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Discrete Mathematics 69 (1988) 165–171
North-Holland

A CHARACTERIZATION OF MATROIDAL SYSTEMS OF INEQUALITIES

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Received 8 April 1986
Revised 1 April 1987

We consider the polyhedron $\text{Conv}\{x \in \{0,1\}^n : Mx \leq b\}$, where M is a $p \times n$ matrix of zeroes and ones and b is a nonnegative integer vector. We give a characterization of such polyhedra whose extreme points are the incidence vectors of the family of independent sets of a matroid and extend our result to polyhedra which are the convex hull of integral polymatroids. We also introduce some new classes of integral matroid polyhedra which extend a result of Edmonds.

We assume familiarity with the basic matroid theory. We recall here the main definitions and refer to [3] for an extensive treatment. Let E be a finite set. A collection $\mathcal{I}$ of subsets of E is called an *independence system* if $\mathcal{I} \neq \emptyset$ and for all $I, J \subseteq E$, $I \in \mathcal{I}$ and $J \subseteq I$ imply $J \in \mathcal{I}$. If $\mathcal{I}$ also satisfies the following exhange property: for all $I, J \in \mathcal{I}$ such that $|I| < |J|$, there exists an element $e \in J - I$ such that $I \cup e \in \mathcal{I}$, then $\mathcal{I}$ is called a *matroid* and its members are called the *independent sets* of the matroid. The *rank* $r(A)$ of a subset A of E is the maximum cardinality of an independent subset of A and the *rank function* r is the set function associating to every subset A of E its rank $r(A)$. A subset F of E is called *closed* or *flat* if $r(F \cup e) > r(F)$ for all $e \in E - F$. The *closure operator* σ is the set function associating with every subset A of E the subset $\sigma(A) = \{e \in E : r(A \cup e) = r(A)\}$; if $\mathcal{I}$ is a matroid, then $\sigma(A)$ is the smallest flat containing A. For further properties of the independent sets, the rank function, the flats, the closure operator, see also [3].

Let $\mathcal{A}$ be a collection of p nonempty subsets of a finite set E, $|E| = n$, b be a vector of $\mathbb{N}^p$. Consider the family of $0, 1$ vectors x which are solution to the following system:

$$Mx \leq b, \tag{1}$$

where M is the $p \times n$ incidence matrix of the collection $\mathcal{A}$. The solutions of (1) are the incidence vectors of the elements of the following independence system: $\mathcal{I}(\mathcal{A}, b) = \{I \subseteq E : |I \cap A| \leq b(A)\ \forall A \in \mathcal{A}\}$. The rank function r of $\mathcal{I}(\mathcal{A}, b)$ is defined by $r(S) = \text{Max}\{|I| : I \in \mathcal{I}(\mathcal{A}, b),\ I \subseteq S\}$ for every subset S of E. It is well known (see [3]) that $\mathcal{I}(\mathcal{A}, b)$ is a matroid if and only if the function r is submodular on 2^E, that is:

$$r(S \cup T) + r(S \cap T) \leq r(S) + r(T) \quad \text{for all subsets } S, T \text{ of } E. \tag{2}$$

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Our first result shows that it is in fact enough to suppose that (2) is satisfied for every pair of members of $\mathscr{A}$.

Theorem 3. *The following statements are equivalent*:

(i) $\mathscr{I}(\mathscr{A}, b)$ *is a matroid,*

(ii) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *for all* $A, B \in \mathscr{A}$ *such that* $A \cap B \neq \emptyset$,

(iii) $r(A \cup B) + r(A \cap B) \leq b(A) + b(B)$ *for all* $A, B \in \mathscr{A}$ *such that* $A \cap B \neq \emptyset$.

Proof. The implication (i)$\to$(ii) follows from (2) and (ii)$\to$(iii) follows from the fact that $b(A) \geq r(A)$ for all $A \in \mathscr{A}$. Hence, it suffices to show that (iii)$\to$(i). Let us suppose by contradiction that $\mathscr{I}(\mathscr{A}, b)$ is not a matroid; we are going to construct a pair $(A, B) \in \mathscr{A}$ contradicting (iii).

Since $\mathscr{I}(\mathscr{A}, b)$ is not a matroid, there exists I, J in $\mathscr{I}(\mathscr{A}, b)$ such that $|I| < |J|$ and $I \cup e \notin \mathscr{I}(\mathscr{A}, b)$ for all $e \in J - I$. Let us choose such a pair (I, J) for which $|I \Delta J|$ is minimum where $I \Delta J = (I - J) \cup (J - I)$. It is easy to see that we have: $|J| = |I| + 1$ and $|I \Delta J| \geq 3$.

