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One-third-integrality in the max-cut problem

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Abstract

Given a graph $G=(V,E)$, the metric polytope $\mathcal{S}(G)$ is defined by the inequalities $x(F)-x(C\setminus F)\leqslant|F|-1$ for $F\subseteq C$, $|F|$ odd, C cycle of G, and $0\leqslant x_e\leqslant 1$ for $e\in E$. Optimization over $\mathcal{S}(G)$ provides an approximation for the max-cut problem. The graph G is called $1/d$-integral if all the vertices of $\mathcal{S}(G)$ have their coordinates in $\{i/d \mid 0\leqslant i\leqslant d\}$. We prove that the class of $1/d$-integral graphs is closed under minors, and we present several minimal forbidden minors for $\frac{1}{3}$-integrality. In particular, we characterize the $\frac{1}{3}$-integral graphs on seven nodes. We study several operations preserving $1/d$-integrality, in particular, the k-sum operation for $0\leqslant k\leqslant 3$. We prove that series parallel graphs are characterized by the following stronger property. All vertices of the polytope $\mathcal{S}(G)\cap\{x \mid \ell\leqslant x\leqslant u\}$ are $\frac{1}{3}$-integral for every choice of $\frac{1}{3}$-integral bounds ℓ, u on the edges of G.

Keywords: Max-cut; Cut polytope; Metric polytope; Linear relaxation; One-third-integrality; Box one-third-integrality; Forbidden minor

1. Introduction

We study a system of inequalities associated with the max-cut problem (see below for a definition). Given a graph $G=(V,E)$, the inequalities are of the form

$$x(F)-x(C\setminus F)\leqslant|F|-1,\quad \text{for } F\subseteq C,\ \ |F| \text{ odd},\ \ C \text{ cycle of } G, \tag{1}$$

$$0\leqslant x_e\leqslant 1,\quad \text{for } e\in E. \tag{2}$$

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Each of these inequalities is valid for all cut vectors. The polytope $\mathcal{S}(G)$ defined by these inequalities is called the *metric polytope* of the graph G. Barahona and Mahjoub [4] characterized the graphs G for which the metric polytope $\mathcal{S}(G)$ is integral as those having no K_5 minor.

In this paper we study the graphs G for which each vertex of the metric polytope $\mathcal{S}(G)$ is $1/d$-integral. We call these graphs $1/d$-integral. The minimum d for which a graph is $1/d$-integral serves as a certain measure of approximation of the max-cut problem by the above system of inequalities. As shown later, there are no $\frac{1}{2}$-integral graphs. Hence the first case after integrality is that of $\frac{1}{3}$-integral graphs.

We present several results on $1/d$-integral graphs. We show in Section 3 that this class is preserved by sum operations: the 0-sum and 1-sum of two $1/d$-integral graphs is $1/d$-integral, and the 2-sum and 3-sum, with some restriction in the latter case, of a $1/d$-integral graph and an integral graph is $1/d$-integral. (In several cases, the requirements on the two summands are different.) In consequence, the class is closed also under subdivisions of edges and, with some restriction, under the ΔY-operation.

The class of $1/d$-integral graphs is closed under minors. We present in Section 4 four minimal forbidden minors for $\frac{1}{3}$-integrality. In particular, all subgraphs of K_6 are $\frac{1}{3}$-integral and we characterize the $\frac{1}{3}$-integral graphs on seven nodes. We also include the full description of $\mathcal{S}(K_n)$ for $n \leqslant 6$.

In Section 5 we characterize the graphs G for which all the vertices of the polytope $\mathcal{S}(G) \cap \{x \mid \ell \leqslant x \leqslant u\}$ are $\frac{1}{3}$-integral for every choice of $\frac{1}{3}$-integral vectors ℓ and $u \in \mathbb{R}^E$; they are the series parallel graphs.

Section 2 contains some tools and operations. We recall how the polytope $\mathcal{S}(G)$ arises as projection of the metric polytope on the edge set of G. We consider some operations on the vertices of $\mathcal{S}(G)$ which are intensively used later, namely switching, the 0- and 1-extension, and the union operation.

Let us mention that the result of Section 5 on box $\frac{1}{3}$-integral graphs has been extended in the context of binary clutters by Gerards and Laurent [7]. Box $1/d$-integral binary clutters are characterized there in terms of forbidden minors for any integer $d \geqslant 2$. In fact, the case $d = 1$ corresponds to the clutters with the $\mathbb{Q}_+$-max-flow min-cut property whose characterization is the object of a conjecture by Seymour [19].

One encounters the polytope $\mathcal{S}(G)$ in connection with various problems. We briefly describe some of them.

The max-cut problem

The polytope $\mathcal{S}(G)$ was introduced in [4] as a linear relaxation of the cut polytope $\mathcal{P}(G)$. Indeed, the $(0,1)$-valued vertices of $\mathcal{S}(G)$ are precisely the characteristic vectors of the cuts of G. Hence, the optimum of the linear program

$$\max c^{\mathrm{T}}x, \quad x \in \mathcal{S}(G), \tag{3}$$

always provides an upper bound on the optimum of

$$\max c^{\mathrm{T}}x, \quad x \in \mathcal{P}(G). \tag{4}$$

Since the max-cut problem is NP-hard, it is important to study for which objective functions c the linear program (3) provides a good approximation for (4). We show that (3) provides a $\frac{4}{3}$-approximation of (4) for any $\frac{1}{3}$-integral graph with nonnegative weight function. The relation between the linear programs (3) and (4) has been studied also in [15, 16] in the case when the objective function is given by $c_e = 1$ for $e \in E(G)$ where G is a graph. In the latter paper it is shown that the expected value of the ratio between (3) and (4) tends to $\frac{4}{3}$ for a random graph with fixed edge probabilities, and the ratio can be arbitrarily close to 2 on a class of sparse graphs.

Recently, a nonpolyhedral relaxation of the cut polytope has been investigated (see, e.g., [13]); Goemans and Williamson [8] have shown that it provides a 1.138-approximation for the max-cut problem for all graphs with nonnegative weights.

Multicommodity flow problems

Let us denote by $\mathcal{C}(G)$ the cone defined by the homogeneous inequalities from the system (1) and (2), i.e., by the inequalities

$$x(e) - x(C \setminus e) \leqslant 0, \quad \text{for } e \in C, \ C \text{ cycle of } G, \tag{5}$$

$$0 \leqslant x_e, \quad \text{for } e \in E. \tag{6}$$

The cone $\mathcal{C}(G)$ has been considered in connection with multicommodity flow problems. By the so-called *Japanese theorem* [10], it is the dual cone to the set of feasible multiflows.

Seymour [20] has shown that the graphs G for which all the extreme rays of $\mathcal{C}(G)$ are $(0,1)$-valued are the graphs with no K_5 minor. Schwärzler and Sebő [18] have characterized the graphs G for which all extreme rays of $\mathcal{C}(G)$ are $(0,1,2)$-valued. Actually, all of them are $\frac{1}{3}$-integral (see Remark 4.6).

The metric cone and polytope

Let $n \geqslant 3$. The metric cone $\mathcal{MC}_n$ is the cone defined by the inequalities

$$x_{ij} - x_{ik} - x_{jk} \leqslant 0, \tag{7}$$

for all triples $\{i,j,k\} \subseteq V = \{1,\ldots,n\}$. Its extreme rays were studied in [1,2,9,14]. The metric polytope $\mathcal{MP}_n$ is the polytope defined by the inequalities (7) and

$$x_{ij} + x_{ik} + x_{jk} \leqslant 2, \tag{8}$$

for all triples $\{i,j,k\} \subseteq V = \{1,\ldots,n\}$. The inequalities (7) and (8) are called the *triangle inequalities*. The metric polytope enjoys a lot of interesting geometrical properties which have been investigated in [6]. Several classes of vertices, mainly arising from graphs, have been constructed and studied in [11]. It has been confirmed that all the vertices considered in that paper are adjacent to integral vertices (see our conjecture in Section 4.1).

It is well known that $\mathcal{S}(K_n)$ and $\mathcal{MP}_n$ coincide and, moreover, $\mathcal{S}(G)$ is the projection of $\mathcal{MP}_n$ on the edge set of G. The analogous statement holds for the cones $\mathcal{C}(K_n)$ and $\mathcal{MC}_n$. We recall the details in Section 2.1. For this reason we call $\mathcal{S}(G)$ the *metric polytope* of the graph G and $\mathcal{C}(G)$ the *metric cone* of G.

Some notation

Alternatively, we let $K(V)$ denote the complete graph on a vertex set V, and $\mathcal{MP}(V)$ denote the corresponding metric polytope. If $x \in \mathbb{R}^E$ is a vector indexed by the edges of a graph $G = (V,E)$, we denote its coordinates alternatively by x_e, $x(e)$, x_{ij}, or $x(i,j)$, for an edge $e = (i,j)$ of G.

Let $G_t = (V_t, E_t)$ be a graph, for $t = 1, 2$. When the subgraph induced by $V_1 \cap V_2$ is a clique on k nodes in both G_1 and G_2, we define the *k-sum* of G_1 and G_2 as the graph $G = (V,E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

A vector is said to be *integral* if all its coordinates are integers. Given an integer $d \geqslant 2$, a vector x is called *$1/d$-integral* if dx is integral; if d is the smallest such integer, we also say that x has *denominator d*. A vector x is called *fully fractional* if none of its coordinates is integral. In particular, the terminology will be used in connection with the vertices of a polytope, i.e., we will speak about $1/d$-fractional vertices, fully fractional vertices, integral vertices, etc. We say that a vector $c \in \mathbb{R}^{\binom{n}{2}}$ is *supported* by a graph $G = (V,E)$ (or, with support in G) if $c_{ij} = 0$ for all $ij \notin E$.

2. Operations

The purpose of this section is to recall several useful operations on the polytope $\mathcal{S}(G)$.

