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by

Hideyuki Takamizawa

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UNIVERSITY OF TSUKUBA
Department of Economics
1-1-1 Tennodai
Tsukuba, Ibaraki 305-8571
JAPAN

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Hideyuki Takamizawa

Graduate School of Humanities and Social Sciences,
University of Tsukuba
Tsukuba Ibaraki 305-8571, JAPAN
E-mail: takamiza[atmark]social.tsukuba.ac.jp

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Abstract

This study proposes an approximation of European option prices under arbitrary diffusion processes of the spot price. The key is to approximate the characteristic function of the log spot price process as the solution to ordinary differential equations. The option price is then obtained by the inverse Fourier transform. Numerical experiments, using a model that has the constant elasticity of volatility specification in both the spot price and volatility processes, confirm reasonable accuracy of the approximation, except when the volatility process exhibits high variation.

Keywords: Approximation, Conditional moment, European option, Fourier transform, Stochastic volatility.

1 Introduction

The Fourier Transform (FT) approach to the pricing of options, originally introduced by Heston (1993), allows for the extension of a class of models under which an efficient computation of option prices is possible. This is because the FT approach works when the characteristic function of a (log) spot price process has a closed form even though the density function does not. This weaker requirement leads to the development of more realistic models of the spot price process. Examples include affine jump-diffusion models [Bakshi et al. (1997), Bates (1996, 2000), Duffie et al. (2000), Pan (2002), and Scott (1997)]; variance-gamma models [Madan and Seneta (1990), and Madan et al. (1998)]; the normal inverse Gaussian model [Barndorff-Nielsen (1998)]; the finite moment log-stable model [Carr and Wu (2003a)]; the CGMY model [Carr et al. (2002)]; and discrete-time GARCH models [Heston and Nandi (2000)].

Nevertheless, it seems that other realistic models are worth pursuing. One notable example is a model that accommodates both stochastic volatility (SV) and constant elasticity of volatility (CEV), as proposed by Jones (2003), and Melino and Turnbull (1990) in the option-pricing literature, and by Andersen and Lund (1997), Brenner et al. (1996), and Gallant and Tauchen (1998) in the time-series literature. This volatility specification can generate skewness and leptokurtosis in the conditional distribution of the spot price, which are fundamental for many securities in capturing not only time-series properties of the spot price but also cross-sectional properties of the option prices observed as implied-volatility smiles or smirks. Furthermore, in pricing options written on an asset that is sensitive to interest rates, the stochastic risk-free rate needs to be incorporated, possibly with the instantaneous correlation between the spot price and the risk-free rate. In the above cases, however, even the characteristic function of the spot price process is unavailable in closed form, which makes it difficult to achieve an efficient computation of option prices based on these models.

This study proposes an approximation of European option prices under general diffusion processes of the spot price. The key is to approximate the characteristic function of the log price process using a method originally proposed by Shoji (2002) and recently applied to bond pricing by Takamizawa and Shoji (2009). The method approximates conditional moments of diffusion processes as the solution to ordinary differential equations. Since the characteristic function is basically given by the expectation, this method can be applicable. Once an analytical expression of the characteristic function is available, the option price is obtained by Fourier inversion.

Using an SV model that has the CEV specification in both the spot price and volatility processes, the accuracy of the approximation is examined by numerical experiments, where benchmark option prices are computed by the Monte Carlo (MC) method. The numerical results are summarized as follows. (i) The third-order approximation generally achieves high accuracy, except when the volatility process exhibits high variation. (ii) When the mean reversion of the volatility process is relatively fast, the second-order approximation performs reasonably well. Actually, the higher accuracy of the third-order approximation is achieved at the cost of computational complexity. This study also explores the possibility of reducing this complexity without much reducing the accuracy.

Section 2 explains the method of approximating conditional moments, whose application within the FT approach is presented in Section 3. Section 4 examines the accuracy of the approximation. Section 5 identifies conditional third moments of the underlying processes that significantly affect option prices, which is aimed at reducing computational burden. Section 6 provides concluding remarks.

2 An Approximation Formula of Conditional Moments

A method of approximating conditional moments, originally proposed by Shoji (2002), is widely applicable to the computation of up to conditional n-th moments, if they exist, for a d-dimensional diffusion process. An important feature of the method is that all the moments considered are computed simultaneously as the solution to a system of ordinary differential equations. Here, a more specific explanation of the method is presented, taking the application to the pricing of options into consideration. First is the case with $(n, d) = (1, 2)$, i.e., the conditional first moments of a two-dimensional process, followed by the cases with up to $(n, d) = (3, 3)$, i.e., the conditional third moments of a three-dimensional process. As seen below, n can be considered as the order of approximation.

2.1 Conditional first moments of a two-dimensional process

Let $X_t = (x_{1,t}\ x_{2,t})'$ be a two-dimensional stochastic process, which evolves according to the following stochastic differential equation (SDE):

$$dx_{1,t} = f_1(X_t)dt + \xi_1(X_t)'dW_t \ , \tag{1}$$

$$dx_{2,t} = f_2(X_t)dt + \xi_2(X_t)'dW_t \ , \tag{2}$$

where W_t is the two-dimensional Brownian motion, and the drift and diffusion functions, f_i and ξ_i $(i = 1, 2)$, satisfy certain technical conditions for the solutions to Eqs.(1) and (2) to exist for an arbitrary X_0. It is also assumed that f_i and $g_{ij} = \xi_i'\xi_j$ $(i, j = 1, 2)$ are appropriately smooth with respect to X_t. Specifically, according to Shoji (2002), they belong to C^{n+1}-class in computing up to conditional n-th moments by the method, which is needed to prove the convergence of approximate moments to the true ones.

Let $\Psi_{s,t}$ be a vector consisting of the first moments of (an increment of) X_t conditioned on time $s \le t$:

$$\Psi_{s,t} = E_s \left(x_{1,t} - x_{1,s} \quad x_{2,t} - x_{2,s} \right)' \ ,$$

where E_s stands for the conditional expectation. By integrating Eq.(1) and taking the conditional expectation,

$$E_s[x_{1,t} - x_{1,s}] = E_s\left[\int_s^t f_1(X_u)du\right] . \tag{3}$$

By applying the Taylor expansion to $f_1(X_u)$ around X_s up to the first order and substituting this into Eq.(3),

$$\begin{aligned} E_s[x_{1,t} - x_{1,s}] &= f_1(X_s)(t-s) \\ &+E_s\left[\int_s^t \left\{f_1^{(1,0)}(X_s)(x_{1,u} - x_{1,s}) + f_1^{(0,1)}(X_s)(x_{2,u} - x_{2,s})\right\} du\right] + R_1 , \end{aligned} \tag{4}$$

where $f^{(i,j)} = \frac{\partial^{i+j} f}{\partial x_1^i \partial x_2^j}$, and R_1 is a residual term. Eq.(4) can be expressed in a vector form as

$$E_s[x_{1,t} - x_{1,s}] = f_1(X_s)(t-s) + \begin{pmatrix} f_1^{(1,0)}(X_s) & f_1^{(0,1)}(X_s) \end{pmatrix} \int_s^t \Psi_{s,u} du + R_1 . \tag{5}$$

Similarly,

$$E_s[x_{2,t} - x_{2,s}] = f_2(X_s)(t-s) + \begin{pmatrix} f_2^{(1,0)}(X_s) & f_2^{(0,1)}(X_s) \end{pmatrix} \int_s^t \Psi_{s,u} du + R_2 . \tag{6}$$

Expressing Eqs.(5) and (6) together in a vector form leads to

$$\Psi_{s,t} = A(X_s) \int_s^t \Psi_{s,u} du + b(X_s)(t-s) + R , \tag{7}$$

where

$$A(X_s) = \begin{pmatrix} f_1^{(1,0)}(X_s) & f_1^{(0,1)}(X_s) \\ f_2^{(1,0)}(X_s) & f_2^{(0,1)}(X_s) \end{pmatrix} , \qquad b(X_s) = \begin{pmatrix} f_1(X_s) \\ f_2(X_s) \end{pmatrix} ,$$

and $R = (R_1\ R_2)'$. Eq.(7) can be developed to

$$\Psi_{s,t} = \int_s^t e^{A(X_s)(t-u)} b(X_s) du + \hat{R} , \tag{8}$$

where $\hat{R}$ is a residual vector. If, in addition, A is invertible,

$$\Psi_{s,t} = A^{-1}(X_s)\{e^{A(X_s)(t-s)} - I\} b(X_s) + \hat{R} . \tag{9}$$

As seen below, Eqs.(7)–(9) hold for any (n, d) with modification to $A(X_s)$ and $b(X_s)$. Omitting the residual vector leads to the approximation. According to Shoji (2002), both R and $\hat{R}$ have order of $O((t-s)^{(n+3)/2})$ when up to conditional n-th moments are computed. Thus, n can be considered as the order of approximation. In this particular case, it is $O((t-s)^2)$ for $n = 1$. Indeed, the conditional moments computed by the method are more accurate than those computed by the conventional Euler method. To illustrate this, suppose a one-dimensional lognormal process X_t, which evolves according to the following SDE: $dX_t = aX_t dt + bX_t dW_t$. We know that $E_s[X_t - X_s] = X_s\{e^{(a-\frac{1}{2}b^2)(t-s)} - 1\}$. On the other hand, it is approximated by the Euler method as $E_s[X_t - X_s] = aX_s(t-s)$, and by the proposed method as $E_s[X_t - X_s] = X_s\{e^{a(t-s)} - 1\}$ by substituting $A(X_s) = a$ and $b(X_s) = aX_s$ into Eq.(9) (without the residual term). Obviously, the latter is closer to the true value. When b is large, however, the approximation error of the proposed method may not be negligible. A more important message from this simple example, therefore, is that even though the focus is on the computation of lower-order moments, it is more appropriate to include information on higher-order moments.

