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Hypercube embedding of generalized bipartite metrics

Michel Deza*, Monique Laurent

CNRS, LIENS, Ecole Normale Superieure, 45, rue d'Ulm, 75230 Paris Cedex 05, France

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Abstract

A metric d is h-embeddable if it can be isometrically embedded in some hypercube. Equivalently, d is h-embeddable if d can be written as a nonnegative integer combination of cut metrics. The problem of testing h-embeddability is NP-complete (Chvátal, 1980). A good characterization of h-embeddability permitting a polynomial-time algorithm was given for several classes of metrics, in particular, for metrics on $n \leqslant 5$ points (Deza, 1961), for path metrics of graphs (Djokovic, 1973), for metrics with values in $\{1,2\}$ (Assouad and Deza, 1980), for metrics on $n \geqslant 9$ points with values in $\{1,2,3\}$ (Avis, 1990). We consider here generalized bipartite metrics, i.e., the metrics d for which $d(i,j) = 2$ for all distinct $i,j \in S$ or $i,j \in T$ for some bipartition (S,T) of the points. We characterize h-embeddable generalized bipartite metrics and derive a polynomial recognition algorithm.

1. Introduction

Given a finite set $V := \{1,\ldots,n\}$ and a mapping $d\colon V^2 \to \mathbb{R}_+$, d is called a *metric* if it satisfies $d(x,x) = 0$, $d(x,y) = d(y,x)$ for all $x,y \in V$ and

$$d(x,y) - d(x,z) - d(y,z) \leqslant 0 \quad \text{for all } x,y,z \in V. \tag{1.1}$$

Note that zero distances between distinct points are allowed. (Hence, we use the word "metric" for denoting what is usually called a semimetric) $Q_N := \{0,1\}^N$ denotes the hypercube of dimension N. The *Hamming distance* between two binary vectors of Q_N is the number of positions where their coordinates differ. A metric d on V is said to be *hypercube embeddable*, *h-embeddable* for short, if their exist n vectors $v_1,\ldots,v_n \in Q_N$ (for some integer N) such that $d(x,y)$ is equal to the Hamming distance between v_x, v_y for all $x,y \in V$. Clearly, if x,y are distinct points of V at distance $d(x,y) = 0$, then the vectors v_x and v_y should coincide. Therefore, zero distances may be ignored when studying h-embeddable metrics.

* Corresponding author.

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It is an NP-complete problem to decide whether a metric is h-embeddable. In fact, this problem is already NP-complete when restricted to the class of metrics having a point at distance 3 from all other points and taking all their other values in $\{2,4,6\}$ [5]. Nevertheless, several classes of metrics are known for which the hypercube embedding problem admits a good characterization yielding a polynomial-time algorithm. This is the case, in particular, for the following classes of metrics d:

(a) d is a metric on $n \leqslant 5$ nodes [6, 8],

(b) d takes only the values 2, 4 and some point is at distance 2 from all other points [5],

(c) d is the shortest path metric of a graph [12],

(d) d takes only the values 1, 2 [1],

(e) d is on $n \geqslant 9$ points and takes only the values 1, 2, 3 [3],

(f) d is a metric whose extremal graph is either a complete graph on 4 nodes, or a cycle of length 5, or the union of two stars ([13], see precise definition of extremal graph there).

In this paper, we extend the cases for which the hypercube embedding problem is polynomially solvable, namely to the class of generalized bipartite metrics. Given a partition of V into $V = S \cup T$, we consider the metrics d such that $d(x,y) = 2$ for all distinct $x, y \in S$ and all distinct $x, y \in T$. We call such a metric a *generalized bipartite metric*. Note that the path metric of the complete bipartite graph with node bipartition (S, T) is indeed of this form (with $d(x,y) = 1$ for all $(x,y) \in S \times T$). For instance, every h-embeddable metric whose values are either odd or equal to 2 is a generalized bipartite metric (this includes the above cases (d), (e)).

The problem of embedding metrics in the hypercube is related to the study of the cut cone in the following way. For any subset A of V, let $\delta(A)$ denote the *cut metric*, defined by $\delta(A)(x,y) = 1$ if $|A \cap \{x,y\}| = 1$ and $\delta(A)(x,y) = 0$ otherwise. Clearly, $\delta(A) = \delta(V \setminus A)$ holds. The cone generated by the $2^{n-1} - 1$ nonzero cut metrics is called the *cut cone* and is denoted by $\mathscr{C}_n$. In fact, $\mathscr{C}_n$ consists of all the metrics on V that are l_1-embeddable [2]. Recall that a metric d on V is said to be *l_1-embeddable* if there exists n vectors $v_1, \ldots, v_n \in \mathbb{R}^N$ (for some N) such that $d(x,y) = \| v_x - v_y \|$ for all $x, y \in V$. For $v \in \mathbb{R}^N$, $\|v\|$ denotes its l_1-norm $\sum_{1 \leqslant h \leqslant N} |v_h|$; if $u, v \in \{0,1\}^N$, then $\|u - v\|$ coincides with their Hamming distance. Similarly, h-embeddable metrics admit the following characterization. A metric d on V is h-embeddable if and only if d is a nonnegative integer combination of cut metrics, i.e, $d = \sum_{A \subseteq V} \lambda_A \delta(A)$ for some $\lambda_A \in \mathbb{Z}_+$. Therefore, every h-embeddable metric belongs to the cut cone $\mathscr{C}_n$. Note that the problem of testing membership in the cut cone is also NP-complete [13]. (Several classes of facets of $\mathscr{C}_n$ are known, yielding necessary conditions for l_1-embeddability and thus for h-embeddability; see, e.g., [10].)