2.1. Projection of the metric polytope

Let $G = (V,E)$ be a graph with node set V and edge set E. Given a subset S of V, $\delta_G(S)$ denotes the cut in G determined by S, i.e., the set $\delta_G(S) = \{ij \in E \mid i \in S, j \notin S\}$. The *cut polytope* $\mathcal{P}(G) \subset \mathbb{R}^E$ is defined as the convex hull of the incidence vectors of the cuts of G. The inequalities (1) and (2) are valid for the cut polytope $\mathcal{P}(G)$ [4].

It is easy to see that the nonredundant inequalities (1) are for the chordless cycles C of G, and the nonredundant inequalities (2) are for the edges e that do not belong to any triangle of G. In particular, the polytope $\mathcal{S}(K_n)$ coincides with the metric polytope $\mathcal{MP}_n$. In fact, in general, the polytope $\mathcal{S}(G)$ is the projection of $\mathcal{MP}_n$ on the space $\mathbb{R}^E$ [3]. More precisely, the following can be easily checked.

Lemma 2.1. *Let $G = (V,E)$ be a graph and let e be an edge of $K(V)$ which does not belong to G. Let $G + e$ denote the graph obtained by adding the edge e to G.*

(i) *If* $x \in \mathcal{MP}(V)$, *then the projection* x_E *of* x *on* $\mathbb{R}^E$ *belongs to* $\mathcal{S}(G)$.

(ii) *If* $y \in \mathcal{S}(G)$, *then there exists* $x \in \mathcal{S}(G+e)$ *whose projection* x_E *on* $\mathbb{R}^E$ *coincides with* y. *Moreover, if* y *is a* $1/d$*-integral vertex of* $\mathcal{S}(G)$, *then there exists such* x *which is a* $1/d$*-integral vertex of* $\mathcal{S}(G+e)$.

Corollary 2.2. *Given a graph* G *on* n *nodes, the following are equivalent.*

(i) G *is* $1/d$*-integral, i.e., all the vertices of the polytope* $\mathcal{S}(G)$ *are* $1/d$*-integral.*

(ii) *For every objective function* c *supported by* G, *the program* $\max(c^{\mathrm{T}}x \mid x \in \mathcal{MP}_n)$ *admits a* $1/d$*-integral optimizing vector.*

2.2. The switching operation

Given a cut $\delta_G(S)$, we define the *switching reflection* $r_{\delta_G(S)}$ of $\mathbb{R}^E$ by $y = r_{\delta_G(S)}(x)$, where $y_{ij} = 1 - x_{ij}$ if $ij \in \delta_G(S)$ and $y_{ij} = x_{ij}$ if $ij \in E \setminus \delta_G(S)$. The switching reflection preserves the cut polytope [4]; indeed, $r_{\delta_G(S)}$ maps the cut $\delta_G(T)$ to the cut $\delta_G(S \Delta T)$. In particular, the switching reflection $r_{\delta_G(S)}$ preserves faces and facets of the cut polytope $\mathcal{P}(G)$. Given $v \in \mathbb{R}^E$, $v_0 \in \mathbb{R}$, suppose that the inequality $v^{\mathrm{T}}x \leqslant v_0$ defines a face of $\mathcal{P}(G)$. Define $v^S \in \mathbb{R}^E$ by $v^S_{ij} = -v_{ij}$ if $ij \in \delta_G(S)$ and $v^S_{ij} = v_{ij}$ otherwise. By applying the switching reflection $r_{\delta_G(S)}$, we obtain the inequality $(v^S)^{\mathrm{T}}x \leqslant v_0 - \sum_{e \in \delta_G(S)} v_e$ which defines a face of $\mathcal{P}(G)$ of the same rank. Clearly, the inequalities (1) are preserved under any switching. Note also that the inequalities (1) are obtained from the inequalities (5) by switching. Therefore, the switching reflections preserve the polytope $\mathcal{S}(G)$. Thus we have the following lemma.

Lemma 2.3. *If* $x \in \mathcal{S}(G)$, *then* $y = r_{\delta(S)}(x) \in \mathcal{S}(G)$; *moreover,* y *is a vertex of* $\mathcal{S}(G)$ *whenever* x *is a vertex of* $\mathcal{S}(G)$.

In the case of the complete graph $G = K_n$, $n \neq 4$, it was proved that the switching reflections together with the permutations of the nodes are the only symmetries of the cut polytope $\mathcal{P}(K_n)$ [5] and of the metric polytope $\mathcal{S}(K_n)$ [11].

2.3. Extension and projection of vertices in $\mathcal{S}(G)$

If $x \in \mathcal{S}(G)$ and $G' = (V, E')$ is a subgraph of G, i.e., $E' \subseteq E$, then the projection $x_{E'}$ of x on $\mathbb{R}^{E'}$ belongs to $\mathcal{S}(G')$; we also say that x is an extension of $x_{E'}$.

In general, vertices are not preserved by projection. However, a nice feature of the polytope $\mathcal{S}(G)$ is that, essentially, we may always assume to deal with fully fractional vertices, since a vertex of $\mathcal{S}(G)$ with some coordinate 0 or 1 is the extension of a vertex x' of $\mathcal{S}(G')$, where G' comes from G by contracting the edge corresponding to the integral coordinate of x.

Let $G = (V, E)$ be defined on the n nodes $1, \ldots, n$ and suppose that $e = (1, n)$ is an edge of G. Let $G' = (V', E')$ denote the graph obtained by contracting the edge e in G; so, $V' = V \setminus \{n\}$. Let V_1, V_n denote, respectively, the set of nodes of $V \setminus \{1, n\}$ that are adjacent to the node 1 and n. Then, $E' = E \setminus \{(n,i) \mid i \in V_n\} \cup \{(1,i) \mid i \in V_n \setminus V_1\}$. Given $x' \in \mathbb{R}^{E'}$, we define its *0-extension* $x \in \mathbb{R}^E$ by

$$x_{ij} = \begin{cases} x'_{1j}, & \text{for } i = 1, \quad j \in V_1, \\ x'_{1j}, & \text{for } i = n, \quad j \in V_n, \\ 0, & \text{for } i = 1, \quad j = n, \\ x'_{ij}, & \text{elsewhere.} \end{cases} \tag{9}$$

Conversely, if $x \in \mathcal{S}(G)$ with $x_{1n} = 0$, then, by the triangle inequalities (7), $x_{1j} = x_{nj}$ holds for all $j \in V_1 \cap V_n$. Hence, defining $x' \in \mathbb{R}^{E'}$ as the projection of x on E', we have that x is the 0-extension of x' as defined by the above relation (9).

Similarly, we define the *1-extension* y of x' by

$$y_{ij} = \begin{cases} x'_{1j}, & \text{for } i = 1, \quad j \in V_1, \\ 1 - x'_{1j}, & \text{for } i = n, \quad j \in V_n, \\ 1, & \text{for } i = 1, \quad j = n, \\ x'_{ij}, & \text{elsewhere.} \end{cases} \tag{10}$$

Moreover, if $y \in \mathcal{S}(G)$ with $y_{1n} = 1$, then y is the 1-extension of its projection x' on E'.

Proposition 2.4. *Let $x \in \mathbb{R}^E$ be the 0-extension of $x' \in \mathbb{R}^{E'}$, i.e., x, x' satisfy (9). Then, $x \in \mathcal{S}(G)$ if and only if $x' \in \mathcal{S}(G')$; moreover, x is a vertex of $\mathcal{S}(G)$ if and only if x' is a vertex of $\mathcal{S}(G')$. The same holds also for x' and its 1-extension y.*

Proof. It is easy to check that $x \in \mathcal{S}(G)$ if and only if $x' \in \mathcal{S}(G')$. Let x' be a vertex of $\mathcal{S}(G')$. Let $\mathcal{B}'$ be a family of $|E'|$ linearly independent inequalities (1) and (2) that are satisfied at equality by x'. The inequalities $x_{1n} \geqslant 0$ and $x_{1j} - x_{1n} - x_{jn} \leqslant 0$, $2 \leqslant j \leqslant n-1$, are satisfied at equality by x. Together with $\mathcal{B}'$, we obtain a set of $|E|$ equalities for x which are linearly independent. Therefore, x is a vertex of $\mathcal{S}(G)$.

Assume now that x is a vertex of $\mathcal{S}(G)$. Let $\mathcal{B}$ be a family of $|E|$ linearly independent equalities chosen among (1) and (2) satisfied by x. We can suppose that $\mathcal{B}$ contains the equalities $x_{1n} = 0$ and $x_{1j} - x_{1n} - x_{nj} = 0$ for $j \in V_1 \cap V_n$. Then, the remaining equalities of $\mathcal{B}$ do not use the edge $(1,n)$; hence, they yield equalities for x'. Therefore, x' is a vertex of $\mathcal{S}(G')$.

The statement about y follows by applying switching and using Lemma 2.3. □

As a consequence, for many questions, we may restrict ourselves to fully fractional vertices. An easy application is that $\mathcal{S}(G)$ has no fractional $\frac{1}{2}$-integral vertices. Two other applications are formulated in Propositions 2.6 and 2.7.