2.2 Up to conditional second moments of a two-dimensional process

The moment vector, $\Psi_{s,t}$, consists of up to the conditional second moments. It is here a five-dimensional vector as

$$\Psi_{s,t} = E_s \Big(x_{1,t} - x_{1,s} \quad x_{2,t} - x_{2,s} \quad (x_{1,t} - x_{1,s})^2 \quad (x_{2,t} - x_{2,s})^2 \quad (x_{1,t} - x_{1,s})(x_{2,t} - x_{2,s}) \Big)' .$$

By applying the Taylor expansion to $f_i(X_u)$ $(i = 1, 2)$ around X_s up to the second order,

$$\begin{aligned} f_i(X_u) &= f_i(X_s) \\ &\quad + f_i^{(1,0)}(X_s)(x_{1,u} - x_{1,s}) + f_i^{(0,1)}(X_s)(x_{2,u} - x_{2,s}) + \frac{1}{2} f_i^{(2,0)}(X_s)(x_{1,u} - x_{1,s})^2 \\ &\quad + \frac{1}{2} f_i^{(0,2)}(X_s)(x_{2,u} - x_{2,s})^2 + f_i^{(1,1)}(X_s)(x_{1,u} - x_{1,s})(x_{2,u} - x_{2,s}) + e_i , \end{aligned} \tag{10}$$

where e_i is a residual term. By substituting Eq.(10) into the right-hand side of $E_s[x_{i,t} - x_{i,s}] = E_s[\int_s^t f_i(X_u)du]$,

$$\begin{aligned} E_s[x_{i,t} - x_{i,s}] &= f_i(t-s) \\ &\quad + \Big(f_i^{(1,0)} \quad f_i^{(0,1)} \quad \frac{1}{2} f_i^{(2,0)} \quad \frac{1}{2} f_i^{(0,2)} \quad f_i^{(1,1)} \Big) \int_s^t \Psi_{s,u} du + R_i , \end{aligned} \tag{11}$$

where X_s is abbreviated for notational convenience.

Next, by applying the Ito formula to $(x_{1,t} - x_{1,s})^2$ and taking the conditional expectation,

$$E_s[(x_{1,t} - x_{1,s})^2] = E_s \left[\int_s^t \{2f_1(X_u)(x_{1,u} - x_{1,s}) + g_{11}(X_u)\} du \right] , \tag{12}$$

where $g_{11} = \xi_1' \xi_1$. By applying the Taylor expansion to $f_1(X_u)$ and $g_{11}(X_u)$ around X_s up to the first and second orders, respectively, the integrand of Eq.(12) becomes

$$\begin{aligned} & 2f_1(X_u)(x_{1,u} - x_{1,s}) + g_{11}(X_u) \\ &\quad = g_{11}(X_s) + \{2f_1(X_s) + g_{11}^{(1,0)}(X_s)\}(x_{1,u} - x_{1,s}) + g_{11}^{(0,1)}(X_s)(x_{2,u} - x_{2,s}) \\ &\quad + \{2f_1^{(1,0)}(X_s) + \frac{1}{2} g_{11}^{(2,0)}(X_s)\}(x_{1,u} - x_{1,s})^2 + \frac{1}{2} g_{11}^{(0,2)}(X_s)(x_{2,u} - x_{2,s})^2 \\ &\quad + \{2f_1^{(0,1)}(X_s) + g_{11}^{(1,1)}(X_s)\}(x_{1,u} - x_{1,s})(x_{2,u} - x_{2,s}) + e_{11} , \end{aligned} \tag{13}$$

where $g^{(i,j)}$ is defined analogously with $f^{(i,j)}$ and e_{11} is a residual term. By substituting Eq.(13) into Eq.(12),

$$
\begin{aligned}
E_s[(x_{1,t}-x_{1,s})^2] &= g_{11}(t-s) \\
&+ \left(2f_1 + g_{11}^{(1,0)} \quad g_{11}^{(0,1)} \quad 2f_1^{(1,0)} + \frac{1}{2}g_{11}^{(2,0)} \quad \frac{1}{2}g_{11}^{(0,2)} \quad 2f_1^{(0,1)} + g_{11}^{(1,1)}\right) \\
&\times \int_s^t \Psi_{s,u} du + R_{11} \; . \qquad (14)
\end{aligned}
$$

A similar manipulation is applied to $E_s[(x_{2,t}-x_{2,s})^2]$ and $E_s[(x_{1,t}-x_{1,s})(x_{2,t}-x_{2,s})]$. Expressing the resulting equations together in a vector form leads to Eq.(7), where

$$
A = \begin{pmatrix}
f_1^{(1,0)} & f_1^{(0,1)} & \frac{1}{2}f_1^{(2,0)} & \frac{1}{2}f_1^{(0,2)} & f_1^{(1,1)} \\
f_2^{(1,0)} & f_2^{(0,1)} & \frac{1}{2}f_2^{(2,0)} & \frac{1}{2}f_2^{(0,2)} & f_2^{(1,1)} \\
2f_1 + g_{11}^{(1,0)} & g_{11}^{(0,1)} & 2f_1^{(1,0)} + \frac{1}{2}g_{11}^{(2,0)} & \frac{1}{2}g_{11}^{(0,2)} & 2f_1^{(0,1)} + g_{11}^{(1,1)} \\
g_{22}^{(1,0)} & 2f_2 + g_{22}^{(0,1)} & \frac{1}{2}g_{22}^{(2,0)} & 2f_2^{(0,1)} + \frac{1}{2}g_{22}^{(0,2)} & 2f_2^{(1,0)} + g_{22}^{(1,1)} \\
f_2 + g_{12}^{(1,0)} & f_1 + g_{12}^{(0,1)} & f_2^{(1,0)} + \frac{1}{2}g_{12}^{(2,0)} & f_1^{(0,1)} + \frac{1}{2}g_{12}^{(0,2)} & f_1^{(1,0)} + f_2^{(0,1)} + g_{12}^{(1,1)}
\end{pmatrix},
$$

$$
b' = \begin{pmatrix} f_1 & f_2 & g_{11} & g_{22} & g_{12} \end{pmatrix} .
$$

Note that the residual terms, R_i and R_{ij} $(i,j=1,2)$, contain (the conditional expectation of) derivatives of f_i higher than the first order and derivatives of g_{ij} higher than the second order. Then, if f_i and g_{ij} are linear and quadratic, respectively, there is no residual term. In other words, the conditional moments computed by the method are exact. Even in this case, it may be beneficial to use this method when the derivation of the closed form conditional moments is demanding.