Let $\mathscr{L}_n$ denote the *cut lattice*, consisting of all integer combinations of cut metrics. One can easily characterize the members of $\mathscr{L}_n$. Namely, for d integral, $d \in \mathscr{L}_n$ if and only if d satisfies the following condition, called the *even condition*:

$$d(x,y) + d(x,z) + d(y,z) \text{ is even} \quad \text{for all } x, y, z \in V. \tag{1.2}$$

Clearly, every h-embeddable metric on V belongs to the cut lattice, i.e., satisfies the even condition (1.2). In summary, we have the following implication:

$$d \text{ is } h\text{-embeddable} \Rightarrow d \in \mathscr{C}_n \cap \mathscr{L}_n.$$

This necessary condition is, in general, not sufficient. We will see in Remark 3.1 an example of a generalized bipartite metric on $n \geqslant 6$ points which belongs to $\mathscr{C}_n \cap \mathscr{L}_n$ but is not h-embeddable. In contrast, for the classes (a)–(f) of metrics mentioned above, it was shown that membership in $\mathscr{C}_n \cap \mathscr{L}_n$ suffices for ensuring h-embeddability. (For the class (a), this result can be rephrased as saying that the family of cut metrics on $n \leqslant 5$ points is a Hilbert basis of the cone $\mathscr{C}_n$.) (In fact, the following stronger result was shown: within the classes (a)–(f), the even condition (1.2) and the hypermetric condition suffice for ensuring h-embeddability; see, e.g., [4, 10] for definitions.)

The paper is organized as follows. In Section 2, we give a characterization of the generalized bipartite metrics that are h-embeddable. This characterization is then used in Section 3 for deriving a polynomial-time recognition algorithm. We give in the last Section 4 several additional results on h-embeddable metrics. In particular, we characterize h-embeddability within the class of metrics which admit a projection on all points but two that can be uniquely written as a positive combination of star cut metrics (see Propositions 4.5 and 4.10).

We conclude this section with some preliminary results and definitions that we need in the remainder of the paper.

Let d be a metric on V which is h-embeddable, i.e., can be decomposed as a nonnegative integer combination of cut metrics. Any such representation: $d = \sum_{A \subseteq V} \lambda_A \delta(A)$ with $\lambda_A \in \mathbb{Z}_+$, is called a *$\mathbb{Z}_+$-realization* of d. An h-embeddable metric is said to be *rigid* if it admits a unique $\mathbb{Z}_+$-realization (i.e., it has an essentially unique embedding in a hypercube).

let $\mathbb{1}_n$ denote the metric on V that takes the value 1 on each pair of distinct points. Given $\alpha \in \mathbb{Z}_+$, the *equidistant metric* $2\alpha\mathbb{1}_n$, which takes the value 2α on each pair of distinct points, is clearly h-embeddable. Indeed,

$$2\alpha\mathbb{1}_n = \sum_{1 \leqslant x \leqslant n} \alpha\delta(\{x\})$$

is a $\mathbb{Z}_+$- realization of $2\alpha\mathbb{1}_n$, called its *star realization.* The following result shows that, for n large enough, the metric $2\alpha\mathbb{1}_n$ is rigid, i.e., the star realization is the only decomposition of $2\alpha\mathbb{1}_n$ as a nonnegative integer sum of cut metrics. This result will play a crucial role in our treatment.

Theorem 1.1 (Deza [7]). *If $n \geqslant \alpha^2 + \alpha + 3$, then the metric $2\alpha\mathbb{1}_n$ is rigid.*

For instance, the metric $2\mathbb{1}_n$ is rigid for any $n \neq 4$. It is easy to see that $2\mathbb{1}_4$ admits exactly two distinct $\mathbb{Z}_+$-realizations, namely, $2\mathbb{1}_4 = \sum_{1 \leqslant x \leqslant 4} \delta(\{x\}) = \sum_{2 \leqslant x \leqslant 4} \delta(\{1, x\})$.

We will also use the next theorem which gives an asymptotic result about the rigidity of the more general class of metrics of the form: $\sum_{1 \leqslant x \leqslant n} \alpha_x \delta(\{x\})$ with $\alpha_1, \ldots, \alpha_n$ positive integers. It is a reformulation of Theorem 7(i) from [9].

Theorem 1.2. *Consider the metric $d = \sum_{1 \leqslant x \leqslant n} \alpha_x \delta(\{x\})$ where $\alpha_1, \ldots, \alpha_n$ are positive integers. If n is large with respect to* $\max(\alpha_1, \ldots, \alpha_n)$, *then d is rigid.*

2. The structure of *h*-embeddable generalized bipartite metrics

In this section, we characterize the generalized bipartite metrics that are h-embeddable. For this, we completely describe the structure of their distance matrices and we shall use it in the next section in order to derive a polynomial recognition algorithm.

Let d be a generalized bipartite metric on $V = \{1, \ldots, n\}$ with bipartition (S, T), i.e., $d(x, y) = 2$ for all $(x, y) \in S^2 \cup T^2$ with $x \neq y$. Let D denote the $|S| \times |T|$ matrix with entries $d(x, y)$ for $x \in S$, $y \in T$; D is called the *(S, T)-distance matrix* of d. We start with an easy observation.

Lemma 2.1. *Let d be a generalized bipartite metric with bipartition (S, T). If d is h-embeddable, then there exists an integer α such that $d(x, y) \in \{\alpha, \alpha + 2, \alpha + 4\}$ for all $(x, y) \in S \times T$.*

Proof. Let α, β denote the smallest and largest value taken by $d(x, y)$ for $(x, y) \in S \times T$; say $\alpha = d(x, y)$, $\beta = d(x', y')$ for $x, x' \in S$, $y, y' \in T$. Using the metric condition (1.1), we obtain that $\beta = d(x', y') \leqslant d(x', x) + d(x, y) + d(y, y') \leqslant 4 + \alpha$. Moreover, α, β have the same parity by (1.2). □

Set $s := |S|$ and $t := |T|$. Let d_S (resp. d_T) denote the projection of d on $S \times S$ (resp. on $T \times T$). Then, $d_S = 2\mathbb{1}_s$ and $d_T = 2\mathbb{1}_t$. The main idea is based on the following simple observation. If $d = \sum_{A \subseteq V} \lambda_A \delta(A)$ is a $\mathbb{Z}_+$-realization of d, then its projection on S, namely $\sum_{A \subseteq V} \lambda_A \delta(A \cap S)$, is a $\mathbb{Z}_+$-realization of d_S. Similarly, its projection on T is a $\mathbb{Z}_+$-realization of d_T. Recall that $2\mathbb{1}_n$ is rigid for all $n \neq 4$. Therefore, if $s \neq 4$, then the realization $\sum_{A \subseteq V} \lambda_A \delta(A \cap S)$ of d_S must be the star realization, i.e., it must coincide with $\sum_{x \in S} \delta(\{x\})$. Recall also that the metric $2\mathbb{1}_4$ has two realizations, namely, the star realization: $\sum_{1 \leqslant x \leqslant 4} \delta(\{x\})$ and the *special realization*: $\delta(\{1,2\}) + \delta(\{1,3\}) + \delta(\{1,4\})$. Therefore, if $s = 4$, we have two alternatives for the realization $\sum_{A \subseteq V} \lambda_A \delta(A \cap S)$ of d_S. The same reasoning applies for d_T.