Corollary 2.5. *The metric polytope has no fractional $\frac{1}{2}$-integral vertices.*

Proof. If $\mathcal{MP}_n$ has a fractional $\frac{1}{2}$-integral vertex, then there would exist a vertex of $\mathcal{MP}_m$, for some $m \leqslant n$, with all coordinates equal to $\frac{1}{2}$. But such vector satisfies none of the inequalities (7) and (8) at equality. □

Proposition 2.6. *If G is $1/d$-integral, then any minor of G is $1/d$-integral.*

Proof. Let G be a $1/d$-integral graph and let $e = (1, n)$ be an edge of G. It is obvious that the graph $G - e$ obtained by deleting the edge e is $1/d$-integral. We show that the graph G/e obtained by contracting the edge e is $1/d$-integral. We take the same notation as above for V_1, V_n and $G' = G/e$. Let w' be an objective function with support in G'. Define the objective w with support in G by

$$w_{ij} = \begin{cases} w'_{1j}, & \text{for } i = 1, \quad j \in V_1, \\ w'_{1j}, & \text{for } i = n, \quad j \in V_n, \\ -M, & \text{for } i = 1, \quad j = n, \\ w'_{ij}, & \text{elsewhere.} \end{cases} \tag{11}$$

By assumption, the linear program $\max(w^{\mathrm{T}}x \mid x \in \mathcal{MP}_n)$ admits a $1/d$-integral optimizing vector x. If we choose the constant M large enough, then $x_{1n} = 0$. Let x' denote the projection of x on $\mathbb{R}^{E'}$. Hence, x' is $1/d$-integral. It is easy to check that x' is an optimizing vector for the linear program $\max(w'^{\mathrm{T}}z \mid z \in \mathcal{MP}_{n-1})$. Therefore, the graph G' is $1/d$-integral. □

Proposition 2.7. *Assume G is $\frac{1}{3}$-integral. Then, for every objective $c \in \mathbb{R}^E_+$,*

$$\max(c^{\mathrm{T}}x \mid x \in \mathcal{S}(G)) \leqslant \tfrac{4}{3}\mathrm{mc}(G, c),$$

where $\mathrm{mc}(G, c)$ *denotes the maximum cut of the graph G with the weights c.*

Proof. The proof is by induction on n, the number of nodes of G. The statement holds trivially if $n \leqslant 2$. Let G be a $\frac{1}{3}$-integral graph on $n \geqslant 3$ nodes and let c be a nonnegative objective function supported by G. Let x be a vertex of $\mathcal{S}(G)$ which optimizes the program $\max(c^{\mathrm{T}}x \mid x \in \mathcal{S}(G))$.

If x is fully fractional, then $x_e = \frac{2}{3}$ for all edges. Therefore, $c^{\mathrm{T}}x = \frac{2}{3}\sum_{e\in E} c_e$. On the other hand, a trivial lower bound for the maximum cut in G is $\mathrm{mc}(G, c) \geqslant \frac{1}{2}\sum_{e\in E} c_e$. Therefore, Proposition 2.7 holds.

Suppose that $x_e = 0$ for some edge $e = (1, n)$. Let x' denote the projection of x on $\mathbb{R}^{E'}$, where E' is the edge set of $G' = G/e$. Consider the objective $c' \in \mathbb{R}^{E'}$ defined by

$$c'_{ij} = \begin{cases} c_{1j}, & \text{for } i = 1, \quad j \in V_1 \setminus V_n, \\ c_{nj}, & \text{for } i = n, \quad j \in V_n \setminus V_1, \\ c_{1j} + c_{nj}, & \text{for } i = 1, \quad j \in V_1 \cap V_n, \\ c_{ij}, & \text{elsewhere.} \end{cases} \tag{12}$$

It is easy to see that x' optimizes the objective function c' over $\mathcal{S}(G')$. By the induction hypothesis, the following inequality holds:

$$\max(c'^{\mathrm{T}}z \mid z \in \mathcal{S}(G')) \leqslant \tfrac{4}{3}\mathrm{mc}(G', c').$$

But, $\mathrm{mc}(G', c') \leqslant \mathrm{mc}(G, c)$ holds. Therefore, Proposition 2.7 holds.

Suppose now that $x_f \neq 0$ for all edges f of G, but $x_e = 1$ for some edge $e = (1,n)$. Let $G' = G - \{1,n\}$ with edge set E'. Let c', x' denote the projection of c, x on $\mathbb{R}^{E'}$, respectively. Since G' is $\frac{1}{3}$-integral, by the induction hypothesis, we have

$$\max(c'^{\mathrm{T}}z \mid z \in \mathcal{S}(G')) \leqslant \tfrac{4}{3}\mathrm{mc}(G',c').$$

This implies $c'^{\mathrm{T}}x' \leqslant \frac{4}{3}\mathrm{mc}(G',c')$. Let $\delta_{G'}(S)$ be an optimizing cut in G' for the weights c'. We have

$$\begin{aligned}\mathrm{mc}(G,c) &\geqslant \tfrac{1}{2}c^{\mathrm{T}}(\chi^{\delta_G(S\cup\{1\})} + \chi^{\delta_G(S\cup\{n\})}) \\ &= \mathrm{mc}(G',c') + c_{1n} + \frac{1}{2}\sum_{u\neq 1,n}(c_{1u} + c_{nu}).\end{aligned}$$

But, $x_{1u}, x_{nu} \leqslant \frac{2}{3}$ for all nodes $u \neq 1,n$ and $\mathrm{mc}(G',c') \geqslant \frac{3}{4}c'^{\mathrm{T}}x'$. Therefore,

$$\mathrm{mc}(G,c) \geqslant \tfrac{3}{4}c'^{\mathrm{T}}x' + c_{1n} + \frac{3}{4}\sum_{u\neq 1,n}(c_{1u}x_{1u} + c_{nu}x_{nu}).$$

We deduce that $\mathrm{mc}(G,c) \geqslant \frac{3}{4}c^{\mathrm{T}}x$. Therefore, Proposition 2.7 holds. □

Finally we observe how a new vertex of the metric polytope $\mathcal{S}(G)$ can be constructed by "gluing" together two given vertices of smaller metric polytopes. Let $G_i = (V_i, E_i)$ be a graph for $i = 1,2$ and assume that the subgraph induced by $V_1 \cap V_2$ is a clique on $k = |V_1 \cap V_2|$ nodes in both G_1 and G_2. Let $G = (V,E)$ denote the k-sum of G_1 and G_2. Let $x_i \in \mathbb{R}^{E_i}$, $i = 1,2$, such that x_1 and x_2 coincide on the edges of the common clique $K(V_1 \cap V_2)$. We can define $x \in \mathbb{R}^E$ by concatenating x_1 and x_2, i.e., setting $x(e) = x_i(e)$ for $e \in E_i$, $i = 1,2$. The vector x is called the *k-union* of x_1 and x_2. This operation will be used for proving results on k-sums of graphs in Sections 3 and 4.

Proposition 2.8. (i) *$x \in \mathcal{S}(G)$ if and only if $x_i \in \mathcal{S}(G_i)$ for $i = 1,2$.*
(ii) *If x_i is a vertex of $\mathcal{S}(G_i)$ for $i = 1,2$, then x is a vertex of $\mathcal{S}(G)$.*

Proof. The part (i) is clear. We verify (ii). Let x_i be a vertex of $\mathcal{S}(G_i)$, $i = 1,2$. We show that x is a vertex of $\mathcal{S}(G)$. Assume $x = \alpha y + (1-\alpha)z$ for some $0 < \alpha < 1$ and $y, z \in \mathcal{S}(G)$. Denote by y_i, z_i the projection of y, z on E_i for $i = 1,2$. We obtain that $x_i = \alpha y_i + (1-\alpha)z_i$, implying that $x_i = y_i = z_i$ for $i = 1,2$. Hence $x = y = z$ holds, yielding that x is a vertex. □

In particular, if x_i is a vertex of the metric polytope $\mathcal{MP}(V_i)$, for $i = 1,2$, such that x_1 and x_2 coincide on the edges of $K(V_1 \cap V_2)$, then their k-union x is a vertex of $\mathcal{S}(G)$, G denoting the k-sum of $K(V_1)$ and $K(V_2)$. By Lemma 2.1, x can be extended to a vertex y of the metric polytope $\mathcal{MP}(V_1 \cup V_2)$. Moreover, if x_1 and x_2 are $1/d$-integral, then y can be chosen $1/d$-integral. Such y is a common extension of both x_1 and x_2.

3. Sums with integral graphs

In this section, we study $1/d$-integrality with respect to the k-sum operation for graphs; d is an integer, $d \geqslant 3$. We prove the following results.

- $1/d$-integrality is preserved by 0- and 1-sums.
- The 2-sum of a $1/d$-integral graph and an integral graph is $1/d$-integral.
- The 3-sum of an integral graph and a rich $1/d$-integral graph (for the definition of a rich graph, see Definition 3.5 below) is $1/d$-integral.

Theorem 3.1. *The* 0- *and* 1-*sum operations preserve* $1/d$-*integrality.*

Proof. Let $G_i = (V_i, E_i)$ be a $1/d$-integral graph, for $i = 1,2$. We suppose first that G_1 and G_2 have no common node and let $G = (V, E)$ denote their 0-sum. Let x be a vertex of $\mathcal{S}(G)$ and let x_{E_i} denote the projection of x on $\mathbb{R}^{E_i}$ for $i = 1,2$. Let $\mathcal{B}$ be a system of $|E|$ linearly independent inequalities from the system (1), (2) that are satisfied at equality by x. Let $\mathcal{B}_i$ denote the subset of $\mathcal{B}$ consisting of the equations supported by G_i, for $i = 1,2$. Then, $|B| = |E| = |B_1| + |B_2| = |E_1| + |E_2|$, implying that $|B_i| = |E_i|$ for $i = 1,2$. Therefore, x_i is a vertex of $\mathcal{S}(G_i)$ and thus is $1/d$-integral, for $i = 1,2$. This shows that x is $1/d$-integral.

The proof is identical when G_1 and G_2 have one node in common. □

Theorem 3.2. *Let* G_1 *and* G_2 *be two graphs having an edge in common. If* G_1 *is* $1/d$-*integral and* G_2 *is integral, then their* 2-*sum is* $1/d$-*integral.*

Proof. Take $G_i = (V_i, E_i)$, for $i = 1,2$, and let f denote the common edge of G_1 and G_2. Let $G = (V, E)$ denote the 2-sum of G_1 and G_2. We show that G is $1/d$-integral, i.e., that every vertex of $\mathcal{S}(G)$ is $1/d$-integral. Let x be a vertex of $\mathcal{S}(G)$ and let x_{E_i} denote the projection of x on $\mathbb{R}^{E_i}$, for $i = 1,2$. If $x_f = 0$ or 1, then we can contract the edge f. Namely, then x is a trivial extension of a vertex y of $\mathcal{S}(G/f)$. But, the graph G/f can be seen as the 0-sum of the graphs G_1/f and G_2/f. By Theorem 3.1, y is $1/d$-integral. Therefore, x is $1/d$-integral.