2.3 Up to conditional third moments of a three-dimensional process

Since the derivation is basically the same, only the resulting $A(X_s)$ and $b(X_s)$ are presented with a brief explanation of notations. Let $X_t = (x_{1,t} \; x_{2,t} \; x_{3,t})'$ be a three-dimensional stochastic process. The SDE for $x_{3,t}$ is specified analogously, where the drift and diffusion functions are denoted as $f_3(X_t)$ and $\xi_3(X_t)$, respectively, with $g_{ij} = \xi_i' \xi_j$ $(i,j=1,2,3)$. Let $\psi_{pqr,s,t} = E_s[(x_{1,t}-x_{1,s})^p (x_{2,t}-x_{2,s})^q (x_{3,t}-x_{3,s})^r]$, and a moment vector consisting of up to the conditional second moments can be expressed as

$$
\Psi_{s,t}^{n=2} = (\psi_{100,s,t} \quad \psi_{010,s,t} \quad \psi_{001,s,t} \quad \psi_{200,s,t} \quad \psi_{020,s,t} \quad \psi_{002,s,t} \quad \psi_{110,s,t} \quad \psi_{101,s,t} \quad \psi_{011,s,t})' , \qquad (15)
$$

where the order of approximation, $n=2$, is indicated in the upper suffix here. A moment vector for $n=3$, $\Psi_{s,t}^{n=3}$, is then obtained by augmenting $\Psi_{s,t}^{n=2}$ with a vector consisting of the conditional third moments:

$$
\begin{aligned}
\Psi_{s,t}^{n=3} = \; & (\Psi_{s,t}^{n=2\,\prime} \quad \psi_{300,s,t} \quad \psi_{030,s,t} \quad \psi_{003,s,t} \\
& \psi_{210,s,t} \quad \psi_{201,s,t} \quad \psi_{120,s,t} \quad \psi_{111,s,t} \quad \psi_{102,s,t} \quad \psi_{021,s,t} \quad \psi_{012,s,t})' , \qquad (16)
\end{aligned}
$$

Table 1 presents the elements of $A(X_s)$ and $b(X_s)$ for $\Psi_{s,t}^{n=3}$. The corresponding matrix and vector for $\Psi_{s,t}^{n=2}$ are the 9×9 upper-left submatrix of $A(X_s)$ and the 9×1 subvector of $b(X_s)$ for $\Psi_{s,t}^{n=3}$, respectively. As seen later, when a two-factor model is considered in which the spot price and volatility are stochastic, Table 1 is used for the reference. Which n to choose depends on the extent to which accuracy one requires. This is examined by numerical experiments in Section 4. When a three-factor model is considered in which the risk-free rate is also stochastic or the volatility is driven by two stochastic factors, the conditional moments of a four-dimensional process are required. The derivation of $A(X_s)$ and $b(X_s)$ for a four-dimensional process is a straightforward extension of the lower-dimensional processes.

3 Application of the Approximation Formula

To clarify how the proposed method is utilized within the FT approach to the pricing of options, the outline of this approach is first provided, followed by the application of the approximation formula.

3.1 Fourier transform of the option value

Let S_t be the spot price at time t. The price of a European put option at time 0 with a maturity date t and a strike price K is given by

$$P_0(K,t) = E_0[e^{-\int_0^t r_u du}(K - S_t)1_{\{S_t<K\}}] \ . \tag{17}$$

The price of a call option is immediately obtained by the put-call parity. S_t and r_t may depend on some state variables, however, they are suppressed at present.

Let s_t and k be logarithm of S_t and K, respectively: $s_t = \ln S_t$ and $k = \ln K$. Define

$$G_0(k;t,a) = E_0[e^{-\int_0^t r_u du} e^{a s_t} 1_{\{s_t<k\}}] \ , \tag{18}$$

and the put option price is given by

$$P_0(k,t) = e^k G_0(k;t,0) - G_0(k;t,1) \ . \tag{19}$$

Thus, the pricing of European options is equivalent to the evaluation of $G_0(k;t,a)$.

Following Duffie et al. (2000), the Fourier transform of $G_0(k;t,a)$ is given by

$$H_0(\theta;t,a) = \int e^{i\theta k} dG_0(k;t,a) = E_0[e^{-\int_0^t r_u du} e^{(a+i\theta)s_t}] \ . \tag{20}$$

Define

$$\phi_0(\alpha;t) = E_0[e^{-\int_0^t r_u du} e^{\alpha s_t}] \ , \tag{21}$$

and $H_0(\theta; t, a) = \phi_0(a + i\theta; t)$. The option value is then recovered by the inverse Fourier transform as

$$G_0(k; t, a) = \frac{1}{2}\phi_0(a; t) - \frac{1}{\pi}\int_0^\infty \frac{1}{\theta} Im[e^{-i\theta k}\phi_0(a + i\theta; t)]d\theta \ , \tag{22}$$

where $Im(x) = \beta$ for $x = \alpha + \beta i$. The feasibility of this FT approach depends on the availability of an analytical expression of $\phi_0(\alpha; t)$.

3.2 Approximation to $\phi_0(\alpha; t)$

Suppose the risk-neutral dynamics of the log spot price, s_t, is given by the following SDE:

$$ds_t = \mu(s_t, Y_t)dt + \sigma(s_t, Y_t)dW_t \ , \tag{23}$$

where W_t is the Brownian motion under the risk-neutral measure, and the drift and diffusion functions possibly depend on a state vector, Y_t. By the absence of arbitrage, $\mu(s_t, Y_t) = r(Y_t) - \frac{1}{2}\sigma^2(s_t, Y_t)$ holds, where the instantaneous risk-free rate, $r(Y_t)$, may also depend on (some elements of) Y_t.

Now, define

$$Z_t = e^{-\int_0^t r_u du} e^{\alpha s_t} \ , \tag{24}$$

and $\phi_0(\alpha; t)$ is the conditional first moment of Z_t, i.e., $\phi_0(\alpha; t) = E_0[Z_t]$. By applying the Ito formula to Z_t,

$$dZ_t = \mu_z(s_t, Y_t, Z_t)dt + \sigma_z(s_t, Y_t, Z_t)dW_t \ , \tag{25}$$

where

$$\mu_z(s_t, Y_t, Z_t) = \left\{-r(Y_t) + \alpha\mu(s_t, Y_t) + \frac{1}{2}\alpha^2\sigma^2(s_t, Y_t)\right\} Z_t \ , \tag{26}$$

$$\sigma_z(s_t, Y_t, Z_t) = \alpha\sigma(s_t, Y_t)Z_t \ . \tag{27}$$

Based on the SDE for Z_t together with those for s_t and Y_t, the elements of A and b in Eq.(7) are determined.

4 Accuracy of the Approximation

4.1 Setup

The MC method is employed to obtain benchmark prices of European put options, where the number of repetitions is set to one million with antithetic variates. In generating sample paths, the SDEs under the risk-neutral measure are discretized by the Euler method with a step size of $1/1,000$. The accuracy is then measured by the relative pricing error, $p^{ap}/p^{mc} - 1$, where p^{ap} and p^{mc} denote the approximate and MC prices, respectively.

The following SV model is considered, which has the CEV specification in both the spot price and volatility processes, and thus is labeled SV-CEV:

$$\frac{dS_t}{S_t} = r_t dt + \sqrt{V_t}\left(\frac{S_t}{S_0}\right)^{\gamma_1 - 1} dW_{1,t} \ , \tag{28}$$

$$dV_t = \kappa_2(\theta_2 - V_t)dt + \sigma_2 V_t^{\gamma_2} dW_{2,t} \ , \tag{29}$$

where $W_{i,t}$ $(i = 1,2)$ are the Brownian motions under the risk-neutral measure with $E_t[dW_{1,t}\, dW_{2,t}] = \rho_{12}dt$. The SV-CEV model nests both the Heston (1993) and Hull and White (HW) (1987) models: the former is obtained by $\gamma_1 = 1$ and $\gamma_2 = 0.5$, whereas the latter is by $\gamma_1 = \gamma_2 = 1$ and $\theta_2 = 0$.

Note that the diffusion term of the instantaneous rate of return on S_t depends on the power of S_t/S_0, i.e., the normalized price. This specification is convenient in the following reason. For small t, which corresponds to an option maturity, S_t/S_0 is likely to be around unity, and so is $(S_t/S_0)^{\gamma_1-1}$ with $\gamma_1 \neq 1$. Then, V_t can be interpreted as (nearly) the instantaneous variance of the return, as is the case of $\gamma_1 = 1$. This helps select appropriate parameter values of the SDE for V_t together with its initial value, V_0. In applying the approximation formula, the SDE for the log spot price, $s_t = \ln S_t$, is actually required. By the Ito formula,

$$ds_t = \left(r_t - \frac{1}{2}V_t\, e^{2(\gamma_1-1)(s_t-s_0)}\right)dt + \sqrt{V_t}\, e^{(\gamma_1-1)(s_t-s_0)} dW_{1,t} \ . \tag{30}$$

As for the instantaneous risk-free rate, both cases of deterministic and stochastic rates are considered. Since the accuracy results are similar between the two cases, only the former is reported while the latter is available upon request.

The input values for option prices are as follows: $S_0 = 1$; $K = 0.9,\ 1.0,\ 1.1$; $t = 0.5$ (half a year); $V_0 = 0.15^2$; and $r_t = 0$ (constant). These values are basically determined following HW (1987, Table II).

