv 2 \pmod 8$ for all $4 \le i \le n$ except at most one such index i. From Claim 4.10, each of d_{12} and d_{13} is 0 or 4 (mod 8) and d_{23} is 2 or 6 (mod 8). We distinguish two cases.

We suppose first that $d_{1i} \equiv 2 \pmod 8$ for all $4 \le i \le n$. There are six possible cases, according to the possible value of $(d_{12}, d_{13}, d_{23}) \pmod 8$; we examine below all possibilities for this triple.

(i) $(0,0,2)$, then $d \approx c$ and thus $d \in LE_n$

(ii) $(0,0,6)$, then $d \approx c + 4e_{23}$, in contradiction with the fact that d satisfies (4.7)

(iii) $(4,0,2)$, then $d \approx c + 4e_{12}$, yielding a contradiction as above

(iv) $(4,0,6)$, then $d \approx c + 4e_{12} + 4e_{23} \approx c$ and thus $d \in LE_n$

(v) $(4,4,2)$, then $d \approx c$ and thus $d \in LE_n$

(vi) $(4,4,6)$, then $d \approx c + 4e_{23}$, yielding a contradiction.

Finally, we suppose that $d_{1i} \equiv 2 \pmod 8$ for $4 \le i \le n-1$ and $d_{1n} \equiv 6 \pmod 8$. As above, we examine the possibilities for the triple $(d_{12}, d_{13}, d_{23}) \pmod 8$ and obtain, for the cases $(0,0,6)$, $(4,0,2)$ and $(4,4,6)$ that $d \in LE_n$, and for the cases $(0,0,2)$, $(4,0,6)$, $(4,4,2)$ a contradiction with the fact that d satisfies the assumption (4.7). □

Remark 4.13. Given an integer $t \ge 2$, a cut $\delta(S)$ is called a *t-ary* cut if $|S| = 0 \pmod t$ and $n - |S| \equiv 0 \pmod t$ holds; so, even cuts are 2-ary cuts. Analogues of relations (4.3), (4.4) for membership of a vector d in the lattice generated by all t-ary cuts are as follows: $Q_N . d \equiv 0 \pmod{t^2}$ and, for any subset A of $[1,n]$ such that $|A| - n/t$ is odd, setting $A' = [1,n] - A$, $\sum_{i<j,(i,j) \in A \times A} (t-1)^2 d_{ij} + \sum_{i<j,(i,j) \in A' \times A'} d_{ij} - \sum_{i<j, i \in A, j \in A'} (t-1) d_{ij} \equiv 0 \pmod{2t^2}$.

5. Do All Facets "Come" from Triangles?

We give below two properties that we have observed on the classes of facets of P_n known so far. Let $v \in R^{n(n-1)/2}$ and $v_0 \in R$. Let $[1,n] = I_1 \cup \cdots \cup I_p$ be a partition of $[1,n]$ into p parts, define $v' \in R^{p(p-1)/2}$ by $v'_{hk} = \sum_{i \in I_h, j \in I_k} v_{ij}$ for $1 \le h < k \le p$, one says that v' is obtained by *collapsing* v. Collapsing preserves validity, namely, if the inequality $v . x \le v_0$ is valid for P_n, then the inequality $v' . x \le v_0$ is valid for P_p ([5]).

Property 5.1 (parity conjecture). Let $v . x \le v_0$ be an inequality defining a facet of P_n. Then, $v . X^{\delta(S)}$ is an even integer for all cuts $\delta(S)$ or, equivalently, the vector v belongs to the lattice generated by the triangles $e_{ij} + e_{ik} + e_{jk}$ and the double edges $2e_{ij}$ for distinct i, j, k in $[1, n]$.

Property 5.2. Let $v.x \leq 0$ be an inequality defining a facet of P_n. Then, it collapses to some triangle facet.

Some easy observations on Property 5.1.