We can now assume that $x_f \neq 0,1$. Let $\mathcal{B}$ be a family of $|E|$ linearly independent equalities from the system (1), (2) satisfied by x. Let $\mathcal{B}_i$ denote the subset of $\mathcal{B}$ consisting of those equalities that are supported by G_i, for $i = 1,2$. Since $0 < x_f < 1$, the families $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjoint and, thus, $|E| = |\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| = |E_1| + |E_2| - 1$. Therefore, $|E_i| - 1 \leqslant |\mathcal{B}_i| \leqslant |E_i|$, for $i = 1,2$. We distinguish two cases.

First, suppose that $|\mathcal{B}_2| = |E_2|$. Then, x_{E_2} is a vertex of $\mathcal{S}(G_2)$ and, thus, since G_2 is integral, x_{E_2} is $(0,1)$-valued, in contradiction with the assumption that $x_f \neq 0,1$.

Suppose now that $|\mathcal{B}_2| = |E_2|-1$. Then, $\mathcal{B}_1 = |E_1|$; hence, x_{E_1} is a vertex of $\mathcal{S}(G_1)$ and, thus, is $1/d$-integral. On the other hand, since it satisfies $|E_2| - 1$ linearly independent equalities, x_{E_2} can be written as the convex combination of two vertices of $\mathcal{S}(G_2)$. Hence, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)}$, where $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ and $\delta(A)$, $\delta(B)$ are two cuts in G_2. Then, $x_f = \alpha$ or $x_f = \beta$; hence, α, β and, thus, x_{E_2} are $1/d$-integral.

Therefore, x is $1/d$-integral. □

Corollary 3.3. *Every subdivision of a $1/d$-integral graph is $1/d$-integral.*

Proof. Let e be an edge of G which should be subdivided. Consider the 2-sum of G with a triangle along the edge e. Then delete the edge e from the 2-sum. The resulting graph is the required subdivision of G. It is $1/d$-integral by Theorem 3.2 and Proposition 2.6. □

Remark 3.4. The 2-sum operation does not preserve $1/d$-integrality in general. As a counterexample, consider the graph G obtained by taking the 2-sum of two copies of K_5; K_5 is $\frac{1}{3}$-integral, but we construct below a $\frac{1}{6}$-integral vertex of $\mathcal{S}(G)$.

We use the following notation. If $K_{S,T}$ denotes the complete bipartite graph with node sets S, T, then $x(K_{S,T})$ takes the value $\frac{1}{3}$ on the edges of $K_{S,T}$ and the value $\frac{2}{3}$ on the other edges. Recall that $x(K_{S,T})$ is a vertex of $\mathcal{MP}_n$, $n = |S| + |T| \geqslant 5$ [2].

Consider two copies G_1 and G_2 of K_5 defined, respectively, on the node sets $\{1,2,3,4,5\}$ and $\{1,2,6,7,8\}$. G is their 2-sum along the edge $(1,2)$. We define $y \in \mathcal{S}(G)$ as follows: its projection on the edge set of G_1 is $x(K_{\{1,5\},\{2,3,4\}})$ and its projection on the edge set of G_2 is $\frac{1}{2}(x(K_{\{1,2,8\},\{6,7\}}) + \chi^{\delta(\{1,2,6\})})$. So, y takes the values $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{5}{6}$. It is easy to check that y is a vertex of $\mathcal{S}(G)$. Indeed, there are altogether nineteen triangle equalities satisfied by y (ten on G_1 and nine on G_2) and they are linearly independent.

We say that a triangle (i,j,k) supports a triangle equality for a vector x if at least one of the four inequalities (7) or (8) is satisfied as equality by x.

Definition 3.5. Call a graph G *rich* if, for every vertex x of $\mathcal{S}(G)$, each triangle of G supports at least one triangle equality for x.

Clearly, every subgraph of a rich graph is rich. For example, K_6 is rich (see Section 5). Therefore, every graph on at most six nodes is rich. Also, every integral graph is rich (in fact, for every vertex, each triangle supports three triangle equalities!).

Note that a $\frac{1}{3}$-integral graph G is rich if no vertex x of $\mathcal{S}(G)$ satisfies $x_{ij} = x_{ik} = x_{jk} = \frac{1}{3}$, or $x_{ij} = x_{ik} = \frac{2}{3}$, $x_{jk} = \frac{1}{3}$, for some triangle (i,j,k) of G.

Remark 3.6. It follows easily from the proofs of Theorems 3.1 and 3.2 that the 0- and 1-sums of rich $1/d$-integral graphs are $1/d$-integral and rich, while the 2-sum of a rich $1/d$-integral graph and an integral graph is $1/d$-integral and rich.

We see below that Theorem 3.2 can be extended to the 3-sum case if we make the additional assumption that the graphs are rich.

Theorem 3.7. *Let G_1 and G_2 be two graphs having a triangle in common. If G_1 is $1/d$-integral and rich and if G_2 is integral, then their 3-sum is $1/d$-integral and, moreover, rich.*

Proof. Take $G_i = (V_i, E_i)$, for $i = 1,2$, and denote by $\Delta = (1,2,3)$ the common triangle to G_1 and G_2. Let $G = (V, E)$ denote the 3-sum of G_1 and G_2. We show that every vertex of $\mathcal{S}(G)$ is $1/d$-integral. Let x be a vertex of $\mathcal{S}(G)$ and let x_{E_i} denote the projection of x on $\mathbb{R}^{E_i}$, for $i = 1,2$. If $x_e = 0$ or 1 for some edge of Δ, then, by contraction of this edge, we can apply Theorem 3.2 on the 2-sum and deduce that x is $1/d$-integral. Hence, we can now assume that $x_e \neq 0,1$ for each edge $e \in \Delta$. Let $\mathcal{B}$ be a family of $|E|$ linearly independent equalities for x and let $\mathcal{B}_i$ denote the subset of the equalities in $\mathcal{B}$ that are supported by G_i, for $i = 1,2$. We distinguish two cases depending whether Δ supports a triangle equality for x or not.

We first suppose that Δ supports a triangle equality for x. Without loss of generality we can assume that $x_{12} + x_{13} + x_{23} = 2$ (if not, apply switching). We can suppose that this equality belongs to $\mathcal{B}$. Hence, $|E| = |\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| - 1 = |E_1| + |E_2| - 3$, implying that $|E_i| - 2 \leqslant |\mathcal{B}_i| \leqslant |E_i|$, for $i = 1,2$. But $|\mathcal{B}_2| \neq |E_2|$, else x_{E_2} would be a vertex of $\mathcal{S}(G_2)$ and, thus, x_{E_2} would be integral.

If $|\mathcal{B}_2| = |E_2| - 1$, then x_{E_2} is the convex combination of two vertices of $\mathcal{S}(G_2)$, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)}$, where $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ and $\delta(A)$, $\delta(B)$ are two cuts in G_2. Both cuts $\delta(A)$, $\delta(B)$ satisfy the triangle equality: $x_{12} + x_{13} + x_{23} = 2$. Hence, at least one edge e of Δ belongs to both cuts $\delta(A)$, $\delta(B)$, implying that $x_e = 1$, a contradiction.

If $|\mathcal{B}_2| = |E_2| - 2$, then $|\mathcal{B}_1| = |E_1|$; hence, x_{E_1} is a vertex of $\mathcal{S}(G_1)$ and, thus, x_{E_1} is $1/d$-integral. On the other hand, x_{E_2} is the convex combination of three vertices of $\mathcal{S}(G_2)$, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)} + \gamma\chi^{\delta(C)}$, where $\alpha, \beta, \gamma \geqslant 0$, $\alpha + \beta + \gamma = 1$ and $\delta(A)$, $\delta(B)$, $\delta(C)$ are cuts in G_2. From the fact that the three cuts $\delta(A)$, $\delta(B)$, $\delta(C)$ satisfy the equality $x_{12} + x_{13} + x_{23} = 2$ and that $x_e \neq 0,1$ for each edge $e \in \Delta$, we deduce that $\delta(A) \cap \Delta = \{12, 13\}$, $\delta(B) \cap \Delta = \{12, 23\}$ and $\delta(C) \cap \Delta = \{13, 23\}$. Hence, $x_{12} = \alpha + \beta$, $x_{13} = \alpha + \gamma$ and $x_{23} = \beta + \gamma$. Setting $x_{12} = a/d$, $x_{13} = b/d$, $x_{23} = 2 - (a+b)/d$ for some integers a, b, we obtain that $\alpha = (a+b)/d - 1$, $\beta = 1 - b/d$ and $\gamma = 1 - a/d$. Therefore, x_{E_2} and, thus, x are $1/d$-integral.

We now suppose that Δ does not support any triangle equality for x. Hence, $|E| = |\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| = |E_1| + |E_2| - 3$, implying that $|E_i| - 3 \leqslant |\mathcal{B}_i| \leqslant |E_i|$, for $i = 1,2$. But, $|\mathcal{B}_2| \neq |E_2|$, since $x_e \neq 0,1$ for each edge $e \in \Delta$, and $|\mathcal{B}_1| \neq |E_1|$, since G_1 is rich (else, x_{E_1} would be a vertex of $\mathcal{S}(G_1)$ with the triangle Δ supporting no equality for x_{E_1}). Hence, $|\mathcal{B}_2| = |E_2| - 1$ or $|E_2| - 2$.

If $|\mathcal{B}_2| = |E_2| - 1$, then x_{E_2} is the convex combination of two cuts in G_2, implying easily that $x_e = 0$ or 1 for some edge $e \in \Delta$.