Before reporting the numerical results, it is noted that the accuracy of the proposed method improves by the logarithmic transformation of the volatility process, i.e., $v_t = \ln V_t$. This can be explained as follows. Looking at Table 1, where (s_t, V_t, Z_t) is substituted for $(x_{1,t}, x_{2,t}, x_{3,t})$, $E_0[Z_t - Z_0]$ is in the third row of the moment vector, $\Psi_{0,t}$. Then, by Eq.(7), the third row of the matrix A is the most influential for this calculation. The third row of A consists of the partial derivatives of the drift function of Z_t. From Eqs.(26) and (30), the drift function of Z_t is given by

$$f_3(x_{1,t}, x_{2,t}, x_{3,t}) = \mu_z(s_t, V_t, Z_t) = (\alpha - 1)\left(r_t + \frac{1}{2}\alpha V_t e^{2(\gamma_1-1)(s_t-s_0)}\right) Z_t \ . \tag{31}$$

Since μ_z is linear in V_t, the second partial derivative with respect to V_t is zero, i.e., $f_3^{(0,2,0)} = 0$. This is the $(3,5)$-element of A, which multiplies the fifth element of $\Psi_{0,u}$, i.e., $E_0[(V_u - V_0)^2]$, on the right-hand side of Eq.(7). Then, $f_3^{(0,2,0)} = 0$ indicates that the conditional second moment of (an increment of) V_t is not effectively used for the computation of $E_0[Z_t - Z_0]$.

After the logarithmic transformation of V_t, on the other hand, this problem is avoided. After the transformation, μ_z becomes

$$f_3(x_{1,t}, x_{2,t}, x_{3,t}) = \mu_z(s_t, v_t, Z_t) = (\alpha - 1)\left(r_t + \frac{1}{2}\alpha e^{2(\gamma_1 - 1)(s_t - s_0) + v_t}\right) Z_t \ , \tag{32}$$

which is nonlinear in v_t. Thus, the coefficient of $E_0[(v_u - v_0)^2]$, $f_3^{(0,2,0)}$, is nonzero. The numerical results reported below are those after this transformation. For ease of reference to Table 1, the functional forms of f_i and g_{ij} $(i, j = 1, 2, 3)$ for the SV-CEV model are summarized in Table 2.

It is important to note that there are many models proposed in the literature in which the dynamics of the logarithmic volatility are directly specified: see, e.g., Andersen et al. (2002), Chernov et al. (2003), and Scott (1987). An advantage of the logarithmic volatility models is to ensure non-negativity of the volatility process without placing parameter constraints. A difficulty, on the other hand, is to lose analytical tractability of option prices. The proposed method may then have potential to overcome this difficulty.

4.2 Accuracy results for the SV-CEV model

As a base case, the parameter values for the SV-CEV model are set to

$$(\gamma_1,\ \kappa_2,\ \theta_2,\ \sigma_2,\ \gamma_2,\ \rho_{12}) = (0.75,\ 1.0,\ 0.15^2,\ 1.0,\ 1.0,\ -0.5)\ .$$

In particular, $\sigma_2 = \gamma_2 = 1$ is determined with reference also to HW (1987, Table II). Some of these values are then changed to further examine the cases in which the approximation is (in)accurate. It is noted that $\gamma_1 < 1$ and $-1 < \rho_{12} < 0$ contribute to capturing the so-called leverage effect: the volatility of the spot price tends to be high when the spot price falls than rises. Various combinations of $(\gamma_1,\ \rho_{12})$ are also tried, the results of which are similar to those reported below.

Table 3 presents put option prices (multiplied by 100) and relative pricing errors (expressed in %) in various cases. Panel A of Table 3 presents the results of the base case. First, the second-order approximation, AP2, overvalues the out-of-the-money (OTM, $K/S_0 = 0.9$) option by 8.4% and undervalues the at-the-money (ATM, $K/S_0 = 1.0$) option by 2.0%. The approximation error for the OTM option appears large, however, the absolute difference between the AP2 and MC prices is 0.086×10^{-2}, or equivalently 0.086% of the spot price, $S_0 = 1$. It is not surprising that the approximation error for the in-the-money (ITM, $K/S_0 = 1.1$) option is very small, as the option value trivially approaches $e^{-rt}K - S_0$, irrespective of the computation methods involved. Second, the third-order approximation, AP3, achieves high accuracy. Specifically, the approximation error for the OTM option is reduced to around 1% and that for the ATM option is negligibly small.

Panel B of Table 3 presents the results when the maturity length is extended to one year, $t = 1$. In theory, the accuracy of the approximation decreases with increasing maturity. In reality, however, the results are mixed between the ATM and OTM options. While the former is not much affected, the accuracy for the latter becomes worse than that in the base case. Still, AP3 seems effective.

Next, the initial value of the volatility process is increased to $V_0 = 0.3^2$, assuming the case where the current volatility level is unusually high relative to its long-term mean, $\theta_2 = 0.15^2$. Panel C of Table 3 shows that while AP2 becomes worse, especially for the ATM and ITM options, AP3 continues to work well. The decrease in the accuracy due to the larger V_0 is mitigated when both V_0 and θ_2 are increased to 0.3^2, as shown in Panel D of Table 3. The results of Panels C and D indicate that the proposed method tends to be more accurate when the current state values are around the long-term means than when they are distant.

Increasing the volatility of V_t by doubling the value of σ_2 results in the deterioration of the accuracy, as shown in Panel E of Table 3. In fact, this is the most difficult case for the proposed method, where even AP3 undervalues the OTM option by nearly 30%. The increase in σ_2 makes the variation in V_t higher, which in turn makes the leptokurtosis of the return distribution higher. In this case, the difficulty of the approximation arises. A similar problem seems to appear in the HW approximation: HW (1987, p. 294). Note that higher variation in V_t can also be generated by a larger V_0 through the CEV specification. In this case, however, the deterioration of the accuracy can be avoided by the third-order approximation, as shown in Panels C and D of Table 3.

On the other hand, increasing the speed of mean reversion of V_t by doubling the value of κ_2 improves the accuracy, as shown in Panel F of Table 3. In particular, even AP2 performs reasonably well. This improvement occurs because faster mean reversion virtually reduces the variation in V_t and thus the leptokurtosis of the return distribution.

Next is the case where the increase in κ_2 does not necessarily reduce the variation in V_t. Given $\kappa_2 = 2$, the value of σ_2 is raised such that the unconditional variance of V_t, $(\theta_2\sigma_2)^2/(2\kappa_2 - \sigma_2^2)$, remains the same as that in the base case. This is achieved by setting $\sigma_2 = \sqrt{2}$ with θ_2 unchanged. Panel G of Table 3 shows that due to the increase in σ_2, the error magnitude generally increases from that in Panel F of Table 3. Compared to the base case, the error magnitude for the OTM option is larger, indicating that the effect of the large σ_2 (higher variation) dominates that of the large κ_2 (faster mean reversion). Conversely, the error magnitude for the ATM option is smaller and that for the ITM option is similar, indicating that the effect of faster mean reversion prevails.

Although these cases are artificial and limited, they seem to reveal when the proposed method is (in)accurate together with the magnitude of approximation errors. Overall, the good performance of the approximation can be expected when the volatility process is not too volatile. In reality, the volatility process does not seem too volatile when it is estimated from the time-series of the spot price. When estimated from the cross-section of option prices, it is reported to be too volatile: see, e.g., Bakshi et al. (1997), and Bates (1996, 2000). This is because high variation in the volatility process is required for generating high leptokurtosis of the return distribution, which in turn is required for explaining the implied volatility surface, especially at short maturity. This too volatile behavior under the risk-neutral measure may suggest that the diffusion component alone has difficulty in generating a sufficient degree of leptokurtosis. This difficulty is mitigated by introducing jumps. Then, the proposed method may actually be more effective when it is combined with the jump component. It is important to note that even after the inclusion of the jump component, the role of the diffusion component is reported to be decisive for capturing not only time-varying second moment and negative skewness of the return distribution, but also the term structure of implied volatilities: see Das and Sundaram (1999), Carr and Wu (2003b), and Huang and Wu (2004). Besides, diffusion models can be used for describing stochastic variations in the arrival rate of jumps: see Carr et al. (2003), and Huang and Wu (2004). The proposed method may contribute to searching appropriate models of the diffusion component, which is left for future study.

5 Identification of Effective Higher-Order Moments

Based on the previous results that the third-order approximation improves the accuracy, this section examines which of the conditional third moments added to $\Psi_{0,t}^{n=2}$ have a greater contribution to this improvement. It is useful to discriminate such effective moments from those of little significance. The latter moments can be removed from $\Psi_{0,t}^{n=3}$ to reduce the computational burden without much reducing the accuracy. The findings from this analysis are particularly useful for high-dimensional models, where the dimension of a moment vector increases rapidly with the number of factors: in general, the length of $\Psi_{s,t}$ consisting of up to n-th conditional moments of a d-dimensional process is $\binom{n+d}{n}$ $-1 = (n+d)!/(n!d!) - 1$.