The following definitions will be useful in the sequel. A $\mathbb{Z}_+$-realization of d is called a *star–star* realization if both its projections on S^2 and on T^2 are the star realizations of $2\mathbb{1}_s$ and $2\mathbb{1}_t$, respectively. A realization of d is called a *star–special* realization if its projection on S^2 is the star realization of $2\mathbb{1}_s$, but $t = 4$ and its projection on T^2 is the special realization of $2\mathbb{1}_4$. Finally, a realization of d is called a *special–special* realization if $s = t = 4$ and both its projections on S^2 and T^2 are the special realization of $2\mathbb{1}_4$.

In the following, we analyze the structure of the h-embeddable generalized bipartite metrics. For this, we distinguish the cases when a star–star, or a star–special, or a special–special realization exists.

Proposition 2.2. *Let d be a generalized bipartite metric with bipartition (S,T). Then, d admits a star–star realization if and only if there exist a partition $\{A,B,C,D\}$ of S and a partition $\{A',B',C',D'\}$ of T (with possibly empty members) with $|A|=|A'|$ and $|B|=|B'|$ and there exist one-to-one mappings $\sigma: A \to A'$ and $\tau: B \to B'$ and an integer $f \geqslant |B|+|D|+|D'|$ such that*

$$d(x,y)=\begin{cases} f & \text{for } (x,y)\in((A\cup C)\times(B'\cup D'))\cup((B\cup D)\times(A'\cup C')) \\ & \qquad \cup\{(z,\sigma(z))\,|\,z\in A\}\cup\{(z,\tau(z))\,|\,z\in B\}, \\ f+2 & \text{for } (x,y)\in((A\cup C)\times(A'\cup C'))\setminus\{(z,\sigma(z))\,|\,z\in A\}, \\ f-2 & \text{for } (x,y)\in((B\cup D)\times(B'\cup D'))\setminus\{(z,\tau(z))\,|\,z\in B\}. \end{cases} \tag{2.1}$$

Fig. 1 shows the (S,T)-distance matrix of the metric d defined by (2.1). We use the following notation in Fig. 1 and in the next figures: I_a denotes the $a \times a$ identity matrix, J_a the $a \times a$ all ones matrix, and a block marked, say, with f, has all its entries equal to f. As a rule, we denote the cardinality of a set by the same lower-case letter; e.g., $a=|A|$, $a'=|A'|$, etc.

Proof of Proposition 2.2. Let d be a generalized bipartite metric admitting a star–star realization: $d=\sum_{U\in\mathscr{U}}\delta(U)$, where $\mathscr{U}$ is a collection (allowing repetition) on nonempty

$$\begin{array}{c|c|c|c|c|} & A' & C' & B' & D' \\ \hline A & (f+2)J_a-2I_a & f+2 & f & f \\ \hline C & f+2 & f+2 & f & f \\ \hline B & f & f & (f-2)J_b+2I_b & f-2 \\ \hline D & f & f & f-2 & f-2 \\ \hline \end{array}$$

Fig. 1.

	a	b	c	d	c'	d'	m
A	I_a	0	0	0	0	0	0
B	0	I_b	0	0	0	0	0
C	0	0	I_c	0	0	0	0
D	0	0	0	I_d	0	0	0
A'	I_a	1	0	1	0	1	1
B'	0	$J_b - I_b$	0	1	0	1	1
C'	0	1	0	1	$I_{c'}$	1	1
D'	0	1	0	1	0	$J_{d'} - I_{d'}$	1

Fig. 2.

subsets of V. Hence, $|U \cap S| \in \{0, s, 1, s-1\}$ and $|U \cap T| \in \{0, t, 1, t-1\}$ for all $U \in \mathscr{U}$. We can suppose without loss of generality that $|U \cap S| \in \{0, 1\}$ for all $U \in \mathscr{U}$. Let M denote the matrix whose columns are the incidence vectors of the members of $\mathscr{U}$. Combining the above-mentioned two possibilities for $U \cap S$ with the four possibilities for $U \cap T$, we obtain that M has the form shown in Fig. 2. Hence the sets A, B, C, D and A', B', C', D' form the desired partitions of S and T. We can now compute $d(x, y)$ for $(x, y) \in S \times T$ and verify that they satisfy relation (2.1), after setting $f := |B| + |D| + |D'| + m$.

Conversely, suppose that d is defined by (2.1). Set $A = \{x_1, \ldots, x_{|A|}\}$ and $B = \{y_1, \ldots, y_{|B|}\}$. One can easily check that d satisfies:

$$d = \sum_{1 \leqslant i \leqslant |A|} \delta(\{x_i, \sigma(x_i)\}) + \sum_{1 \leqslant i \leqslant |B|} \delta(T \setminus \{\tau(y_i)\} \cup \{y_i\}) + \sum_{x \in C \cup C'} \delta(\{x\})$$

$$+ \sum_{x \in D} \delta(T \cup \{x\}) + \sum_{x \in D'} \delta(T \setminus \{x\}) + (f - |B| - |D| - |D'|)\delta(T).$$

This realization is clearly a star–star realization. □

Proposition 2.3. *Let d be a generalized bipartite metric with bipartition (S, T) and suppose $|T| = 4$. Then, d admits a star–special realization if and only if there exist a partition $\{A, B, C, D\}$ of S and a partition $\{A', B', C', D', E'\}$ of T (with possibly empty* *members) with* $|A| = |A'|$, $|B| = |B'|$ *and* $|E'| = 1$ *and there exist one-to-one mappings* $\sigma: A \to A'$ *and* $\tau: B \to B'$ *and nonnegative integers* f, g, m *such that*

$$\begin{cases} f = |B| + |D| + |D'| + m, \\ g = |A| + |D| + |C'| + m - 1, \end{cases}$$

$$d(x,y) = \begin{cases} g & \text{for } (x,y) \in (A \cup D) \times E', \\ g+2 & \text{for } (x,y) \in (B \cup C) \times E', \end{cases} \tag{2.2}$$

with the values $d(x,y)$ *for* $(x,y) \in S \times (T \setminus E')$ *being given by* (2.1).