We first show that the theorem holds when $|I \Delta J| = 3$. Then we have clearly that $|I - J| = 1$ and $|J - I| = 2$. Let $I_0 = I \cap J$, then $I = I_0 \cup a$ and $J = I_0 \cup \{e, f\}$ where a, e, f are distinct elements of E.

Since $I \cup e \notin \mathscr{I}(\mathscr{A}, b)$, there exists A in $\mathscr{A}$ such that

$$|(I \cup e) \cap A| \geq b(A) + 1.$$

Since $I \in \mathscr{I}(\mathscr{A}, b)$, we also have

$$|I \cap A| \leq b(A).$$

Therefore, we obtain

$$|I \cap A| = b(A) \quad \text{and} \quad e \in A.$$

In the same way, there exists B in $\mathscr{A}$ such that:

$$|I \cap B| = b(B) \quad \text{and} \quad f \in B.$$

We now prove that element a belongs to $A \cap B$. We have

$$b(A) = |I \cap A| = |I_0 \cap A| + |a \cap A|.$$

Since $J \in \mathscr{I}(\mathscr{A}, b)$, we also have

$$b(A) \geq |J \cap A| \geq |(I_0 \cup e) \cap A| = ||I_0 \cap A| + 1.$$

Therefore, $a \in A$ and, by the same argument, $a \in B$. Let us now see that (A, B) contradicts (iii).

$$\begin{aligned} b(A) + b(B) &= |I \cap A| + |I \cap B| \\ &= |I_0 \cap A| + |I_0 \cap B| + |a \cap A| + |a \cap B| \\ &= |I_0 \cap (A \cap B)| + |I_0 \cap (A \cup B)| + 2 \\ &= |I \cap (A \cap B)| + |J \cap (A \cup B)| - 1 \\ &< r(A \cap B) + r(A \cup B). \end{aligned}$$

We now prove that, in fact, $|I \Delta J| = 3$. Suppose for a contradiction that $|I \Delta J| > 3$. Choose some elements a in $I - J$ and b in $J - I$. Hence, we have

$$|J - b| = |I - a| + 1$$

and

$$|(I - a) \Delta (J - b)| = |I \Delta J| - 2 < |I \Delta J|,$$

therefore, by choice of (I, J), there exists an element x in $(J - b) - (I - a)$ such that $I' = I - a + x \in \mathscr{I}$.

Consider the pair (I', J) of $\mathscr{I}$. Since $|J| = |I'| + 1$ and $|J \Delta I'| = |J \Delta I| - 2 < |J \Delta I|$, we deduce again that there exists an element y in $J - I'$ such that $I'' = I' \cup y \in \mathscr{I}$.

Consider now the pair (I, I'') of $\mathscr{I}$. We have $|I''| = |I| + 1$ and $|I'' \Delta I|| = 3 < |I \Delta J|$ by assumption; therefore, we infer again the existence of an element z in $I'' - I$ such that $I \cup z \in \mathscr{I}$. However, since $I'' - I = \{x, y\}$ is contained in J, the assertion: $I \cup z \in \mathscr{I}$ contradicts our assumption on (I, J). □

Corollary 4. *$\mathscr{I}(\mathscr{A}, b)$ is a matroid for all vectors $b \in \mathbb{N}^n$ if and only if $\mathscr{A}$ is a nested family, that is, for all members, A, B of $\mathscr{A}$ such that $A \cap B \neq \emptyset$, then $A \subseteq B$ or $B \subseteq A$.*

Proof. Sufficiency follows trivially from Theorem 3. Suppose now that $\mathscr{A}$ is not a nested family, so there exists A, B in $\mathscr{A}$ such that $A \cap B \neq \emptyset$, $A \nsubseteq B$ and $B \nsubseteq A$. Define the function b on $\mathscr{A}$ by: $b(C) = \mathrm{Max}(1, |C \cap (A \cup B - A \cap B)|)$ for all $C \in \mathscr{A}$. It is easy to see that: $r(A) = |A - B|$, $r(B) = |B - A|$, $r(A \cap B) \geq 1$ and $r(A \cup B) = |A \cup B - A \cap B| = r(A) + r(B)$, and therefore $r(A \cup B) + r(A \cap B) > r(A) + r(B)$. □