The following three moment vectors are considered:

$$\begin{aligned}
\text{AP3–0**}: &\quad (\Psi_{0,t}^{n=2\,\prime} \quad \psi_{300,0,t} \quad \psi_{210,0,t} \quad \psi_{201,0,t} \quad \psi_{120,0,t} \quad \psi_{111,0,t} \quad \psi_{102,0,t})'\,, \\
\text{AP3–*0*}: &\quad (\Psi_{0,t}^{n=2\,\prime} \quad \psi_{030,0,t} \quad \psi_{210,0,t} \quad \psi_{120,0,t} \quad \psi_{111,0,t} \quad \psi_{021,0,t} \quad \psi_{012,0,t})'\,, \\
\text{AP3–**0}: &\quad (\Psi_{0,t}^{n=2\,\prime} \quad \psi_{003,0,t} \quad \psi_{201,0,t} \quad \psi_{111,0,t} \quad \psi_{102,0,t} \quad \psi_{021,0,t} \quad \psi_{012,0,t})'\,.
\end{aligned}$$

The moment vector labeled AP3–0** contains the conditional third moments except those not related to s_t, ψ_{0**}, leading to the abbreviation as –0**. The moment vectors labeled AP3–*0* and AP3–**0 are constructed analogously. The dimension of these vectors is fifteen.

Table 4 presents put option prices (multiplied by 100) and relative pricing errors (expressed in %) for the simplified third-order approximations in the base case: for ease of comparison, the previous results in Panel A of Table 3 are also presented. First, in terms of similarity to the performance of AP3, AP3–*0* ranks first. In fact, the error pattern little changes. The result indicates that the conditional third moments related to v_t have a fundamental role for the improvement of the accuracy. This makes sense, as the accuracy is measured in terms of option prices, which are highly sensitive to the volatility process. Also, the error magnitude for AP3–**0 remains almost the same as that for AP3. On the other hand, while AP3–0** actually improves AP2, the error pattern is more similar to AP2 than AP3. This result indicates that the conditional third moments related to s_t do not contribute much to the improvement of the accuracy.

Further investigation reveals that among the conditional third moments related to v_t (and Z_t), $\psi_{021,0,t} = E_0[(v_t - v_0)^2(Z_t - Z_0)]$ is the most influential. The following moment vector of dimension ten, labeled AP2+021, is considered: $(\Psi_{s,t}^{n=2\,\prime}\ \psi_{021,0,t})'$. The relative pricing errors for AP2+021 are also presented in Table 4, showing that the accuracy is not much decreased from that of AP3 with the computational burden similar to that of AP2. This result is robust to various cases considered in the previous section.

6 Concluding Remarks

This study proposed an approximation of European option prices that can be efficiently computed under flexibly specified diffusion models of the spot price. An analytical approximation to the characteristic function of the log spot price is first obtained as the solution to ordinary differential equations, and then Fourier inversion is applied to obtain option prices. Using a stochastic volatility model in which both the spot price and volatility processes have the constant elasticity of volatility specification, Monte Carlo simulations revealed that the third-order approximation generally achieves high accuracy, except when the volatility process exhibits high variation, and that the second-order approximation is effective in some cases, especially when mean reversion of the volatility process is relatively fast.

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$[A]_{i,j}$	1	2	3	4	5
1	$f_1^{(1,0,0)}$	$f_1^{(0,1,0)}$	$f_1^{(0,0,1)}$	$f_1^{(2,0,0)}/2$	$f_1^{(0,2,0)}/2$
2	$f_2^{(1,0,0)}$	$f_2^{(0,1,0)}$	$f_2^{(0,0,1)}$	$f_2^{(2,0,0)}/2$	$f_2^{(0,2,0)}/2$
3	$f_3^{(1,0,0)}$	$f_3^{(0,1,0)}$	$f_3^{(0,0,1)}$	$f_3^{(2,0,0)}/2$	$f_3^{(0,2,0)}/2$
4	$2f_1 + g_{11}^{(1,0,0)}$	$g_{11}^{(0,1,0)}$	$g_{11}^{(0,0,1)}$	$2f_1^{(1,0,0)} + g_{11}^{(2,0,0)}/2$	$g_{11}^{(0,2,0)}/2$
5	$g_{22}^{(1,0,0)}$	$2f_2 + g_{22}^{(0,1,0)}$	$g_{22}^{(0,0,1)}$	$g_{22}^{(2,0,0)}/2$	$2f_2^{(0,1,0)} + g_{22}^{(0,2,0)}/2$
6	$g_{33}^{(1,0,0)}$	$g_{33}^{(0,1,0)}$	$2f_3 + g_{33}^{(0,0,1)}$	$g_{33}^{(2,0,0)}/2$	$g_{33}^{(0,2,0)}/2$
7	$f_2 + g_{12}^{(1,0,0)}$	$f_1 + g_{12}^{(0,1,0)}$	$g_{12}^{(0,0,1)}$	$f_2^{(1,0,0)} + g_{12}^{(2,0,0)}/2$	$f_1^{(0,1,0)} + g_{12}^{(0,2,0)}/2$
8	$f_3 + g_{13}^{(1,0,0)}$	$g_{13}^{(0,1,0)}$	$f_1 + g_{13}^{(0,0,1)}$	$f_3^{(1,0,0)} + g_{13}^{(2,0,0)}/2$	$g_{13}^{(0,2,0)}/2$
9	$g_{23}^{(1,0,0)}$	$f_3 + g_{23}^{(0,1,0)}$	$f_2 + g_{23}^{(0,0,1)}$	$g_{23}^{(2,0,0)}/2$	$f_3^{(0,1,0)} + g_{23}^{(0,2,0)}/2$
10	$3g_{11}$	0	0	$3f_1 + 3g_{11}^{(1,0,0)}$	0
11	0	$3g_{22}$	0	0	$3f_2 + 3g_{22}^{(0,1,0)}$
12	0	0	$3g_{33}$	0	0
13	$2g_{12}$	g_{11}	0	$f_2 + 2g_{12}^{(1,0,0)}$	$g_{11}^{(0,1,0)}$
14	$2g_{13}$	0	g_{11}	$f_3 + 2g_{13}^{(1,0,0)}$	0
15	g_{22}	$2g_{12}$	0	$g_{22}^{(1,0,0)}$	$f_1 + 2g_{12}^{(0,1,0)}$
16	g_{23}	g_{13}	g_{12}	$g_{23}^{(1,0,0)}$	$g_{13}^{(0,1,0)}$
17	g_{33}	0	$2g_{13}$	$g_{33}^{(1,0,0)}$	0
18	0	$2g_{23}$	g_{22}	0	$f_3 + 2g_{23}^{(0,1,0)}$
19	0	g_{33}	$2g_{23}$	0	$g_{33}^{(0,1,0)}$
$[b]_j$	f_1	f_2	f_3	g_{11}	g_{22}

Table 1:

6	7	8	9
$f_1^{(0,0,2)}/2$	$f_1^{(1,1,0)}$	$f_1^{(1,0,1)}$	$f_1^{(0,1,1)}$
$f_2^{(0,0,2)}/2$	$f_2^{(1,1,0)}$	$f_2^{(1,0,1)}$	$f_2^{(0,1,1)}$
$f_3^{(0,0,2)}/2$	$f_3^{(1,1,0)}$	$f_3^{(1,0,1)}$	$f_3^{(0,1,1)}$
$g_{11}^{(0,0,2)}/2$	$2f_1^{(0,1,0)} + g_{11}^{(1,1,0)}$	$2f_1^{(0,0,1)} + g_{11}^{(1,0,1)}$	$g_{11}^{(0,1,1)}$
$g_{22}^{(0,0,2)}/2$	$2f_2^{(1,0,0)} + g_{22}^{(1,1,0)}$	$g_{22}^{(1,0,1)}$	$2f_2^{(0,0,1)} + g_{22}^{(0,1,1)}$
$2f_3^{(0,0,1)} + g_{33}^{(0,0,2)}/2$	$g_{33}^{(1,1,0)}$	$2f_3^{(1,0,0)} + g_{33}^{(1,0,1)}$	$2f_3^{(0,1,0)} + g_{33}^{(0,1,1)}$
$g_{12}^{(0,0,2)}/2$	$f_1^{(1,0,0)} + f_2^{(0,1,0)} + g_{12}^{(1,1,0)}$	$f_2^{(0,0,1)} + g_{12}^{(1,0,1)}$	$f_1^{(0,0,1)} + g_{12}^{(0,1,1)}$
$f_1^{(0,0,1)} + g_{13}^{(0,0,2)}/2$	$f_3^{(0,1,0)} + g_{13}^{(1,1,0)}$	$f_1^{(1,0,0)} + f_3^{(0,0,1)} + g_{13}^{(1,0,1)}$	$f_1^{(0,1,0)} + g_{13}^{(0,1,1)}$
$f_2^{(0,0,1)} + g_{23}^{(0,0,2)}/2$	$f_3^{(1,0,0)} + g_{23}^{(1,1,0)}$	$f_2^{(1,0,0)} + g_{23}^{(1,0,1)}$	$f_2^{(0,1,0)} + f_3^{(0,0,1)} + g_{23}^{(0,1,1)}$
0	$3g_{11}^{(0,1,0)}$	$3g_{11}^{(0,0,1)}$	0
0	$3g_{22}^{(1,0,0)}$	0	$3g_{22}^{(0,0,1)}$
$3f_3 + 3g_{33}^{(0,0,1)}$	0	$3g_{33}^{(1,0,0)}$	$3g_{33}^{(0,1,0)}$
0	$2f_1 + g_{11}^{(1,0,0)} + 2g_{12}^{(0,1,0)}$	$2g_{12}^{(0,0,1)}$	$g_{11}^{(0,0,1)}$
$g_{11}^{(0,0,1)}$	$2g_{13}^{(0,1,0)}$	$2f_1 + g_{11}^{(1,0,0)} + 2g_{13}^{(0,0,1)}$	$g_{11}^{(0,1,0)}$
0	$2f_2 + g_{22}^{(0,1,0)} + 2g_{12}^{(1,0,0)}$	$g_{22}^{(0,0,1)}$	$2g_{12}^{(0,0,1)}$
$g_{12}^{(0,0,1)}$	$f_3 + g_{13}^{(1,0,0)} + g_{23}^{(0,1,0)}$	$f_2 + g_{12}^{(1,0,0)} + g_{23}^{(0,0,1)}$	$f_1 + g_{12}^{(0,1,0)} + g_{13}^{(0,0,1)}$
$f_1 + 2g_{13}^{(0,0,1)}$	$g_{33}^{(0,1,0)}$	$2f_3 + g_{33}^{(0,0,1)} + 2g_{13}^{(1,0,0)}$	$2g_{13}^{(0,1,0)}$
$g_{22}^{(0,0,1)}$	$2g_{23}^{(1,0,0)}$	$g_{22}^{(1,0,0)}$	$2f_2 + g_{22}^{(0,1,0)} + 2g_{23}^{(0,0,1)}$
$f_2 + 2g_{23}^{(0,0,1)}$	$g_{33}^{(1,0,0)}$	$2g_{23}^{(1,0,0)}$	$2f_3 + g_{33}^{(0,0,1)} + 2g_{23}^{(0,1,0)}$
g_{33}	g_{12}	g_{13}	g_{23}

Table 1 (continued):

10	11	12	13
$f_1^{(3,0,0)}/6$	$f_1^{(0,3,0)}/6$	$f_1^{(0,0,3)}/6$	$f_1^{(2,1,0)}/2$
$f_2^{(3,0,0)}/6$	$f_2^{(0,3,0)}/6$	$f_2^{(0,0,3)}/6$	$f_2^{(2,1,0)}/2$
$f_3^{(3,0,0)}/6$	$f_3^{(0,3,0)}/6$	$f_3^{(0,0,3)}/6$	$f_3^{(2,1,0)}/2$
$f_1^{(2,0,0)} + g_{11}^{(3,0,0)}/6$	$g_{11}^{(0,3,0)}/6$	$g_{11}^{(0,0,3)}/6$	$2f_1^{(1,1,0)} + g_{11}^{(2,1,0)}/2$
$g_{22}^{(3,0,0)}/6$	$f_2^{(0,2,0)} + g_{22}^{(0,3,0)}/6$	$g_{22}^{(0,0,3)}/6$	$f_2^{(2,0,0)} + g_{22}^{(2,1,0)}/2$
$g_{33}^{(3,0,0)}/6$	$g_{33}^{(0,3,0)}/6$	$f_3^{(0,0,2)} + g_{33}^{(0,0,3)}/6$	$g_{33}^{(2,1,0)}/2$
$f_2^{(2,0,0)}/2 + g_{12}^{(3,0,0)}/6$	$f_1^{(0,2,0)}/2 + g_{12}^{(0,3,0)}/6$	$g_{12}^{(0,0,3)}/6$	$f_1^{(2,0,0)}/2 + f_2^{(1,1,0)} + g_{12}^{(2,1,0)}/2$
$f_3^{(2,0,0)}/2 + g_{13}^{(3,0,0)}/6$	$g_{13}^{(0,3,0)}/6$	$f_1^{(0,0,2)}/2 + g_{13}^{(0,0,3)}/6$	$f_3^{(1,1,0)} + g_{13}^{(2,1,0)}/2$
$g_{23}^{(3,0,0)}/6$	$f_3^{(0,2,0)}/2 + g_{23}^{(0,3,0)}/6$	$f_2^{(0,0,2)}/2 + g_{23}^{(0,0,3)}/6$	$f_3^{(2,0,0)}/2 + g_{23}^{(2,1,0)}/2$
$3f_1^{(1,0,0)} + 1.5g_{11}^{(2,0,0)}$	0	0	$3f_1^{(0,1,0)} + 3g_{11}^{(1,1,0)}$
0	$3f_2^{(0,1,0)} + 1.5g_{22}^{(0,2,0)}$	0	$1.5g_{22}^{(2,0,0)}$
0	0	$3f_3^{(0,0,1)} + 1.5g_{33}^{(0,0,2)}$	0
$f_2^{(1,0,0)} + g_{12}^{(2,0,0)}$	$g_{11}^{(0,2,0)}/2$	0	$2f_1^{(1,0,0)} + f_2^{(0,1,0)} + g_{11}^{(2,0,0)}/2 + 2g_{12}^{(1,1,0)}$
$f_3^{(1,0,0)} + g_{13}^{(2,0,0)}$	0	$g_{11}^{(0,0,2)}/2$	$f_3^{(0,1,0)} + 2g_{13}^{(1,1,0)}$
$g_{22}^{(2,0,0)}/2$	$f_1^{(0,1,0)} + g_{12}^{(0,2,0)}$	0	$2f_2^{(1,0,0)} + g_{22}^{(1,1,0)} + g_{12}^{(2,0,0)}$
$g_{23}^{(2,0,0)}/2$	$g_{13}^{(0,2,0)}/2$	$g_{12}^{(0,0,2)}/2$	$f_3^{(1,0,0)} + g_{13}^{(2,0,0)}/2 + g_{23}^{(1,1,0)}$
$g_{33}^{(2,0,0)}/2$	0	$f_1^{(0,0,1)} + g_{13}^{(0,0,2)}$	$g_{33}^{(1,1,0)}$
0	$f_3^{(0,1,0)} + g_{23}^{(0,2,0)}$	$g_{22}^{(0,0,2)}/2$	$g_{23}^{(2,0,0)}$
0	$g_{33}^{(0,2,0)}/2$	$f_2^{(0,0,1)} + g_{23}^{(0,0,2)}$	$g_{33}^{(2,0,0)}/2$
0	0	0	0

Table 1 (continued):