If $|\mathcal{B}_2| = |E_2| - 2$, then x_{E_2} is the convex combination of three vertices of $\mathcal{S}(G_2)$, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)} + \gamma\chi^{\delta(C)}$, where $\alpha, \beta, \gamma \geqslant 0$, $\alpha + \beta + \gamma = 1$ and $\delta(A)$, $\delta(B)$, $\delta(C)$ are cuts in G_2. Since $x_e \neq 0,1$ for each edge $e \in \Delta$, no edge of Δ belongs to all three cuts, and every edge belongs to at least one of them. Hence, we have (up to permutation) only the following two possibilities:

- either $\delta(A) \cap \Delta = \emptyset$, $\delta(B) \cap \Delta = \{12, 13\}$, $\delta(C) \cap \Delta = \{12, 23\}$; then, $x_{12} = \beta + \gamma$, $x_{13} = \beta$, $x_{23} = \gamma$, implying that $x_{12} - x_{13} - x_{23} = 0$;
- or $\delta(A) \cap \Delta = \{12, 13\}$, $\delta(B) \cap \Delta = \{12, 23\}$, $\delta(C) \cap \Delta = \{13, 23\}$; then, $x_{12} = \alpha + \beta$, $x_{13} = \alpha + \gamma$, $x_{23} = \beta + \gamma$, implying that $x_{12} + x_{13} + x_{23} = 2$.

In both cases, we have a contradiction with our assumption that Δ supports no triangle equality for x. This concludes the proof that G is $1/d$-integral.

Finally, we verify that G is rich, i.e., that, for each vertex x of $\mathcal{S}(G)$, every triangle supports an equality for x. Take a vertex x of $\mathcal{S}(G)$. Looking through the above proof, we see that either x is some trivial extension, or x_{E_2} is the convex combination of three cuts of G_2 while x_{E_1} is a vertex of $\mathcal{S}(G_1)$. Hence, each triangle of G supports an equality for x; in the first case, apply Remark 3.6 and, in the second case, check it directly. □

The motivation for the notion of rich graphs comes from the 3-sum operation. Namely, we have the following result.

Proposition 3.8. *Let G be a $\frac{1}{3}$-integral graph. If G is not rich, then the 3-sum of G with K_4 is not $\frac{1}{3}$-integral.*

Proof. If G is not rich, then there exists a vertex x of $\mathcal{S}(G)$ and a triangle $\Delta = (1,2,3)$ of G which supports no equality for x. Up to switching, we can suppose that $x_{12} = x_{13} = x_{23} = \frac{1}{3}$. Consider K_4 on the node set $\{1,2,3,u_0\}$ where $u_0 \notin V(G)$. Let H denote the 3-sum of G and K_4 along Δ. Let $y \in \mathcal{S}(H)$ be defined by $y_e = x_e$ for every edge e of G and $y_{u_01} = y_{u_02} = y_{u_03} = \frac{1}{6}$. Then, y is a vertex of $\mathcal{S}(H)$ which is not $\frac{1}{3}$-integral. □

As an application of the 3-sum operation, we obtain that the ΔY-operation preserves $1/d$-integral rich graphs. The ΔY-operation consists of replacing a triangle $\Delta = (1,2,3)$ in a graph by a claw, i.e., deleting the triangle Δ from G and adding a new node u_0 to G adjacent to the nodes 1, 2 and 3.

Corollary 3.9. *The ΔY-operation preserves the class of $1/d$-integral rich graphs.*

Proof. Let G be a $1/d$-integral rich graph and let $\Delta = (1,2,3)$ be a triangle of G. Consider K_4 defined on the node set $\{1,2,3,u_0\}$. By Theorem 3.7, the 3-sum of G and K_4 along the triangle Δ is $1/d$-integral and rich. Then, delete the edges of the triangle Δ. The resulting graph is $1/d$-integral and rich; it is precisely the ΔY-transform of G. □

For instance, the graph K_6 is $\frac{1}{3}$-integral and rich (see the list of its vertices in Section 4.1). Hence, every graph obtained from K_6 by applying the ΔY-operation is $\frac{1}{3}$-integral and rich. One such graph is the Petersen graph.

4. Forbidden minors for $\frac{1}{3}$-integrality

The purpose of this section is to present some minimal forbidden minors for $\frac{1}{3}$-integrality. As a consequence, we can characterize the $\frac{1}{3}$-integral graphs up to seven nodes. We also give the full description of the metric polytope $\mathcal{MP}_n$ for $n \leqslant 6$.

4.1. Small metric polytopes

We recall the description of the metric polytopes of small dimension.

For $n = 4$, $\mathcal{MP}_4$ has $8 = 2^3$ vertices, all of them integral.

For $n = 5$, $\mathcal{MP}_5$ has 32 vertices consisting of 2^4 integral vertices and 2^4 $\frac{1}{3}$-integral vertices obtained by switching of $(\frac{2}{3}, \ldots, \frac{2}{3})$.

For $n = 6$, $\mathcal{MP}_6$ has 544 vertices consisting of 2^5 integral vertices, 2^5 $\frac{1}{3}$-integral vertices obtained by switching of $(\frac{2}{3}, \ldots, \frac{2}{3})$ and 480 vertices which are the trivial extensions of the $\frac{1}{3}$-integral vertices of $\mathcal{MP}_5$.

For $n = 7$, Grishukhin [9] has computed all the extreme rays of the metric cone $\mathcal{MC}_7$. He found that there are thirteen distinct classes (up to permutation and switching) of extreme rays. We do not know the complete description of the vertices of $\mathcal{MP}_7$.

Clearly, every extreme ray of the metric cone $\mathcal{MC}_n$ determines a vertex of the metric polytope $\mathcal{MP}_n$ which is the intersection of the ray with some triangle facet (8). In [12], it is conjectured that every vertex of $\mathcal{MP}_n$ can be obtained, up to switching, in this way. Equivalently, it is conjectured that every fractional vertex of $\mathcal{MP}_n$ is adjacent to some integral vertex. This conjecture holds for $\mathcal{MP}_n$, $n \leqslant 6$, and for several classes of graphical vertices of $\mathcal{MP}_n$ constructed in [11].

It follows from the explicit description of $\mathcal{MP}_n$, $n = 5, 6$, that K_5 and K_6 are $\frac{1}{3}$-integral and rich. Therefore, every graph on at most six nodes is $\frac{1}{3}$-integral and rich. As a consequence, any graph on seven nodes which has a node of degree at most 3 is $\frac{1}{3}$-integral and rich (from Remark 3.6 and Theorem 3.7). K_7 is not rich; many examples of vertices of $\mathcal{MP}_7$, for which some triangle exists which supports no equality, can be found in the list of vertices from [9].

We conclude with a remark on the possible denominators for the fractional vertices of the metric polytope. By Corollary 2.5, no vertex of $\mathcal{MP}_n$ has denominator 2. On the other hand, vertices can be constructed with arbitrary denominator $d \geqslant 3$.

Proposition 4.1. *For every $d \geqslant 3$ and for every n sufficiently large, e.g., $n \geqslant 3d - 1$, there exists a vertex of $\mathcal{MP}_n$ with denominator d.*

Proof. We first recall a construction from [2]. Let $G = (V, E)$ be a graph and $G' = (V', E')$ be a copy of G, where $V = \{1, \ldots, n\}$ and $V' = \{1', \ldots, n'\}$. Consider the graph G^* with node set $V \cup V' \cup \{u_e \mid e \in E\}$ constructed as follows. The edge set of G^* consists of the edges of G, the edges of G' and the following new edges. Join each node $i \in V$ to its twin $i' \in V'$. For each edge $e = (i, j)$ of G with $i < j$, join i and j' to u_e.

Let d_G denote the path metric of G, where $d_G(i, j)$ is the length of a shortest path from i to j in G, for $i, j \in V$. Set $\tau(G) = \max(d_G(i, j) + d_G(i, k) + d_G(j, k) \mid 1 \leqslant i < j < k \leqslant n)$. Define similarly d_{G^*} and $\tau(G^*)$. It is easy to check that $\tau(G^*) = \tau(G) + 2$ holds.

Define the vector $x_{G^*} \in \mathcal{MP}_N$, $N = 2n + |E|$, by $x_{G^*} = \{2/\tau(G^*)\} d_{G^*}$. Then, it follows from [2] that x_{G^*} is a vertex of $\mathcal{MP}_N$. Its denominator is $\tau(G) + 2$ or

$\frac{1}{2}(\tau(G)+2)$, according to the parity of $\tau(G)$.

Let $d \geqslant 3$ be an integer. Let G be a path on d nodes, then $\tau(G) = 2(d-1)$ and, therefore, x_{G^*} is a vertex of $\mathcal{MP}_{3d-1}$ with denominator d. Trivial extensions of x_{G^*} are vertices of $\mathcal{MP}_n$ with denominator d for all $n \geqslant 3d-1$. □

For instance, the polytope $\mathcal{MP}_7$ has vertices with denominators 3, 4, 5, 6 and 7.

4.2. Forbidden minors

We have shown in Proposition 2.6 that $\frac{1}{3}$-integrality is preserved by taking minors. Robertson and Seymour [17] have proved that, for every minor closed class of graphs, there are only finitely many minimal forbidden minors. Thus arises the problem of finding the minimal forbidden minors for the class of $\frac{1}{3}$-integral graphs. We present four of them. This permits us to characterize the $\frac{1}{3}$-integral graphs on seven nodes.

We first give some preliminary results.