(i) The switching operation preserves Property 5.1, hence it is enough to check Property 5.1 for homogeneous facets, i.e. with $v_0 = 0$.

(ii) Property 5.1 is preserved under collapsing; namely, if a facet inducing inequality $v.x \leq 0$ has property 5.1, then any collapsing of it, $v'.x \leq 0$, has it too. Indeed, if v is integer combination of triangles and double edges, then so is v', because any collapsing of a triangle is a triangle or a double edge.

(iii) Both assumptions of validity and full rank are necessary for Property 5.1. Indeed, take $2p \leq n$ and $v.x = \sum_{(i,j)\in[1,p]\times[1,p]\cup[p+1,2p]\times[p+1,2p]} x_{ij} - \sum_{(i,j)\in[1,p]\times[p+1,2p]} x_{ij}$; then the inequality $v.x \leq 0$ is valid but not facet inducing for P_n and $v.X^{\delta(\{1\})} = -1$ is not even. Also, take $4p \leq n$ and $v.x = \sum_{0\leq i\leq p-1} x_{2i+1,2i+2} - \sum_{p\leq i\leq 2p-1} x_{2i+1,2i+2}$, then the inequality $v.x \leq 0$ is not valid for P_n but there exist $n(n-1)/2 - 1$ linearly independent cut vectors satisfying $v.x = 0$ and $v.X^{\delta(\{1\})} = 1$ is not even.

We checked that Property 5.1 holds for the known classes of facets of P_n (namely, parachute facet [7], CW facets [8], Boros-Hammer facet [4], Poljak-Turzik facet [18]). It is an interesting question to look for a facet of P_n that does not enjoy Property 5.1; a good candidate is some inequality of the form $v.x = \sum_{(i,j)\in E} x_{ij} \leq v_0$ where E is a regular graph of odd degree and v_0 is the maximum size of a cut.

Similarly, we checked that Property 5.2 holds for most known classes of facets. Note that a given facet may collapse on different triangle facets. Also, Property 5.2 does not extend to multicut polytopes.

As illustration of the parity conjecture, we give below the explicit decomposition of some facets as linear combination of triangles and double edges (i.e. degenerated triangles). We use the following notation. We set $T(i,j;k) := x_{ij} - x_{ik} - x_{jk}$. The facets we consider are supported by an inequality of the form $v.x \leq 0$.

Example 5.3. (a switching of) the bicycle odd wheel inequality ([3]). Then,

$$\begin{aligned} v.x &= \left(\sum_{1\leq i\leq 2t+2} x_{i,i+1} + x_{2t+2,1}\right) + x_{2t+4,2t+5} - \sum_{1\leq i\leq 2t+3} (x_{2t+4,i} + x_{2t+5,i}) \\ &= \sum_{1\leq i\leq t+1} (T(i, i+t+1; 2t+4) + T(i, i+t+2; 2t+5)) \\ &\quad + T(t+2, 2t+3; 2t+4) - T(t+2, 2t+5; 2t+4). \end{aligned}$$

Example 5.4. The parachute facet ([7]).

$$v.x = \sum_{(i,j)\in P} x_{ij} - \sum_{1\leq i\leq k-1} (x_{0i} + x_{0i'} + x_{ki'} + x_{k'i}) - x_{kk'},$$

where k is an odd integer and P denotes the edge set of the path $(k, k-1, \ldots, 2, 1, 1', 2', \ldots, (k-1)', k')$ and

$$v.x = \sum_{1 \le i \le k-1} (T(i, i+1; a_{i'}) + T(i', (i+1)'; a_i)) + T(1, 1'; 0) - T(k, k'; 0)$$

where $a_i = k$, $a_{i'} = k'$ for i odd and $a_i = a_{i'} = 0$ for i even.