14	15	16
$f_1^{(2,0,1)}/2$	$f_1^{(1,2,0)}/2$	$f_1^{(1,1,1)}$
$f_2^{(2,0,1)}/2$	$f_2^{(1,2,0)}/2$	$f_2^{(1,1,1)}$
$f_3^{(2,0,1)}/2$	$f_3^{(1,2,0)}/2$	$f_3^{(1,1,1)}$
$2f_1^{(1,0,1)} + g_{11}^{(2,0,1)}/2$	$f_1^{(0,2,0)} + g_{11}^{(1,2,0)}/2$	$2f_1^{(0,1,1)} + g_{11}^{(1,1,1)}$
$g_{22}^{(2,0,1)}/2$	$2f_2^{(1,1,0)} + g_{22}^{(1,2,0)}/2$	$2f_2^{(1,0,1)} + g_{22}^{(1,1,1)}$
$f_3^{(2,0,0)} + g_{33}^{(2,0,1)}/2$	$g_{33}^{(1,2,0)}/2$	$2f_3^{(1,1,0)} + g_{33}^{(1,1,1)}$
$f_2^{(1,0,1)} + g_{12}^{(2,0,1)}/2$	$f_1^{(1,1,0)} + f_2^{(0,2,0)}/2 + g_{12}^{(1,2,0)}/2$	$f_1^{(1,0,1)} + f_2^{(0,1,1)} + g_{12}^{(1,1,1)}$
$f_1^{(2,0,0)}/2 + f_3^{(1,0,1)} + g_{13}^{(2,0,1)}/2$	$f_3^{(0,2,0)}/2 + g_{13}^{(1,2,0)}/2$	$f_1^{(1,1,0)} + f_3^{(0,1,1)} + g_{13}^{(1,1,1)}$
$f_2^{(2,0,0)}/2 + g_{23}^{(2,0,1)}/2$	$f_3^{(1,1,0)} + g_{23}^{(1,2,0)}/2$	$f_2^{(1,1,0)} + f_3^{(1,0,1)} + g_{23}^{(1,1,1)}$
$3f_1^{(0,0,1)} + 3g_{11}^{(1,0,1)}$	$1.5g_{11}^{(0,2,0)}$	$3g_{11}^{(0,1,1)}$
0	$3f_2^{(1,0,0)} + 3g_{22}^{(1,1,0)}$	$3g_{22}^{(1,0,1)}$
$1.5g_{33}^{(2,0,0)}$	0	$3g_{33}^{(1,1,0)}$
$f_2^{(0,0,1)} + 2g_{12}^{(1,0,1)}$	$2f_1^{(0,1,0)} + g_{11}^{(1,1,0)} + g_{12}^{(0,2,0)}$	$2f_1^{(0,0,1)} + g_{11}^{(1,0,1)} + 2g_{12}^{(0,1,1)}$
$2f_1^{(1,0,0)} + f_3^{(0,0,1)} + g_{11}^{(2,0,0)}/2 + 2g_{13}^{(1,0,1)}$	$g_{13}^{(0,2,0)}$	$2f_1^{(0,1,0)} + g_{11}^{(1,1,0)} + 2g_{13}^{(0,1,1)}$
$g_{22}^{(1,0,1)}$	$2f_2^{(0,1,0)} + f_1^{(1,0,0)} + g_{22}^{(0,2,0)}/2 + 2g_{12}^{(1,1,0)}$	$2f_2^{(0,0,1)} + g_{22}^{(0,1,1)} + 2g_{12}^{(1,0,1)}$
$f_2^{(1,0,0)} + g_{12}^{(2,0,0)}/2 + g_{23}^{(1,0,1)}$	$f_3^{(0,1,0)} + g_{13}^{(1,1,0)} + g_{23}^{(0,2,0)}/2$	$f_1^{(1,0,0)} + f_2^{(0,1,0)} + f_3^{(0,0,1)} + g_{12}^{(1,1,0)} + g_{13}^{(1,0,1)} + g_{23}^{(0,1,1)}$
$2f_3^{(1,0,0)} + g_{33}^{(1,0,1)} + g_{13}^{(2,0,0)}$	$g_{33}^{(0,2,0)}/2$	$2f_3^{(0,1,0)} + g_{33}^{(0,1,1)} + 2g_{13}^{(1,1,0)}$
$g_{22}^{(2,0,0)}/2$	$f_3^{(1,0,0)} + 2g_{23}^{(1,1,0)}$	$2f_2^{(1,0,0)} + g_{22}^{(1,1,0)} + 2g_{23}^{(1,0,1)}$
$g_{23}^{(2,0,0)}$	$g_{33}^{(1,1,0)}$	$2f_3^{(1,0,0)} + g_{33}^{(1,0,1)} + 2g_{23}^{(1,1,0)}$
0	0	0

Table 1 (continued):

17	18	19
$f_1^{(1,0,2)}/2$	$f_1^{(0,2,1)}/2$	$f_1^{(0,1,2)}/2$
$f_2^{(1,0,2)}/2$	$f_2^{(0,2,1)}/2$	$f_2^{(0,1,2)}/2$
$f_3^{(1,0,2)}/2$	$f_3^{(0,2,1)}/2$	$f_3^{(0,1,2)}/2$
$f_1^{(0,0,2)} + g_{11}^{(1,0,2)}/2$	$g_{11}^{(0,2,1)}/2$	$g_{11}^{(0,1,2)}/2$
$g_{22}^{(1,0,2)}/2$	$2f_2^{(0,1,1)} + g_{22}^{(0,2,1)}/2$	$f_2^{(0,0,2)} + g_{22}^{(0,1,2)}/2$
$2f_3^{(1,0,1)} + g_{33}^{(1,0,2)}/2$	$f_3^{(0,2,0)} + g_{33}^{(0,2,1)}/2$	$2f_3^{(0,1,1)} + g_{33}^{(0,1,2)}/2$
$f_2^{(0,0,2)}/2 + g_{12}^{(1,0,2)}/2$	$f_1^{(0,1,1)} + g_{12}^{(0,2,1)}/2$	$f_1^{(0,0,2)}/2 + g_{12}^{(0,1,2)}/2$
$f_1^{(1,0,1)} + f_3^{(0,0,2)}/2 + g_{13}^{(1,0,2)}/2$	$f_1^{(0,2,0)}/2 + g_{13}^{(0,2,1)}/2$	$f_1^{(0,1,1)} + g_{13}^{(0,1,2)}/2$
$f_2^{(1,0,1)} + g_{23}^{(1,0,2)}/2$	$f_2^{(0,2,0)}/2 + f_3^{(0,1,1)} + g_{23}^{(0,2,1)}/2$	$f_2^{(0,1,1)} + f_3^{(0,0,2)}/2 + g_{23}^{(0,1,2)}/2$
$1.5g_{11}^{(0,0,2)}$	0	0
0	$3f_2^{(0,0,1)} + 3g_{22}^{(0,1,1)}$	$1.5g_{22}^{(0,0,2)}$
$3f_3^{(1,0,0)} + 3g_{33}^{(1,0,1)}$	$1.5g_{33}^{(0,2,0)}$	$3f_3^{(0,1,0)} + 3g_{33}^{(0,1,1)}$
$g_{12}^{(0,0,2)}$	$g_{11}^{(0,1,1)}$	$g_{11}^{(0,0,2)}/2$
$2f_1^{(0,0,1)} + g_{11}^{(1,0,1)} + g_{13}^{(0,0,2)}$	$g_{11}^{(0,2,0)}/2$	$g_{11}^{(0,1,1)}$
$g_{22}^{(0,0,2)}/2$	$f_1^{(0,0,1)} + 2g_{12}^{(0,1,1)}$	$g_{12}^{(0,0,2)}$
$f_2^{(0,0,1)} + g_{12}^{(1,0,1)} + g_{23}^{(0,0,2)}/2$	$f_1^{(0,1,0)} + g_{12}^{(0,2,0)}/2 + g_{13}^{(0,1,1)}$	$f_1^{(0,0,1)} + g_{12}^{(0,1,1)} + g_{13}^{(0,0,2)}/2$
$2f_3^{(0,0,1)} + f_1^{(1,0,0)} + g_{33}^{(0,0,2)}/2 + 2g_{13}^{(1,0,1)}$	$g_{13}^{(0,2,0)}$	$f_1^{(0,1,0)} + 2g_{13}^{(0,1,1)}$
$g_{22}^{(1,0,1)}$	$2f_2^{(0,1,0)} + f_3^{(0,0,1)} + g_{22}^{(0,2,0)}/2 + 2g_{23}^{(0,1,1)}$	$2f_2^{(0,0,1)} + g_{22}^{(0,1,1)} + g_{23}^{(0,0,2)}$
$f_2^{(1,0,0)} + 2g_{23}^{(1,0,1)}$	$2f_3^{(0,1,0)} + g_{33}^{(0,1,1)} + g_{23}^{(0,2,0)}$	$2f_3^{(0,0,1)} + f_2^{(0,1,0)} + g_{33}^{(0,0,2)}/2 + 2g_{23}^{(0,1,1)}$
0	0	0

Table 1 (continued):

Table 1 (continued): The elements of $A(X_s)$ and $b(X_s)$ for the third-order approximation

The moment vector $\Psi_{s,t}^{n=3}$ consists of up to the conditional third moments of a three-dimensional process, $X_t = (x_{1,t}\, x_{2,t}\, x_{3,t})'$, in the following order: define $\psi_{pqr,s,t} = E_s[(x_{1,t} - x_{1,s})^p (x_{2,t} - x_{2,s})^q (x_{3,t} - x_{3,s})^r]$, and

$$
\begin{aligned}
\Psi_{s,t}^{n=2} &= (\psi_{100,s,t} \quad \psi_{010,s,t} \quad \psi_{001,s,t} \quad \psi_{200,s,t} \quad \psi_{020,s,t} \quad \psi_{002,s,t} \quad \psi_{110,s,t} \quad \psi_{101,s,t} \quad \psi_{011,s,t})' \,, \\
\Psi_{s,t}^{n=3} &= (\Psi_{s,t}^{n=2\,\prime} \quad \psi_{300,s,t} \quad \psi_{030,s,t} \quad \psi_{003,s,t} \quad \psi_{210,s,t} \quad \psi_{201,s,t} \quad \psi_{120,s,t} \quad \psi_{111,s,t} \quad \psi_{102,s,t} \quad \psi_{021,s,t} \quad \psi_{012,s,t})' \,.
\end{aligned}
$$

$A(X_s)$ and $b(X_s)$ for $\Psi_{s,t}^{n=2}$ are the 9×9 upper left submatrix of $A(X_s)$ and the 9×1 subvector of $b(X_s)$ for $\Psi_{s,t}^{n=3}$, respectively. The SDEs for $x_{i,t}$ $(i = 1, 2, 3)$ are $dx_{i,t} = f_i(X_t)dt + \xi_i(X_t)'dW_t$, and the notations in the table are as follows: $g_{ij} = \xi_i'\xi_j$,

$$
f_i^{(k,l,m)} = \frac{\partial f_i^{k+l+m}}{\partial x_1^k \partial x_2^l \partial x_3^m} \,, \qquad \text{and} \qquad g_{ij}^{(k,l,m)} = \frac{\partial g_{ij}^{k+l+m}}{\partial x_1^k \partial x_2^l \partial x_3^m}
$$

.