Lemma 4.2. *Let G be a graph and let x be a fully fractional $\frac{1}{3}$-integral vertex of $\mathcal{S}(G)$. The only inequalities* (1) *which are satisfied at equality by x are those where C is a triangle of G.*

Proof. Let F, C be such that the inequality (1) is satisfied as equality by x. Let a (respectively b) denote the number of edges $e \in F$ (respectively $e \in C \setminus F$) for which $x_e = \frac{1}{3}$. From the equality $x(F) - x(C \setminus F) = |F| - 1$, we deduce that $\frac{1}{3}a + \frac{2}{3}(|F|-a) - \frac{1}{3}b - \frac{2}{3}(|C|-|F|-b) = |F|-1$. We obtain that $|F| = 2|C|+a-b-3$. But, $a \geqslant 0$ and $b \leqslant |C| - |F|$, from which we deduce that $|C| \leqslant 3$, i.e., C is a triangle. □

Lemma 4.3. *Let G be a graph and let x be a fully fractional vertex of $\mathcal{S}(G)$. For each cycle C of G, at most one of the inequalities* (1) *supported by C is satisfied at equality by x.*

Proof. Let C be a cycle of G and let F, F' be two distinct subsets of C of odd cardinality. Let $x \in \mathcal{S}(G)$ satisfy the equalities $x(F) - x(C \setminus F) = |F| - 1$ and $x(F') - x(C \setminus F') = |F'| - 1$. We obtain that $|F \cap F'| - x(F \cap F') + \frac{1}{2}(|F \Delta F'| - 2) + x(C \setminus (F \cup F')) = 0$. Therefore, $|F \cap F'| = x(F \cap F')$, $|F \Delta F'| = 2$ and $x(C \setminus (F \cup F')) = 0$. This implies that $x_e = 1$ for $e \in F \cap F'$ and $x_e = 0$ for $e \in C \setminus (F \cup F')$. If x is fully fractional, then $F \cap F' = \emptyset$, $C = F \cup F'$, implying that $|C| = 2$, a contradiction. □

Corollary 4.4. *Let $G = (V, E)$ be a $\frac{1}{3}$-integral graph on seven nodes. If G has at most $|E|$ distinct triangles, then G is rich.*

Proof. Let x be a vertex of $\mathcal{S}(G)$. We show that each triangle of G supports an equality for x. Suppose first that $x_e = 0$ or 1 for some edge $e \in E$. Let Δ be a triangle of G. If Δ contains the edge e, then Δ trivially supports an equality for x. Otherwise Δ is a triangle

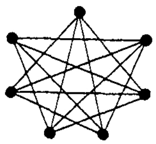

Fig. 1. G_1.

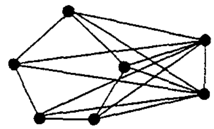

Fig. 2. G_2.

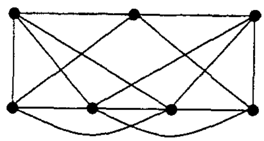

Fig. 3. G_3.

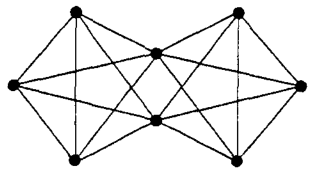

Fig. 4. G_4.

in the graph G/e, obtained by contracting the edge e. Since G/e is on six nodes, it is rich. Hence, Δ supports an equality for the projection of x on G/e. Therefore, Δ also supports an equality for x. We suppose now that x is fully fractional. From Lemmas 4.2 and 4.3, we deduce that G has exactly $|E|$ triangles and each of them supports an equality for x. This shows that G is rich. □

In the following result, we classify the graphs on seven nodes that are $\frac{1}{3}$-integral. If E is a subset of edges of K_7, $K_7 - E$ denotes the graph obtained by deleting from K_7 the edges of E. Set

$$G_1 := K_7 - C_7, \qquad G_2 := K_7 - C_5,$$
$$G_3 := K_7 - (C_4 + P_3), \qquad G_4 := K_8 - (K_{3,3} + K_2).$$

So, G_1, G_2 are, respectively, obtained by deleting a cycle on seven and five nodes from K_7; they are shown in Figs. 1 and 2. The graph G_3 is obtained by taking the 3-sum of two copies of K_5 along a triangle and then deleting two edges of this triangle; it is shown in Fig. 3. The graph G_4 is obtained by taking the 2-sum of two copies of K_5 along an edge and then deleting this edge; it is shown in Fig. 4.

Note that $G_4 - v$ is planar if v is any of the two nodes common to the two K_5's composing G_4. Hence, the suspensions of planar graphs are not $\frac{1}{3}$-integral in general.

Theorem 4.5. (i) *The graphs G_1, G_2, G_3 and G_4 are minimal forbidden minors for the class of $\frac{1}{3}$-integral graphs.*

(ii) *Every graph on seven nodes not containing G_1, G_2 or G_3 is $\frac{1}{3}$-integral and, moreover, rich.*

Proof. The proof of (i) relies partly on computer check. Namely, we checked by computer that G_1, G_2, G_3 are, respectively, $\frac{1}{5}$-, $\frac{1}{5}$-, $\frac{1}{4}$-integral and that the graph $K_7 - C_3$ is $\frac{1}{3}$-integral.

For each of the graphs G_1, G_2, G_3 and G_4, we give below a vertex x of $\mathcal{S}(G)$ which is not $\frac{1}{3}$-integral.

Let $x \in \mathbb{R}^{\binom{7}{2}}$ such that $x_{14} = x_{15} = x_{36} = x_{37} = \frac{1}{5}$, $x_{13} = x_{24} = x_{27} = x_{46} = x_{57} = \frac{2}{5}$, $x_{16} = x_{35} = \frac{3}{5}$, $x_{25} = x_{26} = x_{47} = \frac{4}{5}$. Then, x is a vertex of $\mathcal{S}(G_1)$ where $G_1 = K_7 - C_7$ and C_7 is the cycle $(1,2,3,4,5,6,7)$.

Let $x_{12} = x_{23} = x_{34} = x_{45} = x_{15} = x_{67} = \frac{4}{5}$, $x_{i6} = x_{i7} = \frac{3}{5}$ for $1 \leqslant i \leqslant 5$. Then, x is a vertex of $\mathcal{S}(G_2)$, where $G_2 = K_7 - C_5$ and C_5 is the cycle $(1,2,3,4,5)$.

Let $x_{13} = x_{14} = x_{25} = x_{36} = x_{46} = \frac{1}{4}$, $x_{12} = x_{34} = x_{67} = \frac{2}{4}$ and $x_{15} = x_{23} = x_{24} = x_{37} = x_{47} = x_{57} = \frac{3}{4}$. Then, x is a vertex of $\mathcal{S}(G_3)$, where $G_3 = K_7 - (C_4 + P_3)$, C_4 is the cycle $(1,7,2,6)$ and P_3 is the path $(3,5,4)$.

The graph $K_8 - K_{3,3}$ is obtained by taking the 2-sum of two copies of K_5 along an edge e. We gave in Remark 3.4 a $\frac{1}{6}$-integral vertex x of the polytope $\mathcal{S}(K_8 - K_{3,3})$. In fact, if we project out the edge e, the projection of x remains a vertex of $\mathcal{S}(K_8 - (K_{3,3} + e))$. Therefore, $G_4 = K_8 - (K_{3,3} + e)$ is not $\frac{1}{3}$-integral. On the other hand, it is easily seen that every minor of G_4 is $\frac{1}{3}$-integral.

We now verify that every minor of the graph $G = G_1$, G_2, G_3 is $\frac{1}{3}$-integral. This is clear for a contraction minor, since it is a subgraph of K_6. Let $G - e$ be a deletion minor. If the deleted edge e is adjacent to a node of degree at most 4 in G, then $G - e$ has a node of degree at most 3 and, hence, is $\frac{1}{3}$-integral. Therefore, every minor of G_1 is $\frac{1}{3}$-integral, since G_1 is regular of degree 4. All nodes of G_2 have degree 4 except two adjacent nodes which have degree 6. If e is the edge joining them, then $G_2 - e$ is planar and, therefore, is $\frac{1}{3}$-integral. All the nodes of G_3 have degree 4 except two adjacent nodes which have degree 5. If e is the edge joining them, then $G_3 - e$ is contained in $K_7 - C_3$ and, therefore, is $\frac{1}{3}$-integral. This shows the part (i) of Theorem 4.5.

We prove (ii). Let G be a graph on seven nodes that does not contain any of G_1, G_2, G_3 as a subgraph. If G has a node of degree at most 3, then G is $\frac{1}{3}$-integral and rich. So we can suppose that all the nodes of G have degree at least 4 in G. Hence, all nodes have degree at most 2 in the complement $\bar{G}$ of G, i.e., $\bar{G}$ is a disjoint union of cycles and paths. Since $\bar{G} \not\subseteq C_7$, $\bar{G}$ contains a cycle. If $\bar{G}$ contains a cycle of length 3, then G is contained in $K_7 - C_3$ and, therefore, G is $\frac{1}{3}$-integral. If $\bar{G}$ contains a cycle of length 4, then $\bar{G} = C_4 + C_3$, since $\bar{G}$ is not contained in $C_4 + P_3$. Therefore, G is again contained in $K_7 - C_3$. If $\bar{G}$ contains a cycle of length 5, then $\bar{G} = C_5 + K_2$. Therefore, G is integral since it is planar. If $\bar{G}$ contains a cycle of length 6, then $G = K_7 - C_6$ is $\frac{1}{3}$-integral. Indeed, $K_7 - C_6$ has fourteen chordless cycles (including eleven triangles and three cycles of length 4) and fifteen edges. By Lemma 4.3, every vertex of $\mathcal{S}(K_7 - C_6)$ has some integral coordinate and thus is $\frac{1}{3}$-integral, since it is the trivial extension of a vertex of the cycle polytope of a graph on six nodes.

In order to conclude the proof of (ii), we must show that G is rich. By the above argument, it suffices to verify that both $K_7 - C_3$ and $K_7 - C_6$ are rich. The graph $K_7 - C_6$ has eleven triangles; therefore, it is rich, by Corollary 4.4. We cannot apply Corollary 4.4 to show that $K_7 - C_3$ is rich since this graph has twenty-two triangles and eighteen edges. But it can be checked directly as follows.