Proof. Suppose that d is given by (2.1) and (2.2). Set $A = \{x_1, \ldots, x_{|A|}\}$, $B = \{y_1, \ldots, y_{|B|}\}$ and $E' = \{z'\}$. Then d admits the following star–special realization:

$$d = \sum_{1 \leq i \leq |A|} \delta(\{x_i, \sigma(x_i), z'\}) + \sum_{1 \leq i \leq |B|} \delta(T \setminus \{\tau(y_i), z'\} \cup \{y_i\}) + \sum_{x \in C} \delta(\{x\})$$
$$+ \sum_{x \in C'} \delta(\{x, z'\}) + \sum_{x \in D} \delta(T \cup \{x\}) + \sum_{x \in D'} \delta(T \setminus \{x, z'\}) + m\delta(T).$$

Conversely, suppose that d admits a star–special realization: $\sum_{U \in \mathscr{U}} \delta(U)$, for some collection $\mathscr{U}$ (allowing repetition) of nonempty subsets of V. Let M denote the matrix whose columns are the incidence vectors of the members of $\mathscr{U}$, let $z' \in T$, and let M' denote the submatrix of M obtained by deleting its z'-row. The projection of d on $T \setminus \{z'\}$ is the rigid metric $2\mathbb{1}_3$. Therefore, by the proof of Proposition 2.2, the matrix M' has the form shown in Fig. 2. By assumption, we have the special realization of $2\mathbb{1}_4$ on T, i.e., $|U \cap T| = 0, 2$, or 4 for all $U \in \mathscr{U}$. This observation permits us to determine the z'-row of M. Namely, it has the following form (keeping the notation of Fig. 2.):

	a	b	c	d	c'	d'	m
E'	1	0	0	1	1	0	1

One can now verify that d satisfies (2.1) and (2.2), after setting $f = b + d + d' + m$ and $g = a + d + c' + m - 1$. $\square$

As a consequence of Proposition 2.3, we deduce that any generalized bipartite metric admitting a star–special realization takes, besides the value 2, the following three values:

- $f, f+2, f+4$ if $a + c' = 3$, i.e., $g = f + 2$,
- $f-4, f-2, f$ if $a + c' = 0$, i.e., $g = f - 4$,
- $f-2, f, f+2$ if $a + c' = 2$, (i.e., $g = f$), or if $a + c' = 1$ (i.e., $g = f - 2$).

One can also characterize the generalized bipartite metrics with bipartition (S, T), $|S| = |T| = 4$, that admit a special–special realization. This characterization is analogous to that of Proposition 2.3 for the star–special case. We state the result without proof.

Proposition 2.4. *Let d be a generalized bipartite metric with bipartition (S,T) and suppose that $|S| = |T| = 4$. Then, d admits a special–special realization if and only if there exist a partition $\{A,B,C,D,E\}$ of S and a partition $\{A',B',C',D',E'\}$ of T (with possibly empty members) with $|A| = |A'|$, $|B| = |B'|$ and $|E| = |E'| = 1$ and there exist one-to-one mappings $\sigma: A \to A'$ and $\tau: B \to B'$ and nonnegative integers f,g,h,i,m such that*

$$\begin{cases} f = |B| + |D| + |D'| + m, \\ g = |A| + |D| + |C'| + m - 1, \\ h = |A| + |C| + |D'| + m - 1, \\ i = |B| + |C| + |C'| + m, \end{cases}$$

$$d(x,y) = \begin{cases} h & \text{for } x \in E,\ y \in A' \cup D', \\ h+2 & \text{for } x \in E,\ y \in B' \cup C', \\ i & \text{for } x \in E,\ y \in E', \end{cases} \tag{2.3}$$

with the values $d(x,y)$ for $(x,y) \in (S \setminus E) \times T$ being given by (2.1) *and* (2.2).

In fact, using the fact that $|S \setminus E| = |T \setminus E'| = 3$, we can explicitely describe the generalized bipartite metrics admitting a special–special realization. There are 50 possibilities for the sequence (a,b,c,d,c',d'). Up to permutation on S and T, this gives 9 possibilities for the (S,T)-distance matrix. For example, the two parameter sequences $(2,0,1,0,0,1)$ and $(2,0,0,1,1,0)$ give, respectively, $(f,g,h,i) = (1,1,3,1)$, $(1,3,1,1)$; one can see easily that the corresponding (S,T)-distance matrices are identical up to permutation of the rows and columns. We display in Fig. 3 all the nine distinct (S,T)-distance matrices for generalized bipartite metrics admitting a special–special realization; note that we have substracted the value m from all the entries.

3. Recognition of h-embeddable generalized bipartite metrics

In this section, we see that generalized bipartite metrics can be tested for h-embeddability in polynomial time. Let d be a generalized bipartite metric with bipartition (S,T). In order to check whether d is h-embeddable, one must check whether d admits a star–star, or a star–special, or a special–special realization. Clearly, if $s,t \neq 4$, then only the first situation can occur and the last situation can occur only if $s = t = 4$. In view of Propositions 2.2–2.4, this amounts to check whether d is of the form indicated in (2.1), (2.2) or (2.3). It is quite clear that this can be done in polynomial time. Actually, it can be done in $O(n^2)$ if d is on n points. Though there is no real conceptual difficulty, we give nevertheless, for the sake of completeness, a brief account of the algorithm.

$$
\begin{array}{|cccc|}\hline 3&1&1&1\\1&3&1&1\\1&1&3&1\\1&1&1&3\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 0&2&2&2\\2&0&2&2\\2&2&0&2\\2&2&2&0\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 1&1&1&3\\1&3&3&3\\1&3&3&3\\3&3&3&5\\\hline\end{array}
$$

$$
\begin{array}{|cccc|}\hline 4&4&4&2\\4&2&2&2\\4&2&2&2\\2&2&2&0\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 3&1&1&1\\1&3&1&1\\3&3&3&1\\3&3&1&3\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 0&2&2&2\\2&0&2&2\\2&2&2&4\\2&2&4&2\\\hline\end{array}
$$

$$
\begin{array}{|cccc|}\hline 2&2&2&4\\2&2&2&4\\2&2&2&4\\4&4&4&6\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 4&4&4&2\\4&4&4&2\\4&4&4&2\\2&2&2&0\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 3&3&3&1\\3&3&3&1\\3&3&3&1\\5&5&5&3\\\hline\end{array}
$$

Fig. 3.