Example 5.5. The facet Gr_7 ([7]). Then,

$$\begin{aligned} v.x &= \sum_{1 \le i < j \le 4} x_{ij} + x_{56} + x_{57} - x_{67} - x_{16} - x_{36} - x_{27} - x_{47} - 2\left(\sum_{1 \le i \le 4} x_{5i}\right) \\ &= T(1,2;5) + T(1,3;5) + T(1,4;6) + T(2,3;7) + T(2,4;5) + T(3,4;5) \\ &\quad - T(6,7;5). \end{aligned}$$

Example 5.6. A hypermetric facet ([7]). Then,

$$\begin{aligned} v.x = Hyp_n(-(n-4), -1, 1, \ldots, 1) = &- \sum_{3 \le i \le n} (n-4)x_{1i} + x_{12} \\ &- \sum_{3 \le i \le n} x_{2i} + \sum_{3 \le i < j \le n} x_{ij} \end{aligned}$$

and

$$v.x = -(\lfloor n/2 \rfloor - 2)2x_{12} + \sum_{3 \le i < j \le n} T(i, j; \alpha_{ij}) - T_0,$$

where $\alpha_{ij} = 2$ if $(i, j) = (2t+1, 2t+2)$ for $1 \le t \le \lfloor n/2 \rfloor - 1$, and $\alpha_{ij} = 1$ otherwise, and $T_0 = T(2, n; 1)$ if n is odd and $T_0 = 0$ if n is even.

We consider also $v'.x = (n-4)(x_{12} + x_{1n}) - \sum_{3 \le i \le n-1} (n-4)x_{1i} - \sum_{3 \le i \le n-1} x_{2i} + x_{2n} + \sum_{3 \le i < j \le n} x_{ij} - \sum_{3 \le i \le n-1} x_{in}$. Thus, the inequality $v'.x \le 0$ is a switching of the inequality $v.x \le 0$. Also, $v'.x = -2(\lfloor n/2 \rfloor - 2)x_{12} + \sum_{3 \le i < j \le n} T(i, j; \alpha_{ij}) + \sum_{3 \le i \le n-1} T(n, \alpha_{in}; i) - T_1$, where $T_1 = T(1, n; 2)$ if n is odd and $T_1 = 0$ if n is even.

Actually, in Examples 5.3, 5.4, 5.5 and case $n = 5, 6$ of Example 5.6, we have a "strong" triangulation of the facets, i.e. all coefficients are $+1$ except one coefficient -1 in the linear decomposition. This implies, in particular, that all homogeneous facets of P_6 admit a strong triangulation.

6. The Hypercut Polytope

Given a subset S of $[1, n]$ and $2 \le p \le n-1$, the *p-hypercut* $\delta_p(S)$ is the set of all p-tuples $(i_1, \ldots, i_p)$ of distinct points of $[1, n]$ such that both sets $\{i_1, \ldots, i_p\} \cap S$ and $\{i_1, \ldots, i_p\} \cap ([1, n] - S)$ are not empty. For $p = 2$, the 2-hypercut $\delta_2(S)$ is the usual cut $\delta(S)$. The p-hypercut polytope $HP(p)_n$ is the convex hull of the incidence vectors of the p-hypercuts $\delta_p(S)$ for all subsets S of $[1, n]$, so $HP(p)_n$ is a polytope in R^m where $m = \binom{n}{p} = n!/((n-p)!p!)$. Therefore, $HP(2)_n = P_n$. In fact, as we see below, the 3-hypercut polytope $HP(3)_n$ is the image of the cut polytope P_n under a linear one-to-one mapping. For $n \ge 5$, $p = n-1$, one checks easily that the vertices of $HP(n-1)_n$ are the vectors 0, $1 = (1, \ldots, 1)$, $1 - e_i$ for $1 \le i \le n$, where e_i is the i-th coordinate vector in R^n. For $p = n-2$, the vertices of $HP(n-2)_n$ are the vectors $1 - X^{\delta(\{i\})}$ for $1 \le i \le n$ and $1 - e_{ij}$ for $1 \le i < j \le n$ in $R^{n(n-1)/2}$. Generally, if $p > n/2$, the incidence vector of the cut $\delta_p([p+1, n])$ is $1 - e_{12\ldots p}$ and, therefore, $HP(p)_n$ is full dimensional.