$f_1(s_t, v_t, Z_t)$	$=$	$E_t[ds_t]/dt$	$=$	$r_t - g_{11}/2$
$f_2(s_t, v_t, Z_t)$	$=$	$E_t[dv_t]/dt$	$=$	$\kappa_2(\theta_2 \exp(-v_t) - 1) - g_{22}/2$
$f_3(s_t, v_t, Z_t)$	$=$	$E_t[dZ_t]/dt$	$=$	$(-r_t + \alpha f_1 + \alpha^2 g_{11}/2) Z_t$
$g_{11}(s_t, v_t, Z_t)$	$=$	$E_t[(ds_t)^2]/dt$	$=$	$\exp(2(\gamma_1 - 1)(s_t - s_0) + v_t)$
$g_{22}(s_t, v_t, Z_t)$	$=$	$E_t[(dv_t)^2]/dt$	$=$	$\sigma_2^2 \exp(2(\gamma_2 - 1) v_t)$
$g_{33}(s_t, v_t, Z_t)$	$=$	$E_t[(dZ_t)^2]/dt$	$=$	$(\alpha Z_t)^2 g_{11}$
$g_{12}(s_t, v_t, Z_t)$	$=$	$E_t[ds_t dv_t]/dt$	$=$	$\sigma_2 \rho_{12} \exp((\gamma_1 - 1)(s_t - s_0) + (\gamma_2 - 0.5) v_t)$
$g_{13}(s_t, v_t, Z_t)$	$=$	$E_t[ds_t dZ_t]/dt$	$=$	$\alpha Z_t g_{11}$
$g_{23}(s_t, v_t, Z_t)$	$=$	$E_t[dv_t dZ_t]/dt$	$=$	$\alpha Z_t g_{12}$

Table 2: The functions based on which the elements of A and b are computed The SV-CEV model has the following SDEs for the log spot price process, s_t, and the log volatility process, v_t:

$$\begin{aligned} ds_t &= \left(r_t - \frac{1}{2} e^{2(\gamma_1 - 1)(s_t - s_0) + v_t} \right) dt + e^{(\gamma_1 - 1)(s_t - s_0) + v_t/2} dW_{1,t} \ , \\ dv_t &= \left(\kappa_2 (\theta_2 e^{-v_t} - 1) - \frac{1}{2} \sigma_2^2 e^{2(\gamma_2 - 1) v_t} \right) dt + \sigma_2 e^{(\gamma_2 - 1) v_t} dW_{2,t} \ . \end{aligned}$$

The SDE for Z_t is given by Eqs.(25)–(27). The order of the listed functions corresponds to the moment vector, $\Psi_{s,t}^{n=2}$, given by Eq.(15).

	Put Option Prices (×100)			Relative Pricing Errors (%)		
K/S_0	0.9	1.0	1.1	0.9	1.0	1.1
Panel A: Base case						
AP2	1.115	4.057	10.898	8.43	−2.02	0.31
AP3	1.017	4.136	10.843	−1.17	−0.11	−0.20
MC	1.029	4.141	10.865			
Panel B: $t = 1$						
AP2	2.524	5.703	12.264	13.28	−1.36	1.87
AP3	2.172	5.785	12.000	−2.54	0.06	−0.32
MC	2.229	5.782	12.039			
Panel C: $V_0 = 0.3^2$						
AP2	3.998	7.118	13.839	10.91	−5.51	2.09
AP3	3.558	7.468	13.511	−1.29	−0.86	−0.33
MC	3.605	7.533	13.556			
Panel D: $V_0 = 0.3^2$ and $\theta_2 = 0.3^2$						
AP2	4.394	8.060	14.398	5.23	−2.16	1.26
AP3	4.173	8.244	14.215	−0.05	0.07	−0.02
MC	4.175	8.238	14.218			
Penel E: $\sigma_2 = 2$						
AP2	1.535	3.621	10.840	40.65	−7.63	1.65
AP3	0.769	3.813	10.237	−29.55	−2.71	−4.01
MC	1.092	3.920	10.665			
Panel F: $\kappa_2 = 2$						
AP2	1.071	4.128	10.916	5.90	−0.70	0.20
AP3	1.001	4.161	10.885	−1.02	0.10	−0.09
MC	1.011	4.157	10.895			
Panel G: $\kappa_2 = 2$ and $\sigma_2 = \sqrt{2}$						
AP2	1.177	4.046	10.862	12.26	−1.30	0.39
AP3	1.000	4.098	10.767	−4.63	−0.02	−0.49
MC	1.049	4.099	10.820			

Table 3: Accuracy results in various cases

The relative pricing error is defined as $p^{ap}/p^{mc} - 1$, where p^{ap} and p^{mc} denote the option prices computed by the approximation and MC methods, respectively. The SV-CEV model is originally specified as

$$dS_t/S_t = r_t dt + \sqrt{V_t}\,(S_t/S_0)^{\gamma_1 - 1} dW_{1,t}\ , \quad dV_t = \kappa_2(\theta_2 - V_t)dt + \sigma_2 V_t^{\gamma_2} dW_{2,t}\ ,$$

with $E_t[dW_{1,t}\, dW_{2,t}] = \rho_{12} dt$. The parameter values are set to $(\gamma_1,\ \kappa_2,\ \theta_2,\ \sigma_2,\ \gamma_2,\ \rho_{12}) = (0.75,\ 1.0,\ 0.15^2,\ 1.0,\ 1.0,\ -0.5)$. The input values are as follows: $S_0 = 1$; $K = 0.9, 1.0, 1.1$; $t = 0.5$ (half a year); $V_0 = 0.15^2$; and $r_t = 0$ (constant). These parameter and input values form a base case. Some of them are then changed as indicated in each panel.

	Put Option Prices (×100)			Relative Pricing Errors (%)		
K/S_0	0.9	1.0	1.1	0.9	1.0	1.1
AP2	1.115	4.057	10.898	8.43	−2.02	0.31
AP3	1.017	4.136	10.843	−1.17	−0.11	−0.20
AP3–0**	1.091	4.066	10.878	6.10	−1.82	0.12
AP3–*0*	1.014	4.137	10.845	−1.38	−0.11	−0.19
AP3–**0	1.041	4.156	10.882	1.20	0.35	0.16
AP2+021	1.047	4.147	10.886	1.83	0.14	0.20

Table 4: Accuracy results for the simplified third-order approximations in the base case

The simplified third-order approximations correspond to the following moment vectors:

$$\begin{aligned}
\text{AP3–0**}: &\quad (\Psi_{0,t}^{n=2\,\prime} \quad \psi_{300,0,t} \quad \psi_{210,0,t} \quad \psi_{201,0,t} \quad \psi_{120,0,t} \quad \psi_{111,0,t} \quad \psi_{102,0,t})'\,, \\
\text{AP3–*0*}: &\quad (\Psi_{0,t}^{n=2\,\prime} \quad \psi_{030,0,t} \quad \psi_{210,0,t} \quad \psi_{120,0,t} \quad \psi_{111,0,t} \quad \psi_{021,0,t} \quad \psi_{012,0,t})'\,, \\
\text{AP3–**0}: &\quad (\Psi_{0,t}^{n=2\,\prime} \quad \psi_{003,0,t} \quad \psi_{201,0,t} \quad \psi_{111,0,t} \quad \psi_{102,0,t} \quad \psi_{021,0,t} \quad \psi_{012,0,t})'\,,
\end{aligned}$$

where $\Psi_{0,t}^{n=2}$ is a nine dimensional vector consisting of the conditional second moments of increments of $(s_t,\, v_t,\, Z_t)$ and $\psi_{pqr,0,t} = E_0[(s_t - s_0)^p(v_t - v_0)^q(Z_t - Z_0)^r]$. AP2+021 corresponds to the moment vector as $(\Psi_{0,t}^{n=2\,\prime}\, \psi_{021,0,t})'$.