The description from Proposition 2.2 enables us to design a polynomial algorithm for testing whether a generalized bipartite metric has a star–star realization and finding such a realization if one exists. Let d be a generalized bipartite metric. We consider three cases.

Case 1: $d(x,y) = \alpha$ for all $(x,y) \in S \times T$ (for some $\alpha \in \mathbb{Z}_+$). Suppose first that $\alpha = 1$. If $|T| = 1$, then $d = \sum_{x \in S} \delta(\{x\})$ is a star–star realization of d. If $|S| = |T| = 2$, $S = \{1,2\}$, $T = \{3,4\}$, then $d = \delta(\{1,3\}) + \delta(\{1,4\})$ is a star–star realization of d. Otherwise $|S| \geqslant 3$, $|T| \geqslant 2$ and then d is not h-embeddable (since d does not belong to the cut cone). If $\alpha \geqslant 2$, then $d = \sum_{x \in V} \delta(\{x\}) + (\alpha - 2)\delta(T)$ is a star–star realization of d.

Case 2: $d(x,y)$ takes the two values $\alpha, \alpha + 2$ for $(x,y) \in S \times T$ (for some $\alpha \in \mathbb{Z}_+$). Suppose that d has a star–star realization, i.e., its distance matrix is of the form shown in Fig. 1. Then, one of $\alpha, \alpha + 2$ is equal to the value f from Proposition 2.2. Suppose $\alpha = f$ (the case $\alpha + 2 = f$ is similar). Note that one of the following two conditions holds: either (i) $B = B' = \emptyset$ and $D' = \emptyset$ (or $D = \emptyset$), or (ii) $|B| = |B'| = 1$ and $D = D' = \emptyset$. Let $H = (V_H, E_H)$ (resp. $K = (V_K, E_K)$) denote the graph whose edges are the pairs $(x,y) \in S \times T$ such that $d(x,y) = f$ (resp. $f + 2$). Up to permutation of S and T, at least one of the following is true:

(a) H is a matching of size $|S| = |T|$ (in which case $A = S$ and $A' = T$),
(b) H contains a complete bipartite subgraph with parts $S' \subseteq S$ and T,
(c) H contains a complete bipartite subgraph with parts $S' \subseteq S$ and T, and a complete bipartite subgraph with parts $T' \subseteq T$ and S,
(d) K contains a complete bipartite subgraph with parts $S'' \subseteq S$ and T.

(For example in the above case (ii), (c) occurs with $1 \leqslant |S'| \leqslant 2$ and $1 \leqslant |T'| \leqslant 2$; moreover, if $|S'| = 2$ then $|T'| = 1$, $|T| = 2$ and d has indeed a star–star realization with $A' = T \backslash T'$, $B' = T'$, $A \cup B = S'$, $C = S \backslash S'$, $C' = \emptyset$.) We choose the maximal such sets S', S'', T'. Then, it remains only to discuss a small (in $O(1)$) number of cases. We leave out the details.

Case 3: $d(x,y)$ takes the three values $\alpha - 2, \alpha, \alpha + 2$ for $(x,y) \in S \times T$ (for some $\alpha \in \mathbb{Z}_+$). Suppose that d has a star–star realization. Then, α is equal to the value f from Proposition 2.2. Let $H = (V_H, E_H)$ (resp. $K = (V_K, E_K)$) denote the graph whose edges are the pairs $(x,y) \in S \times T$ such that $d(x,y) = f - 2$ (resp. $f + 2$). Then V_H and V_K are disjoint, $V_H \cap (A \cup C \cup A' \cup C') = \emptyset$, $V_H \subseteq B \cup D \cup B' \cup D'$, and the edges $e = (x,y)$ not belonging to H with $x, y \in V_H$ form a matching and satisfy $d(x,y) = f$. Similarly for K. Note that if an element x belongs to $B \cup D \setminus V_H$ or to $A \cup C \setminus V_K$ then $d(x,y) = f$ for all $y \in T$. Setting $S_f := \{x \in S \mid d(x,y) = f \text{ for all } y \in T\}$, then $V_H \cap S = B \cup D$ and $V_K \cap S = A \cup C$ whenever $S_f = \emptyset$. Defining similarly the set T_f, we have $V_H = B \cup D \cup B' \cup D'$ and $V_K = A \cup C \cup A' \cup C'$ whenever both S_f and T_f are empty. If this is the case, then the properties mentioned above permit us to determine the sets $A, B, C, D, A', B', C', D'$ and to conclude the verification for d. Note, moreover, that $S_f \cap (C \cup D) = T_f \cap (C' \cup D') = \emptyset$, and $|A| = 1$, $C' = \emptyset$ if $S_f \cap A \neq \emptyset$, and $|B| = 1$, $D' = \emptyset$ if $S_f \cap B \neq \emptyset$. Hence, $|S_f|, |T_f| \leqslant 2$. Based on these observations, one can describe all possible cases (whose number is clearly in $O(1)$) and conclude the verification for d.

One can check whether d has a star–special realization in the following way. Suppose $|T| = 4$. Let $z' \in T$ and let d' denote the restriction of d to the set $V \setminus \{z'\}$. If d has a star–special realization then d' has a star–star realization. We see easily that there are $O(1)$ possible star–star realizations for d' and all of them can be found in polynomial time. One then checks whether they can be extended to a star–special realization of d.

Finally, one can verify trivially whether d has a special–special realization. Indeed, this is the case if and only if, for some $m \in \mathbb{Z}_+$, the (S, T)-distance matrix of the metric $d - m\delta(T)$ is one of the nine matrices from Fig. 3 (up to permutation on S and T).

In conclusion, we have shown that one can test in polynomial time whether a generalized bipartite metric is h-embeddable.