Let $G = K_7 - C_3$ be defined on the nodes $\{1,2,3,4,5,6,7\}$ and the deleted triangle C_3 be $(5,6,7)$. Let x be a vertex of $\mathcal{S}(G)$. If x has some integral component, then every triangle of G supports an equality for x. Let x be fully fractional, so its components are $\frac{1}{3}$, $\frac{2}{3}$. Call a triangle Δ of G *bad* if it supports no equality for x, i.e., x takes the values $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, or $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ on the edges of Δ. At most four triangles of G are bad. There are four triangles on the nodes $\{1,2,3,4\}$. Among them, the number of bad triangles can be zero, two or four. If the four triangles on $\{1,2,3,4\}$ are bad, then $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = x_{34} = \frac{1}{3}$ (up to switching). Clearly, no such x exists for which all the remaining eighteen triangles of G support an equality. If two of the triangles on $\{1,2,3,4\}$ are bad then, e.g., $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = \frac{1}{3}$, $x_{34} = \frac{2}{3}$ (up to switching). It is again impossible to find such x for which at most two of the remaining eighteen triangles are bad. Let the four triangles on $\{1,2,3,4\}$ support an equality for x, i.e., $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = x_{34} = \frac{2}{3}$ (up to switching). We look at the possibilities for x_{ij}, $1 \leqslant i \leqslant 4$, $5 \leqslant j \leqslant 7$. Fix $j \in \{5,6,7\}$. If $x_{ij} = \frac{1}{3}$ for exactly one of the edges $1j$, $2j$, $3j$, $4j$, say $x_{1j} = \frac{1}{3}$, then no triangle equality covers the edge $1j$, contradicting the fact that x is a vertex. The same holds if $x_{ij} = \frac{1}{3}$ for three of the edges $1j$, $2j$, $3j$, $4j$. If $x_{ij} = \frac{1}{3}$ for two (respectively four) of the edges $1j$, $2j$, $3j$, $4j$, then four (respectively six) of the six triangles going through node j are bad. This contradicts the fact that x is a vertex since the equalities supported by triangles on $\{1,2,3,4,5,6,7\} \setminus \{j\}$ have rank at most 14. □

Remark 4.6. The class $\mathcal{G}$ consisting of the graphs G for which all extreme rays of the cone $\mathcal{C}(G)$ are $(0,1,2)$-valued has been characterized in [18]. Namely, a graph G belongs to $\mathcal{G}$ if and only if G has no minor H_6 or $K_7 - (K_{3,3} + P_2)$ (recall that H_6 is the graph obtained by equally splitting a node of K_5). Equivalently, a 2-connected graph G belongs to $\mathcal{G}$ if and only if G is the 2-sum of a graph without K_5 minor and of a copy of K_5. Therefore, by Theorems 3.1 and 3.2, every graph in $\mathcal{G}$ is $\frac{1}{3}$-integral.

5. Box $\frac{1}{3}$-integral graphs

We have seen that the 2-sum operation does not preserve $\frac{1}{3}$-integrality. This leads us to the study of a stronger notion, box $\frac{1}{3}$-integrality, which is preserved by 2-sums. Box $\frac{1}{3}$-integrality is a stronger property than $\frac{1}{3}$-integrality. Namely, we ask not only that the polytope $\mathcal{S}(G)$ has all its vertices $\frac{1}{3}$-integral, but also that each slice of $\mathcal{S}(G)$ determined by adding the box constraints $\ell_e \leqslant x_e \leqslant u_e$ for $e \in E$ has only $\frac{1}{3}$-integral vertices, for all choices of $\frac{1}{3}$-integral bounds ℓ and u.

Definition 5.1. The graph G is *box $\frac{1}{3}$-integral* if the polytope

$$\mathcal{S}(G) \cap \{x \mid \ell_e \leqslant x_e \leqslant u_e, e \in E\}$$

is empty or has only $\frac{1}{3}$-integral vertices, for all ℓ and u belonging to $\{0, \frac{1}{3}, \frac{2}{3}, 1\}^E$.

Equivalently, the graph $G = (V, E)$ is box $\frac{1}{3}$-integral if, for every $\ell, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^E$ such that $\mathcal{MP}_n \cap \{x \mid \ell_e \leqslant x_e \leqslant u_e, e \in E\} \neq \emptyset$ and for every objective function c

supported by G, the linear program $\max(c^{\mathrm{T}}x \mid x \in \mathcal{MP}_n, \ell_e \leqslant x_e \leqslant u_e, e \in E)$ admits a $\frac{1}{3}$-integral optimizing vector.

We are able to characterize the class of box $\frac{1}{3}$-integral graphs. Recall that a graph G is said to be *series parallel* if G is a subgraph of a graph which can be obtained by iterated 2-sums of a collection of copies of K_3. Equivalently, G is series parallel if G does not contain any K_4 minor.

Theorem 5.2. *A graph G is box $\frac{1}{3}$-integral if and only if G is series parallel.*

The proof of Theorem 5.2 consists of the following steps:

- box $\frac{1}{3}$-integrality is preserved by 0-, 1- and 2-sums;
- K_3 is box $\frac{1}{3}$-integral, but K_4 is not box $\frac{1}{3}$-integral.

The fact that 0- and 1-sums preserve box $\frac{1}{3}$-integrality is proved in the same way as for $\frac{1}{3}$-integrality. The result about the 2-sum needs two preliminary lemmas.

In the next lemma, we show that every point in a slice of the metric polytope can be rounded to a $\frac{1}{3}$-integral point of the slice.

Lemma 5.3. *Take $\ell, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{\binom{n}{2}}$ such that $\mathcal{MP}_n \cap \{x \mid \ell \leqslant x \leqslant u\} \neq \emptyset$. Given $x \in \mathcal{MP}_n \cap \{x \mid \ell \leqslant x \leqslant u\}$, there exists $y \in \mathcal{MP}_n \cap \{x \mid \ell \leqslant x \leqslant u\}$ such that y satisfies*

(i) $y_e = x_e$ *if* $x_e \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$,
(ii) $y_e = \frac{1}{3}$ *if* $0 < x_e < \frac{1}{3}$,
(iii) $y_e = \frac{2}{3}$ *if* $\frac{2}{3} < x_e < 1$,
(iv) $y_e \in \{\frac{1}{3}, \frac{2}{3}\}$ *if* $\frac{1}{3} < x_e < \frac{2}{3}$.

Proof. We will proceed by induction on $n \geqslant 3$. The statement holds easily if $n = 3$. Let $n \geqslant 4$ be given. We distinguish two cases.

Assume first that $0 < x_e < 1$ for all edges. Then, we define y by

$$y_e = \begin{cases} \frac{1}{3}, & \text{if } 0 < x_e \leqslant \frac{1}{3}, \\ \frac{2}{3}, & \text{if } \frac{2}{3} \leqslant x_e < 1, \\ \frac{1}{3} \text{ or } \frac{2}{3}, & \text{if } \frac{1}{3} < x_e < \frac{2}{3}. \end{cases}$$

Clearly, $y \in \mathcal{MP}_n$ and $\ell \leqslant y \leqslant u$.

Assume now that $x_e = 0, 1$ for some edge e; we can consider only the case of $x_e = 0$ due to switching. Let $e = (1, n)$. Since $x_{1n} = 0$, $x_{1i} = x_{in}$ for all $2 \leqslant i \leqslant n-1$. Set $\ell'_{1i} = \max(\ell_{1i}, \ell_{in})$ and $u'_{1i} = \min(u_{1i}, u_{in})$ for $2 \leqslant i \leqslant n-1$, and $\ell'_{ij} = \ell_{ij}$, $u'_{ij} = u_{ij}$ otherwise. Let x' denote the projection of x in $\mathcal{MP}_{n-1}$. Clearly, x' satisfies $\ell' \leqslant x' \leqslant u'$. By the induction hypothesis, there exists y' satisfying the statement for x' and the bounds ℓ' and u'. Let y be the 0-extension of y'. Then, y satisfies the statement for x and the bounds ℓ and u. □

The following lemma deals with sensitivity of optimization over slices of the metric polytope when the objective function varies on a single edge.

Lemma 5.4. *Take* ℓ *and* $u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{\binom{n}{2}}$ *such that* $\mathcal{MP}_n \cap \{x \mid \ell \leqslant x \leqslant u\} \neq \emptyset$ *and* $c \in \mathbb{R}^{\binom{n}{2}}$. *For* $t \in \mathbb{R}$, *define* $c(t) \in \mathbb{R}^{\binom{n}{2}}$ *by* $c(t)_e = c_e$ *for all edges* e *except* $c(t)_f = c_f + t$ *for a fixed edge* f. *For* $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, *we define the set* M_α *consisting of the scalars* $t \in \mathbb{R}$ *for which the linear program* $\max(c(t)^{\mathrm{T}}x \mid x \in \mathcal{MP}_n,\ \ell \leqslant x \leqslant u)$ *admits a* $\frac{1}{3}$*-integral optimizing vector* x *satisfying* $x_f = \alpha$. *Then, the set* M_α *is a closed interval.*

Proof. We show that M_α is convex. Let $t, t+s \in M_\alpha$ and $0 \leqslant \lambda \leqslant 1$ be given. We show that $t + \lambda s \in M_\alpha$.

Let C_0 (respectively, C_1, C) denote the maximum value for the objective function $c(t)^{\mathrm{T}}x$ (respectively, $c(t+s)^{\mathrm{T}}x$, $c(t+\lambda s)^{\mathrm{T}}x$) optimized over $\mathcal{MP}_n \cap \{x \mid \ell \leqslant x \leqslant u\}$ and let x_0 (respectively, x_1, x) denote the corresponding optimizing vectors. By assumption, we can suppose that $x_0(f) = x_1(f) = \alpha$.

First, note that, for any $y \in \mathbb{R}^{\binom{n}{2}}$, $c(t+\lambda s)^{\mathrm{T}}y = c(t)^{\mathrm{T}}y + \lambda s y_f$ and $c(t+\lambda s)^{\mathrm{T}}y = c(t+s)^{\mathrm{T}}y - (1-\lambda) s y_f$.