Remark 5. Theorem 3 provides an efficient procedure to test whether the family of 0, 1 vectors which solve the system $Mx \leq b$ can be interpreted as the set of incidence vectors of the independent sets of a matroid, provided that a rank oracle for the independence system $\mathscr{I}(\mathscr{A}, b)$ is available. In fact, much less is needed: let $\mathrm{Ub}(A)$ be an upper bound to the value of the rank of the set A that is guaranteed to be tight, i.e., to coincide with $r(A)$, when $\mathscr{I}(\mathscr{A}, b)$ is a matroid. Then, as a consequence of Theorem 3, $\mathscr{I}(\mathscr{A}, b)$ is a matroid if and only if $\mathrm{Ub}(A \cup B) + \mathrm{Ub}(A \cap B) \leq b(A) + b(B)$ for all $A, B \in \mathscr{A}$ such that $A \cap B \neq \emptyset$. However, we do not know any efficient algorithm to compute the values of $\mathrm{Ub}(A \cap B)$, $\mathrm{Ub}(A \cup B)$.

Consider now the following theorem due to Edmonds ([1])

Theorem 6. *If $\mathscr{A}$ is an intersecting family, i.e., $A \cap B \in \mathscr{A}$, $A \cup B \in \mathscr{A}$ for all $A, B \in \mathscr{A}$ such that $A \cap b \neq \emptyset$, and if $b(A \cup B) + b(A \cap B) \leq b(A) + b(B)$ for all $A, B \in \mathscr{A}$ such that $A \cap B \neq \emptyset$, then $\mathscr{I}(\mathscr{A}, b)$ is a matroid and its rank function is given by $r(S) = \min\{\sum b(A_i) + |S - \bigcup A_i| : A_i \in \mathscr{A},\ A_i \cap A_j \neq \emptyset \text{ for all } i \neq j\}$ for every subset S of E.*

Furthermore, the polyhedron spanned by the incidence vectors of $\mathcal{I}(\mathcal{A}, b)$ *is* $P = \{x: 0 \leqslant x \leqslant 1, Mx \leqslant b\}$.

Remark 7. Theorem 3 generalizes the first part of Theorem 6 stating that $\mathcal{I}(\mathcal{A}, b)$ is a matroid (use (iii) and observe that $r(A \cap B) + r(A \cup B) \leqslant b(A \cap B) + b(A \cup B)$ for all intersecting $A, B \in \mathcal{A}$).

Remark 8. Theorem 3 does not extend to the case when the rank function is assumed to be submodular only on crossing pairs, that is, on pairs (A, B) of members of $\mathcal{A}$ such that $A \cap B \neq \emptyset$ and $A \cup B \neq E$. Take, for instance, $E = \{1, 2, 3, 4, 5\}$ and $\mathcal{A} = \{A = \{1, 2, 3\}, B = \{1, 4, 5\}, C = \{1, 2, 3, 5\}\}$ with $b(A) = 2$, $b(B) = 2$ and $b(C) = 3$. The rank function of $\mathcal{I}(\mathcal{A}, b)$ is submodular on the unique crossing pair (A, C), but $\mathcal{I}(\mathcal{A}, b)$ is not a matroid since its bases are $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$ and $\{2,3,4,5\}$.

In the following part, we give a sufficient condition for the integrality of the polyhedron: $P = \{x: 0 \leqslant x \leqslant 1, Mx \leqslant b\}$ when $\mathcal{I}(\mathcal{A}, b)$ is a matroid, generalizing the second part of Theorem 6 of Edmonds.

Let us first recall some results on matroids polyhedra. Let $\mathcal{M}$ be a matroid on the set E with $\mathcal{I}$ as family of independent sets, r as rank function, σ as closure operator. We assume that the following condition holds:

$$\{e\} \in \mathcal{I} \quad \text{for all } e \in E. \tag{9}$$

A subset F of E is called *separable* if F can be partitioned into $F = F_1 \cup F_2$ with $F_1 \neq \emptyset$, $F_2 \neq \emptyset$ and $r(F) = r(F_1) + r(F_2)$. Notice that if F is closed, then F_1, F_2 are also closed; indeed we have $F = \sigma(F_1) \cup \sigma(F_2)$ and therefore we deduce from (2) that $r(F) \leqslant r(\sigma(F_1)) + r(\sigma(F_2)) - r(\sigma(F_1) \cap \sigma(F_2))$, which implies $r(\sigma(F_1) \cap \sigma(F_2)) = 0$ and thus $\sigma(F_1) \cap \sigma(F_2) = \emptyset$ since $r(\sigma(F_1)) = r(F_1)$, $r(\sigma(F_2)) = r(F_2)$) and $r(F) = r(F_1) + r(F_2)$.