Remark 3.1. We give an example of a generalized bipartite metric on $n \geqslant 6$ points which is not h-embeddable, but belongs to the cut lattice $\mathscr{L}_n$ and to the cut cone $\mathscr{C}_n$. Given an integer $k \geqslant 5$, we consider the metric d_{2k} defined on $2k$ points by: $d_{2k}(i, i+k) = 4$ for any $1 \leqslant i \leqslant k$ and $d_{2k}(i,j) = 2$ for all other pairs (i,j), $1 < i \neq j \leqslant 2k$. Hence, d_{2k} is a generalized bipartite metric with bipartition $(\{1, 2, \ldots, k\}, \{k+1, k+2, \ldots, 2k\})$. (Note that $\frac{1}{2}d_{2k}$ is the path metric of the 1-skeleton of the k-dimensional cross polytope β_k, which is defined as the convex hull of the $2k$ vectors $\pm e_i$ $(1 \leqslant i \leqslant k)$, where $e_1, \ldots, e_k$ are the unit vectors in $\mathbb{R}^k$.) It is an easy exercise to verify, for instance using the above procedure, that d_{2k} is not h-embeddable. On the other hand, one verifies easily that d_{2k} belongs to the cut cone $\mathscr{C}_{2k}$. Indeed, for some α, take a $\mathbb{Z}_+$-realization $\sum \lambda_S \delta(S)$ of $2\alpha \mathbb{1}_k$ such that $N := \sum \lambda_S \leqslant 4\alpha$ (e.g., consider $\sum \delta(S)$, where the sum is taken over all subsets S of $\{1, \ldots, k\}$ of cardinality $\lfloor k/2 \rfloor$). Then, $\sum \lambda_S \delta(S \cup \{i + k\colon\ i \notin S\}) + (4\alpha - N)\delta(\{k+1, \ldots, 2k\}) =$

αd_{2k}. Let d_n denote the projection of d_{2k} on the first n elements of the set $\{1, \ldots, 2k\}$ for $1 \leqslant n \leqslant 2k$. One can also verify that d_n is not h-embeddable if $k+1 \leqslant n \leqslant 2k$, while d_n belongs to $\mathscr{C}_n \cap \mathscr{L}_n$. (Note that, for $n = k+1$, $\frac{1}{2} d_n$ is the path metric of the complete graph K_n with one deleted edge.) This example was first given in [1].

4. Some more results on h-embeddable metrics

In this section we give additional results on hypercube embedding that are obtained by application of some extension of the method used in the preceeding sections for studying generalized bipartite metrics.

Let d be a metric on V. Suppose that there exists a bipartition (S, T) of V such that the projections d_S and d_T of d on S and T are of the form:

$$d_S = \sum_{x \in S} \alpha_x \delta(\{x\}), \qquad d_T = \sum_{x \in T} \beta_x \delta(\{x\}) \tag{4.1}$$

for some positive integers α_x, β_x. From Theorem 1.2, we know that d_S and d_T are rigid if $|S|$ is big enough with respect to $\max_{x \in S} \alpha_x$ and $|T|$ is big enough with respect to $\max_{x \in T} \beta_x$. So, theoretically, one could use the same technique as the one used in Proposition 2.2 for studying h-embeddability of these metrics. However, a precise analysis of the structure of the distance matrix of such metrics seems technically much more involved than in the case where all α_x, β_x are equal to 1, considered in Section 2.

The next simplest case to consider after the case of generalized bipartite metrics would be the class of metrics d for which $d(x,y) = 4$ for $x \neq y \in S$ and $d(x,y) = 2$ for $x \neq y \in T$ (i.e., all α_x's are equal to 2 and all β_x's to 1). One can characterize h-embeddability of these metrics by a similar reasoning as was applied to generalized bipartite metrics in Section 2 and, as a consequence, recognize them in polynomial time. Indeed, the metric $4\mathbb{1}_n$ is rigid for $n = 3$ and $n \geqslant 9$ and has exactly three $\mathbb{Z}_+$-realizations: its star realization and two special ones for each $n \in \{4,5,6,7,8\}$ [11]. We do not give the details.

In the following, we give a complete characterization of h-embeddability for the metrics satisfying (4.1) in the case $|T| \leqslant 2$. We first consider the case $|T| = 1$. We introduce some notation.

Let d be defined on the set $\{1, \ldots, n, n+1\}$ and let $\beta, \alpha_x \in \mathbb{Z}$ for $x \in S := \{1, \ldots, n\}$. For $x \in S$, set

$$\sigma_x := \frac{1}{2}\left(\sum_{y \in S} d(y, n+1) - \alpha_y \right) - \frac{n-2}{2}(d(x, n+1) - \alpha_x), \tag{4.2}$$

$$\beta_x := \frac{\sigma_x - \beta}{n-2}, \tag{4.3}$$

$$\begin{aligned} &\sigma := \min(\sigma_x \mid x \in S), \\ &\tau := \min\left(\frac{d(x, n+1) - d(y, n+1) + d(x,y)}{2} \,\middle|\, x \neq y \in S \right). \end{aligned} \tag{4.4}$$

Proposition 4.1. *Let d be a metric one the set $V := \{1, \dots, n, n+1\}$ which satisfies the even condition* (1.2). *Suppose that the projection d_S of d on the subset $S := \{1, \dots, n\}$ satisfies: $d_S = \sum_{1 \leqslant x \leqslant n} \alpha_x \delta(\{x\})$ for some positive integers $\alpha_1, \dots, \alpha_n$ and that d_S is rigid. Then, d is h-embeddable if and only if $\sigma_x \geqslant 0$ for all $x \in S$. Moreover, the $\mathbb{Z}_+$-realizations of d are all the realizations of the form:*

$$d = \beta\delta(\{n+1\}) + \sum_{x \in S} \beta_x \delta(\{x, n+1\}) + (\alpha_x - \beta_x)\delta(\{x\}), \tag{4.5}$$

where β_x $(x \in S)$ are given by (4.3) *and β is a nonnegative integer satisfying*

$$\sigma - (n-2)\tau \leqslant \beta \leqslant \sigma \quad \text{and} \quad \frac{\sigma - \beta}{n-2} \in \mathbb{Z} \tag{4.6}$$