In particular, $c(t+\lambda s)^{\mathrm{T}}x_0 = C_0 + \lambda s\alpha$, and $c(t+\lambda s)^{\mathrm{T}}x_1 = C_1 - (1-\lambda)s\alpha$, implying that $(1-\lambda)C_0 + \lambda C_1 \leqslant C$.

On the other hand, we have that $C = c(t+\lambda s)^{\mathrm{T}}x = c(t)^{\mathrm{T}}x + \lambda s x_f \leqslant C_0 + \lambda s x_f$, and $C = c(t+\lambda s)^{\mathrm{T}}x = c(t+s)^{\mathrm{T}}x - (1-\lambda)s x_f \leqslant C_1 - (1-\lambda)s x_f$, implying that $(1-\lambda)C_0 + \lambda C_1 \geqslant C$.

Therefore, the equality $(1-\lambda)C_0 + \lambda C_1 = C$ holds. In consequence, each of the vectors x_0 and x_1 is an optimizing vector for the program $\max(c(t+\lambda s)^{\mathrm{T}}x \mid x \in \mathcal{MP}_n, \ell \leqslant x \leqslant u)$. Hence, $t + \lambda s \in M_\alpha$.

Using compactness of the set $\mathcal{MP}_n \cap \{x \mid \ell \leqslant x \leqslant u,\ x(f) = \alpha\}$, it is easy to see that the set M_α is closed. $\square$

Theorem 5.5. *The k-sum operation,* $k = 0, 1, 2$, *preserves box* $\frac{1}{3}$*-integrality.*

Proof. For $k = 0, 1$, the proof is identical to that of Theorem 3.1.

We now show that the 2-sum operation preserves box $\frac{1}{3}$-integrality. Take two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, having a common edge f and denote their 2-sum by $G = (V, E)$. We suppose that G_i is box $\frac{1}{3}$-integral for $i = 1, 2$, and we show that G is box $\frac{1}{3}$-integral. Take $c \in \mathbb{R}^E$ and $\ell, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^E$ such that $\mathcal{MP}_n \cap \{x \mid \ell_e \leqslant x_e \leqslant u_e, e \in E\} \neq \emptyset$. Let y be an optimizing vector for the program

$$\text{(P)} \qquad \max(c^{\mathrm{T}}x \mid x \in \mathcal{MP}_n, \ell_e \leqslant x_e \leqslant u_e, e \in E).$$

Observe, first, that we may assume that each interval $[\ell_e, u_e]$ is tight for y, i.e., satisfies $\ell_e = u_e = y_e$ if $y_e \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and $u_e - \ell_e = \frac{1}{3}$ otherwise. Indeed, if it is not the case, define ℓ', u' by the above conditions; then, y is also an optimizing vector for the program $\max(c^{\mathrm{T}}x \mid x \in \mathcal{MP}_n, \ell'_e \leqslant x_e \leqslant u'_e, e \in E)$, and the bounds ℓ', u' are tight for y.

Define $c_i \in \mathbb{R}^{E_i}$ by $c_i(e) = c(e)$ for all edges $e \in E_i$, except $c_1(f) = c(f)$ and $c_2(f) = 0$.

Let us first suppose that $\ell_f = u_f := \alpha$. By the assumption, we know that the program $\max(c_i^{\mathrm{T}} x \mid x \in \mathcal{MP}(V_i), \ell_e \leqslant x_e \leqslant u_e, e \in E_i)$ admits a $\frac{1}{3}$-integral optimizing vector z_i, for $i = 1,2$. Since $z_1(f) = z_2(f) = \alpha$, we can construct the 2-union z of z_1 and z_2. Then, z is a $\frac{1}{3}$-integral optimizing vector for the program (P).

We can now assume that (ℓ_f, u_f) is $(0, \frac{1}{3})$ or $(\frac{1}{3}, \frac{2}{3})$ or $(\frac{2}{3}, 1)$. For $t \in \mathbb{R}$, we consider the translate $c_i(t)$ of the objective function c_i defined by $c_i(t)(e) = c_i(e)$ for all edges $e \in E_i$, except $c_1(t)(f) = c_1(f) + t$ and $c_2(t)(f) = c_2(f) - t$. For $i = 1,2$, and for $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, we define the set M_α^i, consisting of the scalars $t \in \mathbb{R}$ for which the program $\max(c_i(t)^{\mathrm{T}} x \mid x \in \mathcal{MP}(V_i), \ell_e \leqslant x_e \leqslant u_e, e \in E_i)$ admits a $\frac{1}{3}$-integral optimizing vector taking the value α on the edge f. Hence, $M_\alpha^i = \emptyset$ if $\alpha \neq \ell_f, u_f$ and, by Lemma 5.4, $M_{\ell(f)}^i$ and $M_{u(f)}^i$ are two closed intervals covering $\mathbb{R}$, for $i = 1,2$.

Consider the program $\max(c_1(t)^{\mathrm{T}} x \mid x \in \mathcal{MP}(V_1), \ell_e \leqslant x_e \leqslant u_e, e \in E_1)$ for large t, $t \to +\infty$, and then, for small t, $t \to -\infty$. Hence,

$$\mathcal{MP}(V_1) \cap \{x \mid \ell_e \leqslant x_e \leqslant u_e, e \in E_1, x(f) = \ell(f)\} \neq \emptyset \implies M_{\ell(f)}^1 \neq \emptyset,$$

$$\mathcal{MP}(V_1) \cap \{x \mid \ell_e \leqslant x_e \leqslant u_e, e \in E_1, x(f) = u(f)\} \neq \emptyset \implies M_{u(f)}^1 \neq \emptyset;$$

in fact, any t small enough belongs to $M_{\ell(f)}^1$ and any t large enough belongs to $M_{u(f)}^1$. In the same way,

$$\mathcal{MP}(V_2) \cap \{x \mid \ell_e \leqslant x_e \leqslant u_e, e \in E_2, x(f) = \ell(f)\} \neq \emptyset \implies M_{\ell(f)}^2 \neq \emptyset,$$

$$\mathcal{MP}(V_2) \cap \{x \mid \ell_e \leqslant x_e \leqslant u_e, e \in E_2, x(f) = u(f)\} \neq \emptyset \implies M_{u(f)}^2 \neq \emptyset$$

(any t large enough belongs to $M_{\ell(f)}^2$ and any t small enough to $M_{u(f)}^2$). Therefore, we can always find some $t \in M_\alpha^1 \cap M_\alpha^2$ for $\alpha = \ell(f)$ or $u(f)$, except in the cases when $M_{u(f)}^1 = M_{\ell(f)}^2 = \emptyset$ or $M_{\ell(f)}^1 = M_{u(f)}^2 = \emptyset$. But these two cases cannot occur; to see it, we use Lemma 5.3.

Indeed, if $(\ell_f, u_f) = (0, \frac{1}{3})$, then, by Lemma 5.3, we can find a vector y belonging to the set $\mathcal{MP}(V) \cap \{x \mid \ell \leqslant x \leqslant u\}$ such that $y_f = \frac{1}{3}$. By the above observations, we deduce that $M_{u(f)}^1$ and $M_{u(f)}^2$ are both nonempty. Similarly, if $(\ell_f, u_f) = (\frac{2}{3}, 1)$, then Lemma 5.3 produces y with $y_f = \frac{2}{3}$ and, thus, both sets $M_{\ell(f)}^1$ and $M_{\ell(f)}^2$ are nonempty. Also, in the case $(\ell_f, u_f) = (\frac{1}{3}, \frac{2}{3})$, we have such y with, say, $y_f = \frac{1}{3}$ and, then, $M_{\ell(f)}^1$ and $M_{\ell(f)}^2$ are nonempty.

In consequence, we can always find some $t \in M_\alpha^1 \cap M_\alpha^2$, for $\alpha = \ell(f)$ or $u(f)$. Then, for such t, there exists a $\frac{1}{3}$-integral vector z_i satisfying $z_i(f) = \alpha$ and which is optimum for the program $\max(c_i(t)^{\mathrm{T}} x \mid x \in \mathcal{MP}(V_i), \ell_e \leqslant x_e \leqslant u_e, e \in E_i)$. Therefore, we can construct the 2-union z of z_1 and z_2 which is a $\frac{1}{3}$-integral optimizing vector for the program (P). □

Lemma 5.6. *K_3 is box $\frac{1}{3}$-integral.*

Proof. We show that the polytope $\mathcal{MP}_3 \cap \{x \mid \ell \leqslant x \leqslant u\}$ has only $\frac{1}{3}$-integral vertices for every $\ell, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^3$. Let x be a vertex of the polytope $\mathcal{MP}_3 \cap \{x \mid \ell \leqslant x \leqslant u\}$ and let $\mathcal{B}$ be a set of three linearly independent active constraints at x. $\mathcal{B}$ contains some triangle equalities and some bounding equalities: $x_e = \ell_e$ or $x_e = u_e$.

• If $\mathcal{B}$ contains three triangle equalities, then x is a vertex of $\mathcal{MP}_3$ and, thus, x is 0–1-valued.

• If $\mathcal{B}$ contains two triangle equalities, then we deduce that $x_e = 0$ or 1, for some edge e; but $\mathcal{B}$ contains another bounding equality, say on edge f, $f \neq e$. Then, two coordinates of x are $\frac{1}{3}$-integral and, thus, the third one too.

• If $\mathcal{B}$ contains only one triangle equality and two bounding equalities, or if $\mathcal{B}$ contains three bounding equalities, then x is clearly $\frac{1}{3}$-integral. □

Remark 5.7. The graph K_4 is not box $\frac{1}{3}$-integral. For example, consider the vector $x \in \mathcal{MP}_4$ defined by $x_{12} = x_{13} = x_{14} = \frac{1}{6}$ and $x_{23} = x_{24} = x_{34} = \frac{1}{3}$. Then, x is a vertex of the polytope $\mathcal{MP}_4 \cap \{x \mid 0 \leqslant x_{ij} \leqslant \frac{1}{3}, 1 \leqslant i < j \leqslant 4\}$.

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