Let P_0 be the polyhedron spanned by the incidence vectors of the members of $\mathcal{I}$. The following theorem, due to Edmonds, gives the facets of P_0, i.e., the minimal set of linear inequalities necessary to define P_0.

Theorem 10 (see [2]). *The polyhedron* P_0 *is given by*

$$\left\{x \geqslant 0: \sum_{e \in F} x_e \leqslant r(F), \text{ for every subset } F \subseteq E\right\}.$$

Furthermore, let F *be a subset of* E, *then the inequality* $\sum_{e \in F} x_e \leqslant r(F)$ *induces a facet of* P_0 *if and only if* F *is closed and nonseparable.*

Using this result, we prove the following theorem:

Theorem 11. *Suppose $\mathscr{A}$, b satisfy the following conditions:*

(i) *$\mathscr{I}(\mathscr{A}, b)$ is a matroid,*

(ii) *$A \cup B \in \mathscr{A}$ for all $A, B \in \mathscr{A}$ such that $A \cap B \neq \emptyset$,*

(iii) *$b(A) = r(A)$ for all $A \in \mathscr{A}$ which is a nonseparable flat,*

(iv) *$b(A) \geq 1$ for all $A \in \mathscr{A}$.*

Then the polyhedron spanned by the incidence vectors of the members of $\mathscr{I}(\mathscr{A}, b)$ is equal to $P = \{x: 0 \leq x \leq 1, Mx \leq b\}$.

Notice that (iv) is equivalent to the condition (9) mentioned above.

Proof. In view of Theorem 10, it is enough to show that every nonseparable closed set F such that $|F| \geq 2$ is a member of the family $\mathscr{A}$, since, by (iii), we have, in this case, $r(F) = b(F)$; the polyhedron P being integral if and only if all the facets of P, different from $x_e \geq 0$ and $x_e \leq 1$, are contained in the set of linear inequalities given by $Mx \leq b$. Let F be a closed nonseparable subset of E with $|F| \geq 2$. We prove that $F \in \mathscr{A}$ through Lemmas 12, 13, 14 as follows. Define $\mathscr{A}_F = \{A \in \mathscr{A}: A \subseteq F\}$. Lemma 12 implies that $\mathscr{A}_F \neq \emptyset$. Let $A_1, \ldots, A_m$ be the distinct maximal elements of $\mathscr{A}_F$, hence $A_1 \cup \cdots \cup A_m \subseteq F$ and, as a consequence of condition (ii), $A_i \cap A_j = \emptyset$. Lemma 13 proves that $F = A_1 \cup \cdots \cup A_m$. Then Lemma 14 shows that $r(F) = \sum_{i=1}^{m} r(A_i)$ implying that $m = 1$, else F is separable, and thus $F = A_1 \in \mathscr{A}$. □

Lemma 12. *Let I be a maximal independent subset of F and e be an element of $F - I$. Then there exists a subset $A \in \mathscr{A}$, $A \subseteq F$ such that $e \in A$ and $|I \cap A| = b(A)$.*

Proof. Since $e \in F - I$, $I \cup e \notin \mathscr{I}(\mathscr{A}, b)$ which implies the existence of $A \in \mathscr{A}$ such that $e \in A$ and $|I \cap A| = b(A)$. Suppose by contradiction that $A \not\subseteq F$, hence there exists an element $a \in A - F$. Since $a \notin F$, $I \cup a \in \mathscr{I}(\mathscr{A}, b)$ yielding $|(I \cup a) \cap A| \leq b(A)$, which contradicts the assumptions $|I \cap A| = b(A)$ and $a \in A$. □

Notice that, since F is nonseparable, F is not an independent set and therefore we can find an independent set I and an element e as defined in Lemma 12.

Lemma 13. $F = A_1 \cup A_2 \cup \cdots \cup A_m$.