(with σ, σ_x, τ being given by (4.2), (4.4)). *In particular, d is rigid whenever d satisfies some inequality* (1.1) *at equality.*

Proof. Suppose first that d is h-embeddable. Let $\sum_{U \in \mathscr{U}} \delta(U)$ be a $\mathbb{Z}_+$-realization of d. Its projection on S is a $\mathbb{Z}_+$-realization of the metric $d_S = \sum_{x \in S} \alpha_x \delta(\{x\})$, assumed to be rigid. Hence, the sets $U \cap S$ are the singletons $\{x\}$ for $x \in S$, each repeated α_x times, and the empty set repeated, say β times. Denote by β_x the number of sets $U \in \mathscr{U}$ for which $U \cap S = \{x\}$ and $n+1 \in U$. Then, the realization $\sum_{U \in \mathscr{U}} \delta(U)$ can be rewritten as (4.5). Hence, $d(x, n+1) = \beta + \alpha_x - 2\beta_x + \sum_{y \in S} \beta_y$, from which we obtain

$$\sigma_x = \beta + (n-2)\beta_x.$$

This shows that $\sigma_x \geqslant 0$, $\sigma \geqslant \beta$, and (4.3). We check that $\beta \geqslant \sigma - (n-2)\tau$. For this, note that, for $x, x' \in S$,

$$\frac{\sigma_x - \sigma_{x'}}{n-2} = \tfrac{1}{2}(d(x', n+1) - d(x, n+1) - d(x, x')) + \alpha_x. \tag{4.7}$$

Note also the identity:

$$\tfrac{1}{2}(d(x, n+1) - d(y, n+1) + d(x, y)) = \alpha_x - \beta_x + \beta_y.$$

Therefore, there exist $x_0 \neq y_0 \in S$ such that

$$\tau = \alpha_{x_0} - \beta_{x_0} + \beta_{y_0} = \alpha_{x_0} - \beta_{x_0} + \frac{\sigma_{y_0} - \sigma}{n-2} + \frac{\sigma - \beta}{n-2}.$$

Hence, $\tau \geqslant (\sigma - \beta)/(n-2)$ and $(\sigma - \beta)/(n-2) \in \mathbb{Z}$ by (4.7).

Conversely, suppose that $\sigma_x \geqslant 0$ for all $x \in S$, where the σ_x's are given by (4.2). As σ_x can be rewritten as

$$\sigma_x = (n-1)\alpha_x + d(x, n+1) + \sum_{y \in S} \frac{1}{2}(d(y, n+1) - d(x, n+1) - d(x, y)),$$

we deduce that σ_x is an integer. Let β be a nonnegative integer satisfying (4.6) (note that one can always choose $\beta = \sigma$) and let β_x, σ, τ be defined by (4.3) and (4.4). We show that (4.5) is a $\mathbb{Z}_+$-realization of d. Clearly, $\beta_x \geqslant 0$. Choosing $x' \in S$ such that

$\sigma_{x'} = \sigma$, we deduce from (4.8) that $(\sigma_x - \sigma)/(n-2) \in \mathbb{Z}$ and $(\sigma_x - \sigma)/(n-2) \leqslant -\tau + \alpha_x$. Therefore, $\beta_x = (\sigma_x - \beta)/(n-2) = (\sigma_x - \sigma)/(n-2) + (\sigma - \beta)/(n-2) \in \mathbb{Z}$ and $\beta_x \leqslant \alpha_x$. Finally, we check that (4.5) holds. The distances between pairs of points of S agree clearly and it is not difficult to check that

$$\beta + \sum_{y \in S} \beta_y + \alpha_x - 2\beta_x = d(x, n+1).$$

Hence, we have shown that d is h-embeddable and that all its $\mathbb{Z}_+$-realizations are as indicated in Proposition 4.1. In particular, if d satisfies an inequality (1.1) at equality, then $\tau = 0$ which implies that $\beta = \sigma$ and thus all β_x are uniquely determined. Hence, d is rigid. □

The following result can be deduced as an application of Proposition 4.1.

Corollary 4.2. *Let d be defined on the set $V = \{1, \ldots, n, n+1\}$. Suppose that its projection d_S on the subset $S := \{1, \ldots, n\}$ satisfies $d_S = \sum_{1 \leqslant x \leqslant n} \alpha_x \delta(\{x\})$ for some positive integers $\alpha_1, \ldots, \alpha_n$ and that d_S is rigid. Set $\beta := d(1, n+1)$ and suppose that $d(x, n+1) = \beta - d(1, x)$ for $2 \leqslant x \leqslant n$.*

(i) *d satisfies the metric condition* (1.1) *if and only if*

$$\beta \geqslant \alpha_1 + \max(\alpha_x + \alpha_y : 2 \leqslant x < y \leqslant n).$$

(ii) *d satisfies the even condition* (1.2) *if and only if β is an integer.*

(iii) *d is h-embeddable if and only if β is an integer and $\beta \geqslant \sum_{x \in S} \alpha_x$; moreover, d is rigid.*

Let us now turn to an analogue of Proposition 4.1 for the case $|T| = 2$. Let d be defined on the set $V := \{1, \ldots, n, n+1, n+2\}$. let d_S, d', d'' denote the projections of d on the subsets $S := \{1, \ldots, n\}$, $S \cup \{n+1\}$, $S \cup \{n+2\}$, respectively. We suppose that $d_S = \sum_{x \in S} \alpha_x \delta(\{x\})$ for some positive integers α_x and that d_S is rigid. Hence, we can apply Proposition 4.1 for testing h-embeddability of d' and d''. Let $\sigma'_x, \beta'_x, \sigma', \tau'$ be defined by relations (4.2)–(4.4) (where β' is to be determined) when considering the metric d' instead of d. Similarly, let $\sigma''_x, \beta''_x, \sigma'', \tau''$ be defined by (4.2)–(4.4) (where β'' is to be determined) when considering the metric d'' instead of d and the point $n+2$ instead of $n+1$.