Consider the map f from $R^{n(n-1)/2}$ to $R^{n(n-1)(n-2)/6}$ defined by $y = f(x)$ with $y_{ijk} = (x_{ij} + x_{ik} + x_{jk})/2$ for all triples (i, j, k). The map f is one-to-one if $n \geq 5$. Indeed, assume that $y = f(x) = 0$. Take distinct points i, j, k, h, l in $[1, n]$. Then, $x_{ij} + x_{ik} + x_{jk} = x_{ij} + x_{ih} + x_{jh} = 0$, yielding that $x_{ik} + x_{jk} = x_{ih} + x_{jh}$. Similarly, $x_{ih} + x_{ik} = x_{jh} + x_{jk}$ which, together with the preceding relation, implies that $x_{ik} = x_{jh}$. Similarly, $x_{ik} = x_{jl} = x_{ih}$ and thus all components of x are equal, implying that $x = 0$.

It is immediate to see that $y = f(x)$ if x is the incidence vector of the cut $\delta(S)$ and y is the incidence vector of the 3-hypercut $\delta_3(S)$ for any subset S of $[1, n]$. Therefore, $HP(3)_n = f(P_n)$. Hence, for $n \geq 5$, the hypercut polytope $HP(3)_n$ is a polytope of dimension $n(n-1)/2$ in $R^{n(n-1)(n-2)/6}$ and its linear description can be deduced from that of the cut polytope P_n, as we recall in Lemma 6.1 below.

Let f be a one-to-one linear map from R^p to R^q, $q \geq p$. Let A denote the associated $p \times q$ matrix such that $f(x) = Ax$ for $x \in R^p$. Since f is one-to-one, there exists a non singular $p \times p$ submatrix A_1 of A. Assume that the rows of A_1 are indexed by the set L and let A_2 denote the $(q-p) \times p$ submatrix of A formed by the remaining rows, so its rows are indexed by $L' = [1, q] - L$. For $y \in R^q$, set $y_1 = (y_j)_{j \in L}$ and $y_2 = (y_j)_{j \in L'}$, so $y = (y_1, y_2)$. Every row of A_2 is linear combination of the rows of A_1, so $A_2 = BA_1$ for some $(q-p) \times p$ matrix B. One sees easily that $y \in R^q$ belongs to the range of f, i.e. $y = f(x)$ for some x, if and only if $y_2 = By_1$ holds. The following lemma is easy to check.

Lemma 6.1. *Let $P = \{x \in R^p: Mx \leq b\}$ be a polytope in R^p. Then, its image under the linear map f is given by $f(P) = \{y \in R^q: y_2 = By_1$ and $M(A_1)^{-1}y_1 \leq b\}$.*

We conclude with the explicit description of some facets of $HP(3)_n$. Take p such that $2p + 1 \leq n$. Then, the inequality $\sum_{1 \leq i < j \leq 2p+1} x_{ij} \leq p(p+1)$ defines a facet of P_n and, therefore, the inequality $\sum_{1 \leq i < j < k \leq 2p+1} y_{ijk} \leq p(p+1)(2p-1)/2$ defines a facet of $HP(3)_n$, because $\sum_{1 \leq i < j < k \leq 2p+1} y_{ijk} = (2p-1)/2(\sum_{1 \leq i < j \leq 2p+1} x_{ij})$ holds. For instance, for $p = 1$, the triangle facet (1.2) corresponds to the facet $y_{ijk} \leq 1$.

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Received: November 6, 1990
Revised: June 7, 1991

Note added in proof:
Since the paper was not type-set from the last revision, we would like to include some comments concerning Section 2. The following results should be included:
— The cut polytope P_n is not 4-neighbourly
— Any face of P_n of dimension less than or equal to 5 is simplicial.