Proof. Let e be an element of F. We show that e belongs to some of the A_i's. We first prove that $r(F - e) = r(F)$; otherwise we have $r(F) = r(F - e) + 1 = r(F - e) + r(\{e\})$, which contradicts the nonseparability of F. Let I be a maximal independent subset of $F - e$, hence I is also a maximal independent subset of F. From Lemma 12, there exists $A \in \mathscr{A}$ such that $e \in A$ and $A \subseteq F$. Let A_i be a maximal element of $\mathscr{A}_F$ containing A, then $e \in A_i$. □

Lemma 14. *The flat F is separable; in fact, $r(F) = \sum_{i=1}^{m} r(A_i)$.*

Proof. Let I be a maximal independent subset of F and $I_i = I \cap A_i$ for $i \in [1, m]$. It is enough to show that I_i is indeed a maximal independent subset of A_i, that is, $|I_i| = r(A_i)$, yielding therefore

$$r(F) = |I| = \sum_{i=1}^{m} |I_i| = \sum_{i=1}^{m} r(A_i).$$

Suppose by contradiction that there exists an element $e \in A_i - I_i$ such that $I_i \cup e \in \mathscr{I}(\mathscr{A}, b)$. Hence $e \in F - I$ and again Lemma 12 yields the existence of $A \in \mathscr{A}$ such that $e \in A$, $|I \cap A| = b(A)$ and $A \subseteq F$. Since $e \in A \cap A_i$, we have therefore $A \subseteq A_i$; hence $b(A) + 1 = |(I \cup e) \cap A| = |(I \cup e) \cap A \cap A_i| = |(I_i \cup e) \cap A|$ which contradicts the fact that $I_i \cup e$ is independent. □

In this last section, we consider the following polyhedron:

$$\mathrm{Conv}\{x \in \mathbb{N}^n : Mx \leqslant b\}, \tag{15}$$

where M is again the $p \times n$ incidence matrix of a collection $\mathscr{A}$ of p nonemtpy subsets of $E = [1, n]$ and b is a vector of $\mathbb{N}^p$. Let us denote by $P(\mathscr{A}, b)$ the set of integral points of polyhedron (15). We characterize those polyhedra (15) for which $P(\mathscr{A}, b)$ is a polymatroid.

Given a vector $u = (u_1, \ldots, u_n)$ of $\mathbb{R}^n$, let $|u| = \sum_{i=1}^n u_i$. Given two vectors u, v of $\mathbb{R}^n$, define the vector $w = u \vee v$ of $\mathbb{R}^n$ whose components are $w_i = \mathrm{Max}(u_i, v_i)$ for all $i \in [1, n]$. Consider the following partial order on $\mathbb{R}^n$; $x \leqslant y$ if and only if $x_i \leqslant y_i$ for all $i \in [1, n]$. For all $i \in [1, n]$, e_i denotes the vector of $\mathbb{R}^n$ whose components are all equal to zero except the ith component which is equal to one. Let P be a family of vectors of $\mathbb{N}^n$. For every subset S of E, we define the rank of S by:

$$\rho(S) = \mathrm{Max}\Big(\sum_{i \in S} u_i : u \in P\Big) \tag{16}$$

and the rank function is the set function associating with every $S \subseteq E$ its rank $\rho(S)$.

An *integral polymatroid* is a family P of vectors of $\mathbb{N}^n$ such that:

(P1) for all $u \in P$ and $v \in \mathbb{N}^n$, if $v \leqslant u$, then $v \in P$;

(P2) for all $u, v \in P$, if $|v| > |u|$, then there exists $w \in P$ such that $u < w \leqslant u \vee v$.

The vectors of P are called *independent vectors*. The concept of integral polymatroids was introduced by Edmonds in [1] as a generalization of matroids (obtained when P contains only 0,1-vectors; then the independent sets of matroid are precisely the subsets of $\mathbb{N}$ whose 0, 1 incidence vectors belong to P). An introduction to integral polymatroids can be found in [3].

Given a family P of vectors of $\mathbb{N}^n$ satisfying (P1), it is known (see [3]) that P is a polymatroid if and only if its rank function ρ (as defined in (16)) is a submodular set function, that is:

$$\rho(S \cup T) + \rho(S \cap T) \leq \rho(S) + \rho(T) \quad \text{for all } S, T \subseteq E. \tag{17}$$

Our next result shows that it is in fact enough to suppose that (17) is satisfied for each intersecting pair of members of $\mathscr{A}$, thus extending Theorem 3 to polymatroids. The proof, although more involved, is very similar to the one used in Theorem 3, hence it is omitted.

Theorem 18. *The following statements are equivalent*:
(i) *$P(\mathscr{A}, b)$ is a polymatroid,*
(ii) *$\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all $A, B \in \mathscr{A}$ such that $A \cap B \neq \emptyset$,*
(iii) *$\rho(A \cup B) + \rho(A \cap B) \leq b(A) + b(B)$ for all $A, B \in \mathscr{A}$ such that $A \cap B \neq \emptyset$.*

Acknowledgments

We thank Manfred Padberg for improving substantially the presentation of this paper.

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