Proposition 4.3. *Let d be a metric on $V := \{1, \ldots, n, n+1, n+2\}$ that satisfies the even condition* (1.2). *Suppose that its projection d_S on the subset $S := \{1, \ldots, n\}$ is of the form: $d_S = \sum_{x \in S} \alpha_x \delta(\{x\})$ for some positive integers α_x and that d_S is rigid. Then d is h-embeddable if and only if* (i), (ii) *hold.*

(i) *The projection d' (resp. d'') of d on $S \cup \{n+1\}$ (resp. on $S \cup \{n+2\}$) is h-embeddable.*

(ii) $$\begin{cases} d(n+1, n+2) \leqslant \beta' + \beta'' + \sum_{x \in S} \min(\beta'_x + \beta''_x, 2\alpha_x - \beta'_x - \beta''_x), \\ d(n+1, n+2) \geqslant \max(\beta', \beta'') - \min(\beta', \beta'') + \sum_{x \in S} \max(\beta'_x, \beta''_x) - \min(\beta'_x, \beta''_x), \end{cases}$$

where β', β'' are nonnegative integers satisfying $\sigma' - (n-2)\tau' \leqslant \beta' \leqslant \sigma'$, $(\sigma' - \beta')/(n-2) \in \mathbb{Z}$ and $\sigma'' - (n-2)\tau'' \leqslant \beta'' \leqslant \sigma''$, $(\sigma'' - \beta'')/(n-2) \in \mathbb{Z}$.

Proof. Suppose that d is h-embeddable. Then, a $\mathbb{Z}_+$-realization of d is of the form:

$$\begin{aligned} d = & \sum_{x \in S} \gamma_x \delta(\{x\}) + \gamma'_x \delta(\{x, n+1\}) + \gamma''_x \delta(\{x, n+2\}) \\ & + \sum_{x \in S} (\alpha_x - \gamma_x - \gamma'_x - \gamma''_x) \delta(\{x, n+1, n+2\}) \\ & + \gamma' \delta(\{n+1\}) + \gamma'' \delta(\{n+2\}) + \gamma \delta(\{n+1, n+2\}), \end{aligned}$$

where all the coefficients are nonnegative integers. We obtain the following decomposition for d':

$$d' = (\gamma + \gamma')\delta(\{n+1\}) + \sum_{x \in S} (\gamma_x + \gamma''_x)\delta(\{x\}) + (\alpha_x - \gamma_x - \gamma''_x)\delta(\{x, n+1\}).$$

Comparing with (4.5), we deduce that $\beta' := \gamma + \gamma'$ satisfies (4.6), and $\beta'_x = \alpha_x - \gamma_x - \gamma''_x$ for $x \in S$. Similarly, when considering d'', we obtain that $\beta'' := \gamma + \gamma''$ satisfies (4.6), and $\beta''_x = \alpha_x - \gamma_x - \gamma'_x$ for $x \in S$. Therefore,

$$\begin{aligned} d(n+1, n+2) &= \gamma' + \gamma'' + \sum_{x \in S} (\gamma'_x + \gamma''_x) \\ &= \beta' + \beta'' - 2\gamma + \sum_{x \in S} (2\alpha_x - 2\gamma_x - \beta'_x - \beta''_x). \end{aligned}$$

As $\max(\beta'_x, \beta''_x) - \min(\beta'_x, \beta''_x) \leqslant 2\alpha_x - 2\gamma_x - \beta'_x - \beta''_x \leqslant \min(\beta'_x + \beta''_x, 2\alpha_x - \beta'_x - \beta''_x)$ (since $\gamma_x, \gamma'_x, \gamma''_x \geqslant 0$ and $\alpha_x \geqslant \gamma_x + \gamma'_x + \gamma''_x$) and $\max(\beta', \beta'') - \min(\beta', \beta'') \leqslant \beta' + \beta'' - 2\gamma \leqslant \beta' + \beta''$ (since $\gamma, \gamma', \gamma'' \geqslant 0$), we deduce that (ii) holds.

Conversely, suppose that (i), (ii) hold. Then, we can write

$$d(n+1, n+2) = B + \sum_{x \in S} B_x$$

where B, B_x are chosen in the following way: B has the same parity as $\beta' + \beta''$ and satisfies

$$\max(\beta', \beta'') - \min(\beta', \beta'') \leqslant B \leqslant \beta' + \beta''$$

and, for $x \in S$, B_x has the same parity as $\beta'_x + \beta''_x$ and satisfies

$$\max(\beta'_x, \beta''_x) - \min(\beta'_x, \beta''_x) \leqslant B_x \leqslant \min(\beta'_x + \beta''_x, 2\alpha_x - \beta'_x - \beta''_x).$$

For $x \in S$ set

$$a_x := \frac{\beta'_x + \beta''_x - B_x}{2}, \qquad d_x := \frac{2\alpha_x - \beta'_x - \beta''_x - B_x}{2},$$

$$b_x := \frac{B_x + \beta'_x - \beta''_x}{2}, \qquad c_x := \frac{B_x + \beta''_x - \beta'_x}{2}$$

and set

$$a := \frac{\beta' + \beta'' - B}{2}, \qquad b := \frac{\beta' - \beta'' + B}{2}, \qquad c := \frac{\beta'' - \beta' + B}{2}.$$

Hence, $a_x, b_x, c_x, d_x, a, b, c$ are nonnegative integers. One can easily check that

$$d = \left(\sum_{x \in S} a_x \delta(\{x, n+1, n+2\}) + b_x \delta(\{x, n+1\}) + c_x \delta(\{x, n+2\}) + d_x \delta(\{x\})\right)$$
$$+ a\delta(\{n+1, n+2\}) + b\delta(\{n+1\}) + c\delta(\{n+2\}),$$

which shows that d is h-embeddable. □

Finally, let us consider the class of metrics taking their values in $\{1, 2\alpha, 2\alpha + 1\}$ for some integer $\alpha \geqslant 2$. The case $\alpha = 1$ was studied in [3] and the case $\alpha \geqslant 2$ can be easily settled as follows.

Proposition 4.4. *Assume d takes all its values in $\{1, 2\alpha, 2\alpha + 1\}$ for some integer $\alpha \geqslant 2$. Then, d is h-embeddable if and only if d satisfies the metric condition* (1.1) *and the even condition* (1.2).

Recently, h-embeddability was characterized within the class of metrics taking their values in $\{\alpha, \beta, \alpha + \beta\}$, where α, β are nonnegative integers such that at least one of α, β, or $\alpha + \beta$ is odd, yielding a polynomial recognition algorithm [14].

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